Model interactions for Pfaffian paired states based on Chern-Simons field theory description

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On the basis of a Chern-Simons field-theoretical description we propose a simple method for the derivation of model interactions for Pfaffian paired states. We verify the method in the case of the Pfaffian (i.e., Moore-Read) state and derive a general form of the model interaction in the case of the particle-hole (PH) Pfaffian. More than one Landau level is needed to establish the correlations of the PH Pfaffian, and we present the values of relevant three-body pseudopotentials for two Landau levels.

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I. INTRODUCTION

The discovery of the fractional quantum Hall effect (FQHE) at an even-denominator, 5/2, filling factor [1] initiated an intensive search for viable paired states for the explanation of the effect. A new paradigm of Pfaffian paired states (with p-wave Cooper pairs) was introduced [2] that was the most important building block for proposals of topological superconductivity and topological quantum computing. Still, after so many years of intensive experimental and theoretical research, we are not sure which paired state can be associated with the observed even-denominator FQHE. A paired state, more precisely, the theoretical concept of the so-called particle-hole (PH) Pfaffian, seems, according to a recent experiment [3] and other experimental data and theory [4] and theoretical proposals [4–7], very relevant for the solution of the puzzle at 5/2, although another proposal [8] was also made that the measured thermal conductance in the experiment of Banerjee *et al.* [3] is a result of an insufficient equilibration of the edge modes of the anti-Pfaffian (a state related to the Pfaffian state).

The PH Pfaffian topological phase may be a result of disorder-dominated physics [5–7,9] but might also be a result of Landau level (LL) mixing in a system that may be considered uniform [9,10]. It is desirable to understand whether the PH Pfaffian (state) can be supported in a uniform system, as was done and demonstrated in numerical experiments [11–23] for the Pfaffian and its PH conjugated partner, the anti-Pfaffian [24,25], taking into account their model interactions as well [11,26]. The search proved difficult because it was found in numerical experiments [27] that the projection of what is believed to be an appropriate model wave function of the PH Pfaffian, to a fixed LL, represents a gapless state (not the gapped one necessary for FQHE). Therefore, for numerical experiments and in general, it is very desirable to find a model interaction for the uniform PH Pfaffian state, and in this work we will describe its general form.

The proposal of the PH Pfaffian state is connected to the advance [28] in the effective (field-theoretical) description of the Fermi-liquid-like state of composite fermions (underlying quasiparticles) at compressible, even-denominator fractions. Namely, instead of classical composite fermions, we may use Dirac composite fermions to describe these fractions. The PH Pfaffian constitutes a p-wave pairing in the opposite sense of the direction dictated by external magnetic field that is materialized in the chiral motion of the charge on the edge. The PH Pfaffian also makes the most natural (underlying s-wave) pairing of Dirac composite fermions. Thus, the solution of the enigma of the PH Pfaffian may help us to understand better pairing and superconductivity in Dirac systems ranging from the FQHE ones to graphene and topological insulators.

This paper is organized as follows: In Sec. II, the Chern-Simons (CS) description is reviewed, emphasizing its part in the p-wave pairing. In Sec. III, the method is introduced based on the CS description and the special pairing part to establish model interactions for the Pfaffian and PH Pfaffian. The model interaction(s) depend on the sign and strength of a pairing parameter, and in Sec. IV, we analyze and discuss an effective phase diagram and emergent phases as the value of the pairing parameter is varied. The conclusions are given in the same section.

II. CHERN-SIMONS DESCRIPTION AND PFAFFIAN PAIRED STATES

In the following we will briefly review reasons for Dirac composite fermion theory and the role of mass in this theory. As argued in [28], the physics of the PH-symmetric, half-filled, fixed LL of classical electrons may be connected to the physics of the half-filled, $n = 0$ LL of Dirac electrons (in the presence of magnetic field). The PH transformation on real electrons may be considered a charge conjugation and time reversal (CT) transformation in this Dirac system. The symmetry under this (CT) transformation is also realized in its dual theory (with fermions that do not couple directly to the external fields), i.e., Dirac composite fermion theory. The presence of a mass term breaks this symmetry (the mass term transforms into minus itself). Therefore, the mass term

in the Dirac composite fermion theory (i.e., its extension with a mass), breaks the PH symmetry of the beginning (classical) electrons and may mimic LL mixing of real systems.

The Dirac composite fermion theory [28] can provide a framework for an analysis of Pfaffian paired states, as shown in Ref. [10]. The theory predicts, in the absence of the Dirac mass, equal-weight superposition of the Pfaffian and anti-Pfaffian and, for small mass, either the Pfaffian or anti-Pfaffian depending on the sign of the mass. On the basis of the same theory, (i) the *criticality* of the PH Pfaffian (reversed-chirality p wave) for the *zero* Dirac mass case (i.e., in the presence of the PH symmetry) was predicted [10,29], that is, numerically supported (no clear gapped state for the two-body interaction in a fixed LL for PH-symmetric shift on the sphere [30] and high overlap of the projected to the lowest LL usual PH Pfaffian wave function with a composite fermion liquid wave function [27,31]), and (ii) the *stabilization* was predicted for the usual PH Pfaffian, i.e., with a complex-conjugate Pfaffian part of the Pfaffian state, for *nonzero* mass. The nonzero mass means the absence of PH symmetry and can mimic LL mixing. The PH Pfaffian state that follows from the Dirac composite fermion description with mass [10] is

$$\Psi_{ZF} = Pf\left\{\frac{1}{(z_i^* - z_j^*)}\right\}\prod(z_k - z_l)^2. \tag{1}$$

Here

$$Pf\left\{\frac{1}{(z_i^* - z_j^*)}\right\} \sim \sum_P \mathrm{sgn}\, P \prod_{i=1}^{N/2} \frac{1}{(z^*_{P(2i-1)} - z^*_{P(2i)})}, \tag{2}$$

where the sum is over all permutations P of N integers.

The field theories (CS and Dirac composite fermion) via gauge fields encode basic interactions and influence among electrons. By a gauge field we describe and summarize the combined effects of Coulomb interaction and constrained dynamics resulting from the fact that electrons mostly live in a fixed LL. The CS description does not include the projection to a fixed LL, and we will use this description to estimate qualitatively the influence of other LLs [beyond the first order (in the LL mixing parameter) in the perturbation theory] when a system supports a paired state. (Reference [32] gives the first-order corrections in the perturbation theory.)

To set the stage and notation we will first review the CS description at 1/2 filling, i.e., the Halperin-Lee-Read description [33], with nonrelativistic composite fermions. This description may be considered a large mass m limit [34,35] of the Dirac composite fermion description (which is manifestly symmetric under particle-hole exchange). Thus, the nonrelativistic description breaks PH symmetry and includes the LL mixing which promotes the PH Pfaffian (according to Refs. [9,10]) and makes a natural framework for the investigation of the PH Pfaffian.

We start with the one-particle Hamiltonian,

$$H = \frac{\Psi^\dagger(\boldsymbol{p} - \boldsymbol{A})^2\Psi}{2m}, \tag{3}$$

with $c = 1, e = 1$, and $\hbar = 1$, where $A_\alpha = -(1/2)B\epsilon_{\alpha\beta}x_\beta$, i.e., $A_x = -(B/2)y$ and $A_y = (B/2)x$, $\boldsymbol{B} = B\mathbf{z}$. We also take $l_B = \sqrt{\frac{\hbar c}{eB}} = 1$. The one-particle eigenstates are $\Psi_m \sim z^m \exp\{-(1/4)|z|^2\}$, $m = 0, 1, 2, \ldots$, i.e., holomorphic functions (functions of only z) if we do not consider the exponential, $\exp\{-(1/4)|z|^2\}$. The CS transformation introduces gauge field $\mathbf{a}$,

$$H = \frac{\Psi^\dagger_{cf}(\boldsymbol{p} - \boldsymbol{A} - \boldsymbol{a})^2\Psi_{cf}}{2m}, \tag{4}$$

where $\boldsymbol{\nabla} \times \mathbf{a} = -2\,\Psi^\dagger\Psi = -2\,\rho(\mathbf{r})$. In the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{a} = 0$,

$$a_x(\mathbf{r}) = 2\int d\mathbf{r}'\,\frac{y - y'}{|\mathbf{r} - \mathbf{r}'|^2}\rho(\mathbf{r}'), \tag{5}$$

and

$$a_y(\mathbf{r}) = -2\int d\mathbf{r}'\,\frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^2}\rho(\mathbf{r}'). \tag{6}$$

We would like to understand the pairing effect of the so-called statistical interaction term,

$$V_{st} = -\mathbf{a}\cdot\frac{\Psi^\dagger_{cf}(\mathbf{p}\Psi_{cf}) - (\mathbf{p}\Psi^\dagger_{cf})\Psi_{cf}}{2m} = -\mathbf{a}\cdot\mathbf{j}_{cf}. \tag{7}$$

After simple steps and substitutions, which we describe in Appendix A, we arrive at the following expression for the Cooper channel in the inverse space [26], where operators $a_\mathbf{p}$ are associated with the inverse space:

$$V^C_{st} = i\frac{4\pi}{m}\frac{1}{V}\sum_{\mathbf{q},\mathbf{p}}\frac{(\mathbf{p}\times\mathbf{q})}{|\mathbf{p} - \mathbf{q}|^2}a^\dagger_\mathbf{q}a_\mathbf{p}a^\dagger_{-\mathbf{q}}a_{-\mathbf{p}}. \tag{8}$$

Using complex notation for vectors $\mathbf{p}$ and $\mathbf{q}$, we can rewrite the Cooper channel as

$$V^C_{st} = \frac{2\pi}{m}\frac{1}{V}\sum_{\mathbf{q},\mathbf{p}}\frac{(p^*\,q - p\,q^*)}{|\mathbf{p} - \mathbf{q}|^2}a^\dagger_\mathbf{q}a_\mathbf{p}a^\dagger_{-\mathbf{q}}a_{-\mathbf{p}}. \tag{9}$$

The second term in $(p^*\,q - p\,q^*)$ with a negative sign has the potential to develop pairing instability. We can rewrite that term as

$$\delta V^C_{st} = -\sum_{\mathbf{q},\mathbf{p}}\exp\{i(\theta_p - \theta_q)\}Fa^\dagger_\mathbf{q}a_\mathbf{p}a^\dagger_{-\mathbf{q}}a_{-\mathbf{p}}, \tag{10}$$

where F is a positive function. Doing the mean-field analysis as in Ref. [26] with the effective interaction,

$$\delta_{mf}V^C_{st} = -\sum_{\mathbf{q},\mathbf{p}}\exp\{i(\theta_p - \theta_q)\}F\{\langle a^\dagger_\mathbf{q}a^\dagger_{-\mathbf{q}}\rangle a_{-\mathbf{p}}a_\mathbf{p} + a^\dagger_\mathbf{q}a^\dagger_{-\mathbf{q}}\langle a_{-\mathbf{p}}a_\mathbf{p}\rangle\}, \tag{11}$$

and using the form of the BCS reduced interaction as in the Ref. [36],

$$\delta_{mf}V^C_{st} = \sum_\mathbf{p}\{\Delta^*_\mathbf{p}a_{-\mathbf{p}}a_\mathbf{p} + \Delta_\mathbf{p}a^\dagger_\mathbf{p}a^\dagger_{-\mathbf{p}}\}, \tag{12}$$

we find for the wave function of the Cooper pair the following behavior:

$$g(\mathbf{p}) \sim \frac{1}{\Delta^*_\mathbf{p}} \sim \frac{1}{p}, \tag{13}$$

and in the real space,

$$g(\mathbf{r}) \sim \frac{1}{z}. \tag{14}$$

This leads to a holomorphic Pfaffian state because the basis of the single-particle states is also holomorphic (up to the exponential factor, functions of only z, not z^*). If there were a minus sign in front of the pairing interaction in (8) or a plus sign instead of a minus sign in (7), we would have the antiholomorphic pairing part, and this would lead to the PH Pfaffian state (1).

III. MODEL INTERACTIONS FOR THE PFAFFIAN AND PH PFAFFIAN

Following what was done in Ref. [26], we start with a BCS-like description of the effective pairing interaction for ordinary, non-Dirac composite fermions. Thus, we start with classical composite fermions assuming that the effective mass is large (considerable) and includes the particle-hole symmetry breaking necessary for stabilization (development) of the Pfaffian (PH Pfaffian). The effective description we assume is a possible reduction of the CS description with higher-order terms when a p-wave state (topological superconductivity) of composite fermions is established. Thus, our beginning Hamiltonian is

$$H^{ef}_{BCS} = \frac{1}{2m}\Psi^\dagger_{cf}(\mathbf{p})^2\Psi_{cf} + \lambda\delta\mathbf{a}\mathbf{j}_{cf}, \tag{15}$$

where $\delta\mathbf{a} = \mathbf{A} + \mathbf{a}$ and $\mathbf{a}$ is described in Eqs. (5) and (6). Note that we included the regularized form, discussed in Appendix A, of the statistical interaction, i.e., the form with $\delta\mathbf{a}$ instead of the one with $\mathbf{a}$ in Sec. II. This amounts to the subtraction of the zero-point energy, i.e., (orbital) cyclotron energy due to the motion in a constant magnetic field, which we do not expect to be present in the description of pairing. λ is an effective coupling which can be negative in the case of the Pfaffian and positive for the PH Pfaffian (compare with the discussion in Sec. II). (The effective BCS interaction in the case of the PH Pfaffian, with $\lambda > 0$, is the one that was derived in the scope of the Dirac composite fermion theory [28,34,35] in the presence of a large Dirac mass (i.e., LL mixing, of both signs) in Ref. [10].) We rewrite H^{ef}_{BCS} as

$$H^{ef}_{BCS} = \frac{1}{2m}\Psi^\dagger_{cf}(\mathbf{p}-\delta\mathbf{a})^2\Psi_{cf} - \frac{1}{2m}(\delta\mathbf{a})^2\Psi^\dagger_{cf}\Psi_{cf} + (1+\lambda)\delta\mathbf{a}\mathbf{j}_{cf}. \tag{16}$$

Now we apply the CS transformation in reverse, from composite fermions to the electron representation, to arrive at

$$H^{el}_{BCS} = \frac{1}{2m}\Psi^\dagger(\mathbf{p}-\mathbf{A})^2\Psi - \frac{1}{2m}(\delta\mathbf{a})^2\Psi^\dagger\Psi + (1+\lambda)\delta\mathbf{a}\mathbf{J}_{el} + (1+\lambda)\frac{1}{m}(\delta\mathbf{a})^2\Psi^\dagger\Psi. \tag{17}$$

To get H^{el}_{BCS} from H^{ef}_{BCS} we used the fact that the CS is a unitary phase transformation [37] on fermion fields, which transforms the first (kinetic) term in (16) back to the first term in (17) [compare (3) and (4) in the previous section] and does not change the form of the second term in (16). The third term in (16) has the composite fermion current $\mathbf{j}_{cf}$ described in (7), which transforms as

$$\mathbf{j}_{cf} \to \mathbf{j}_{el} + \frac{\mathbf{a}}{m}\Psi^\dagger\Psi. \tag{18}$$

Applying this transformation, we get the two last terms in (17), where

$$\mathbf{J}_{el} = \frac{-i}{2m}\Psi^\dagger(\nabla + i\mathbf{A})\Psi - [(\nabla + i\mathbf{A})\Psi]^\dagger\Psi \tag{19}$$

is the gauge-invariant electron current. The Hamiltonians $H^{el}_{BCS}(\lambda)$ describe the effective interactions of electrons that lead to paired states.

The CS description is, in essence, the Laughlin ansatz (or organization of the solution) translated into the language of field theory and thus refers mostly to the lowest-LL physics. Our goal is a representation of the effective Hamiltonian for paired states in a fixed LL, and in the following we will use the lowest LL as a stage for that goal. Thus, we will model interactions also for the paired states in the second LL, like the Pfaffian, by effective parameters that we will find in the lowest LL.

To begin modeling in a fixed LL we neglect (i.e., consider as a constant) the first term (the kinetic term) in Eq. (17). The remaining term, i.e., an effective interaction that we will project to the lowest LL, is

$$V^{el}_{BCS}(\lambda) = (1+\lambda)\delta\mathbf{a}\mathbf{J}_{el} + (1/2+\lambda)\frac{1}{m}(\delta\mathbf{a})^2\Psi^\dagger\Psi. \tag{20}$$

In the following we will consider the resulting two-body interactions from (20) less important for the physics of paired states and thus concentrate on the resulting three-body interaction. The contributions from the two-body part have to be carefully calculated, and in Appendix B, the contribution from the first term in (20), $\sim \delta\mathbf{a}\mathbf{j}_{el}$, where $\mathbf{j}_{el} = \frac{-i}{2m}\{\Psi^\dagger\nabla\Psi - \nabla\Psi^\dagger\Psi\}$, can be found.

The three-body interaction from the complete effective interaction in (20) is

$$V^3_{BCS}(\lambda) = (1/2+\lambda)\frac{1}{m} :(\mathbf{a})^2\Psi^\dagger\Psi: . \tag{21}$$

Plugging in the expressions for $\mathbf{a}$ in (5) and (6), we get

$$V^3_{BCS}(\lambda) = (1/2+\lambda)\frac{4}{m}\int d\mathbf{r}_1 \times \int d\mathbf{r}_2 \frac{(\mathbf{r}_3-\mathbf{r}_1)(\mathbf{r}_3-\mathbf{r}_2)}{|\mathbf{r}_3-\mathbf{r}_1|^2|\mathbf{r}_3-\mathbf{r}_2|^2} :\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rho(\mathbf{r}_3):; \tag{22}$$

that is, the three-body interaction in the coordinate representation is

$$V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = (1/2+\lambda)\frac{4}{m}\frac{(\mathbf{r}_3-\mathbf{r}_1)(\mathbf{r}_3-\mathbf{r}_2)}{|\mathbf{r}_3-\mathbf{r}_1|^2|\mathbf{r}_3-\mathbf{r}_2|^2}. \tag{23}$$

The fully antisymmetric wave functions for three particles are given in [38], and they are

$$\Psi_{k,l}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \frac{1}{Z_{kl}}\left(z_a^2+z_b^2\right)^k\left[\frac{(z_a+iz_b)^{3l}-(z_a-iz_b)^{3l}}{2i}\right] \times \exp\left\{-\frac{1}{4}(|z_a|^2+|z_b|^2+|z_c|^2)\right\}, \tag{24}$$

where integers $k \geqslant 0$, $l \geqslant 1$ and the total angular momentum of the state is $M = (2k+3l)$. The normalization factor

TABLE I. Matrix elements in the lowest Landau level.

	M							
	3	5	6	7	8	9		10
Δ_M	1/24	1/48	7/240	1/80	2/105	$\frac{221/10080}{1/(240\sqrt{21})}$	$\frac{1/(240\sqrt{21})}{1/120}$	3/224
$\frac{\Delta_M}{\Delta_{M=3}}$	1	0.5	0.7	0.3	∼0.475	$\frac{\sim 0.526}{\sim 0.022}$	$\frac{\sim 0.022}{0.2}$	∼0.321

$Z_{kl} = 2^{3l+2k+1}[\pi^3(3l+k)!k!]^{1/2}$, and the complex coordinates are

$$z_a = \sqrt{\frac{2}{3}}\left(\frac{z_1+z_2}{2} - z_3\right), \quad z_b = \frac{z_1 - z_2}{\sqrt{2}}, \quad z_c = \frac{z_1+z_2+z_3}{\sqrt{3}}. \tag{25}$$

Thus, $\sum_{i=1}^{3} |z_i|^2 = |z_a|^2 + |z_b|^2 + |z_c|^2$.

To describe relevant three-body pseudopotentials (PPs) we introduce matrix elements of a rescaled three-body interaction: $\frac{V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)}{\Lambda}$, where $\Lambda \equiv (1/2+\lambda)4/m$. The diagonal matrix elements are defined as

$$\Delta_{M=2k+3l} = \int d\mathbf{r}_1 \int d\mathbf{r}_2 \int d\mathbf{r}_3 \frac{V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)}{\Lambda} |\Psi_{k,l}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|^2. \tag{26}$$

The resulting three-body PPs for a fixed λ are

$$V_M(\lambda) = \Lambda\Delta_M = (1/2+\lambda)(4/m)\Delta_M. \tag{27}$$

The matrix elements relevant for the interaction in the lowest LL are listed in Table I.

Table I shows the rescaled values of the three-body PPs of the interaction defined in (23), i.e., $\Delta_M = \frac{V_M(\lambda)}{\Lambda}$, as functions of the total angular momentum M. There are two (orthogonal) wave functions for three fermions at $M = 9$, and thus, the corresponding cases with $l = 3, k = 0$ and $l = 1, k = 3$ are in the two columns, respectively, for $M = 9$ with common off-diagonal (mutual overlap) elements.

Remarkably, ratios among the first three values of three-body PPs, $\frac{V_{M=5}}{V_{M=3}} = \frac{\Delta_{M=5}}{\Delta_{M=3}} = 0.5$ and $\frac{V_{M=6}}{V_{M=3}} = \frac{\Delta_{M=6}}{\Delta_{M=3}} = 0.7$, are quite close to the ones obtained by the first-order perturbation theory in the second LL [32], ∼0.4 and ∼0.7, respectively. We should notice that in this case the corresponding unit can be expressed as $\frac{e^2}{l_B}\frac{1}{\kappa}$, where the LL mixing factor, $\kappa = (\frac{e^2}{l_B})/(\hbar\frac{eB}{mc})$, divides the expressions contrary to the case in the perturbation theory, and thus, again, we should be aware that we work with systems in which the LL mixing is considerable (no $\kappa \to 0$ limit).

But we can use the identified correspondence in ratios to conclude that the field theory correctly predicts the main features of a model interaction for a Pfaffian. Namely, it predicts negative values of three-body PPs, $V_M(\lambda) = \Lambda\Delta_M = (1/2+\lambda)(4/m)\Delta_M$, for $M = 3, 5$, and 6, to be crucial for the establishment of a Pfaffian according to the expression in Eq. (20), where we take $\lambda \lesssim -1$ in the Pfaffian case. This prediction is in complete accordance with the numerical work in Ref. [19]. In Fig. 3 of that work we see that the negative values of three-body PPs, with the specified ratios (based on the first-order perturbation theory in the second LL), are crucial in obtaining the Pfaffian state. (We remind the reader of the comments in the paragraph below Eq. (19) that we model paired states in a fixed, lowest LL and this model interaction is relevant for any fixed LL, including the second LL.) In Table I we also list calculated matrix element values for higher angular momenta ($M = 7, 8, 9, 10$), and (after the simple rescaling) they follow the basic trend of the first-order perturbation theory for the second LL, which favors Pfaffian and anti-Pfaffian states as analyzed in Ref. [22].

The generation of three-body terms due to the LL mixing using the perturbation theory has a long history [39–42]. It led to the identification of relevant three-body parameters for effective Pfaffian physics. Here we find a similar description of an effective interaction for a Pfaffian using only field-theoretical arguments.

It is an interesting question whether the field theory may also predict a model interaction for the anti-Pfaffian (or differentiate the anti-Pfaffian from the Pfaffian). Field theory can describe the anti-Pfaffian as a Pfaffian pairing instability of a Fermi liquid of composite holes [43] and, for that case, can come up with the three-body interaction in (22). Therefore, we should consider that three-body interaction in a fixed LL and apply PH transformation to get the model interaction for the anti-Pfaffian.

Next, according to formula (20) field theory predicts that in the case of the PH Pfaffian, $\lambda > 0$, positive three-body PPs are necessary for its establishment. This is quite expected due to the role of the negative ones in the establishment of the Pfaffian and anti-Pfaffian (Fig. 3 in Ref. [19]; see also Ref. [22]), and thus, the positive values will suppress the tendency to the Pfaffian and anti-Pfaffian.

Therefore, the main feature of the model interaction for a PH Pfaffian in a fixed LL (the projection of the PH Pfaffian) according to the field theory arguments is the series of positive three-body PPs, $V_M(\lambda)$ for ratios $\frac{V_M}{V_{M=3}} = \frac{\Delta_M}{\Delta_{M=3}}$, specified in Table I. Certainly, the question remains whether a real system will slip into a Fermi liquid state, and we will discuss this in the next section.

We comment that according to the main features of the effective (model) interaction for the PH Pfaffian (in a fixed LL) listed above, we do not expect more than three-body

TABLE II. Matrix elements in the lowest Landau level with regularization.

	M							
	3	5	6	7	8	9		10
Δ_M	0.03309	0.01944	0.02487	0.01186	0.01592	0.01946	0.00897	0.01047
$\frac{\Delta_M}{\Delta_{M=3}}$	1	∼0.587	∼0.752	∼0.358	∼0.481	∼0.588	∼0.271	∼0.316

(additional) PPs to be relevant. The main features are derived on the basis of the BCS reduction we described in Eq. (15), i.e., the reduction we believe is a faithful description of paired states. On the other hand, on the basis of the perturbation theory in κ we would also expect more than three-body PPs. If, indeed, the PH Pfaffian physics is present, for some $\kappa \gtrsim 1$, it may be preceded by a distinct phase for which the perturbation theory in κ is valid. On the other hand, the PH Pfaffian would be based on a nonperturbative (nonanalytic) in κ description of the LL mixing.

IV. BEYOND THE PROJECTION TO A FIXED LANDAU LEVEL

We have argued for the main features of a model interaction for a PH Pfaffian in a fixed LL. But we should also note that there are strong arguments that the projection will represent a gapless state:

(i) Let's assume that the projection (i.e., associated wave function) is PH symmetric. According to Refs. [10,29], i.e., arguments in Sec. II of Ref. [10] based on Dirac composite fermion (manifestly PH symmetric) theory (more precisely, the Bogoliubov description of the pairing of Dirac composite fermions which encapsulates the BCS ground state), such a state must be critical (gapless).

But we derived a model Hamiltonian (interaction) in a fixed LL that has an explicit three-body interaction which breaks PH symmetry, and thus, we should also consider the possibility that the projection is not PH symmetric.

(ii) If the projection is not PH symmetric and represents a gapped state, the state based on the projection and its corresponding partner, a distinct phase, that we get by the PH exchange have the same shift, i.e., an integer, a topological number that characterizes the state of the system on a curved background, such as a sphere. This is certainly not a sign of two distinct phases. Moreover, the numerical results in Ref. [27] for the overlap of the projection and its partner under the PH exchange are very high for system sizes up to $N = 12$ despite the fact that the overlap must decay to zero in the thermodynamic limit, irrespective of the presence of the PH symmetry. (Thus, either the projection is PH symmetric and we are back to the preceding case, or the state is gapless.)

Our model interaction for the PH Pfaffian in Eq. (17) [or (20)] is defined in the space organized by LLs, and a question is whether we will capture the nature and physics of the PH Pfaffian if we consider only one LL for which the rescaled magnitudes of the three-body PPs [divided by $\Lambda = (1/2 + \lambda)4/m$] are specified in Table I. We can take the effective LL mixing parameter in this system to be $|1/2 + \lambda|$. Thus, in the case of the Pfaffian when $\lambda = -1$, we can stay in a fixed LL, while if λ is of the same magnitude but opposite sign, i.e., when we have the case of the PH Pfaffian with $\lambda = 1$, it seems we need to consider an additional LL.

To assess the role of higher LLs we concentrate on the three-body interaction, more precisely, diagonal matrix elements of the three-body interaction when one, two, or three electrons are in one higher LL (the second LL). We considered the wave functions that we get by applying the raising operators

$$a_i^\dagger = \sqrt{2}(-\partial_{z_i} + z_i^*/4),$$

$i = 1, 2, 3$, to the lowest-LL wave functions in Eq. (24); we considered applying (i) $\frac{1}{\sqrt{3}}(a_1^\dagger + a_2^\dagger + a_3^\dagger)$ (one electron of three electrons in the second LL, equivalent to a center of mass excitation), (ii) $\frac{1}{3}(a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_1^\dagger a_3^\dagger)$ (two electrons of three electrons in the second LL), and (iii) $a_1^\dagger a_2^\dagger a_3^\dagger$ (all three electrons in the second LL). While calculating these elements, we encountered ultraviolet divergences because of

TABLE III. Matrix elements for states with two particles in the lowest Landau level and one particle in the second Landau level.

	M							
	2	4	5	6	7	8		9
Δ_M	0.03279	0.01882	0.02528	0.01136	0.01933	0.01880	0.00844	0.01212
$\frac{\Delta_M}{\Delta_{M=2}}$	1	∼0.574	∼0.771	∼0.346	∼0.590	∼0.573	∼0.257	∼0.370

TABLE IV. Matrix elements for states with one particle in the lowest Landau level and two particles in the second Landau level.

	M							
	1	3	4	5	6	7		8
Δ_M	0.02880	0.02132	0.02302	0.01402	0.01430	0.01791	0.00880	0.01046
$\frac{\Delta_M}{\Delta_{M=1}}$	1	∼0.740	∼0.799	∼0.487	∼0.497	∼0.622	∼0.305	∼0.363

the limitations of the effective CS theory and its inability to capture short-range physics. Therefore, we had to regularize the interaction in Eq. (23) (of the effective CS description). Instead of $|\mathbf{r}_3-\mathbf{r}_1|^2\cdot|\mathbf{r}_3-\mathbf{r}_2|^2$ in the denominator of Eq. (23), we took $(|\mathbf{r}_3-\mathbf{r}_1|^2+a^2)\cdot(|\mathbf{r}_3-\mathbf{r}_2|^2+a^2)$, where a is a short-distance cutoff. We checked that if the denominator is modified into $|\mathbf{r}_3-\mathbf{r}_1|^2\cdot|\mathbf{r}_3-\mathbf{r}_2|^2+a^4$, the values of implied PPs for $a\lesssim l_B$ do not change significantly. The values of implied matrix elements when $a=1=l_B$ are given in Tables II–V.

When a value for a certain matrix element was missing, the numerical error was substantial.

V. DISCUSSION AND CONCLUSIONS

We need to analyze more closely the role of the parameter λ. The sign of λ in the composite fermion picture (15) determines the chirality of the underlying p-wave topological superconductivity of the composite fermions, and after the CS transformation into the electron picture, in (17), the parameter λ determines whether we are in the Pfaffian, composite Fermi liquid (CFL) [33], or PH Pfaffian phase. Even without the help of exact diagonalizations, we can come up with a schematic phase diagram of the electron system as a function of λ (see Fig. 1). Two insights lead to the phase diagram as a function of λ: (i) effective three-body coupling is $(1/2+\lambda)$ (this determines its sign), and (ii) the effective LL mixing parameter is $|1/2+\lambda|$ (this determines how many LLs we need to include). [We consider two-body interactions, which are likely positive and monotonically decreasing with momenta, irrelevant for (PH) Pfaffian pairing. The paired states are expected to be stabilized by three-body interactions.]

As we change λ from negative to positive values, more precisely, for $\lambda\gtrsim-1$, we can estimate (due to the effective LL mixing $\sim|1/2+\lambda|$) that we need an extra LL to describe the electron system with underlying BCS p-wave pairing of composite fermions at (around) $\lambda=1/2$. Prior to that value of λ, for $-1\lesssim\lambda<-1/2$, we expect a Pfaffian instability of electrons; this is based on our expectation (corroborated by numerics in [19,22]) that the negative values of three-body interaction with specific ratios for lower angular momenta (Table I and Fig. 2) will support Pfaffian physics. Pfaffian physics largely occurs in a fixed LL, and the ground-state wave function of the Pfaffian phase can be described by a completely holomorphic expression in the (fixed) lowest LL. The fixed LL physics is scale invariant in a special way; the characteristic length is present only in the exponential factor, which does not change as correlations in a fixed LL change. To describe Pfaffian physics it is sufficient to stay in the lowest LL and use the (unregularized, negative values of) PPs described in Fig. 2 The physics does not depend on any length scale, and although it is surprising at first sight that the field theory can come up with finite matrix elements, we can use them without any need to regularize. As we increase λ, for $-1/2<\lambda\lesssim1/2$, the effective three-body interaction is positive because the value of the effective coupling, $(1/2+\lambda)$, changes sign to positive at $\lambda=-1/2$. Thus, in this effective description a phase transition may occur at $\lambda=-1/2$. We expect an entrance into the (compressible) CFL phase. For $-1/2<\lambda\lesssim1/2$ the LL mixing, $\sim|1/2+\lambda|$, is not large, and we may consider also in this region only PPs of the lowest LL. At $\lambda=0$, in the composite fermion representation as well as the electron representation, we have a Fermi liquid phase. Furthermore, for a whole interval, $-1/2<\lambda\lesssim1/2$, we expect a CFL phase (in the electron system) because for $\lambda=-1/2$ there is an abrupt change in the sign of the three-body interaction accompanied by an oscillatory behavior in the positive values of PPs as a function of the total angular momentum of three fermions. The oscillatory behavior of the (positive) values of PPs might be a sign of the compressible

TABLE V. Matrix elements in the second Landau level.

M	0	2	3	4	5	6		7
Δ_M	0.02488	0.02113	0.01978	0.01427	0.01256	0.01474	0.01012	0.01119
$\frac{\Delta_M}{\Delta_{M=0}}$	1	∼0.849	∼0.795	∼0.573	∼0.505	∼0.592	∼0.407	∼0.450

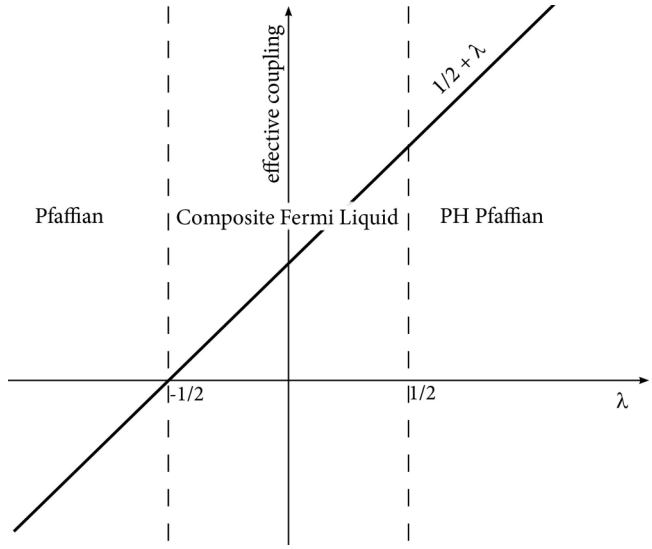

FIG. 1. Schematic phase diagram as the pairing parameter λ is varied.

correlations in the phase that we expect to be the CFL; a state of three fermions may reduce its angular momentum without resistance (or a significant increase in energy). Thus, a (single) series with positive oscillatory values of three-body PPs in the lowest LL (see Table I and Fig. 2) may be a hallmark of the whole region, $-1/2 < \lambda \lesssim 1/2$, in which the topological pairing at weak coupling of composite fermions in (15) is suppressed under the CS transformation into the electron representation. This system, at $0 < \lambda \lesssim 1/2$, as we will discuss below, confined to a single LL also represents a projection of the PH Pfaffian to a fixed LL: any attempt to confine the description of the PH Pfaffian in a fixed LL will produce a gapless state, a state close to the CFL [10,27].

But we should note and emphasize that the particular oscillatory behavior of PPs, as described in the bottom part of Fig. 2, with positive values, may promote the pairing necessary for a PH Pfaffian, in which any three electrons

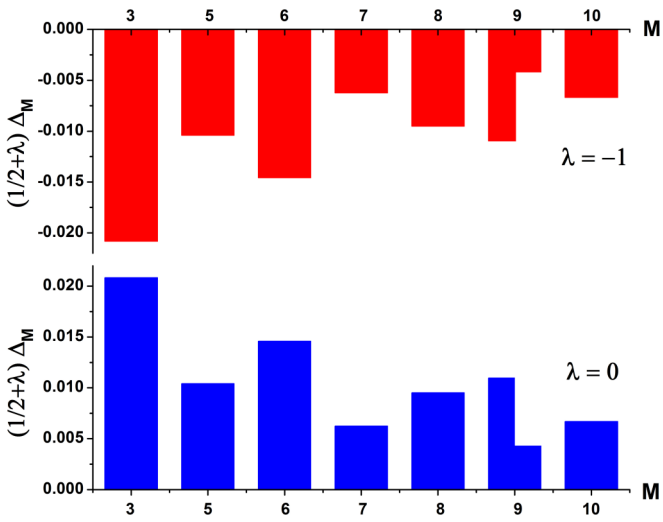

FIG. 2. Matrix elements of three-body pseudopotentials in the lowest Landau level for $\lambda = -1$ (above) and $\lambda = 0$ (bottom).

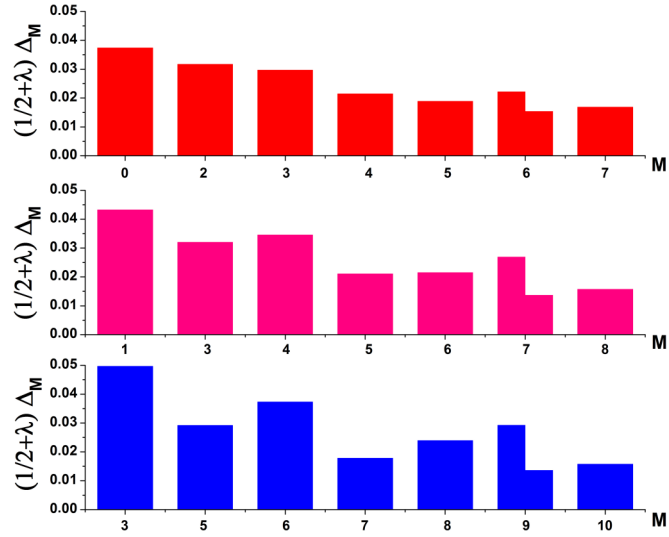

FIG. 3. Three-body pseudopotential matrix elements for $\lambda = 1$ (PH Pfaffian case) in the second Landau level (top), for states with two particles in the second Landau level and one in the lowest Landau level (middle), and in the lowest Landau level (bottom).

can mostly correlate (efficiently minimize their energy) in the total angular momentum equal to $M = 7$, which is the characteristic angular momentum for PH Pfaffian pairing. (Recall that the characteristic angular momentum for a Pfaffian is $M = 5$ [44].) Nevertheless, the compressible, Fermi-liquid-like behavior results from the projection to a fixed LL due to phase-space constraints which prohibit the pairing. Namely, the expected leading term in the projection of the paired state is equal to zero if the (unprojected) pairing function is $g(z) \sim 1/z^*$. We will come back to this point in the concluding remarks.

Because the LL mixing parameter in our model Hamiltonian, (17), can be estimated to be equal to $|1/2 + \lambda|$, above $\lambda \sim 1/2$ we need to include at least one additional LL (the second LL) to capture the underlying p-wave pairing state of the electron system. This brings the natural scale (for distance), the magnetic length l_B as an external, fixed scale that we used to regularize the three-body PPs, as described in the previous section. The regularized intra-LL PPs are given in that section. What we notice is that if we confine our description to the lowest LL, the PPs are still characterized by oscillatory behavior (see Table II and Fig. 3), and this can lead to the compressible state. On the other hand, interestingly, the three-body intra-LL PPs for the higher, second LL are characterized by monotonically decreasing (with total angular momentum) positive values (Table V and Fig. 3). More importantly, in Table IV and Fig. 3, in the case of three electrons, of which two are in the second LL, we see an abrupt decrease in the values of repulsive PPs that occurs at $M = 5$ (effectively, $M = 7$ in the lowest LL), a characteristic angular momentum for the PH Pfaffian pairing ($M = 5$ in the case of the Pfaffian, in the lowest LL). This opens up the possibility for PH Pfaffian pairing correlations, which by definition are antiholomorphic, to form and also exist in the higher band, the second LL. At least two LLs are needed to establish the PH Pfaffian. This is not surprising given the

fact that the Pfaffian antisymmetrized product of Cooper pairs with the projected pairing function $1/z^*$ to the lowest LL $g_{LLL}(z) \sim z$ is zero. The same pairing function, projected to the second LL, is $g_{sLL}(z, z^*) \sim (|z|^2 - 4)\, z$, and thus, the extra factor, $(|z|^2 - 4)$, brings the (magnetic) length scale into the description and enables a nontrivial pairing to develop and exist at short distances. But we should note that the values of calculated PPs for electronic correlations do not lead immediately to an expectation for the existence of a gapped paired state; the transformation to the electron representation may lead to a compressible state with some pairing correlations due to the inclusion of the second LL. Further numerical investigations are necessary to probe the existence of a gapped state with the PH Pfaffian topological characterization on the basis of the calculated PPs for the lowest two LLs. In this work we developed a general framework, a model interaction that can be used in the investigation of the PH Pfaffian.

In the field-theoretical approach that we considered, besides using the mean-field (classical) equations of motion for fields in the effective composite fermion theory, we did not do any further approximations. Our approach clearly calls for a necessary inclusion of other LLs in numerical approaches in the quest for the PH Pfaffian. By working in a fixed LL, no matter how well the influence of other LLs is included, one cannot access the PH Pfaffian pairing. We described the pertinent three-body parameters for two LLs.

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APPENDIX A: STATISTICAL INTERACTION

In this Appendix the details of the calculations that lead from the statistical interaction in Eq. (7) to the Cooper channel expression in Eq. (8) will be explained. If $\Psi_{cf} = \sum \frac{1}{\sqrt{V}} \exp\{i\mathbf{k}\cdot\mathbf{r}\}\, a_{\mathbf{k}}$ and

$$\mathbf{j}_{cf}(\mathbf{r}) = \frac{1}{2m}[\Psi^\dagger_{cf}(-i\nabla\Psi_{cf}) + (i\nabla\Psi^\dagger_{cf})\Psi_{cf}], \tag{A1}$$

then

$$\mathbf{j}^{cf}_{\mathbf{p}} = \int d\mathbf{r} \exp\{i\mathbf{p}\cdot\mathbf{r}\}\, \mathbf{j}_{cf}(\mathbf{r}) = \frac{1}{2m}\sum_{\mathbf{k}} a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{k}} (2\mathbf{k}+\mathbf{p}). \tag{A2}$$

Using the solutions in (5) and (6), it follows that

$$\int \frac{\mathbf{j}_{cf}\cdot\mathbf{a}}{2} = \int d\mathbf{r} \int d\mathbf{r}' \left\{ j^{cf}_x(\mathbf{r}) \frac{y-y'}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r}') - j^{cf}_y(\mathbf{r}) \frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r}') \right\}. \tag{A3}$$

If we introduce

$$\mathbf{j}_{cf}(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} \exp\{i\mathbf{k}\mathbf{r}\}\mathbf{j}_{cf}(-\mathbf{k}) \tag{A4}$$

and

$$\rho(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} \exp\{-i\mathbf{k}\mathbf{r}\}\rho(\mathbf{k}), \tag{A5}$$

we can rewrite the above expression as

$$\int \frac{\mathbf{j}_{cf}\cdot\mathbf{a}}{2} = \frac{1}{V}\sum_{\mathbf{k}} i\left\{ \frac{k_y}{|\mathbf{k}|^2} j^{cf}_x(-\mathbf{k}) - \frac{k_x}{|\mathbf{k}|^2} j^{cf}_y(-\mathbf{k}) \right\} \rho(\mathbf{k})(2\pi). \tag{A6}$$

With

$$\rho(\mathbf{k}) = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}}, \tag{A7}$$

the statistical interaction becomes

$$\int V_{st} = (-i)\frac{2\pi}{m}\frac{1}{V}\sum_{\mathbf{k},\mathbf{p},\mathbf{q}} 2\frac{(q_x k_y - q_y k_x)}{|\mathbf{k}|^2} a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} a^\dagger_{-\mathbf{k}+\mathbf{q}} a_{\mathbf{q}}. \tag{A8}$$

If we consider only the Cooper channel, i.e., $\mathbf{q} = -\mathbf{p}$,

$$V^C_{st} = (-i)\frac{2\pi}{m}\frac{1}{V}\sum_{\mathbf{k},\mathbf{p}} (-2)\frac{(p_x k_y - p_y k_x)}{|\mathbf{k}|^2} a^\dagger_{\mathbf{p}+\mathbf{k}} a_{\mathbf{p}} a^\dagger_{-\mathbf{k}-\mathbf{p}} a_{-\mathbf{p}}. \tag{A9}$$

Introducing $\mathbf{q} = \mathbf{k} + \mathbf{p}$, we get Eq. (8) in the main text.

To have a regularized, nondivergent statistical interaction, we should exclude the $\mathbf{k} = 0$ point in the summation, in (A8), which amounts to an elimination of the part of $\mathbf{a}$ that describes the external, constant magnetic field, i.e., $V_{st} = -\mathbf{a}\cdot\mathbf{j}_{cf} \to V_{st}(\text{regularized}) = -\delta\mathbf{a}\cdot\mathbf{j}_{cf}$. This does not influence the Cooper channel description, and we omitted this regularization in Sec. II, but it is important for the discussion in Sec. III.

APPENDIX B: TWO-BODY PSEUDOPOTENTIALS

In this Appendix the details of the calculation of the PPs for the two-body interaction, $\int d\mathbf{r}\, \mathbf{a}\cdot\mathbf{j}_{el}$, in the lowest and second LLs will be presented. The projection of the first term in (20),

$$(1+\lambda)\delta\mathbf{a}\mathbf{j}_{el}, \tag{B1}$$

can be found by considering its second quantized form with field operators that belong (are projected) to a fixed LL. We consider

$$\delta\mathbf{a}\mathbf{j}_{el} = \frac{\phi_0}{\pi}\int d\mathbf{r}' \frac{1}{2i}\frac{\bar{j}_{el}(z-z') - j_{el}(\bar{z}-\bar{z}')}{|z-z'|^2}\delta\rho(\mathbf{r}'), \tag{B2}$$

and thus,

$$\int d\mathbf{r}\ \mathbf{aj}_{el} = -\frac{1}{2m}\int d\mathbf{r}\frac{\phi_0}{\pi}\int d\mathbf{r}'\frac{\left\{\left[\Psi^\dagger\frac{\partial}{\partial z}\Psi - \frac{\partial}{\partial z}\Psi^\dagger\ \Psi\right](z-z') - \left[\Psi^\dagger\frac{\partial}{\partial \bar{z}}\Psi - \frac{\partial}{\partial \bar{z}}\Psi^\dagger\ \Psi\right](\bar{z}-\bar{z}')\right\}}{|z-z'|^2}\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}'). \tag{B3}$$

By partial integration and using $\frac{\partial}{\partial z}(\frac{1}{\bar{z}}) = \frac{\partial}{\partial \bar{z}}(\frac{1}{z}) = 2\pi\delta^2(\mathbf{r}-\mathbf{r}')$, in the usual units,

$$\int d\mathbf{r}\ \mathbf{aj}_{el} = -\frac{2}{m}\int d\mathbf{r}\int d\mathbf{r}'\frac{\left\{\left[\Psi^\dagger\frac{\partial}{\partial z}\Psi\right](z-z') - \left[\Psi^\dagger\frac{\partial}{\partial \bar{z}}\Psi\right](\bar{z}-\bar{z}')\right\}}{|z-z'|^2}\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}'), \tag{B4}$$

with

$$\Psi(\mathbf{r}) = \sum\langle\mathbf{r}|\Psi_n\rangle a_n = \sum\mathcal{M}_n z^n\exp\left\{-\frac{1}{4}|z|^2\right\}a_n, \tag{B5}$$

where

$$\mathcal{M}_n = \mathcal{N}_n = \frac{1}{\sqrt{2\pi\,2^n n!}}, \tag{B6}$$

$n = 0, 1, 2, 3, \ldots$, in the lowest LL and

$$\mathcal{M}_n = \tilde{\mathcal{N}}_n f_n, \tag{B7}$$

with $n = -1, 0, 1, 2, 3, \ldots$, and

$$\tilde{\mathcal{N}}_n = -\frac{1}{\sqrt{2\pi\,2^{n+2}(n+1)!}}, \tag{B8}$$

with

$$f_n = (2n+2-\bar{z}z), \tag{B9}$$

in the second LL.

In the lowest LL,

$$\int d\mathbf{r}\ \mathbf{aj}_{el} = \sum_{\overline{n},n,\overline{m},m} -\frac{2}{m}\frac{(2\pi)^2}{2}\mathcal{N}_{\overline{n}}\,\mathcal{N}_n\,\mathcal{N}_{\overline{m}}\,\mathcal{N}_m\,I(\overline{n},n,\overline{m},m)\ :a^\dagger_{\overline{n}}\,a_n\,a^\dagger_{\overline{m}}\,a_m:, \tag{B10}$$

and we have to use $\tilde{\mathcal{N}}_n$'s instead of $\mathcal{N}_n$'s and calculate $\tilde{I}(\overline{n},n,\overline{m},m)$ in the place of $I(\overline{n},n,\overline{m},m)$ for the effective interaction in the second LL.

We will consider the following diagonal elements with respect to states: $a^\dagger_r a^\dagger_s|0\rangle$, with $r = 0$ and $s = l$ in the lowest LL and $r = -1$ and $s = l-1$ in the second LL. In this way we can extract the PPs W_l^2, $l = 1, 3, 5, \ldots$, and $\tilde{W}_l^2$, $l = 1, 3, 5, \ldots$, in the lowest and second LLs, respectively:

$$W_l^2 = -\frac{(2\pi)^2}{m^*}\mathcal{N}_l^2\,\mathcal{N}_0^2\,\{I(l,l,0,0) + I(0,0,l,l) - I(0,l,l,0) - I(l,0,0,l)\} \tag{B11}$$

and

$$\tilde{W}_l^2 = -\frac{(2\pi)^2}{m^*}\tilde{\mathcal{N}}_{l-1}^2\,\tilde{\mathcal{N}}_{-1}^2\,\{\tilde{I}(l-1,l-1,-1,-1) + \tilde{I}(-1,-1,l-1,l-1) - \tilde{I}(-1,l-1,l-1,-1) - \tilde{I}(l-1,-1,-1,l-1)\}. \tag{B12}$$

In the above formulas we used m^* to denote the effective mass. The explicit expressions for I and $\tilde{I}$, with $\Delta = m - \overline{m} - n + \overline{n}$, are

$$\begin{aligned} I(\overline{n},n,\overline{m},m) = \int_0^\infty dr_1 r_1\Bigg\{\int_0^{r_1} dr_2 r_2\exp\left[-\frac{1}{2}(|r_1|^2+|r_2|^2)\right]r_2^{\overline{m}+m+\frac{\Delta}{2}}r_1^{\overline{n}+n-\frac{\Delta}{2}-2}\left(n-\frac{1}{4}r_1^2\right) \\ + \int_{r_1}^\infty dr_2 r_2\exp\left[-\frac{1}{2}(|r_1|^2+|r_2|^2)\right]r_2^{\overline{m}+m-\frac{\Delta}{2}-2}r_1^{\overline{n}+n+\frac{\Delta}{2}}\left(-\frac{1}{4}r_2^2\right)\Bigg\} \end{aligned} \tag{B13}$$

and

$$\tilde{I}(\overline{n},n,\overline{m},m)=\int_0^\infty dr_1 r_1\left\{\int_0^{r_1} dr_2 r_2 \exp\left[-\frac{1}{2}(|r_1|^2+|r_2|^2)\right] r_2^{\overline{m}+m+\frac{\Delta}{2}} r_1^{\overline{n}+n-\frac{\Delta}{2}-2}\right.$$
$$\times\left[n f_{\overline{n}}(r_1)f_n(r_1)f_{\overline{m}}(r_2)f_m(r_2)-\frac{1}{4}r_1^2 f_{\overline{n}}(r_1)\tilde{f}_n(r_1)f_{\overline{m}}(r_2)f_m(r_2)\right]$$
$$\left.+\int_{r_1}^\infty dr_2 r_2 \exp\left[-\frac{1}{2}(|r_1|^2+|r_2|^2)\right] r_2^{\overline{m}+m-\frac{\Delta}{2}-2} r_1^{\overline{n}+n+\frac{\Delta}{2}} f_{\overline{n}}(r_1)\tilde{f}_n(r_1)f_{\overline{m}}(r_2)f_m(r_2)\left(-\frac{1}{4}r_2^2\right)\right\}, \tag{B14}$$

where $f_l(r)=2l+2-r^2$ and $\tilde{f}_l(r)=2l+6-r^2$.

Explicitly, for the lowest LL,

$$I_1=I(l,l,0,0)=l!\left\{-\frac{1}{2}+\frac{2^l}{2}-\frac{2^l}{4}\right\}, \tag{B15}$$

$$I_2=I(0,0,l,l)=l!\left\{-\frac{2^l}{4}\right\}, \tag{B16}$$

$$I_3=I(0,l,l,0)=l!\left\{-\frac{1}{4}\right\}, \tag{B17}$$

$$I_4=I(l,0,0,l)=l!\left\{-\frac{1}{4}\right\}. \tag{B18}$$

Therefore,

$$W_l^2\sim(-I_1-I_2+I_3+I_4)=0 \tag{B19}$$

for any $l=1,3,5,\ldots$.

Explicitly, for the second LL,

$$\tilde{I}_1=\tilde{I}(l-1,l-1,-1,-1)=l!\left\{-\frac{l^2+2l+5}{2}+2^l\right\}, \tag{B20}$$

$$\tilde{I}_2=\tilde{I}(-1,-1,l-1,l-1)=l!\left\{-\frac{l+1}{2}+2^l\right\}, \tag{B21}$$

$$\tilde{I}_3=\tilde{I}(-1,l-1,l-1,-1)=l!\left\{-\frac{3}{2}+\frac{5}{4}l-\frac{l^2}{4}+2^{l+1}\right\}, \tag{B22}$$

$$\tilde{I}_4=\tilde{I}(l-1,-1,-1,l-1)=l!\left\{-\frac{l^2-5l+6}{4}\right\}. \tag{B23}$$

Therefore,

$$\tilde{W}_l^2=\frac{(-\tilde{I}_1-\tilde{I}_2+\tilde{I}_3+\tilde{I}_4)}{2^{l+3}l!}=\frac{l}{2^{l+1}} \tag{B24}$$

for a given $l=1,3,5,\ldots$.

[1] R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Observation of An Even-Denominator Quantum Number in the Fractional Quantum Hall Effect, Phys. Rev. Lett. **59**, 1776 (1987).

[2] G. Moore and N. Read, Nonabelions in the fractional quantum hall effect, Nucl. Phys. B **360**, 362 (1991).

[3] M. Banerjee, M. Heiblum, V. Umansky, D. E. Feldman, Y. Oreg, and A. Stern, Observation of half-integer thermal Hall conductance, Nature (London) **559**, 205 (2018).

[4] K. K. W. Ma and D. E. Feldman, The sixteenfold way and the quantum Hall effect at half-integer filling factors, Phys. Rev. B **100**, 035302 (2019).

[5] D. F. Mross, Y. Oreg, A. Stern, G. Margalit, and M. Heiblum, Theory of Disorder-Induced Half-Integer Thermal Hall Conductance, Phys. Rev. Lett. **121**, 026801 (2018).

[6] C. Wang, A. Vishwanath, and B. I. Halperin, Topological order from disorder and the quantized Hall thermal metal: Possible applications to the $\nu=5/2$ state, Phys. Rev. B **98**, 045112 (2018).

[7] B. Lian and J. Wang, Theory of the disordered $\nu=5/2$ quantum thermal Hall state: Emergent symmetry and phase diagram, Phys. Rev. B **97**, 165124 (2018).

[8] S. H. Simon, Interpretation of thermal conductance of the $\nu=5/2$ edge, Phys. Rev. B **97**, 121406(R) (2018).

[9] P. T. Zucker and D. E. Feldman, Stabilization of the Particle-Hole Pfaffian Order by Landau-Level Mixing and Impurities That Break Particle-Hole Symmetry, Phys. Rev. Lett. **117**, 096802 (2016).

[10] L. Antonić, J. Vučičević, and M. V. Milovanović, Paired states at 5/2: Particle-hole Pfaffian and particle-hole symmetry breaking, Phys. Rev. B **98**, 115107 (2018).

[11] E. H. Rezayi and F. D. M. Haldane, Incompressible Paired Hall State, Stripe Order, and the Composite Fermion Liquid Phase in Half-Filled Landau Levels, Phys. Rev. Lett. **84**, 4685 (2000).

[12] M. R. Peterson, K. Park, and S. Das Sarma, Spontaneous Particle-Hole Symmetry Breaking in the $\nu = 5/2$ Fractional Quantum Hall Effect, Phys. Rev. Lett. **101**, 156803 (2008).

[13] M. R. Peterson, T. Jolicoeur, and S. Das Sarma, Finite-Layer Thickness Stabilizes the Pfaffian State for the 5/2 Fractional Quantum Hall Effect: Wave Function Overlap and Topological Degeneracy, Phys. Rev. Lett. **101**, 016807 (2008).

[14] A. E. Feiguin, E. Rezayi, C. Nayak, and S. Das Sarma, Density Matrix Renormalization Group Study of Incompressible Fractional Quantum Hall States, Phys. Rev. Lett. **100**, 166803 (2008).

[15] H. Wang, D. N. Sheng, and F. D. M. Haldane, Particle-hole symmetry breaking and the $\nu = 5/2$ fractional quantum Hall effect, Phys. Rev. B **80**, 241311(R) (2009).

[16] A. Wojs, C. Toke, and J. K. Jain, Landau-Level Mixing and the Emergence of Pfaffian Excitations for the 5/2 Fractional Quantum Hall Effect, Phys. Rev. Lett. **105**, 096802 (2010).

[17] M. Storni, R. H. Morf, and S. Das Sarma, Fractional Quantum Hall State at $\nu = 5/2$ and the Moore-Read Pfaffian, Phys. Rev. Lett. **104**, 076803 (2010).

[18] E. H. Rezayi and S. H. Simon, Breaking of Particle-Hole Symmetry by Landau Level Mixing in the $\nu = 5/2$ Quantized Hall State, Phys. Rev. Lett. **106**, 116801 (2011).

[19] K. Pakrouski, M. R. Peterson, T. Jolicoeur, V. W. Scarola, C. Nayak, and M. Troyer, Phase Diagram of the $\nu = 5/2$ Fractional Quantum Hall Effect: Effects of Landau-Level Mixing and Nonzero Width, Phys. Rev. X **5**, 021004 (2015).

[20] M. P. Zaletel, R. S. K. Mong, F. Pollmann, and E. H. Rezayi, Infinite density matrix renormalization group for multicomponent quantum Hall systems, Phys. Rev. B **91**, 045115 (2015).

[21] A. Tylan-Tyler and Y. Lyanda-Geller, Phase diagram and edge states of the $\nu = 5/2$ fractional quantum Hall state with Landau level mixing and finite well thickness, Phys. Rev. B **91**, 205404 (2015).

[22] E. H. Rezayi, Landau Level Mixing and the Ground State of the $\nu = 5/2$ Quantum Hall Effect, Phys. Rev. Lett. **119**, 026801 (2017).

[23] W. Hutzel, J. J. McCord, P. T. Raum, B. Stern, H. Wang, V. W. Scarola, and M. R. Peterson, Particle-hole-symmetric model for a paired fractional quantum Hall state in a half-filled Landau level, Phys. Rev. B **99**, 045126 (2019).

[24] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Particle-Hole Symmetry and the $\nu = 5/2$ Quantum Hall State, Phys. Rev. Lett. **99**, 236807 (2007).

[25] M. Levin, B. I. Halperin, and B. Rosenow, Particle-Hole Symmetry and the Pfaffian State, Phys. Rev. Lett. **99**, 236806 (2007).

[26] M. Greiter, X. G. Wen, and F. Wilczek, Paired Hall states, Nucl. Phys. B **374**, 567 (1992).

[27] R. V. Mishmash, D. F. Mross, J. Alicea, and O. I. Motrunich, Numerical exploration of trial wave functions for the particle-hole-symmetric Pfaffian, Phys. Rev. B **98**, 081107(R) (2018).

[28] D. T. Son, Is the Composite Fermion a Dirac Particle?, Phys. Rev. X **5**, 031027 (2015).

[29] M. V. Milovanović, Paired states in half-filled Landau levels, Phys. Rev. B **95**, 235304 (2017).

[30] N. Regnault (unpublished).

[31] A. C. Balram, M. Barkeshli, and M. S. Rudner, Parton construction of a wave function in the anti-Pfaffian phase, Phys. Rev. B **98**, 035127 (2018).

[32] I. Sodemann and A. H. MacDonald, Landau level mixing and the fractional quantum Hall effect, Phys. Rev. B **87**, 245425 (2013).

[33] B. I. Halperin, P. A. Lee, and N. Read, Theory of the half-filled Landau level, Phys. Rev. B **47**, 7312 (1993).

[34] A. C. Potter, M. Serbyn, and A. Vishwanath, Thermoelectric Transport Signatures of Dirac Composite Fermions in the Half-Filled Landau Level, Phys. Rev. X **6**, 031026 (2016).

[35] C. Wang, N. R. Cooper, B. I. Halperin, and A. Stern, Particle-Hole Symmetry in the Fermion-Chern-Simons and Dirac Descriptions of a Half-Filled Landau Level, Phys. Rev. X **7**, 031029 (2017).

[36] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect, Phys. Rev. B **61**, 10267 (2000).

[37] S.-C. Zhang, The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect, Int. J. Mod. Phys. B **6**, 25 (1992).

[38] R. B. Laughlin, Quantized motion of three two-dimensional electrons in a strong magnetic field, Phys. Rev. B **27**, 3383 (1983).

[39] W. Bishara and C. Nayak, Effect of Landau level mixing on the effective interaction between electrons in the fractional quantum Hall regime, Phys. Rev. B **80**, 121302(R) (2009).

[40] M. R. Peterson and C. Nayak, More realistic Hamiltonians for the fractional quantum Hall regime in GaAs and graphene, Phys. Rev. B **87**, 245129 (2013).

[41] S. H. Simon and E. H. Rezayi, Landau level mixing in the perturbative limit, Phys. Rev. B **87**, 155426 (2013).

[42] R. E. Wooten, J. H. Macek, and J. J. Quinn, Including Landau level mixing in numerical studies of the quantum Hall effect, Phys. Rev. B **88**, 155421 (2013).

[43] M. Barkeshli, M. Mulligan, and M. P. A. Fisher, Particle-hole symmetry and the composite Fermi liquid, Phys. Rev. B **92**, 165125 (2015).

[44] S. H. Simon, E. H. Rezayi, and N. R. Cooper, Pseudopotentials for multiparticle interactions in the quantum Hall regime, Phys. Rev. B **75**, 195306 (2007).