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Rapid Communications

Odd-frequency Berezinskii superconductivity in Dirac semimetals

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We formulate a general framework for addressing both odd- and even-frequency superconductivity in Dirac semimetals and demonstrate that the odd-frequency or the Berezinskii pairing can naturally appear in these materials because of the chirality degree of freedom. We show that repulsive frequency-dependent interactions favor the Berezinskii pairing while an attractive electron-electron interaction allows for the BCS pairing. In the case of Dirac and Weyl semimetals at the charge neutrality point, both the conventional BCS and odd-frequency Berezinskii pairings require critical coupling. Since these pairings could originate from physically different mechanisms, our findings pave the way for controlling the realization of the Berezinskii superconductivity in topological semimetals. We also present the density of states with several cusplike features that can serve as an experimentally verifiable signature of the odd-frequency gap.

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Introduction. The odd-frequency (OF) superconductivity, which was first suggested by Berezinskii in the 1970's as a possible order parameter for superfluid ^{3}He [1], continues to be a challenging and interesting problem both from the theoretical and experimental perspective (for reviews of the OF superconductivity, see Refs. [2–4]).

Following the initial attempts with the phonon-mediated interactions in Refs. [5,6], it was proposed that the spin-dependent fluctuations might lead to the realization of the Berezinskii pairing [6,7]. Later, the OF pairing was considered in the Hubbard models for strong-coupling electron-phonon systems [8–10]. However, the fundamental question about the microscopic mechanism of the intrinsic OF superconductivity still awaits clarification. It is known that the OF pairing requires an electron-electron interaction with strong frequency dependence. Compared with the infrared divergence governing an even-frequency (EF) pairing, an OF one seems to be disfavored in the bulk of conventional metals.

There are now numerous platforms where the OF or Berezinskii superconducting states might appear in various condensed matter systems including heterostructures [11–13], multiband [4,14,15] and driven [16] systems, vortices in the type-II superconductors [17], to name but a few examples.

In this Rapid Communication, we provide a general theoretical scheme to study the OF superconductivity in nodal Dirac and Weyl systems. First, by using the effective action approach [18,19], we consider the possibility of intrinsic OF pairing in Dirac semimetals [20–23], which are a natural platform for topologically nontrivial superconductivity [24–26]. Next, we propose a scenario for realizing OF Cooper pairs due to a *repulsive* frequency-dependent interaction. This possibility is plausible since the OF gap is determined by the derivative of the potential with respect to frequency rather than the potential itself. Therefore, the *repulsive* pairing potential can lead to an *effectively attractive* potential for the Berezinskii pairing. This allows for a completely different pairing mechanism, which is usually absent for the EF case. Finally, we demonstrate that the generation of both even- and odd-frequency gaps in Dirac semimetals at the charge neutrality point requires values of the potential strengths that exceed the critical ones. This provides the possibility to rule out the BCS type of superconductivity that is ubiquitous in conventional materials. As an experimental signature, the distinctive cusplike features in the density of states (DOS) are identified. We believe also that our work will stimulate the study of superconductivity in dynamical systems. In particular, it is important in view of a possible realization of OF states in interacting organic Dirac materials reported in Ref. [27] and in twisted bilayer graphene [28].

Model. OF pairing is essentially a time-dependent pairing state, where the inclusion of retardation effects is crucial. Therefore, we choose not to use the effective Hamiltonian language that is challenging and debated [18,19,29]. Instead, we employ the effective action approach. While the derivation is provided in Sec. II of the Supplemental Material (SM) [30], here we present only the key details. The inverse Green's function $G_{\rm N}^{-1}(x_1 - x_2)$ is

$$G_{\rm N}^{-1}(x_1 - x_2) = [i\partial_{t_1} - \hat{H}_{\rm N}(x_1)]\delta(x_1 - x_2) - \frac{\tau_+}{2}\Delta_{\rm MF}(x_1 - x_2) - \frac{\tau_-}{2}\Delta_{\rm MF}^{\dagger}(x_1 - x_2), \quad (1)$$

where $\boldsymbol{\tau}$ are the Pauli matrices acting in the Nambu space, $\tau_\pm = \tau_x \pm i\tau_y$, $x = (t, \mathbf{r})$ is the time-space four-vector, $\Delta_{\rm MF}(x_1 - x_2)$ is the mean-field (MF) gap, and throughout this Rapid Communication, we set $\hbar = c = k_B = 1$. The Nambu

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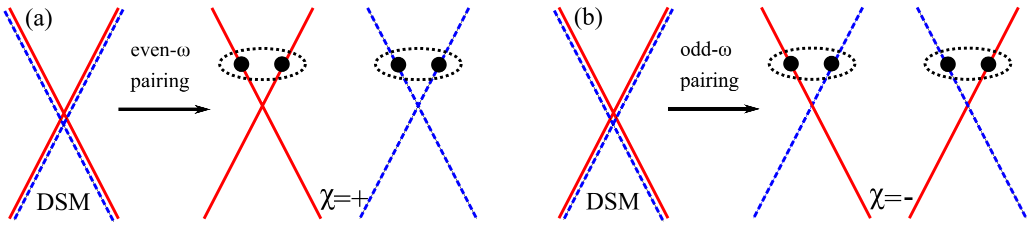

FIG. 1. The spin-singlet, s-wave pairing channels in Dirac semimetals (DSMs) in the case of (a) even- and (b) odd-frequency pairings. The EF gap corresponds to pairing of the quasiparticles of the same chirality (intranode channel). The OF Cooper pairs are allowed by the interchirality pairing. In both panels, red and blue dashed lines correspond to the right- and left-handed quasiparticles.

Hamiltonian $\hat{H}_{\rm N}(x)$ is defined as

$$\hat{H}_{\rm N}(x) = \frac{\mathbb{1}_2+\tau_z}{2}\hat{H}(\mathbf{r}) - \frac{\mathbb{1}_2-\tau_z}{2}\hat{\Theta}\hat{H}(\mathbf{r})\hat{\Theta}^{-1}, \quad (2)$$

where $\hat{H}(\mathbf{r})$ for three-dimensional (3D) Dirac semimetals (DSMs) is defined in Eq. (4) below.

The MF gap equation in the momentum space reads

$$\hat{\Delta}(\omega,\mathbf{k}) = -i\int\frac{d\omega' d^3k'}{(2\pi)^4}\hat{V}(\omega-\omega';\mathbf{k}-\mathbf{k}')\hat{F}(\omega',\mathbf{k}'), \quad (3)$$

where $\hat{F}(\omega',\mathbf{k}')$ is the causal anomalous Green's function. (The normal and anomalous Green's functions are defined in Sec. II of the SM [30].) As usual, the gap equation follows from the extremum of the MF action with respect to $\Delta^{\dagger}_{\rm MF}$.

To study the possibility of OF superconductivity in 3D DSMs, we employ a minimal model with a single Dirac point. The case of a 2D DSM is considered in Sec. VI of the SM [30]. The explicit form of the low-energy Hamiltonian for free electrons reads

$$\hat{H}(\mathbf{r}) = -\mu\mathbb{1}_4 - iv_F\gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}). \quad (4)$$

Here, μ is the electric chemical potential, v_F is the Fermi velocity, and γ^0 and $\boldsymbol{\gamma}$ are mutually anticommuting gamma matrices. The eigenvalues of the $\gamma_5 \equiv i\gamma^0\gamma_x\gamma_y\gamma_z$ matrix correspond to the chirality χ degree of freedom.

Let us discuss the structure of both even- and odd-frequency gaps. In general, two distinctive types of superconducting pairing of chiral fermions can be considered in Weyl and Dirac semimetals [31–38], which are schematically described in Figs. 1(a) and 1(b). For simplicity, we consider the following gaps that are odd and even in frequency,

$$\hat{\Delta}_{\rm odd}(\omega) = i\sigma_y\otimes\mathbb{1}_2\Delta_{\rm odd}(\omega), \quad (5)$$

$$\hat{\Delta}_{\rm even}(\omega) = \mathbb{1}_2\otimes\mathbb{1}_2\Delta_{\rm even}(\omega), \quad (6)$$

respectively. Here, the first matrix in the tensor product acts in the chirality space and the second one is in the pseudospin space. The gap properties are summarized in Table I.

Gap equation. The gap equation (3) is an integral equation that usually determines an unknown gap function $\Delta(\omega,\mathbf{k})$ for a predefined potential. In the case of OF superconductivity, we find it convenient to reformulate the gap equation as an equation for the pairing potential itself. In what follows, a theoretical scheme that is both able to restore the pairing potential via the known gap and determine the gap via the known potential is provided.

For simplicity, we consider only the case of the s-wave pairing, where the dependence on momentum can be omitted, although the approach can be generalized to other cases. Below we concentrate only on the case of vanishing temperature $T\to 0$ and electric chemical potential $\mu\to 0$. In addition, we will consider pairing potentials that do not grow at $\omega\to\infty$, which is indeed the case for physical potentials. Then, performing the transformation $\omega\to i\omega$ in Eq. (3), one obtains the following gap equation,

$$\hat{\Delta}(\omega) = \int_{-\infty}^{\infty} d\omega'\hat{V}(\omega-\omega')\hat{f}(\omega'), \quad (7)$$

where

$$\hat{f}(\omega') \equiv \int\frac{d^3k'}{(2\pi)^4}\hat{F}(\omega',\mathbf{k}'), \quad (8)$$

with the momentum integral taken up to a cutoff Λ_k. The explicit form of $\hat{f}(\omega')$ in the case of odd- and even-frequency pairings is given in Sec. III of the SM [30]. Since the matrix structure of $\hat{f}(\omega')$ coincides with that in $\hat{\Delta}(\omega)$ and $\hat{V}\propto\mathbb{1}_8$, the matrix structures can be omitted.

A convenient method to solve the integral equation (7) is to transform it into the differential one with the appropriate boundary conditions. By factorizing the matrix structure, the gap equation (7) can be approximated as follows [see also Eq. (5) and Sec. III A of the SM [30]),

$$\Delta_{\rm odd}(\omega) = -2\int_0^{\omega} d\omega'\omega' V'(\omega) f_{\rm odd}(\omega') - 2\int_{\omega}^{\infty} d\omega'\omega V'(\omega') f_{\rm odd}(\omega'). \quad (9)$$

We note that only the derivative of the potential with respect to frequency $V'(\omega)\equiv\partial_\omega V(\omega)$ enters the gap equation for the

TABLE I. The odd- and even-frequency gaps given in Eqs. (5) and (6) as well as their symmetry $SP^*\chi T^*$ classification. Here, S, P^*, χ, and T^* denote the symmetry properties with respect to the spin, relative coordinate, chirality, and relative time permutations, respectively. The $SP^*\chi T^* = -1$ rule, which is analogous to the $SP^*OT^* = -1$ in multiorbital systems [3], is satisfied for spin-singlet, s-wave, odd-frequency pairing due to the chirality degree of freedom. (For details, see Sec. I of the SM [30].)

Δ	S	P^*	χ	T^*	Total
$\Delta_{\rm odd}(\omega)$	−	+	−	−	−
$\Delta_{\rm even}(\omega)$	−	+	+	+	−

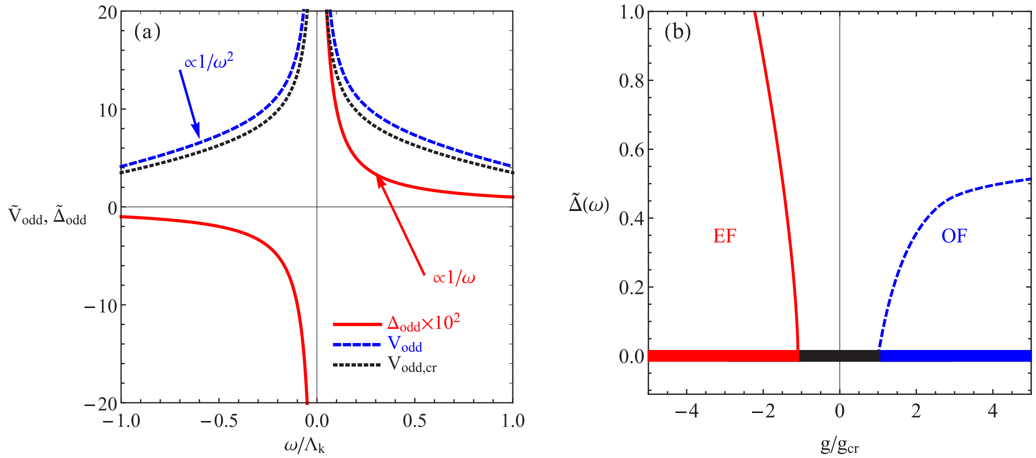

FIG. 2. (a) The OF gap (red solid lines) as well as the pairing potential (blue dashed lines) as functions of frequency at $\alpha = 10^{-2}\Lambda_k$. Black dotted lines represent the critical value of the pairing potential obtained at $\alpha \to 0$. (b) The dependence of the gaps on the coupling constant for the EF (red solid line) and OF (blue dashed line) gaps. The 1D phase diagram is shown by the thick red, blue, and black lines at $\tilde{\Delta} = 0$. For the sake of definiteness, $\omega = 10^{-2}\Lambda_k$. In addition, $\tilde{\Delta}(\omega) = \Delta(\omega)/\Lambda_k$ and $\tilde{V}(\omega) = V(\omega)/|V_{crit}|$, where V_{crit} is given in Eq. (17). Note also that we used a finite frequency cutoff $\Lambda_\omega = 2\Lambda_k$ in the numerical calculations.

OF gap. Thus, we conclude that the Berezinskii pairing can be supported by a wide range of potentials that have a suitable derivative. The differential form of the gap equation is derived in Sec. III A in the SM [30].

The equation for the pairing potential $V(\omega)$ that allows for the OF pairing reads (for the details of the derivation, see Sec. III A of the SM [30])

$$\omega V''(\omega) - V'(\omega) = -\frac{\omega\Delta'_{odd}(\omega) - \Delta_{odd}(\omega)}{2\int_0^\omega d\omega'\omega' f_{odd}(\omega')}. \quad (10)$$

The boundary condition is

$$V'(\omega)\big|_{\omega\to\infty} = -\frac{\Delta_{odd}(\omega)}{2\int_0^\omega d\omega'\omega' f_{odd}(\omega')}\bigg|_{\omega\to\infty}. \quad (11)$$

In addition, in order to fix the pairing potential itself, it is physically reasonable to demand that it vanishes at large frequencies.

Next, let us consider the case of EF pairing. The gap equation (7) can be approximated as

$$\Delta_{even}(\omega) = 2\int_0^\omega d\omega' V(\omega) f_{even}(\omega') + 2\int_\omega^\infty d\omega' V(\omega') f_{even}(\omega'). \quad (12)$$

Unlike the case of OF pairing, the EF gap is sensitive to the potential $V(\omega)$ itself. For the details of the derivation as well as the differential gap equation, see Sec. III B of the SM [30].

The equation for the EF pairing potential is

$$V'(\omega) = \frac{\Delta'_{even}(\omega)}{2\int_0^\omega d\omega' f_{even}(\omega')}, \quad (13)$$

with the boundary condition

$$V(\omega)\big|_{\omega\to\infty} = \frac{\Delta_{even}(\omega)}{2\int_0^\omega d\omega' f_{even}(\omega')}\bigg|_{\omega\to\infty}. \quad (14)$$

Pairing potentials. Let us illustrate the proposed framework by calculating the pairing potentials for a few representative gap *Ansätze*. The simplest EF gap is

$$\Delta_{even}(\omega) = \alpha. \quad (15)$$

As for the OF gap, let us consider an *Ansatz* that produces a vanishing at $\omega \to \infty$ potential,

$$\Delta_{odd}(\omega) = \alpha\frac{\Lambda_k}{\omega}. \quad (16)$$

Due to the complicated form of the functions $f_{odd}(\omega)$ and $f_{even}(\omega)$, which is provided in Sec. III of the SM [30], the solutions for the potentials are obtained numerically. The OF gap as well as the corresponding pairing potential are presented in Fig. 2(a). We found that the potential diminishes approximately as $\propto 1/\omega^2$ at large frequencies and diverges at small ones.

Next, let us discuss the dependence of the even- and odd-frequency gaps on the coupling constant g. The corresponding results are shown in Fig. 2(b). We found that the potential for the EF gap does not depend on frequency and its critical value, which separates the normal and EF superconducting phases, reads as

$$V_{crit} = g_{cr}^{even} = -\frac{8\pi^2 v_F^3}{\Lambda_k^2}. \quad (17)$$

As expected, the pairing potential is attractive in this case. The coupling constant g, which is just a potential itself in this case, should exceed the critical value $|g_{cr}^{even}|$ to allow for an EF gap. Indeed, this is a well-known result in quantum mechanics when the DOS vanishes at the Fermi level (see, e.g., Ref. [39]).

As for the pairing potential for the OF case, it also should be sufficiently strong to generate the gap, i.e., $|g| \geqslant |g_{cr}^{odd}|$ (see also the results in Ref. [40]). This critical value is determined at $\alpha \to 0$ assuming that the potential V can be factorized as $V(\alpha, \omega) = g(\alpha)\tilde{V}(\omega)$. Then, $g/g_{cr}^{odd} = V(\alpha, \omega)/V(0, \omega)$. In

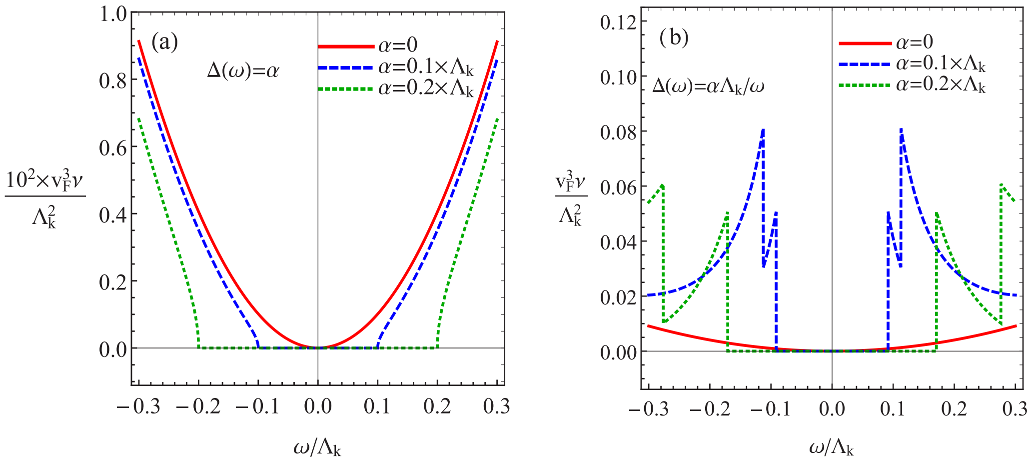

FIG. 3. The dependence of the electron DOS $\nu(\omega)$ on frequency at a few values of the gap strength α for (a) $\Delta(\omega)=\alpha$ and (b) $\Delta(\omega)=\alpha\Lambda_k/\omega$. As expected, the generation of the EF gap pushes the states away from the region of small frequencies, where the spectral weight is recovered at the energy cutoff $\omega\approx\Lambda_k$ (see also Fig. S2 in the SM [30]). The case of the Berezinskii pairing is qualitatively different and is manifested in the formation of the cusplike features at small ω. These features originate from the additional branches of the energy dispersion induced by the OF gap. (The details are provided in Sec. V of the SM [30].)

general, such a separation is not exact, which, however, does not change our main qualitative conclusions. The gap as well as the dependence of the corresponding potentials on frequency are summarized in Table II. Another key result of this study is that the pairing potential for the OF gap (16) can be *repulsive*, $V(\omega)>0$. Microscopically, such pairing potentials might correspond to the dynamically screened Coulomb potential and long-time-tail effects in electronic correlations of disordered systems [41–44]. The results for 2D DSMs, which are presented in Sec. VI in the SM [30], are qualitatively the same.

As an experimental signature of the Berezinskii pairing, we propose to study the DOS, where characteristic double-cusp features universally appear. The corresponding results are shown and discussed in Figs. 3(a) and 3(b) (see also Sec. V of the SM [30]). The cusplike features originate from the poles of the spectral function that are found self-consistently for the frequency-dependent gaps. The main controlling parameter of the splitting is the magnitude of the gap or strength of the interaction potential.

Conclusions. We solved the gap equation for both odd- and even-frequency superconductivity pairings in Dirac semimetals. By using the effective action approach, we derive the integral gap equation and show how to convert it into a differential one with the appropriate boundary conditions. There are two ways how the proposed framework can be utilized. The first is to determine the superconductivity gap via a given potential. The second way is to consider the inverse problem in which the pairing potential is determined via the predefined gap. We start by noting that both even- and odd-frequency pairings require values of the coupling potential that exceed some critical values in Dirac semimetals at the charge neutrality point. This allows us to rule out the ubiquity of the former and puts both pairings on an equal footing.

We show that the gap equation for an EF pairing is determined by the pairing potential itself. On the other hand, the OF gap depends only on the derivatives from the potential with respect to frequency. Thus, the OF pairing can be generated by a *repulsive* potential with an appropriate derivative.

The proposed scheme is illustrated for spin-singlet, *s*-wave gaps. In agreement with the general consideration, the pairing potential for the OF gap is indeed repulsive. The corresponding derivative, which enters the gap equation, is, however, negative. This finding should be contrasted to the case of the EF pairing, where the gap exists only for an attractive interaction. Thus, we suggest another scenario in the search for superconductors that support Berezinskii pairing due to a strongly frequency-dependent repulsive pairing potential.

We compare the key aspects of even- and odd-frequency superconductivity in conventional metals and DSMs in Table III. While the explicit calculations were performed in the case of the DSM within a spin-singlet and *s*-wave channel, we believe that the proposed scenario is quite general and

TABLE II. The even- and odd-frequency gaps as well as the qualitative dependence of the pairing potentials on frequency.

$\Delta(\omega)$	$V(\omega)$
α	const <0
$\alpha\Lambda_k/\omega$	$\propto 1/\omega^2>0$

TABLE III. The key aspects of even- and odd-frequency superconductivity in conventional metals and DSMs. Unlike the BCS superconductivity in metals, the generation of both odd- and even-frequency gaps requires the pairing potential to exceed a certain critical value. Further, OF pairing might be possible for repulsive interactions.

	BCS, EF	OF
Metal: $\epsilon_k=k^2/(2m)-\mu$	$g_{\text{crit}}\to 0$	$\lvert g\rvert>\lvert g_{\text{crit}}\rvert$
DSM: $\epsilon_k=v_F k$	$g<g_{\text{crit}}^{\text{even}}<0$	$g>g_{\text{crit}}^{\text{odd}}>0$

could be realized in various systems. A rigorous study of other pairing channels and their competition will be reported elsewhere.

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