Rapid Communications

Random network models with variable disorder of geometry

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Recently it was shown [I. A. Gruzberg, A. Klümper, W. Nuding and A. Sedrakyan, Phys. Rev. B **95**, 125414 (2017)] that taking into account random positions of scattering nodes in the network model with $U(1)$ phase disorder yields a localization length exponent 2.37 ± 0.011 for plateau transitions in the integer quantum Hall effect. This is in striking agreement with the experimental value of 2.38 ± 0.06. Randomness of the network was modeled by replacing standard scattering nodes of a regular network by pure tunneling (respectively, reflection) with probability p where the particular value $p = 1/3$ was chosen. Here we investigate the role played by the strength of the geometric disorder, i.e., the value of p. We consider random networks with arbitrary probability $0 < p < 1/2$ for extreme cases and show the presence of a line of critical points with varying localization length indices having a minimum located at $p = 1/3$.

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Introduction. The physics of plateau transitions in the integer quantum Hall effect (IQHE) poses a crucial condensed matter problem potentially necessitating a new understanding of quantum criticality. It relates not only to chiral systems where time-reversal symmetry (TRS) is broken, but also to topological insulators (TIs) with TRS. This transition is an example of a metal-insulator transition in two dimensions where TRS is broken due to the presence of a magnetic field. In their seminal paper [1] Chalker and Coddington suggested a phenomenological model (CC model) for edge excitations in magnetic fields, where the disorder potential creates a scattering network based on quantum tunneling between Fermi levels of neighbor Fermi "puddles" in the ground state. For simplicity, the authors suggested that the scattering nodes in the landscape of the random potential are disposed regularly, while the information about randomness is coded in random $U(1)$ phases associated with the links of the network. During the last 30 years there were huge activities [2–19] in understanding the CC model, its continuum limit, and links to conformal field theories [19–22]. A spin Hall analog of the CC model was formulated [23] and investigated in [24]. It appeared, however, that the position of scattering nodes on a regular lattice loses an essential part of the randomness of the potential. Numerical calculations of the Lyapunov exponent in the CC model give a localization length index $\nu = 2.56 \pm 0.011$ [25–30], which is well separated from the experimental value $\nu = 2.38 \pm 0.06$ [11,31,32]. Recently, alternatives to the CC model approach give values $\nu \simeq 2.58$ in Ref. [33] and $\nu = 2.48 \pm 0.02$ in [34] with only the latter being just compatible with the experimental result.

The discrepancy between the experimental value of ν and the CC model prediction may be due to the importance of electron-electron interactions studied in papers [35–39]. However, another solution of this problem was proposed in the paper [40], based on the observation that randomness of the relative positions of nearest-neighbor scattering nodes has to be taken into account. For a depiction of a disorder potential with nonregular positioned saddle points, see Fig. 1. This randomness of the network leads to the appearance of curvature in 2D space and may be regarded as the induction of quenched 2D gravity, which changes the universality class of the problem. In order to generate disordered networks in the transfer-matrix formalism, a new model was formulated, where the regular scattering with S matrix $S = \left(\begin{smallmatrix} r & t \\ -t & r \end{smallmatrix}\right)$ at the saddle points is randomly replaced by two other extreme events. Here, the S matrix takes the form of complete reflection, $(t, r) = (0, 1)$, with probability p_1 or the form of complete tunneling, $(t, r) = (1, 0)$, with probability p_2 as presented in Fig. 2. The probability of regular scattering events is $p_3 = 1 - p_1 - p_2$. The two extreme scattering events eliminate links in the scattering network. They perform a kind of "surgery" to a flat network where n faces with $n = 3, 5, 6, \ldots$ appear in the lattice. Examples of such surgery are presented in Fig. 2. Following this procedure we can formulate a hopping model of fermions on a random Manhattan lattice (ML), as is presented in Fig. 3, which corresponds to the landscape of the potential presented in Fig. 1.

The appearance of n faces in the ML means that our 2D geometry is not flat anymore and contains local Gaussian curvature $R_n = \frac{\pi}{2}(4 - n)$ for each n face with $n \neq 4$. This is the discrete analog of the Gaussian curvature integrated over a face $\int_{\text{face}} R\sqrt{g}\, d^2\xi$. Hence, the average over randomness of the saddle points leads to the average over all configurations of the curved space [41,42], with yet to be determined functional measure. The field, which is characterizing different surfaces and by use of which one can ensure reparametrization invariance of the model is the metric, while the corresponding theory is 2D gravity. This indicates that we have a noncritical

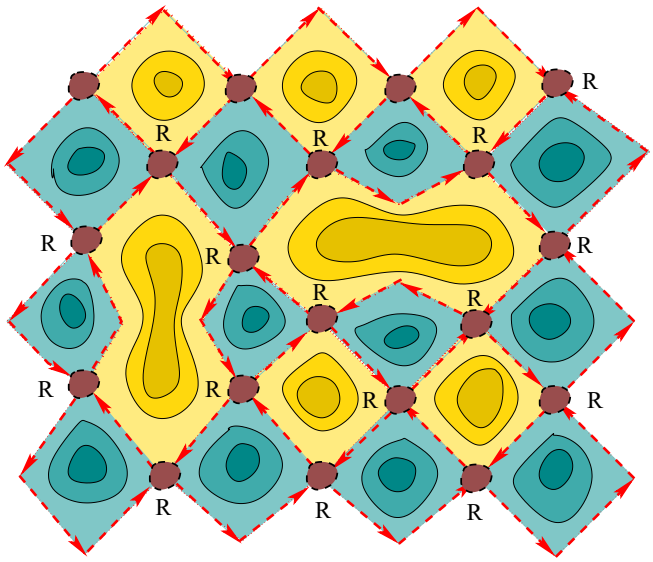

FIG. 1. Modified CC network with two "open" nodes, one in the vertical and one in the horizontal direction.

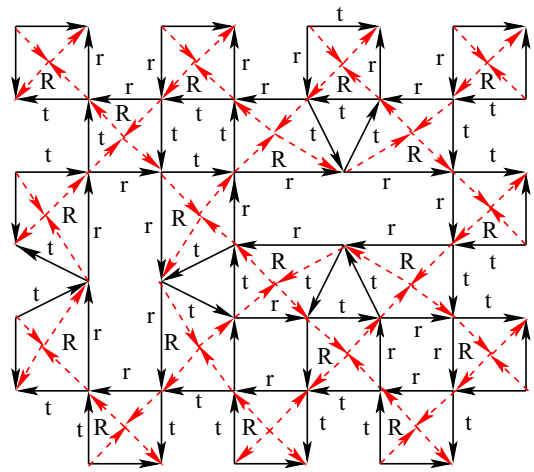

FIG. 3. The modified medial lattice corresponding to the network shown in Fig. 1. We see two hexagons (vertically and horizontally oriented) appearing in neighborhood with two triangles each. Red dotted lines here correspond to red dotted lines in Fig. 1.

string model, where all physical variables should be invariant under arbitrary coordinate transformations. The appearance of this new field and symmetry is the reason for the changes of the critical indices of the flat problem. And, as it appeared [40], by taking equal probability $1/3$ for each of the three nodes (complete reflection, complete transmission, regular scattering), the localization length index becomes $\nu = 2.37 \pm 0.011$, very close to the experimental value. In passing, we like to remark that recently the problem of the fractional quantum Hall effect on arbitrary gravitational background has attracted considerable interest [43–45].

In Refs. [33,34] the regular tight-binding lattice model in a magnetic field with random site energies was numerically analyzed. In [33] the authors considered the one-particle Green's function and got $\nu = 2.58(4)$ for the correlation length index, while in [34] the density of states around zero energy was analyzed yielding $\nu = 2.48 \pm 0.02$. The first paper confirms the result for the standard CC model. Our modified CC model differs essentially from this model because it contains information about the geometry of filled Landau levels, which form "lakes" in a random potential background. Assuming

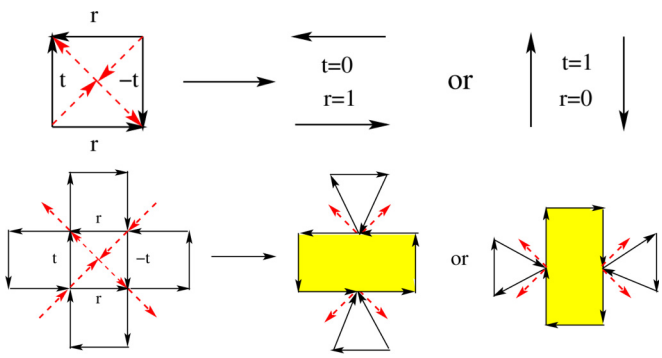

FIG. 2. Top: Graphical illustration of the opening of a node. Bottom: The resulting modifications of the medial lattice.

that the numerical analysis of both models was done on sufficiently large lattices we have to conclude that the models with different critical indices belong to different universality classes.

Another important question appearing here is the validity of the Harris criterion [46,47]. According to it, for $d\nu > 2$ with d being the spatial dimension, any new disorder cannot change the critical index ν of the system. For the CC model we find $\nu \sim 2.56$ [25–30] so the above condition is fulfilled. Therefore one naively may expect that disorder connected with randomness of the network cannot change the CC model localization length index.

The fundamental arguments leading to the Harris criterion are based on the following observations (see, for instance, [48]): We consider a system at some temperature T close to the critical temperature T_c of the ordered bulk. We then divide the space into correlated blocks of the size of the correlation length $\xi(T)$. Each block i has its own realization of disorder and has a corresponding transition temperature $T_{c,i}$. If the deviation of the critical temperatures, thanks to the central limit theorem of the order $\Delta T_i \sim \xi^{-d/2}$, is smaller than the distance of the actual temperature T from the critical point $T - T_c \sim \xi^{-1/\nu}$, then a uniform phase transition happens and the disorder is irrelevant. In the other case, different blocks may stay on different sides of the critical point T_c and far from it which will change the critical behavior. This is the case of geometric disorder involving a finite fraction of extreme nodes with $(t, r) = (0, 1)$ or $(t, r) = (1, 0)$ deviating considerably from the CC critical point $r_c = t_c = 1/\sqrt{2}$. In general, it appears questionable if the RG perturbative reasoning applies to strong disorder. Investigations concerning this issue are available [47]. In summary, the presented arguments cannot be considered as proof and the influence of geometric disorder on the applicability of Harris' criterion needs further investigation.

A natural question that appears is, what is the meaning of probabilities p_i, $i = 1, 2, 3$ and do the critical indices of the model depend on them? In this Rapid Communication

we consider the model with singular blocks [see Figs. 2(b) and 2(c)] appearing in the network with equal probabilities $p_1 = p_2 = p$, while the regular scattering has the probability $1 - 2p$. It is clear that $p \leqslant 1/2$.

Construction and simulation of random networks. For the calculation of the correlation length index of our model we used a variant of the transfer-matrix method formulated in Refs. [49,50] and further developed in Refs. [1,26]. We calculate the product

$$\mathcal{T}_L = \prod_{j=1}^{L} T_{1j} U_{1j} T_{2j} U_{2j} \tag{1}$$

of layers of transfer matrices $M_1 U_{1j} M_2 U_{2j}$ corresponding to two columns T_{1j} and T_{2j} of vertical sequences of 2×2 scattering nodes,

$$T_{1j} = \begin{pmatrix} T^1_{\alpha_1} & 0 & \cdots & 0 \\ 0 & T^1_{\alpha_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T^1_{\alpha_M} \end{pmatrix} \tag{2}$$

and

$$T_{2j} = \begin{pmatrix} [T^2_{\alpha_1}]_{22} & 0 & \cdots & 0 & [T^2_{\alpha_1}]_{21} \\ 0 & T^2_{\alpha_2} & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & T^2_{\alpha_{M-1}} & 0 \\ [T^2_{\alpha_1}]_{12} & 0 & \cdots & 0 & [T^2_{\alpha_1}]_{11} \end{pmatrix}. \tag{3}$$

Here the index $\alpha_i = 1, 2, 3$ should be randomly fixed; $\alpha = 1$ with probability $1 - 2p$ for regular scatterings

$$T^1_1 = \begin{pmatrix} 1/t & r/t \\ r/t & 1/t \end{pmatrix}, \quad T^2_1 = \begin{pmatrix} 1/r & t/r \\ t/r & 1/r \end{pmatrix}, \tag{4}$$

or $\alpha = 2, 3$ with probability p, for surgery operations, i.e., "extremal scatterings,"

$$T^{1/2}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T^{1/2}_3 = \begin{pmatrix} 1/\epsilon & \sqrt{1-\epsilon^2}/\epsilon \\ \sqrt{1-\epsilon^2}/\epsilon & 1/\epsilon \end{pmatrix}. \tag{5}$$

The parameter ϵ here is a regularization parameter, which ideally should be set to zero after the calculation of the Lyapunov exponent.

This choice of the transfer matrices corresponds to periodic boundary conditions in the transverse direction. In other words, these transfer matrices describe the random network model on a cylinder.

The U matrices have a simple diagonal form with independent phase factors $U_{nm} = \exp(i\phi_n)\,\delta_{nm}$ for $U = U_{1j}$ and U_{2j}. The parameters t and r of the regular scattering are the transmission and reflection amplitudes at each node and we parametrize them as in the previous paper [40]:

$$t = \frac{1}{\sqrt{1+e^{2x}}} \quad \text{and} \quad r = \frac{1}{\sqrt{1+e^{-2x}}}. \tag{6}$$

The parameter x corresponds to the Fermi energy measured from the Landau band center scaled by the Landau band width. Following paper [40] we expect that the critical point of the model of arbitrary p is still given by the value $t_c^2 = 1/2$ as for the regular nodes corresponding to $x = 0$. The phases ϕ_n are random variables uniformly distributed in the range $[0, 2\pi)$, reflecting that the phase of an electron approaching a saddle point of the random potential is arbitrary.

To extract the exponent ν for random networks, we numerically estimate the Lyapunov exponent γ defined as the smallest positive eigenvalue of

$$\frac{1}{2L} \ln[T_L T_L^\dagger] \tag{7}$$

in the limit $L \to \infty$. In the standard transfer-matrix method one multiplies many transfer matrices for a single realization of disorder and relies on the self-averaging property of Lyapunov exponents. This property in the limit of infinite length of the sample is the subject of the central-limit-type theorem for products of random matrices due to Oseledec [51]. The modification of Ref. [26] that we use here, however, is based on another central-limit-type theorem for products of random matrices due to Tutubalin [52]. This theorem states that the Lyapunov exponents of products of a finite number of random matrices are random numbers whose distribution approaches Gaussian for large sample lengths.

These theorems allow us to simulate ensembles of $N_r = 624$ strips of height M (the number of nodes per column, varying from 20 to 200) in the case of $q = 1/3$ and length $L = 5 \times 10^6$. This is equivalent [26] to the standard transfer-matrix simulation of a single sample of effective length $L_{\text{eff}} = N_r \times L > 3 \times 10^9$, exceeding the longest previously reported sample lengths. Moreover, this method allows for an estimate of the precision of the calculated Lyapunov exponents by means of the standard deviation of those ensembles. The range of the parameter x we have considered is $x \in [0, 0.08]$ which encodes deviations of t from t_c. Then we fit all data of the Lyapunov exponent for pairs of the parameters (M, x) extracting the localization index ν. For each ensemble of the random network we check that the histogram of the Lyapunov exponents is close to a Gaussian.

We use the so-called LU decomposition of transfer matrices [30], because it is faster than the standard QR decomposition approach. Since t and r appear in the denominators

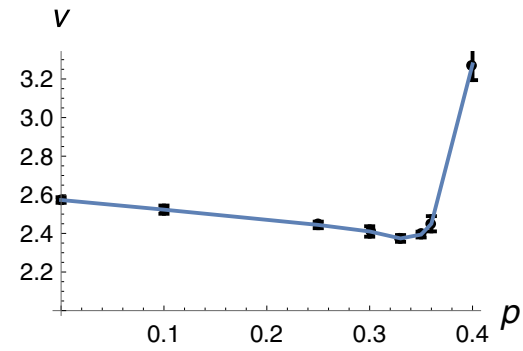

FIG. 4. Localization length index ν_p versus probability p of singular blocks.

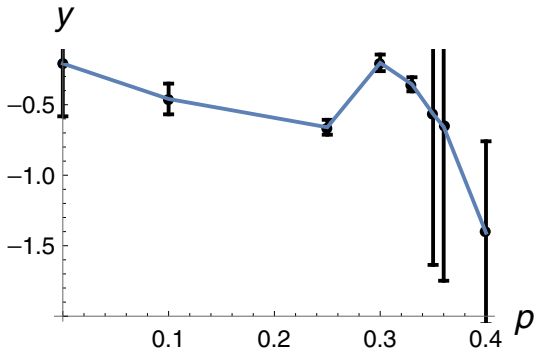

FIG. 5. Subleading index versus probability p of singular blocks.

of the matrix elements of transfer matrices, making them zero is a singular procedure, related to the disappearance of two horizontal channels upon opening a node in the vertical direction (see Fig. 2). To overcome this difficulty, following [40] we take for every open node either t or r to be equal to $\varepsilon \ll 1$. It appears that the result for the Lyapunov exponent is unchanged within our error 10^{-3} in a range from $\varepsilon = 10^{-5}$ up to ε to 10^{-7}. For even smaller ε the results start changing again. This is to be expected because the large differences of values in the entries of transfer matrices cause numerical instabilities for the LU decomposition. Interestingly, we found that the results for the Lyapunov exponents for longer chains depend less on the value of ϵ than for shorter chains. We have chosen $\varepsilon = 10^{-6}$ for our calculations.

As usual, the Lyapunov exponent γ is expected to have the following finite-size scaling behavior:

$$\gamma M = \Gamma[M^{1/\nu} u_0(x), M^{y} u_1(x)]. \tag{8}$$

Here $u_0(x)$ is the relevant field and $u_1(x)$ is the leading irrelevant field. The relevant field vanishes at the critical point and $y < 0$. The fitting and the error analysis of our numerical data are described in the Supplemental Material [53]. We chose the fitting result by goodness of χ^2 and the Akaike information criterion [54]. The results of the analysis as functions of the disorder parameter p are presented in Fig. 4 for the localization length exponent ν, in Fig. 5 for the exponent y of the irrelevant field, and in Fig. 6 for the parameter $\Gamma_c = \pi(\alpha_0 - 2)$ related to the multifractal exponent α_0 [55]. In Table I we present these results as numbers.

TABLE I. Numerical values for the exponents ν, y and the multifractal parameter Γ_c and their uncertainties. Different cases of the disorder parameter $p \in [0, 1[$ are considered.

p	Γ_c	$\Delta\Gamma_c$	ν	$\Delta\nu$	y	Δy
0	0.7823	0.05695	2.573	0.0145	−0.2078	0.3744
0.1	0.816	0.00595	2.523	0.0213	−0.4592	0.1089
0.25	0.8489	0.00295	2.444	0.017	−0.6598	0.0527
0.3	0.08974	0.07275	2.41	0.027	−0.2028	0.0588
1/3	0.864	0.0568	2.374	0.0175	−0.355	0.05
0.35	0.8728	0.04895	2.394	0.015	−0.5661	1.7
0.36	0.8859	0.04395	2.45	0.0395	−0.6562	1.9235
0.4	0.95	0.00465	3.276	0.082	−1.408	0.6487

Figure 4 shows an interesting behavior of ν versus the probability p. We see that a minimum is achieved precisely at $p = 1/3$ which may very likely correspond to the plateau transitions in IQHE. The value $p = 0$ gives $\nu = 2.56$ for the Chalker-Coddington model, just as expected. At $p = 1/2$, where we do not have regular scattering nodes at all the x dependence of γ should disappear. Therefore, one can expect $\nu = \infty$, because precisely in this situation the critical behavior of the Lyapunov exponent of the form x^{ν} will produce zero. As we see from Fig. 4, the index sharply increases close to $p = 1/2$.

Results and summary. In summary, we have considered the possibility that a certain type of geometric disorder, previously missing in the study of the integer QH transition, changes its universality class. Our numerical simulations support this idea. We see that the random occurrence of singular blocks in the network with some probability p leads to a geometry with curvature. The network model has a critical index ν_p that apparently changes continuously with p, i.e., it realizes a line of critical points with different universality classes at different points. The p_1-p_2 phase diagram of the model is presented in Fig. 7, where the diagonal line from zero to A is a line of critical points.

The minimal value of ν at $p = 1/3$ corresponds to the value expected for the exponent of the IQH transitions. The meaning of the other models as well as the meaning of the parameter p remains an open question at the moment. It would not be surprising, if the approaches presented in papers [33] and [34] were related to different pameters p in our model.

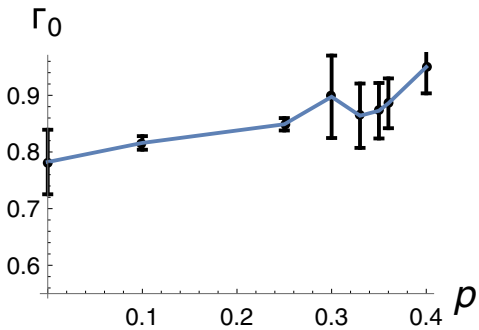

FIG. 6. Coefficient $\Gamma_c = \pi(\alpha_0 - 2)$ related to the multifractal exponent α_0 versus probability p of singular blocks.

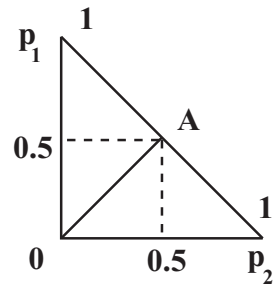

FIG. 7. The (p_1, p_2) phase diagram of the model $(p_1 + p_2 \leqslant 1)$. The segment $[0, A]$ is a line of critical points.

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