Phase transitions in the Rényi entropies of a 2 + 1D large-N interacting vector model

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We consider the reduced density matrix for a disk in the ground state of the interacting fixed points of the large-N $O(N)$ vector model in $2+1$ dimensions. Using the map to the free energy on $\mathbb{H}_2 \times S^1$, we show that there is an instability in the Rényi entropies at $n = 1$ in the $N \to \infty$ limit, indicating a phase transition at or near this Rényi parameter. We close with a discussion of the finite-N case.

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I. INTRODUCTION

Consider a quantum system with Hilbert space $\mathcal{H}$ with a factorization $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Given a pure state $|\Psi\rangle$ such as the ground state, the spectrum of the reduced density-matrix $\rho_A = \mathrm{Tr}_B|\Psi\rangle$ is a measure of the entanglement between $\mathcal{H}_A$ and $\mathcal{H}_B$ in the state ψ. This can contain much useful information about the system at hand, particularly for quantum field theories. A judicious choice of factorization provides order parameters sensitive to zero-temperature quantum phase transitions [1–4]. In the case of two-dimensional (2D) conformal field theories, there is some evidence that the entanglement spectrum corresponding to the ground state, for a suitable set of factorizations, can completely characterize the theories [5], at least, away from limits with duals described by classical Einstein gravity. In the case of theories with a dual gravitational description, the entanglement properties of a given state appear to be key to constructing the dual geometry [6–8].

The entanglement spectrum can be extracted from the entanglement Rényi entropies $S_n(A) = \frac{1}{1-n} \ln \mathrm{Tr}\, \rho^n$ if we are able to compute the latter for all real and positive n. These entropies have a further use, in that if we are able to take the limit $n \to 1$, we can compute the von Neumann entropy $S(A) = -\mathrm{Tr}\, \rho \ln \rho$, which has useful general thermodynamic-type properties. The path-integral implementation of this limit, as a kind of replica trick, is a powerful technique for computing the von Neumann entropies, especially, for conformal field theories [9–11].

We can always write $\rho_A = e^{-2\pi H_A}$ for some "modular Hamiltonian" H_A acting on $\mathcal{H}_A$. In which case, continuously tuning n is equivalent to tuning the temperature for the system with Hilbert space $\mathcal{H}_A$ and Hamiltonian H_A in the canonical ensemble. Extracting the entanglement spectrum from the Rényi entropies $S_n(A)$ is the equivalent to transforming to the microcanonical ensemble from the canonical ensemble in this system. This presentation then begs the question of whether there are any phase transitions as a function of n. Such transitions can point to generic features of the entanglement spectrum, perhaps following the discussion in Ref. [12]. In principle, they could provide an obstruction to computing the von Neumann entropy if the transition is at $n = 1$, although we will point out in the Conclusions that this is not necessarily the case. Such transitions have been found, for example, in the ϵ expansion around $d = 3$ spatial dimensions [13], and a characterization of the onset of a class of phase transitions has been performed for holographic theories in Refs. [14–16].[1]

In this paper, we will consider the interacting infrared (IR) fixed point of the $O(N)$ vector model in $d = 2$ spatial dimensions when A corresponds to the region inside a two-dimensional disk of radius R.[2] In this simple case, H_A can be written as the Hamiltonian on two-dimensional hyperbolic space $\mathbb{H}_2$ [20]. In this background, and in the $N \to \infty$ limit, the classic Hohenberg-Mermin-Wagner-Coleman theorem [21–23] does not immediately pertain [24–26]. We will find that there is an instability in the theory for H_A at $T = \frac{1}{2\pi n} = 1/2\pi$ after the fashion of the instabilities studied in a holographic context [14,15]. This instability is towards a symmetry-breaking phase, indicating a possible second-order phase transition there. At present, we cannot completely rule out the possibility that there is, instead, a first-order phase transition at a higher temperature, although we will argue in the Conclusions that the results of Ref. [27] make this unlikely. That said, the existence of an instability is already interesting and nontrivial and comprises a nontrivial analytic result for the entanglement structure of an interacting quantum field theory. The story at finite N is also unknown to us, but we discuss some issues and possibilities in Sec. IV.

The outline of our paper will be as follows. In Sec. II, we review the map between the disk and the hyperbolic space in order to relate Rènyi entropies and free energies. In Sec. III, we begin with a review of the interacting $O(N)$ model in flat space and proceed to study the theory on hyperbolic space

[1]There is also some evidence for a phase transition for the Rényi entropies for the entanglement of two intervals with their complement in 2D conformal field theories (CFTS) [17,18].

[2]Note that we do not perform a singlet projection as opposed to Ref. [19], so the higher-spin dual in that work is not directly relevant to us.

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$\mathbb{H}_2$. We find that the theory develops an instability at $T = \frac{1}{2\pi}$, which is our main result. In Sec. IV, we argue that combining our results with the work of Ref. [27] points to the transition being second order, and that this transition at $n = 1$ does not invalidate the replica trick calculation. We discuss the possible physics at finite N and outline possible future work. The appendices contain some detailed calculations whose results we use in the main body of the paper.

II. REVIEW: RÉNYI ENTROPIES AS FREE ENERGIES

In this paper, we consider quantum field theories in $d = 2$ spatial dimensions on $\mathbb{R}^2$. We are interested in the entanglement properties of the ground state as measured by the reduced density matrix for degrees of freedom inside a circular disk D_R of radius R.

Although for general quantum theories, we can always write $\rho_A = e^{-2\pi H_A}$, there is no guarantee for a general subregion A in a general quantum field theory (QFT) that the modular Hamiltonian H_A is anything nice or local. However, for CFTs in d-spatial dimensions, when $\mathcal{H}_A$ corresponds to the degrees of freedom inside a d-dimensional spherical ball of radius R, H_A can be written as the Hamiltonian in d-dimensional hyperbolic space $\mathbb{H}_d$ with radius of curvature R [20]. Note that as defined, H_A is dimensionless: We will replace $H_A \to RH_A$ in this formula so that $\rho = e^{-2\pi RH_A}$ is the canonical ensemble density operator for a theory on hyperbolic space with temperature $T = \frac{1}{2\pi R}$.

The map between the density matrices is induced by a conformal transformation. Here, we recall the conformal transformation from the spherical ball to $\mathbb{H}_d$ used by Ref. [20] to establish the relationship between density operators. This maps the domain of dependence of the spherical ball to the hyperbolic space and, consequently, relates the modular Hamiltonian on the causal diamond of the disk to the generator of time translations in hyperbolic space.

One first starts with the flat-space metric,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2. \tag{2.1}$$

Our spherical subregion sits at $r = R,\ t = 0$. We now make the following conformal transformation:

$$t = R\frac{\sinh\left(\frac{\tau}{R}\right)}{\cosh\,\rho + \cosh\left(\frac{\tau}{R}\right)}, \quad r = R\frac{\sinh\,\rho}{\cosh\,\rho + \cosh\left(\frac{\tau}{R}\right)}, \tag{2.2}$$

and obtain the metric on $R \times \mathbb{H}_d$, but for a conformal prefactor,

$$ds^2 = \left[\cosh\,\rho + \cosh\left(\frac{\tau}{R}\right)\right]^{-2} \times [-d\tau^2 + R^2(d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2)], \tag{2.3}$$

where the first term in the square brackets the conformal prefactor in parentheses and the second term inside the square brackets is the metric on $\mathbb{H}_d \times R$.

Given this identification,

(1) Equal-time correlation functions on D_R are equivalent, up to conformal factors, to correlators at equal Euclidean time for the theory on $\mathbb{H}_2 \times S^1_{2\pi R}$—that is, for equilibrium correlators of the CFT on $\mathbb{H}_2$ at temperature $T = 1/2\pi R$.

(2) The Rényi entropies are

$$\frac{1}{1-n}\ln\,\mathrm{Tr}\,\rho^n = \frac{2\pi Rn}{n-1}F(\beta = 2\pi nR), \tag{2.4}$$

where F is the free energy at inverse temperature β.

Our focus will be on the second point: We will use the first point to nail down our renormalization prescription.

For a QFT, most of the entanglement between a spatial region and its complement is carried by ultraviolet degrees of freedom near the boundary line/point/surface between them. An explicit calculation on the Rényi or von Neumann entropies, thus, requires a cutoff. In the case at hand, the cutoff and the radius R are the only dimensional parameters. If we write the cutoff Λ in units of momentum so that the UV corresponds to large Λ, we expect that [13,28]

$$S_n = s_{-1,n}R\Lambda + s_{0,n} + O\left(\frac{1}{\Lambda R}\right) + \cdots. \tag{2.5}$$

The finite piece s_0 is thought to be universal.

Under the conformal transformation, the UV cutoff near the entangling surface becomes an IR cutoff on the volume of $\mathbb{H}_2$. We will treat this with a hard radial cutoff following Refs. [20,29]; the upshot is that one can replace any (divergent) factors of $\mathrm{Vol}(\mathbb{H}_2)$ with -2π [29].

The leading divergence in the entanglement entropies is physical in continuum QFT (as opposed to theories of quantum gravity, in which case, the divergence can be absorbed into the renormalization of Newton's constant). In addition, any explicit representation of the path integral has additional UV divergences. These are standard QFT divergences, removed from physical quantities via a renormalization prescription for the vacuum energy and the couplings in order to properly define the path integral in the first place. We will treat them separately via proper time regularization.

III. THE INTERACTING LARGE-N $O(N)$ VECTOR MODEL

A. Review: the $O(N)$ model on $\mathbb{R}^2$

The theory we will focus on is the interacting IR fixed point of the large-N $O(N)$ vector model in $d = 2$ spatial dimensions. We open with a review of this theory on flat Minkowski spacetime $\mathbb{R}^3$ to introduce the techniques we will use in a more familiar context. This theory can be written as the infrared fixed point of the theory,[3]

$$S = \int d^3x\left[\frac{1}{2}(\partial\vec{\phi})^2 - \frac{h}{4N}(\vec{\phi}^2)^2\right], \tag{3.1}$$

where $\vec{\phi}$ is a N-dimensional vector. The infrared critical point corresponds to the limit $h \to \infty$ (with the limit taken through positive values). We will embed the above Lagrangian in the larger set of theories,

$$S = \int d^3x\left[\frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}m^2\vec{\phi}^2 - \frac{h}{4N}(\vec{\phi}^2)^2\right]. \tag{3.2}$$

[3]Note that there are several presentations of the underlying physics, corresponding to different UV embeddings. See, for example, Refs. [30–33]. In this paper, we use the UV embedding described in Refs. [31,32] and references therein.

We do this because, in our renormalization scheme, quantum fluctuations will push the IR fixed point away from $m^2 = 0$. Furthermore, as we will see, it is interesting to study the theory as a function of m^2.

The theory of Eq. (3.2) supports a phase transition at zero temperature. The equilibrium finite-temperature theory can be studied via a Euclidean path integral on $\mathbb{R}^2 \times S^1_{\beta=1/T}$. At long-distances $\delta x \gg \beta$, we may dimensionally reduce the theory to $d = 2$, and the Hohenberg-Mermin-Wagner-Coleman theorem [21–23] prevents a phase transition in this model. At large N, this implies an infrared singularity in the gap equation. At zero temperature, however, a second-order phase transition can be reached by dialing m^2, and corresponds to an ordering transition in which $\vec{\phi}$ acquires a vacuum expectation value. To set ourselves up for a discussion of the theory on $\mathbb{H}_2$, we review this zero-temperature phase transition.

We start by unpacking the quartic coupling term via a Hubbard-Stratanovich transformation: In Euclidean space, we write

$$S[\vec{\phi}, \sigma] = \int d^3x\sqrt{g}\left(\frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}(m^2 + h\sigma)\vec{\phi}^2 - \frac{hN}{4}\sigma^2\right). \tag{3.3}$$

Integrating out σ in the path integral yields the original action in Eq. (3.2). Note that since the action is quadratic in σ, this is the same as setting $\sigma = \frac{\phi^2}{N}$, that is, the saddle-point value.

Classically, when $m^2 > 0$, the classical vacuum is at $\vec{\phi} = 0$, whereas, for $m^2 < 0$, there is a S^{N-1} family of vacua with $N - 1$ massless Goldstone modes. Quantum effects are important and spoil this transition at finite temperature. At $T = 0$, however, there is a transition at a specific value of m^2, although quantum effects also push this away from $m^2 = 0$.

To find this point, let us first assume we are in the symmetry-broken phase for which $\vec{\phi}$ has an expectation value. We will determine the values of m^2 for which this is self-consistent. We can use the $O(N)$ symmetry to rotate $\vec{\phi}$ into the first component,

$$\vec{\phi} = (\sqrt{N}\phi + \delta\phi, \pi_a), \quad a = 1 \cdots N - 1, \tag{3.4}$$

here, $\phi \geqslant 0$ is the expectation value; $\delta\phi$ the "Higgs mode" when $\phi \neq 0$; and π_a the putative Goldstone modes. Because the Hubbard-Stratanovich transformation renders the action Gaussian in π, we can integrate these modes out, to arrive at the partition function,

$$\begin{aligned} Z[\sigma] = \int [d\sigma][d\phi] \exp\bigg[-N \int d^3x\bigg(\frac{1}{2}(\partial\phi)^2 \\ + \frac{(m^2 + h\sigma)}{2}\phi^2 - \frac{h\sigma^2}{4}\bigg) \\ + \frac{(N-1)}{2}\text{Tr}\ \ln(-\partial^2 + m^2 + h\sigma)\bigg]. \end{aligned} \tag{3.5}$$

The second line is the exponentiated one-loop determinant coming from the Gaussian integral. Note that $m^2_{\text{eff}} = m^2 + h\sigma$ is the effective mass of the putative Goldstone modes π. Equation (3.5) is a large-N result: The effects of integrating out $\delta\phi$ are subleading in N (cf. Ref. [32]).

In the $N \to \infty$ limit, the theory can be solved via the saddle-point method. The result is the pair of gap equations,

$$\begin{gathered} (-\partial^2 + m^2 + h\sigma)\phi = 0, \\ \phi^2 - \sigma + \frac{1}{V}\text{Tr}\left(\frac{1}{-\partial^2 + m^2 + h\sigma}\right) = 0, \end{gathered} \tag{3.6}$$

where V is the regularized volume of $\mathbb{R}^3$. The trace can be computed explicitly,

$$\begin{aligned} \text{Tr}\left(\frac{1}{-\partial^2 + m^2 + h\sigma}\right) &= V\int_{\Lambda^{-2}}^{\infty} ds \int \frac{d^3p}{(2\pi)^{3/2}} e^{-s(p^2+m^2+h\sigma)} \\ &= \frac{V}{2^{3/2}}\int_{\Lambda^{-2}}^{\infty} \frac{ds}{s^{3/2}} e^{-s(m^2+h\sigma)}. \end{aligned} \tag{3.7}$$

We assume that Lorentz invariance will not be broken, in which case we can search for solutions to these equations with constant ϕ, σ. In this case, the first equation in Eq. (3.6) has two branches. In one branch, $\phi = 0$ and $m^2_{\text{eff}} = m^2 + h\sigma$ is unconstrained, although stability of the theory demands that $m^2_{\text{eff}} \geqslant 0$ for the minimum of the free energy. In the other branch, $\phi \neq 0$ and $m^2 + h\sigma = 0$. As we started, self-consistently, with the assumption $\phi > 0$, we will look for solutions in this class. Since $m^2 + h\sigma = 0$, the trace is simple to compute

$$\frac{1}{V}\text{Tr}\left(\frac{1}{-\partial^2 + m^2 + h\sigma}\right) = \frac{\Lambda}{\sqrt{2}}. \tag{3.8}$$

Thus, there is a solution for

$$\phi^2 = \sigma - \frac{\Lambda}{\sqrt{2}} = -\frac{m^2}{h} - \frac{\Lambda}{\sqrt{2}}. \tag{3.9}$$

This has a nonzero solution when $m^2 < m_c^2 = -h\frac{\Lambda}{\sqrt{2}}$.

At $m^2 = m_c^2$, $\phi^2 = 0$, and $m^2_{\text{eff}} = 0$. At long distances, or equivalently when $h \to \infty$ for fixed distances, this describes the conformal point of the model, the interacting $O(N)$ CFT at large N.

For $m^2 > m_c^2$, we must set $\phi^2 = 0$. The fluctuations are now $O(N)$ symmetric, and the remaining action is given in Eq. (3.5) with $N - 1$ replaced by N. At large N, the saddle-point equations are still Eq. (3.6) with the effective-mass $m^2 + h\sigma > 0$. Performing the proper time integral in Eq. (3.7) and discarding terms which vanish as $\Lambda \to \infty$, the saddle-point equation becomes

$$\sigma = \frac{\Lambda}{\sqrt{2}} - \sqrt{\frac{\pi}{2}}\sqrt{m^2 + h\sigma}. \tag{3.10}$$

We can simplify this by defining $\sigma = \frac{\Lambda}{\sqrt{2}} + \delta\sigma$ and $m^2 = m_c^2 + \delta m^2$. Squaring both sides,

$$\delta\sigma^2 = \frac{\pi}{2}(\delta m^2 + h\,\delta\sigma), \tag{3.11}$$

which always has a real solution for $\delta\sigma$; the negative branch $\delta\sigma = \frac{1}{4}(\pi h - \sqrt{\pi^2h^2 + 8\pi\,\delta m^2})$ vanishes as $\delta m^2 \to 0$ as it should.

B. The $O(N)$ model on $\mathbb{H}_2$: setup

We now pass to the system of interest: the ($D = 3$)-dimensional Euclidean $O(N)$ model on the manifold

$\mathbb{H}_2 \times S^1$, where $\mathbb{H}_2$ is the two-dimensional hyperbolic space with metric,

$$ds^2 = R^2(d\rho^2 + \sinh^2 \rho \, d\phi^2), \tag{3.12}$$

where $\rho \in [0, \infty)$ and $\phi \in [0, 2\pi)$. Here, R is the radius of $\mathbb{H}_2$ and is equal to the radius of the disk whose reduced density matrix we are trying to compute.

We work with the action,

$$S[\vec{\phi}] = \int d^3x\sqrt{g}\left[\frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}\left(m^2 + \frac{1}{8}\mathcal{R}\right)\vec{\phi}^2 + \frac{h}{2N}(\vec{\phi}^2)^2\right], \tag{3.13}$$

where g is the determinant of the metric on $\mathbb{H}_2 \times S^1$, $\mathcal{R}$ is the Ricci scalar, and the indices are contracted using the curved space metric. We have included an explicit curvature coupling appropriate for a conformally coupled scalar. For this spacetime, it is constant (and negative) and could be absorbed into m^2: We often simply use the combination $M^2 = m^2 + \frac{1}{8}\mathcal{R}$. As before, this is a theory with a dimension-1 coupling h. In the present case, we can form a dimensionless coupling hR or $h\beta$; the IR corresponds to $h \to \infty$ at fixed R, β or to R, β, and all distances probed scaled to infinity at fixed h. Note that the family of QFTs labeled by h are not, for general h, simply related to the same family in flat space as the coupling is not exactly marginal. We are, nonetheless, assuming that the IR limit $h \to \infty$ corresponds to the interacting $O(N)$ model on $\mathbb{H}_2 \times S^1$. This is manifestly true at distances smaller than R, β; we are, thus, assuming that the effects of the background curvature and topology do not drive Eq. (3.13) to a different fixed point.

Again, we use the Hubbard-Stratonovich trick to rewrite the action using an auxiliary field σ which is also integrated over,

$$S[\vec{\phi}, \sigma] = \int d^3x\sqrt{g}\left(\frac{1}{2}(\partial\vec{\phi})^2 + \frac{1}{2}(M^2 + h\sigma)\vec{\phi}^2 - \frac{hN}{4}\sigma^2\right). \tag{3.14}$$

Our goal is to understand the phase structure of the IR CFT as a function of temperature. We will do this as before by studying the saddle-point equations as $N \to \infty$ after integrating out fluctuations of the scalars $\vec{\phi}$,

$$\begin{gathered}(-\nabla^2_{\mathcal{H}} + M^2 + h\sigma)\phi = 0,\\ \phi^2 - \sigma + \frac{1}{V}\mathrm{Tr}\left(\frac{1}{-\nabla^2_{\mathcal{H}} + M^2 + h\sigma}\right) = 0\end{gathered} \tag{3.15}$$

[up to terms of order $O(1/N)$ as before] where V is the regularized volume of $\mathbb{H}_2 \times S^1$ and $\mathcal{H} = \mathbb{H}_2 \times S^1$. To solve these, we need to choose the proper boundary conditions for σ, ϕ, appropriate for the Rényi entropy calculation. Furthermore, the trace contains UV divergences, and we will find that we must select the counterterm for the mass of $\vec{\phi}$ to get the desired IR physics. We now turn to each of these issues.

C. Boundary conditions on $\mathbb{H}_2$

If we were simply quantizing a *free* scalar field on $\mathbb{H}_2$, we would choose boundary conditions so that the modes are normalizable. Such modes will fall off exponentially in ρ. This should also be the correct choice for the Rényi entropy. One argument starts with the observation in Ref. [34] that the reduced density matrix on the disk for a free scalar field in its ground state breaks up into superselection sectors, each of which corresponds to a different Dirichlet condition on the scalar. A Dirichlet condition on the boundary of the disk becomes, under the conformal transformation to $\mathbb{H}_2 \times R$, exponential falloff. The boundary value of the scalar maps to the amplitude of the decaying mode with the decay rate being governed by the conformal transformation. In particular, the constraint $\phi(|\vec{x}| = R, t = 0) = \phi_0$ for a free field on the boundary of a disk of radius R becomes $\phi(\rho, \tau = 0) = \frac{1}{\sqrt{\cosh \rho + 1}}\phi_0$ as $\rho \to \infty$ on $\mathbb{H}_2 \times R$ after making the conformal transformation in Eq. (2.2). A constant field mode is not in the Hilbert space of the free theory. We will also not allow it as a saddle point. The reader may object that this is also true in flat space for which expectation values of a scalar field label superselection sectors, yet in studying statistical mechanics, one may vary the constant mode. For the free conformal scalar in hyperbolic space, this would run us into trouble as an arbitrarily large constant mode would lower the free energy indefinitely due to the negative m^2 from the conformal coupling to negative curvature. A direct computation of the free energy reveals no such pathology [29], so we simply forbid nonzero constant values of $\vec{\phi}$.

We, therefore, expect that for the interacting theory, a constant mode for $\vec{\phi}$ should also not be an allowed saddle point.[4] On the other hand, we believe that σ can have a constant expectation value. Recall that $\sigma = \vec{\phi}^2$ is the saddle-point equation of motion for σ before integrating out $\vec{\phi}$. In the free theory, it is perfectly possible for $\vec{\phi}^2$ to have a constant expectation value at finite temperature, even if ϕ falls off exponentially. This was shown via path-integral techniques in Ref. [35]; in Appendix A, we show this directly using the operator approach in order to see how thermally excited exponentially decaying modes can combine to form a constant value of $\vec{\phi}^2$.

D. Renormalization

The goal of our calculation is to study the phase structure of the interacting $O(N)$ CFT on $\mathbb{H}_2 \times S^1_\beta$ as a function of β. This is based on the identification in Sec. II, which includes the identification between equal-time correlators on the disk and thermal correlators on $\mathbb{H}_2 \times S^1_R$. In the discussion above, we are working with a nonconformal theory under the assumption that the desired behavior emerges in the IR limit $h \to \infty$. Note that away from the UV or IR fixed points, Eq. (3.13) is not invariant under conformal transformations. We will assume that the limits $h \to 0, \infty$ match, in particular, the correlators at equal Euclidean time should map into each other under conformal transformations in this limit [20]. Note that the two theories *do* match for distances $\ll R$, so we are making an assumption that the space-time curvature is not introducing a new fixed point.

[4]See Ref. [14] for a related discussion in the context of holography.

Starting with this assumption, we must fix the the bare mass m^2—and, thus, the effective-mass M'^2—by a renormalization prescription, to ensure that we hit the correct IR fixed point. The identification of this prescription was, at least, to us, nontrivial. $M'^2 = m^2 + h\sigma + \frac{1}{8}\mathcal{R}$ is the effective mass for the Goldstone modes in the symmetry-broken phase and of all of the scalar fluctuations in the symmetry-unbroken phase. But, the saddle-point solution for σ depends on the temperature. One might have stated that the modular Hamiltonian is meant to be the CFT Hamiltonian on $\mathbb{H}_2$, in which case, it would make sense to demand that the effective-mass M'^2 takes the conformal value $\frac{1}{8}\mathcal{R}$ at zero temperature. Instead, our guide will be the fact that ground-state equal-time correlation functions in the disk map to thermal correlators on $\mathbb{H}_2$ for $\beta = R$ under the conformal transformation in Ref. [20]. We use this fact to nail down the correct value of m^2. We focus on correlators of $O_2 = \frac{1}{\sqrt{N}}\vec{\phi}^2$, a conformal primary with operator dimension 2 in the interacting fixed point and dimension 1 in the free $h = 0$ theory.

Our starting point is the following observation. We can consider our interacting CFT as the IR fixed point of the free theory perturbed by the double-trace operator hO_2^2. Let us write the two-point function F of O_2 in the UV theory as $F(x, y) = \langle x|\hat{F}|y\rangle$, where $|x\rangle$ is the position eigenstate for a quantum-mechanical particle on $\mathbb{H}_2 \times S^1$ and $\hat{F}$ is an operator on that space. In the free theory, $\vec{\phi}$'s are conformally coupled scalars on $\mathbb{H}_2 \times S^1_R$, and we can use Wick's theorem to write

$$F(x, y) = \left(\langle x|\frac{1}{-\partial^2 - \frac{1}{4R^2}}|y\rangle\right)^2, \tag{3.16}$$

where ∂^2 is the Laplacian on $\mathbb{H}_2 \times S^1_R$ and the appearance of $-1/(4R^2)$ is the result of the conformal coupling to the background curvature. It is straightforward to show that Eq. (3.16) is equivalent to the result one would get from a conformal transformation from correlators on the disk D_R in an equal-time slice of $\mathbb{R}^3$.

The IR two-point function can be written as $G = \langle x|\widehat{G}|y\rangle$. We can run the arguments in Sec. 2 of Ref. [36], through Eq. (7), in any background space-time, to find

$$\widehat{G} = \frac{\widehat{F}}{1 + h\widehat{F}} \tag{3.17}$$

for any h. In the limit $h \to \infty$, $h\widehat{F} \gg 1$ for any matrix elements of $\widehat{F}$ in spectral or position space, the correlators can be written as

$$\widehat{G}_{IR} \sim \frac{1}{h} - \frac{1}{h^2\widehat{F}}. \tag{3.18}$$

In flat $\mathbb{R}^3$, the second term in Eq. (3.18) G_{IR} is the correlator of O_2 in the interacting CFT [36] with the first being a contact term. We will assume that, on $\mathbb{H}_2 \times S^1_R$, when $F(x, y)$ is the UV correlator in Eq. (3.16), $\widehat{G} = -1/(h^2\hat{F})$ is the correlator of the IR fixed point.

To ensure this, we compute G_{IR} directly and explicitly in our model using large-N techniques and fix the renormalization prescription to ensure that it is equal to the right-hand side of Eq. (3.18). We consider the action Eq. (3.14) after rescaling $\sigma \to \sigma/(h\sqrt{N})$ and including a coupling $JO_2 = J\frac{\vec{\phi}^2}{\sqrt{N}}$ in order to match the discussion in Ref. [36] and, thus, Eqs. (3.17) and (3.18). Integrating over $\vec{\phi}$, we find

$$S[\sigma] = \frac{N}{2}\mathrm{Tr}\ \ln\left(-\partial^2 + M^2 + \frac{\sigma + J}{\sqrt{N}}\right) - \int d^3x\sqrt{g}\frac{1}{4h}\sigma^2. \tag{3.19}$$

Next, we write σ as a sum of its (constant) divergent part σ_0, which solves the gap equation when $J = 0$, and a finite part $\delta\sigma$ and expand Eq. (3.19) to keep only terms which are $O(1)$ or larger in the large-N expansion to find

$$\begin{aligned} S[\sigma] = {}& f(\sigma_0) + \frac{\sqrt{N}}{2V}\mathrm{Tr}\left(\frac{1}{-\partial^2 + M'^2}\right)J \\ & - \frac{1}{2V}\mathrm{Tr}\left(\frac{1}{-\partial^2 + M'^2}\right)^2(\delta\sigma + J)^2 \\ & + \int d^3x\sqrt{g}\left(-\frac{\delta\sigma^2}{4h}\right), \end{aligned} \tag{3.20}$$

where $f(\sigma_0) = -\frac{V\sigma_0^2}{4h} + \frac{N}{2}\mathrm{Tr}\ \ln(\partial^2 + M'^2)$ are the J-independent terms and V is the (regularized) volume of $\mathbb{H}_2 \times S^1$.

In the limit $h \to 0$, corresponding to the UV, the saddle point for $\delta\sigma$ is $\delta\sigma = 0$; inserting this in Eq. (3.20), we find that the two-point function of O_2 is

$$\widetilde{F} = \left(\langle x|\frac{1}{-\partial^2 + (M')^2}|y\rangle\right)^2 \equiv \langle x|\hat{\widetilde{F}}|y\rangle, \tag{3.21}$$

where the tilde denotes the computation using our large-N action. In the IR limit $h \to \infty$, the two-point function is

$$\hat{\widetilde{G}} = \frac{\hat{\widetilde{F}}}{1 + h\hat{\widetilde{F}}}. \tag{3.22}$$

Again, for distance scales such that $h\hat{\widetilde{F}} \gg 1$ (where we understand $\hat{\widetilde{F}}$ to correspond to matrix elements in some appropriate basis), we find

$$\hat{\widetilde{G}} = \frac{1}{h} - \frac{1}{h^2\hat{\widetilde{F}}} + \cdots. \tag{3.23}$$

This matches Eqs. (3.17) and (3.18) if we set $(M')^2 = -\frac{1}{4R^2}$ at $\beta = R$, which we will do below.

E. Solving the gap equation near $\beta = R$

We now focus our attention on the gap equations in Eq. (3.15). In the case that σ is constant, we can write the trace explicitly using the results in Appendix B,

$$\begin{aligned} &(-\nabla^2_{\mathcal{H}} + M^2 + h\sigma)\phi \\ &\quad = 0,\ \phi^2 - \sigma + \frac{1}{(2\pi)R^2\beta}\int_\epsilon^\infty ds \\ &\quad \times \sum_\ell \int_0^\infty d\lambda_0\lambda_0 \tanh(\pi\lambda_0) \\ &\quad \times \exp\left[-s\left(\frac{\ell^2}{\beta^2} + \frac{\lambda^2}{R^2} + M^2 + h\sigma\right)\right] = 0, \end{aligned} \tag{3.24}$$

where we have regularized the divergent λ_0 integral with a cutoff $\epsilon = \frac{1}{\Lambda^2}$. In these equations, $\phi = 0$ is always a solution. Solutions with $\phi \neq 0$ may exist, but will be nonconstant on $\mathbb{H}_2$ and will require nonconstant σ: The trace in Eq. (3.15) is then considerably more challenging to compute. We will focus on the simpler question of when the $\phi = 0$ solution is a local saddle point or a local minimum, indicating a possible phase transition; as in the former case, there are directions in field space which will continuously lower the free energy. As we will see, once we fix m^2 with our renormalization prescription, $\beta = R$ is the boundary in temperature between stable and unstable free energy at $\phi = 0$.

At $\beta = R$, we fix $M'^2 = M^2 + h\sigma = -\frac{1}{4R^2}$. In this case, the integral in the second line of Eq. (3.24) can be evaluated analytically (see Appendix B), yielding the saddle-point solution,

$$\sigma = \alpha\Lambda, \quad \alpha = \frac{\sqrt{\pi}}{(2\pi)}. \tag{3.25}$$

This fixes $M^2 = -\alpha\Lambda - \frac{1}{4R^2}$. Note that there is no IR divergence in this calculation, even at the conformal point at which the eigenvalue spectrum of the kinetic operator is continuous down to $\lambda = 0$. The essential point is that the infrared behavior of scalars in $\mathbb{H}_2$ is weaker than in $\mathbb{R}^2$ [24] and does not, in and of itself, give an obstruction to a finite-temperature phase transition.

The trace in the gap equation (3.15) is difficult to compute explicitly for general $\beta \neq R$, even for constant σ. We consider small perturbations in β around $\beta = R$. In Appendix B, we also show that to linear order in $\delta\beta = \beta - R$, the saddle-point equations at $\phi = 0$ yield $(M')^2 + \frac{1}{4R^2} = -c\,\delta\beta$, where c is a positive real number. Using this fact, we can show that there is an instability that sets in for $\beta > R$.

For $\beta > R$ or $\delta\beta > 0$, we can expand the action Eq. (3.14) about the saddle-point solution $\phi = 0$, $M^2 + h\sigma = -\frac{1}{4R^2} - c\,\delta\beta$. Integrating by parts, the action for $\delta\vec{\phi}$ takes the form

$$S_{\delta\phi} = \int d^3x\, \delta\vec{\phi} \cdot \left(-\nabla^2_{\mathbb{H}_2} - \partial^2_{S^1} - \frac{1}{4R^2} - c\,\delta\beta\right)\delta\vec{\phi}. \tag{3.26}$$

The essential point is that with our boundary conditions, the operator $-\nabla^2_{\mathbb{H}_2} - \frac{1}{4R^2} - \partial^2_{S^1}$ has eigenvalues which are positive, continuous, and extend to zero, as is implied by Eq. (B3). Thus, the kinetic operator for $\delta\vec{\phi}$ will have negative eigenvalues bounded by $-c\,\delta\beta$ when $\delta\beta > 0$, and so there are directions in field space which will lower the free energy.

By a similar token, for $\delta\beta < 0$, the action quadratic in $\delta\vec{\phi}$ is strictly positive about the $\vec{\phi} = 0$ saddle point. We have not been able to show whether there is or is not another local minimum for nonzero $\vec{\phi}$ in this range, much less what the free energy might be. The solutions to Eq. (3.15) require nonzero ϕ and, thus, nonzero σ. Without finding or ruling out a solution, we cannot, at this time, say whether there is a first- or second-order transition as a function of temperature in this model. We close by noting that this treatment works strictly in the $N \to \infty$ limit for which the saddle-point equations in ϕ, σ capture the theory completely.

IV. CONCLUSIONS

We have shown that with a few plausible conjectures, there is an instability in the large-N $O(N)$ model on $\mathbb{H}_2 \times S^1_\beta$ at $\beta = R$, indicating a phase transition in the Rényi entropy for the reduced density matrix describing the spatial region D_R of the same theory in $\mathbb{R}^3$ in the vacuum state. A number of questions arise:

(1) One conjecture we rely on, which is commonly made for the $O(N)$ model on other space-times (e.g., Refs. [32,37]), that the IR fixed point of Eq. (3.13) at $\beta = R$ with m^2 chosen according to our renormalization prescription is, in fact, conformal to the IR fixed point of the same action on the domain of dependence of D_R in $\mathbb{R}^3$. It is certainly true that we will get the correct behavior at distances $\ll R$, so it is a question of whether the space-time curvature might generate a fixed point distinct from the one we wish to reach. Further study along the lines of Refs. [38,39] would be of interest.

(2) We have made an assumption about the boundary conditions for our model based on those for the UV theory, despite the fact that the RG trajectories will not completely match up. This would be worth checking.

(3) We have found an instability for the theory at $\beta = R$. What we do not know is whether this corresponds to a second-order phase transition or rather indicates a kind of spinodal point of a first-order transition as appears to be the case with the Hagedorn transition in string theory [40,41]. Note that the same question pertains to the holographic discussion of Rényi phase transitions in Refs. [14,15]. It is notable that the extension of calculations of Ref. [14] to bulk theories with higher-curvature corrections to Einstein gravity [16] exhibits first-order transitions in the Rényi entropies of putative duals (at $n > 1$).

(4) To understand this, one would need to solve (at least, in some controlled approximation) the gap equation Eq. (3.15) for nonzero values of $\vec{\phi}$. This will be more challenging than the case of flat space as the boundary conditions require that nontrivial solutions be nonconstant (as also pointed out in Ref. [14]).

(5) Finally, we would like to better understand finite-N corrections. In other contexts [25,26], for flat spatial dimensions in the thermodynamic limit, finite-N effects can change the nature of the phase transition to one of Berezinski-Kosterlitz-Thouless type [42,43]. Hyperbolic space tames the IR divergences that would appear in flat space and can change the location of phase transitions [24]. An additional complication here is that the potential ordering is not via constant expectation values for the scalar but via normalizable modes on $\mathbb{H}_2$ for which fluctuations between different directions in the field space $\mathbb{R}^n$ have finite action.

We close with a comment on the validity of the replica trick and with an argument that the transition at hand is, indeed, second order. These comments are motivated by the replica trick calculation of Ref. [27], which successfully reproduces the direct calculation of the entanglement entropy of Ref. [37]. One might worry that this result is in tension with our claim of a transition at $n = 1$.

However, this is not necessarily the case. Let us imagine first that there was a first-order phase transition at $n = 1$. This means that there would be a kink in the free energy. At large

N, this occurs because two local minima of the free-energy exchange dominance. If we expand the free energy along either branch in a power series on $(n-1)$, it will be analytic, and the $n \to 1$ limit still gives the correct free energy. In the calculation of Ref. [27], this could happen if the function f in Eq. (16) changes values as $n-1$ changes sign. But that term drops out of the entanglement entropy calculation.

If the transition is second order, this is still not an obstruction to the use of the replica trick in Ref. [27]. That work relies on the first two terms $F \sim F_0 + (n-1)F_1 + \cdots$, where the further terms vanish faster than linearly. The free energy for a system with a second-order transition could have the expansion,

$$F = f(n-1) + (n-1)^a g(n-1) + \cdots, \tag{4.1}$$

where f, g are analytic functions of $n-1$ and $a > 1$ is noninteger. As $n \to 1$, the constant and linear term would dominate the calculation, and the replica trick would still give the right answer.

Furthermore, the fact that the replica trick calculation [27] gets the right answer at leading order in $1/N$ provides a good argument that there is not a first-order transition at $n > 1$ as this would occur via a different solution to the gap equations which would have lower free energy, and the free energy in Ref. [27] would give the wrong result. In principle, there could be a first-order transition at $n = 1$, but a distant saddle point becoming dominant at $n = 1$ just as the $\phi = 0$ saddle becomes unstable is a highly nongeneric (if not completely ruled out) situation.

All this being said, the final two questions listed above are important for understanding the implications of the instability we found here for the the structure of the entanglement spectrum.

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APPENDIX A: THERMAL EXPECTATION VALUE OF $:\vec{\phi}^2:$ on $\mathcal{H}$

Let us consider a conformally coupled free-scalar field on $\mathcal{H}$ defined by the Euclidean action,

$$S[\phi] = \int d^3\bar{x}\left[-\frac{1}{2}\sqrt{g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m'^2\phi^2 + \frac{1}{8}R^{(3)}\phi^2\right)\right]. \tag{A1}$$

The equation of motion is given by

$$\left(\nabla^2 - m'^2 - \frac{R^{(3)}}{8}\right)\phi = 0. \tag{A2}$$

In order to quantize the scalar field, let us recast the above equation as a wave equation,

$$\left(\partial_t^2 - \frac{\lambda^2}{R_H^2} - M^2\right)\phi = 0, \tag{A3}$$

where the constant Ricci scalar has been absorbed in the mass term. We now define the frequencies $\omega_{\lambda_0}^2 = \frac{\lambda^2}{R_H^2} + M^2$ (recall that $\lambda^2 = \lambda_0^2 + \frac{1}{4}$) and using the eigenbasis (B2) express the scalar in terms of creation and annihilation operators,

$$\phi(x) = \int d\lambda_0 \sum_n \frac{1}{\sqrt{2\omega_{\lambda_0}}}\mu_n(\lambda_0) \times \left[a_{n,\lambda_0}\theta_{n,\lambda_0}(\rho,\psi) + a_{n,\lambda_0}^\dagger(t)\theta_{n,\lambda_0}^*(\rho,\psi)\right], \tag{A4}$$

where $\theta_{n,\lambda_0}(\rho,\phi) = e^{in\psi}P^{|n|}_{-(1/2)+i\lambda_0}(\cosh\,\rho)$ and the density of Laplacian eigenstates $\mu_n(\lambda_0)$ is given by

$$\mu_n(\lambda_0) = \frac{1}{2\pi}\frac{\left|\Gamma\left(\frac{1}{2} - |n| + i\lambda_0\right)\right|^2}{|\Gamma(i\lambda_0)|^2} = \frac{\lambda_0\tanh(\pi\lambda_0)}{2\pi}\frac{1}{\left|\left(\frac{1}{2} + i\lambda_0\right)\cdots\left(i\lambda_0 + |n| - \frac{1}{2}\right)\right|^2}. \tag{A5}$$

The conjugate field $\pi(x,t)$ becomes

$$\pi(x) = \int d\lambda_0 \sum_n \frac{-\sqrt{\omega_{\lambda_0}}}{\sqrt{2}}\mu_n(\lambda_0) \times \left[a_{n,\lambda_0}\theta_{n,\lambda_0}(\rho,\psi) - a_{n,\lambda_0}^\dagger\theta_{n,\lambda_0}^*(\rho,\psi)\right]. \tag{A6}$$

Now, it is easy to check that imposing the commutation relations,

$$\left[a_{n,\lambda_0}(t), a^\dagger_{n',\lambda_0'}(t)\right] = \frac{\delta_{nn'}\delta(\lambda_0 - \lambda_0')}{\mu_n(\lambda_0)}, \tag{A7}$$

on the creation and annihilation operators is equivalent to imposing the following canonical commutation relations on the fields,

$$[\phi(x,t), \pi(x',t)] = \frac{\delta(\psi - \psi')\delta(\lambda_0 - \lambda_0')}{R_H^2\sinh(\rho)}. \tag{A8}$$

With these results, we wish to show that composite operators can have constant expectation values on $\mathbb{H}_2$, even if the boundary conditions on $\vec{\phi}$ forbid it. Let us consider the operator ϕ^2. We normal order to ensure that the two-point function vanishes

$$\langle 0|{:}\phi^2(x,t){:}|0\rangle = \langle 0|\int d\lambda_0 d\lambda_0' \sum_{nn'}\frac{\mu_{n'}(\lambda_0')\mu_n(\lambda_0)}{\sqrt{\omega_{\lambda_0}\omega_{\lambda_0'}}} \times a^\dagger_{n',\lambda_0'}a_{n,\lambda_0}\theta_{n,\lambda_0}(\rho,\psi)\theta^*_{n,\lambda_0'}(\rho,\psi)|0\rangle = 0. \tag{A9}$$

Here, $|0\rangle$ refers to the vacuum on $\mathbb{H}_2 \times R$. We now make the system thermal by making the time dimension compact,

$$\langle :\phi^2(x): \rangle_\beta = \frac{\mathrm{Tr}[e^{-\beta H} :\phi^2(x):]}{\mathrm{Tr}(e^{-\beta H})}, \tag{A10}$$

where H is the modular Hamiltonian given by

$$H(\pi, \phi) = \int d^2x \frac{\sqrt{g}}{2}[\pi^2 + (\partial_i \phi)^2 + m^2\phi^2]. \tag{A11}$$

In order to compute Eq. (A10), we need to express the modular Hamiltonian Eq. (A11) in terms of the creation and annihilation operators using Eqs. (A4) and (A6). Using the completeness relation for the θ's, we find the following normal-ordered Hamiltonian:

$$:H: = \int d\lambda_0 \sum_n \mu_n(\lambda_0)\omega_{\lambda_0} a^\dagger_{n,\lambda_0} a_{n,\lambda_0}. \tag{A12}$$

We can see that the energy eigenvalues ω_{λ_0} are independent of n. The thermal expectation value of $:\phi^2(x):$ is found to be

$$\langle :\phi^2(x): \rangle_\beta = \int \lambda_0 \sum_n \mu_n(\lambda_0) \frac{1}{e^{\beta\omega_{\lambda_0}} - 1} \langle n, \lambda_0 | :\phi^2(x): | n, \lambda_0 \rangle, \tag{A13}$$

where the exponential prefactor arises because we are working with a boson. After a bit of algebra, this expression becomes

$$\langle :\phi^2(x): \rangle_\beta = \int_0^\infty \frac{d\lambda_0}{2\omega_{\lambda_0}} \sum_n \mu_n(\lambda_0) |\theta_{\lambda_0,n}(x)|^2 \frac{1}{e^{\beta\omega_{\lambda_0}} - 1}. \tag{A14}$$

At this point, it is useful to use the identity [44],

$$\sum_{n=-\infty}^{\infty} \frac{\left|P^{|n|}_{-(1/2)+i\lambda_0}[\cosh(\rho)]\right|^2}{\left|\left(\frac{1}{2} + i\lambda_0\right) \cdots \left(i\lambda_0 + |n| - \frac{1}{2}\right)\right|^2} = 1, \tag{A15}$$

which, together with Eq. (A5) brings the expectation value to the form

$$\langle :\phi^2(x): \rangle_\beta = \int_0^\infty \frac{d\lambda_0}{2\omega_{\lambda_0}} \sum_n \frac{\lambda_0 \tanh(\pi\lambda_0)}{2\pi} \frac{1}{e^{\beta\omega_{\lambda_0}} - 1}, \tag{A16}$$

which is manifestly constant.

APPENDIX B: THE SADDLE-POINT INTEGRAL

In this Appendix, we analyze the gap equations of Eq. (3.15) on $\mathbb{H}_2 \times S^1$ in detail. Before we get into this discussion, it is useful to briefly review the Laplacian and its eigenfunctions on hyperbolic space. We start with the Laplacian on $\mathcal{H}$ given by $\nabla^2_{\mathcal{H}}$,

$$\nabla^2_{\mathcal{H}} = \frac{1}{R^2}\left[\partial^2_\rho + \coth(\rho)\partial_\rho + \frac{1}{\sinh^2(\rho)}\partial^2_\phi\right] + \frac{1}{\beta^2}\partial^2_t. \tag{B1}$$

Here, β is the radius of the thermal circle, R is the radius of $\mathbb{H}_2$, and the variables ρ, t, and ϕ are the embedding coordinates in the metric of Eq. (3.12). The corresponding eigenfunctions are

$$\langle \lambda n \ell | \rho\phi t \rangle = \Phi_{\lambda_0,n,\ell}(t, \phi, \rho) = e^{i\ell t} e^{in\phi} P^{|n|}_{-(1/2)+i\lambda_0}(\cosh\rho), \tag{B2}$$

where $P^{|n|}_{-(1/2)+i\lambda_0}(\cosh\rho)$ are associated Legendre functions [44]. The eigenfunctions satisfy the equation,

$$\nabla^2_{\mathcal{H}} \Phi_{\lambda_0,n,\ell} = -\left(\frac{\lambda^2}{R^2} + \frac{\ell^2}{\beta^2}\right)\Phi_{\lambda,n,\ell}, \quad \lambda^2 = \lambda_0^2 + \frac{1}{4}. \tag{B3}$$

Here, λ is the eigenvalue of the Laplacian on the the hyperbolic space, and λ_0 is the associated wave number. Thus, the eigenspectrum of the hyperbolic Laplacian is $\lambda^2 \in [\frac{1}{4}, \infty)$ [44].

The position eigenkets are normalized as follows:

$$\langle \bar{x} | \bar{y} \rangle = \langle \rho\phi t | \rho'\phi' t' \rangle = \frac{\delta(\rho - \rho')\delta(\phi - \phi')\delta(t - t')}{\beta R^2 \sinh(\rho)}. \tag{B4}$$

We now have all the ingredients to evaluate the trace term in Eq. (3.24). Using the eigenbasis Eq. (B2) to express the trace, we get

$$I(\epsilon) = \frac{1}{(2\pi)R^2\beta} \int_{\epsilon=\Lambda^{-2}}^\infty ds \sum_\ell \int_0^\infty d\lambda_0 \lambda_0 \tanh(\pi\lambda_0) \times \exp\left[-s\left(\frac{\ell^2}{\beta^2} + \frac{\lambda^2}{R^2} + M'^2\right)\right], \tag{B5}$$

where we have introduced a cutoff Λ and $M^2 + h\sigma = M'^2$ is the effective mass.

Let us evaluate the integral Eq. (B5) at the conformal point $R = \beta$. We set the effective mass to its conformal value $M'^2 = -\frac{1}{4R^2}$ to find

$$I(\epsilon) = \frac{1}{2\pi R} \int_{\epsilon=(\Lambda R)^{-2}}^\infty ds \sum_\ell \int_0^\infty d\lambda_0 \lambda_0 \tanh(\pi\lambda_0) e^{-s(\ell^2+\lambda_0^2)} \tag{B6}$$

$$= c_{-1}\Lambda + \frac{c_0}{R} + \frac{1}{R}\sum_{n\geqslant 1} c_n \epsilon^{n/2}, \tag{B7}$$

where we have scaled the variable $s \to \frac{s}{R^2}$. In the limit $\Lambda \to \infty$, c_n, with $n \geqslant 1$ vanish. Therefore, let us focus on the divergent part (c_{-1}) and the finite part (c_0). In this regard, it is useful to note that the sum over ℓ can be expressed in terms of the third Jacobi θ function,

$$\sum_\ell e^{-s\ell^2} = \theta_3(0, q = e^{-s}). \tag{B8}$$

In order to extract the divergent part of $I(\epsilon)$, we consider the limit $s \to 0$. Here, we will use the identity,

$$\theta_3(0, e^{-s}) = \sqrt{\frac{\pi}{s}}\theta_3(0, e^{-\pi^2/s}), \tag{B9}$$

to get

$$\sum_\ell e^{-s\ell^2} = \sqrt{\frac{\pi}{s}} \sum_n e^{-\pi^2 n^2/s}. \tag{B10}$$

It is clear that, as $s \to 0$, the leading contribution comes only from $n = 0$. Using this fact in Eq. (B6) and noting that the divergent part comes from $\lambda_0 \to \infty$, we can replace $\tanh(\pi\lambda_0) \to 1$. We can now perform the λ_0 integral to find

$$\frac{\sqrt{\pi}}{2R} \int_\epsilon^\infty \frac{ds}{s^{3/2}}, \tag{B11}$$

where we have suppressed a factor of 2π. It is now easy to perform the integral over s, and we find that $c_{-1} = \sqrt{\pi}$.

To get the finite part of Eq. (B6), let us define

$$c_0 = \int_0^\infty ds \int d\lambda_0 \lambda_0 \left[\sum_{n=-\infty}^{\infty} \tanh(\pi\lambda_0) e^{-s(n^2+\lambda_0^2)} - \int d\eta\, e^{-s(\lambda_0^2+\eta^2)} \right], \tag{B12}$$

where we have subtracted the divergent part. We first perform the s integral and using the identities,

$$\int_{-\infty}^{\infty} d\eta \frac{1}{\eta^2+\lambda_0^2} = \frac{\pi}{\lambda_0}, \tag{B13}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+\lambda_0^2} = \frac{\pi\, \coth(\pi\lambda_0)}{\lambda_0}, \tag{B14}$$

we see that the finite part of Eq. (B12) vanishes!

In order to know the corrections to the trace-term Eq. (B6) slightly away from the conformal point $\beta = R$, we compute its derivative with respect to β,

$$\partial_\beta I(\epsilon)|_{\beta=R} = \frac{1}{2\pi R^2} \int_{(\Lambda R)^{-2}}^{\infty} ds \int d\lambda_0 \lambda_0 \tanh(\pi\lambda_0) e^{-s\lambda^2} \times \left[-1 + 2\sum_{\ell=1}^{\infty} (2\ell^2 s - 1) e^{-s\ell^2} \right]. \tag{B15}$$

We can recast the term inside square brackets as

$$\sum_{\ell=-\infty}^{\infty} (2\ell^2 s - 1) e^{-s\ell^2} = -(2s\partial_s + 1)\theta_3(0, q = e^{-s}) = (2se^{-s}\partial_q - 1)\theta_3(0, q = e^{-s}), \tag{B16}$$

where we have used Eq. (B8). Using the power series expansion of θ_3, we find that as $s \to \infty$, $q \to 1$, the derivative of θ_3 is finite, and the sum in Eq. (B16) becomes equal to -1. For $s = 0$, θ_3 diverges. To understand this better, let us use the modular transformation to write Eq. (B16) as

$$-\lim_{s\to 0}(2s\partial_s + 1)\sqrt{\frac{\pi}{s}}\theta_3(0, q' = e^{-(\pi^2/s)}) = -\lim_{s\to 0} 2\sqrt{\frac{\pi}{s}}\left[2q'\frac{\pi^2}{s} + O(q'^2) \right]. \tag{B17}$$

Thus, we see that the sum in Eq. (B16) is negative for all nonzero values of s and converges to 0 as $s \to 0$. This, in particular, implies that Eq. (B15) is negative.

Using these results in Eq. (3.24), the second gap equation becomes

$$\phi^2 - \sigma + \frac{\sqrt{\pi}}{2\pi}\Lambda + I_{\text{finite}} = 0, \tag{B18}$$

where $I_{finite} = -c(\beta - R)$ with c being a real positive number. When $\phi = 0$, this implies that

$$\sigma = \frac{\sqrt{\pi}}{2\pi}\Lambda - c(\beta - R). \tag{B19}$$

[1] A. Hamma, R. Ionicioiu, and P. Zanardi, Ground state entanglement and geometric entropy in the kitaev model, Phys. Lett. A **337**, 22 (2005).

[2] A. Kitaev and J. Preskill, Topological Entanglement Entropy, Phys. Rev. Lett. **96**, 110404 (2006)

[3] M. Levin and X.-G. Wen, Detecting Topological Order in a Ground State Wave Function, Phys. Rev. Lett. **96**, 110405 (2006)

[4] S. T. Flammia, A. Hamma, T. L. Hughes, and X.-G. Wen, Topological Entanglement Rényi Entropy and Reduced Density Matrix Structure, Phys. Rev. Lett. **103**, 261601 (2009)

[5] M. Headrick, A. Lawrence, and M. Roberts, Bose-Fermi duality and entanglement entropies, J. Stat. Mech.: Theory Exp. (2013) P02022.

[6] M. Van Raamsdonk, Building up spacetime with quantum entanglement, Gen. Relativ. Gravit. **42**, 2323 (2010); Building up spacetime with quantum entanglement, Int. J. Mod. Phys. D **19**, 2429 (2010).

[7] J. Maldacena and L. Susskind, Cool horizons for entangled black holes, Fortschr. Phys. **61**, 781 (2013)

[8] V. Balasubramanian, P. Hayden, A. Maloney, D. Marolf, and S. F. Ross, Multiboundary Wormholes and Holographic Entanglement, Class. Quant. Grav. **31**, 185015 (2014).

[9] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech.: Theory Exp. (2004) P06002.

[10] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A: Math. Theor. **42**, 504005 (2009).

[11] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. B **424**, 443 (1994).

[12] A. Hüller, First order phase transitions in the canonical and the microcanonical ensemble, Z. Phys. B **93**, 401 (1994).

[13] M. A. Metlitski, C. A. Fuertes, and S. Sachdev, Entanglement Entropy in the O(N) model, Phys. Rev. B **80**, 115122 (2009).

[14] A. Belin, A. Maloney, and S. Matsuura, Holographic phases of Renyi entropies, J. High Energy Phys. 12 (2013) 050.

[15] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R. C. Myers, and T. Sierens, Holographic charged Renyi entropies, J. High Energy Phys. 12 (2013) 059.

[16] V. G. M. Puletti and R. Pourhasan, Non-analyticity of holographic Rényi entropy in Lovelock gravity, J. High Energy Phys. 08 (2017) 002.

[17] A. Belin, C. A. Keller, and I. G. Zadeh, Genus two partition functions and Rényi entropies of large c conformal field theories, J. Phys. A: Math. Theor. **50**, 435401 (2017).

[18] X. Dong, S. Maguire, A. Maloney, and H. Maxfield, Phase transitions in 3D gravity and fractal dimension, J. High Energy Phys. 05 (2018) 080.

[19] I. R. Klebanov and A. M. Polyakov, AdS dual of the critical O(N) vector model, Phys. Lett. B **550**, 213 (2002).

[20] H. Casini, M. Huerta, and R. Myers, Towards a derivation of Holographic Entanglement Entropy, J. High Energy Phys. 05 (2011) 036.

[21] N. D. Mermin and H. Wagner, Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models, Phys. Rev. Lett. **17**, 1133 (1966).

[22] P. C. Hohenberg, Existence of long-range order in one and two dimensions, Phys. Rev. **158**, 383 (1967).

[23] S. Coleman, There are no goldstone bosons in two dimensions, Commun. Math. Phys. **31**, 259 (1973).

[24] C. G. Callan, Jr. and F. Wilczek, Infrared Behavior at Negative Curvature, Nucl. Phys. B **340**, 366 (1990).

[25] E. Witten, Chiral Symmetry, the 1/n Expansion, and the SU(N) Thirring Model, Nucl. Phys. B **145**, 110 (1978).

[26] D. Anninos, S. A. Hartnoll, and N. Iqbal, Holography and the Coleman-Mermin-Wagner theorem, Phys. Rev. D **82**, 066008 (2010).

[27] S. Whitsitt, W. Witczak-Krempa, and S. Sachdev, Entanglement entropy of the large N Wilson-Fisher conformal field theory, Phys. Rev. B **95**, 045148 (2017).

[28] H. Liu and M. Mezei, A Refinement of entanglement entropy and the number of degrees of freedom, J. High Energy Phys. 04 (2013) 162.

[29] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, Renyi entropies for free field theories, J. High Energy Phys. 04 (2012) 074.

[30] A. M. Polyakov, Gauge fields and strings, Contemp. Concepts Phys. **3**, 1 (1987).

[31] M. Moshe and J. Zinn-Justin, Quantum Field Theory in the Large N Limit: A Review, Phys. Rep. **385**, 69 (2003).

[32] S. A. Hartnoll and S. P. Kumar, The $O(N)$ model on a squashed S^3 and the Klebanov-Polyakov correspondence, J. High Energy Phys. 06 (2005) 012.

[33] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 2011).

[34] C. A. Agon, M. Headrick, D. L. Jafferis, and S. Kasko, Disk entanglement entropy for a Maxwell field, Phys. Rev. D **89**, 025018 (2014).

[35] C. P. Herzog and T. Nishioka, The edge of entanglement: getting the boundary right for non-minimally coupled scalar fields, J. High Energy Phys. 12 (2016) 138.

[36] S. S. Gubser and I. R. Klebanov, A universal result on central charges in the presence of double-trace deformations, Nucl. Phys. B **656**, 23 (2003).

[37] I. R. Klebanov, S. S. Pufu, and B. R. Safdi, F-theorem without supersymmetry, J. High Energy Phys. 10 (2011) 038.

[38] H. Osborn, Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories, Nucl. Phys. B **363**, 486 (1991).

[39] X. Dong, B. Horn, E. Silverstein, and G. Torroba, Perturbative critical behavior from spacetime dependent couplings, Phys. Rev. D **86**, 105028 (2012).

[40] J. J. Atick and E. Witten, The hagedorn transition and the number of degrees of freedom of string theory, Nucl. Phys. B **310**, 291 (1988).

[41] M. Kruczenski and A. Lawrence, Random walks and the Hagedorn transition, J. High Energy Phys. 07 (2006) 031.

[42] V. L. Berezinsky, Destruction of long range order in one-dimensional and two-dimensional systems having a continuous symmetry group. 1. Classical systems, Zh. Eksp. Teor. Fiz. **59**, 907 (1971) [Sov. Phys. JETP **32**, 493 (1971)].

[43] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, J. Phys. C **6**, 1181 (1973).

[44] N. Balazs and A. Voros, Chaos on the pseudosphere, Phys. Rep. **143**, 109 (1986).