Rapid Communications

Hidden Chern number in one-dimensional non-Hermitian chiral-symmetric systems

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We consider a class of one-dimensional non-Hermitian models with a special type of a chiral symmetry which is related to pseudo-Hermiticity. We show that the topology of a Hamiltonian belonging to this symmetry class is determined by a hidden Chern number described by an effective two-dimensional Hermitian Hamiltonian $H^{\text{eff}}(k,\eta)$, where η is the imaginary part of the energy. This Chern number manifests itself as topologically protected in-gap end states at zero real part of the energy. We show that the bulk-boundary correspondence coming from the hidden Chern number is robust and immune to the non-Hermitian skin effect. We introduce a minimal model Hamiltonian supporting topologically nontrivial phases in this symmetry class, derive its topological phase diagram, and calculate the end states originating from the hidden Chern number.

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Open quantum systems with loss (dissipation) and gain (coherent amplification) are described by non-Hermitian (NH) Hamiltonians [1,2] and have unexpected properties which often depend on the symmetries of the system [1–11]. Adding a non-Hermitian component to a Hamiltonian does not only broaden the resonances and allow the eigenstates to decay, but the eigenmodes can merge with each other at exceptional points, which are topological defects where not only are the eigenvalues degenerate but also the eigenvectors are parallel to each other [1,12–15]. The flexibility to engineer gain and loss in a controllable manner, for example, in optics, (opto)mechanics, plasmonics, superconducting quantum circuits, dissipative Bose-Einstein condensates, exciton-polariton condensates, and cold atom systems [1,2,12,16–32] has naturally raised the interest to study the symmetry and topology in non-Hermitian physics systematically [33–37] with potential applications, for example, in the design of topologically protected laser modes [38–43].

Because in NH systems the energies are complex, the Altland-Zirnbauer symmetry classes support new types of winding number and $\mathbb{Z}_2$ invariants determined by the complex spectra [33], leading to NH topological phases with no Hermitian counterparts [33,35,36]. The classification is further enriched because for a NH Hamiltonian the transpose and complex conjugation are not equivalent so that the ten Altland-Zirnbauer symmetry classes need to be extended [34] to 38 non-Hermitian (nonspatial) symmetry classes [35,36]. Furthermore, Hermitian Hamiltonians are gapped if the energy bands do not cross the Fermi energy, but non-Hermitian systems feature two different types of complex-energy gaps, so-called point (line) gaps where bands do not cross a point (line) in the complex-energy plane, giving rise to further ramifications of the topological classification [36]. Various models and realizations of the different NH topological phases have been proposed [44–58]. For Hermitian Hamiltonians the bulk-boundary correspondence guarantees that the topological invariants for periodic boundary conditions predict the presence of boundary states for open boundary conditions, but this is typically not the case for NH systems [59,60], where the experimental consequences of topological invariants are less clear because the bulk-boundary correspondence can be typically established either on the level of singular value spectra [61] or on the level of biorthogonal density [62,63]. One of the reasons for the breakdown of the bulk-boundary correspondence in NH systems is that the bulk states are often localized in the vicinity of the boundary (NH skin effect) so that the boundary effects are not described by the bulk Bloch Hamiltonian [64–70]. Nevertheless, for particular NH symmetry classes, such as the pseudo-Hermitian Hamiltonians, the bulk-boundary correspondence can be established [36].

From the viewpoint of experiments, particularly relevant one-dimensional (1D) NH Hamiltonians can be constructed by considering Hermitian hopping Hamiltonians with on-site gain and loss terms. Assuming a chiral-symmetric hopping Hamiltonian this type of NH Hamiltonians $\mathcal{H}_k$ satisfy a special kind of *NH chiral symmetry*,

$$\mathcal{S}\mathcal{H}_k\mathcal{S} = -\mathcal{H}_k^\dagger, \tag{1}$$

where $\mathcal{S}$ is a Hermitian unitary operator. Furthermore, in the following we assume that $\mathcal{S}$ is traceless so that the unit cell contains an even number of lattice sites. In the topological classification discussed in Ref. [36], the NH chiral-symmetric Hamiltonians are discussed and the topological invariant and bulk-boundary correspondence has been established [36,71]. They can also be considered in the framework of pseudo-Hermitian Hamiltonians because $i\mathcal{H}_k$ is a pseudo-Hermitian matrix. In this Rapid Communication we establish another perspective on the topological invariants and bulk-boundary correspondence for these Hamiltonians. Namely, we show that the topology of a 1D NH chiral-symmetric Hamiltonian satisfying Eq. (1) is described by an effective 2D Hermitian Hamiltonian $H^{\text{eff}}(k,\eta)$, where η is the imaginary part of the energy. Moreover, we show that $H^{\text{eff}}(k,\eta)$ supports the Chern number as a topological invariant which determines the existence of boundary states also for the non-Hermitian Hamiltonian $\mathcal{H}_{\text{obc}}$ with open boundary conditions via the bulk-boundary correspondence of Hermitian Hamiltonians

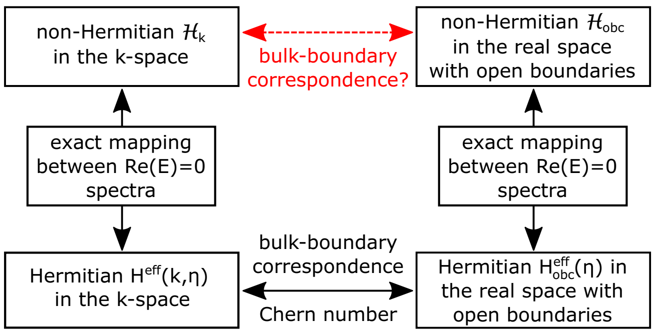

FIG. 1. Schematic representation of the hidden Chern number and bulk-boundary correspondence for the 1D non-Hermitian chiral-symmetric Hamiltonians $\mathcal{H}_k$ satisfying relation (1). The bulk topology of $\mathcal{H}_k$ is described by an effective 2D Hermitian Hamiltonian $H^{\rm eff}(k,\eta)$, where η is the imaginary part of the energy. The 2D Hermitian Hamiltonian $H^{\rm eff}(k,\eta)$ supports the Chern number as a topological invariant which determines the existence of boundary modes for the corresponding Hermitian Hamiltonian $H^{\rm eff}_{\rm obc}(\eta)$ with open boundary conditions. The $H^{\rm eff}_{\rm obc}(\eta)$ determines the existence of boundary states for the non-Hermitian Hamiltonian $\mathcal{H}_{\rm obc}$ with open boundary conditions.

(see Fig. 1). Finally, we introduce a minimal model Hamiltonian supporting topologically nontrivial phases in this symmetry class, derive its topological phase diagram, and calculate the end states originating from the hidden Chern number. The hidden Chern numbers agree with the topological invariant considered in Ref. [36].

By inspecting the characteristic polynomial of $\mathcal{H}_k$ satisfying Eq. (1) we note that if λ is the eigenvalue of $\mathcal{H}_k$, then also is $-\lambda^\star$. Every square matrix can be easily decomposed into Hermitian and anti-Hermitian parts, namely, $\mathcal{H}_k = \mathcal{H}_k^h + \mathcal{H}_k^a$ with $\mathcal{H}_k^{h/a} = \frac{1}{2}(\mathcal{H}_k \pm \mathcal{H}_k^\dagger)$. The chiral relation (1) means that

$$\begin{aligned} \mathcal{S}\mathcal{H}_k^h\mathcal{S} &= -\mathcal{H}_k^h, \\ \mathcal{S}\mathcal{H}_k^a\mathcal{S} &= \mathcal{H}_k^a, \end{aligned} \tag{2}$$

so $\mathcal{S}$ anticommutes with the Hermitian and commutes with the anti-Hermitian part of the Hamiltonian. Moreover, the NH Hamiltonian $\mathcal{H}_k$ satisfies the relation with a Hermitian Hamiltonian H_k (see Ref. [72]),

$$\mathcal{S}\mathcal{H}_k\mathcal{S} = -\mathcal{H}_k^\dagger \iff \mathcal{H}_k = i\mathcal{S}H_k,\ H_k^\dagger = H_k. \tag{3}$$

For traceless $\mathcal{S}$ we find that $\mathcal{H}_k$ satisfying Eq. (1) has a generic block structure in the eigenbasis of $\mathcal{S}$. Namely,

$$\mathcal{H}_k = \begin{pmatrix} iP_k & Q_k^\dagger \\ Q_k & iR_k \end{pmatrix}, \tag{4}$$

where P_k and R_k are $N \times N$ Hermitian matrices and

$$\mathcal{S} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{5}$$

It is worth noticing that any 2×2 real traceless Hamiltonian $\mathcal{H}_k$ can be put in the form of Eq. (4) so it satisfies the chiral symmetry (1), as shown in Ref. [72].

Now consider the real-space version of this Hamiltonian with open boundary conditions $\mathcal{H}_{\rm obc}$. If $\mathcal{H}_k$ satisfies (1) then also $\mathcal{H}_{\rm obc}$ satisfies it with $\mathcal{S}_{\rm obc} = \mathbb{1}_L \otimes \mathcal{S}$, where L is the number of unit cells stacked along the chain. We are interested in the end states of $\mathcal{H}_{\rm obc}$ with zero real part of the energy. Thus, we demand that there exists $\eta \in \mathbb{R}$ such that

$$\mathcal{H}_{\rm obc}|\psi\rangle = i\eta|\psi\rangle. \tag{6}$$

Using (3) we obtain $\mathcal{H}_{\rm obc} = i\mathcal{S}_{\rm obc}H_{\rm obc}$ and $H_{\rm obc}^\dagger = H_{\rm obc}$, so that we get from Eq. (6)

$$(H_{\rm obc} + \eta\mathcal{S}_{\rm obc})|\psi\rangle = 0. \tag{7}$$

Notice that this is a zero-energy eigenproblem for an effective *Hermitian* Hamiltonian,

$$H^{\rm eff}_{\rm obc}(\eta) \equiv H_{\rm obc} + \eta\mathcal{S}_{\rm obc} = \mathcal{S}_{\rm obc}(\eta - i\mathcal{H}_{\rm obc}). \tag{8}$$

We can use this to define a k-space effective Hermitian Hamiltonian,

$$H^{\rm eff}(k,\eta) \equiv H_k + \eta\mathcal{S} = \mathcal{S}(\eta - i\mathcal{H}_k). \tag{9}$$

The above Hamiltonian is defined in a two-dimensional (k,η) space and its Chern number is quantized if it is gapped and it can be compactified in η. A nontrivial Chern number C of $H^{\rm eff}(k,\eta)$ at half filling implies that we have C chiral boundary modes of $H^{\rm eff}_{\rm obc}(\eta)$ crossing the energy gap and it means that we have C solutions of the eigenproblem (7) or (6). Mapping the non-Hermitian Hamiltonians with open and periodic boundary conditions to the same Hermitian problem guarantees that if $\mathcal{H}_k$ has a gapped real spectrum, then $\mathcal{H}_{\rm obc}$ also has a gapped real spectrum, meaning that the non-Hermitian skin effect does not lead to a breakdown of the bulk-boundary correspondence. Moreover, if $H^{\rm eff}(k,\eta)$ is gapped and topological then $H^{\rm eff}_{\rm obc}(\eta)$ supports boundary states, and therefore also $\mathcal{H}_k$ is gapped and $\mathcal{H}_{\rm obc}$ supports end states with zero real part of the energy (see Fig. 1 for the schematic view of this induced bulk-boundary correspondence).

Now what remains is the question of quantization of the Chern number of $H^{\rm eff}(k,\eta)$. We know that it is quantized as long as $H^{\rm eff}(k,\eta)$ is periodic in k and η. Periodicity in k is obvious but in the canonical basis where Eqs. (4) and (5) are satisfied we obtain

$$H^{\rm eff}(k,-\infty) = -H^{\rm eff}(k,+\infty). \tag{10}$$

We can overcome this problem by defining a compactified version of Hamiltonian $H^{\rm eff}(k,\eta)$ given by

$$H^{\rm eff}_{\rm cp}(k,\eta) = \mathcal{R}_\eta H^{\rm eff}(k,\eta)\mathcal{R}_\eta^\dagger, \tag{11}$$

where

$$\mathcal{R}_\eta = \exp\left[i\frac{\pi}{4}(1+\tanh\eta)\mathcal{G}\right], \quad \mathcal{G} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{12}$$

This way $\mathcal{R}_{-\infty} = \mathbb{1}$ and $\mathcal{R}_{+\infty} = i\mathcal{G}$ so that $H^{\rm eff}_{\rm cp}(k,\eta)$ is compactified in η as

$$H^{\rm eff}_{\rm cp}(k,\eta\to-\infty) = H^{\rm eff}_{\rm cp}(k,\eta\to+\infty) = -|\eta|\mathcal{S}. \tag{13}$$

Note that the spectrum of $H^{\rm eff}_{\rm cp}(k,\eta)$ and $H^{\rm eff}(k,\eta)$ is the same. Now, the Chern number C for $H^{\rm eff}_{\rm cp}(k,\eta)$ can be obtained using the Kubo formula [73,74],

$$C = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\eta \int_0^{2\pi} dk\,\Omega_{k,\eta}, \tag{14}$$

where the Berry curvature $\Omega_{k,\eta}$ is given by

$$\Omega_{k,\eta} = \sum_{\substack{n\leqslant n_F\\ m>n_F}} \Im \frac{2\langle\psi^n_{k,\eta}|\partial_k H^{\rm eff}_{\rm cp}|\psi^m_{k,\eta}\rangle\langle\psi^m_{k,\eta}|\partial_\eta H^{\rm eff}_{\rm cp}|\psi^n_{k,\eta}\rangle}{\left(E^{(n)}_{k,\eta} - E^{(m)}_{k,\eta}\right)^2}. \quad (15)$$

Here, $|\psi^p_{k,\eta}\rangle$ are eigenstates of $H^{\rm eff}_{\rm cmp}(k,\eta)$ (sorted in ascending order of eigenenergy) and n_F is the number of occupied bands.

In what follows, we focus on a minimal tight-binding model with uniform hopping t supporting nontrivial phases. In this model we have a four-site unit cell with gain and loss terms g_1, g_2, g_3, and g_4, so that the NH Hamiltonian is

$$\mathcal{H}_k = \begin{pmatrix} ig_1 & t & 0 & te^{-ik} \\ t & ig_2 & t & 0 \\ 0 & t & ig_3 & t \\ te^{ik} & 0 & t & ig_4 \end{pmatrix}. \quad (16)$$

Without loss of generality we can assume that

$$g_4 = -g_1 - g_2 - g_3. \quad (17)$$

This model satisfies chiral relation (1) with $\mathcal{S} = \mathbb{1} \otimes \sigma_z$. From Eq. (9) we find that the effective Hermitian Hamiltonian for our model is

$$H^{\rm eff}(k,\eta) = \begin{pmatrix} \eta+g_1 & -it & 0 & -ite^{-ik} \\ it & -\eta-g_2 & it & 0 \\ 0 & -it & \eta+g_3 & -it \\ ite^{ik} & 0 & it & -\eta-g_4 \end{pmatrix}, \quad (18)$$

and similarly from Eq. (8) we can get $H^{\rm eff}_{\rm obc}(\eta)$. The phase diagram of $H^{\rm eff}(k,\eta)$ is given in Fig. 2. The gapped phases are located mainly in the plane $g_3 = -g_1$ and we find two nontrivial ones with hidden Chern numbers $C = -1$ along the direction $g_1 = -g_2$ and two trivial ones with $C = 0$ along the direction $g_1 = g_2$. For better clarity Fig. 3 shows two-dimensional phase diagrams in the planes $g_3 = -g_1$ and $g_3 = -g_1 + 0.2t$ (see also Ref. [72]).

In Fig. 4 we show the spectra of the Hamiltonian $H^{\rm eff}_{\rm obc}(\eta)$ with open boundary conditions as a function of η for the gapped phases shown in Fig. 3(a). In the nontrivial phase [Fig. 4(a)] we can see that the gap around zero energy is crossed by two states connecting the valence and conduction band. Each of them crosses the zero-energy level once and values of η at which this happens correspond to two solutions of Eq. (6), so that in the case of the original non-Hermitian Hamiltonian $\mathcal{H}_{\rm obc}$ the real part of the eigenenergy is zero. In the trivial case [Fig. 4(b)] these states are missing.

Figure 5 shows energy spectra for a topologically nontrivial non-Hermitian system with open boundary conditions. We find two end states with zero real part of the energy and the imaginary parts of the energies stick out of the bulk spectrum. As one can see from Fig. 5(c) the states with zero real part of the energy are strongly localized at the left or right end of the chain. The rest of the states are delocalized in the bulk.

We have also computed the topological invariant for the model Hamiltonian [Eq. (16)] using the approach discussed

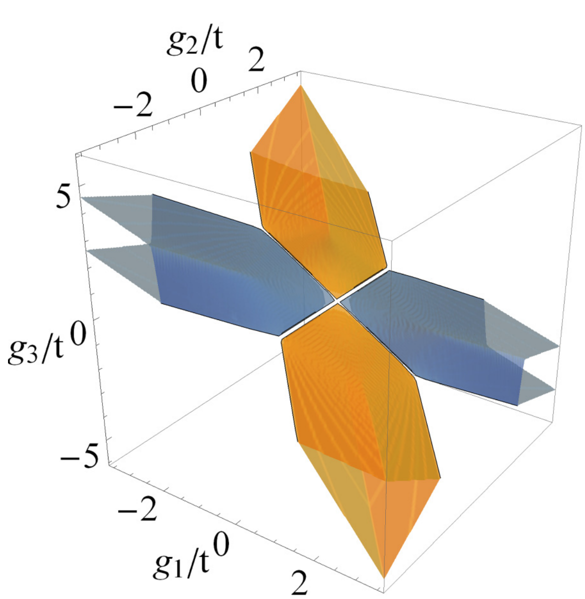

FIG. 2. Phase diagram of the effective Hermitian model (18). Inside the colored surfaces there are gapped phases with Chern numbers $C = -1$ (orange) and $C = 0$ (blue). Outside the colored surfaces there are gapless phases with indirect gap closings.

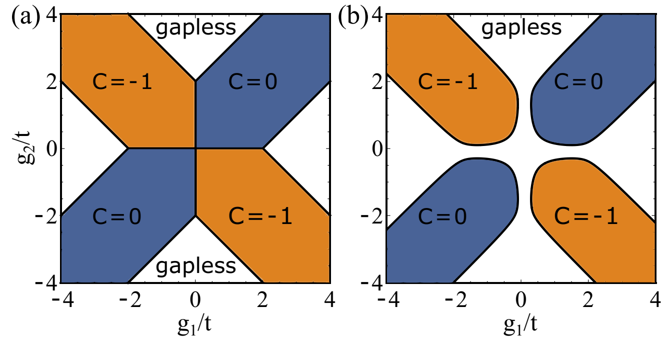

FIG. 3. Phase diagram of the effective Hermitian model (18) for (a) $g_3 = -g_1$ and (b) $g_3 = -g_1 + 0.2t$. Inside the colored surfaces there are gapped phases with Chern numbers $C = -1$ (orange) and $C = 0$ (blue). Outside the colored regions there are gapless phases with indirect gap closings.

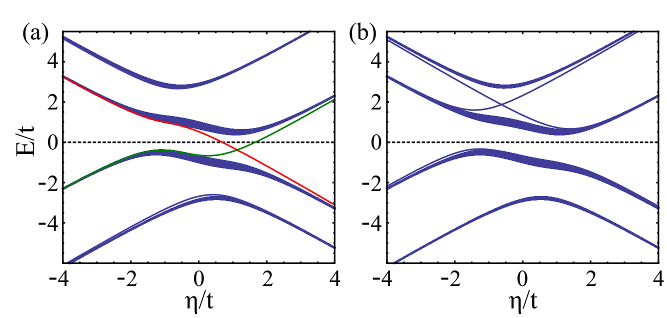

FIG. 4. Spectrum of the Hermitian Hamiltonian $H^{\rm eff}_{\rm obc}(\eta)$ as a function of η for (a) $g_1 = 2t$, $g_2 = -t$ ($C = -1$) and (b) $g_1 = -2t$, $g_2 = -t$ ($C = 0$). In both cases $g_3 = -g_1$. The zero-energy level is marked with a dashed line and the boundary states localized at the opposite ends of the chain are shown in red and green.

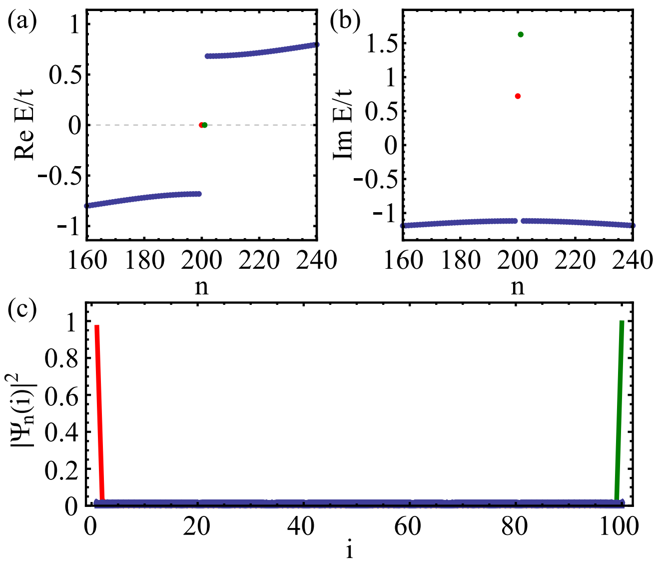

FIG. 5. (a), (b) Real and imaginary parts of the eigenenergy spectrum of the non-Hermitian Hamiltonian $\mathcal{H}_{\rm obc}$ (16) with open boundary conditions. States are ordered by increasing real part of the energy and n enumerates states. (c) Local density of states for each eigenstate. Parameters are $g_1 = t$, $g_2 = -2.2t$, and $g_3 = -g_1 + 0.2t$, and the system length is $L = 100$ sites. Red/green dots or lines correspond to the boundary states localized at the opposite ends of the chain.

in Ref. [36]. We find that nontrivial invariants appear exactly in the same gapped regions where our hidden Chern number is nonzero. Therefore, we conclude that the hidden Chern number gives an alternative perspective on topological invariants and bulk-boundary correspondence of NH chiral-symmetric Hamiltonians.

In summary, we have shown that one-dimensional non-Hermitian chiral-symmetric models support a hidden Chern number as a topological invariant. The Chern number determines the number of end states with zero real part of the energy and the end states are immune to the non-Hermitian skin effect. Moreover, we have calculated the topological phase diagram and end states for a minimal 4×4 gain and loss model that supports a nontrivial topological phase. Our approach gives another perspective on the topological invariants and bulk-boundary correspondence of non-Hermitian systems and the idea can be generalized to various dimensions and symmetries. Although there exist several proposals for realizing topological invariants of Hermitian systems in lower-dimensional non-Hermitian systems [75,76], the idea to utilize the imaginary part of the energy as another dimension has not been explored so far. We point out that also gapless phases can support hidden Chern numbers, and, in particular, the gapless phase of the non-Hermitian Hamiltonian (16) carries a hidden Chern number $C = -1$ in the vicinity of topologically nontrivial gapped phase, leading to localized topological end states [72]. The end states can be utilized, for example, in laser modes which have a topologically protected frequency stability.

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