Adiabatic almost-topological pumping of fractional charges in noninteracting quantum dots

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We use exact techniques to demonstrate theoretically the pumping of fractional charges in a single-level noninteracting quantum dot, when the dot-reservoir coupling is adiabatically driven from weak to strong coupling. The pumped charge averaged over many cycles is quantized at a fraction of an electron per cycle, determined by the ratio of Lamb shift to level broadening; this ratio is imposed by the reservoir band structure. For uniform density of states, half an electron is pumped per cycle. We call this *adiabatic almost-topological pumping*, because the pumping's Berry curvature is sharply peaked in the parameter space. Hence, so long as the pumping contour does not touch the peak, the pumped charge depends only on how many times the contour winds around the peak (up to exponentially small corrections). However, the topology does not protect against nonadiabatic corrections, which grow linearly with pump speed. In one limit, the peak becomes a delta function, so the adiabatic pumping of fractional charges becomes entirely topological. Our results show that quantization of the adiabatic pumped charge at a fraction of an electron does not require fractional particles or other exotic physics.

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I. INTRODUCTION

Since the seminal work of Thouless on quantum pumping [1], there have been many pumping and turnstile protocols discussed in nanoscale systems [2–12] and cold atom experiments [13,14]. In recent years, there has been great interest in exotic systems which exhibit topological pumping of *fractional charges*, meaning that any two driving contours with the same topology will drive the same fractional charge. Such fractional charge pumping has been found in models of Coulomb-blockaded quantum dots [15,16], topological insulators [17–20], systems with fractional quantum Hall physics [19,21], fermionic gases with short-range interactions [22], fractional levitons [23], and the Bose-Hubbard model [24]. These models have either strong interaction effects or nontrivial topological properties (nonzero Chern numbers, or similar). This makes us ask if either are necessary; can a noninteracting topologically trivial system also exhibit fractional pumping of a topological nature?

We consider a noninteracting single-level quantum dot at low temperatures. Using the fact that it is an exactly soluble model, we consider adiabatic pumping in this model without approximation (particularly without assuming weak dot-reservoir coupling). Our results show the adiabatic pumping of a fraction of an electron in an *almost topological* manner. That is to say, any pumping contour with the same topology [see Fig. 1(b)] will pump the same fractional charge (up to exponentially small corrections), if the pumping is slow enough to be adiabatic. However, the topology does not protect against nonadiabatic corrections which go like one over the pumping period. This is much less robust than many of the topological pumps mentioned above, in which the topology also means that the nonadiabatic corrections decay exponentially with increasing pumping period.

The fractional pumping that we present here occurs when the dot-reservoir couplings, $K_{\rm L}$ and $K_{\rm R}$, are adiabatically driven from weak to strong coupling and back around the pumping cycle, with the dot level fixed at energy ϵ_d. We take ϵ_d to be above the reservoirs' electrochemical potential, μ. Here "strong" coupling means that it induces a level broadening larger than $(\epsilon_d - \mu)$, so the dot level becomes a resonance that spreads across the electrochemical potential. The pumped charge is given by the integral over the Berry curvature inside the contour, which is sharply peaked and decays exponentially away from the peak. Formally, the adiabatic pumping would be topological if this peak was a Dirac δ function. Here the peak has a finite extent, so we refer to the pumping as *almost topological*, because it depends only on how many times the contour winds around the peak—up to exponentially small corrections—for any pumping contour that does not impinge on the peak. Half an electron is pumped per cycle, if the reservoirs have a uniform density of states (and so impose no Lamb shift of the quantum dot). However, in general, the fraction of an electron pumped per cycle (between zero and one) is given by the ratio of the Lamb shift imposed by the reservoirs to the level broadening. This ratio is entirely determined by the reservoirs' density of states, which is imposed by their band structure.

Earlier works on pumping of dot-reservoir coupling —with direct driving of the dot-level [25], a Lamb shift induced by the reservoir band structure [25,26], Coulomb blockade effects [27,28], or nonadiabatic driving [29]—did not investigate large level broadening, and so did not find the quantized pumping of fractional charges.

Note that we consider the *average* charge per cycle. There are no fractionally charged quasiparticles in our noninteracting system, so we expect that there is a certain probability that n electrons are pumped (for integer $n = 0, \pm1, \pm2, \ldots$)

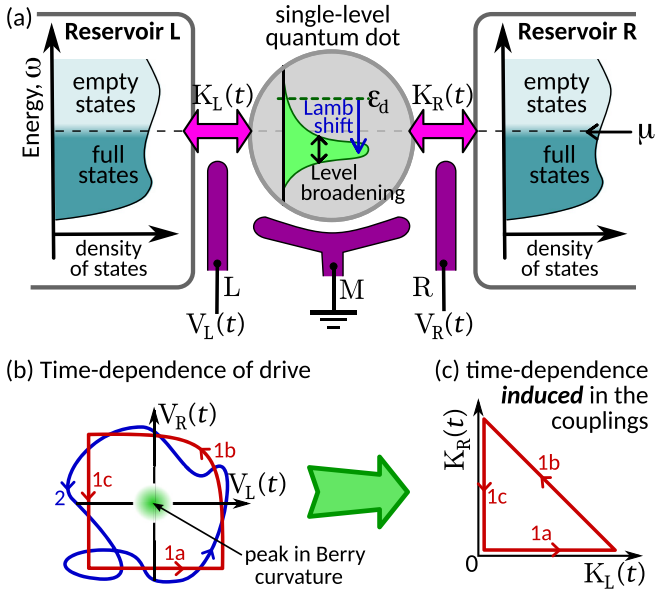

FIG. 1. (a) A quantum dot with tunnel couplings $K_L(t)$ and $K_R(t)$ to the reservoirs, controlled by gates voltages, $V_L(t)$ and $V_R(t)$. These are slowly varied around the cycle in (b), with gate M ensuring the dot level is fixed at energy ϵ_d. Any contour enclosing the Berry curvature peak in (b) without touching it (e.g., contours 1 and 2) pumps the same fraction of an electron per cycle, up to exponentially small corrections. The couplings induce level broadening and a Lamb shift on the dot. Since $K_L(t)$ and $K_R(t)$ depend exponentially on $V_{L,R}(t)$, contour 1 in (b) maps to contour 1 in (c).

in any given cycle. Yet these probabilities are such that the average over many cycles will reveal itself as a fraction per cycle. Hence the observation of a topological fractional average charge per cycle in adiabatic pumping does not require the existence of fractionally charged quasiparticles, or other exotic physics. It is not yet clear to us if there is a connection to the fractional charges recently discussed in Ref. [30].

A. Organisation of this work

Section II introduces our model Hamiltonian, and Sec. III A outlines our main result about adiabatic almost-topological pumping of fractional charges. Section IV shows this is half an electron per cycle for readers familiar with scattering theory (others can skip this section). Section V explains that the pumping is not simply related to as changes in the dot occupation. Sections VI and VII use the Keldysh formalism to get our main result, Eq. (40). Section VIII discusses the nonadiabatic corrections. Section IX gives our conclusions.

II. MODEL HAMILTONIAN

We consider a noninteracting single-level quantum dot connected to two electron reservoirs with time-dependent couplings, described by the Hamiltonian

$$H = \epsilon_d d^\dagger d + \sum_{i,k} [\epsilon_k c_{ik}^\dagger c_{ik} + \gamma_i(t)(d^\dagger c_{ik} + c_{ik}^\dagger d)], \quad (1)$$

often called the Fano-Anderson model [31,32]. Here, $d^\dagger$ and d are creation and annihilation operator of the dot state, which has energy ϵ_d, while $c_{ik}^\dagger$ and c_{ik} are those for the state with wave number k and energy ϵ_k in the reservoir $i = L, R$. The tunnel-coupling between the system and the mode k in reservoir i is $\gamma_i(t)$, which is taken to vary slowly with time. This model neglects electron-electron interactions on the dot; the simplest experimental implementation is discussed in Sec. III D. The fact this model is quadratic in the creation and annihilation operators means that it is exactly soluble. As a result, we will get its adiabatic pumping properties without making any approximations (in particular, we will not need to assume weak dot-reservoir coupling).

We take the reservoirs to have a continuum of states, and assume they both have the same density of states $\rho(\omega)$. In general, this density of states may have energy (ω) dependence, band gaps, etc. The system's coupling to each reservoir is described in terms of the time-dependent function

$$\Gamma_i(\omega, t) = K_i(t)\,\rho(\omega), \quad (2)$$

where the coupling parameter $K_i(t) = |\gamma_i(t)|^2$. A second crucial quantity for the physics of this model is

$$\Lambda_i(\omega, t) = K_i(t) \int\!\!\!\!\!\!- \, d\varepsilon \, \frac{\rho(\varepsilon)}{\omega - \varepsilon}, \quad (3)$$

where the integral is the principal value. For compactness of what follows, we also define

$$\Gamma(\omega, t) = \Gamma_L(\omega, t) + \Gamma_R(\omega, t), \quad (4)$$

$$\Lambda(\omega, t) = \Lambda_L(\omega, t) + \Lambda_R(\omega, t). \quad (5)$$

We refer to $\Gamma_i(\omega, t)$ as *level broadening*, and to $\Lambda_i(\omega, t)$ as a *Lamb shift*. This is a slight abuse of terminology, but it is justified by the dot's local density of states [31,32] being $\Gamma(\omega)/[(\omega - \epsilon_d - \Lambda(\omega))^2 + \Gamma^2(\omega)]$. So if Γ and Λ are ω-independent, then they are the level broadening and Lamb shift, respectively. We simply keep this terminology for cases where Γ and Λ have an ω dependence.

In what follows, our results will be simplest if K_i is written in terms of the dimensionless coupling X_i, which measures the level broadening in units of the distance of the dot level from the electrochemical potential;

$$X_i = \frac{\rho(\mu)\,K_i}{2(\epsilon_d - \mu)} \quad \text{for } i = L, R, \quad (6)$$

where $\rho(\mu)$ is the density of states at the electrochemical potential, and the factor of two makes formulas compact.

We drive the gate voltages V_i, not the couplings K_i, so we need a relation between them. Typically, the dot is coupled to reservoir i through tunnel barriers of height E_i and width L_i, so $K_i \sim \exp[-\kappa_i]$ with $\kappa_i = \sqrt{2mE_i(V_i)}\,L_i(V_i)/\hbar$. For large L_i and E_i, even small changes in V_i will make large percentage changes in X_i, so we can expand up to linear order in V_i about $X_i = 1$. Since electrons are negatively charged, this gives

$$X_i = \exp[\alpha_i V_i] \quad (7)$$

with $\alpha_i = -(d\kappa/dV_i) > 0$, where $V_i = 0$ is chosen to coincide with $X_i = 1$. We mainly work with Eq. (7), but the almost topological fractional pumping also holds for $X_i = \exp[f_i(V_i)]$

for any $f_i(V_i)$ which is very positive for $V_i \to \infty$, and is very negative for $V_i \to -\infty$ (see the end of Sec. VII A). This covers many physical systems.

III. ADIABATIC ALMOST-TOPOLOGICAL PUMPING OF A FRACTION OF AN ELECTRON PER CYCLE

Let us now briefly overview our main results, with the detailed calculations postponed to Sec. VII. Firstly, for a dot coupled to reservoirs without band structure, there is a topological pumping at half an electron per cycle. Secondly, one can choose the reservoir band structure to ensure the pumped charge is topologically quantized at an arbitrary fraction of an electron per cycle.

A. Half an electron per cycle

Here we consider a situation where the reservoir density of states is energy independent (ω-independent), which is know as the wide-band limit, and so $\rho(\omega) = \rho$. Then the reservoir induces a level broadening of the quantum dot's energy level, but induces no Lamb shift; $\Lambda(\omega; \boldsymbol{K}) = 0$ in Eq. (5). Our calculations (using scattering theory in Sec. IV or Keldysh theory in Sec. VI) show that this control of the level broadening allows the pumping of half an electron per cycle in the low-temperature limit.

The dot level is taken to be above the reservoir's electrochemical potential, $(\epsilon_d - \mu) > 0$, and the pumping cycle is taken to be cycle 1 in Figs. 1(b) and 1(c), with neither ϵ_d nor μ change during the pumping cycle. The basic physical process, sketched in Fig. 2 is the following. (1) Loading [segment 1a in Fig. 1(c)]. The dot starts weakly coupled to the reservoirs ($V_{\rm L}$ and $V_{\rm R}$ very negative) so the dot's level broadening is much less than $(\epsilon_d - \mu)$, as a result the dot's occupation is negligible. The coupling to reservoir L is increased ($V_{\rm L}$ increased), so that the reservoir wave functions spread into the dot [as in Fig. 2(a)], as the dot state hybridizes with reservoir states. The dot thus absorbs a charge of $\Delta Q_{\rm load}$. Once the level broadening is much more than $(\epsilon_d - \mu)$, one reaches the limit where half the broadened level is below the reservoir's Fermi energy. In this limit, there is half an electron in the dot, $\Delta Q_{\rm load} \to \frac{1}{2}$; in other words a 50% chance of finding the dot level occupied.

(2) Moving [segment 1b in Fig. 1(c)]. The coupling to reservoir L is slowly reduced to zero, while that to reservoir R is slowly increased to its maximum value ($V_{\rm L}$ reduced and $V_{\rm R}$ increased), in such a way that the sum of the two couplings remains constant. Thus the wave functions of reservoir R spread more into the dot, while those of reservoir L spread less into the dot. The occupation of the dot remains the same, but the dot state's hybridization moves from reservoir L to reservoir R.

(3) Unloading [segment 1c in Fig. 1(c)]. The coupling to R is reduced ($V_{\rm R}$ reduced) so the level broadening again becomes much less than $(\epsilon_d - \mu)$. As a result, the dot level empties into reservoir R, as the reservoir wave functions spread into the dot become negligible, and one returns the dot to its initial state.

This cycle transfers a charge of ΔQ from reservoir L to reservoir R, with $\Delta Q \neq \Delta Q_{\rm load}$. When the coupling is large enough that the level broadening in step 1b is much more than $(\epsilon_d - \mu)$, then $\Delta Q \to \Delta Q_{\rm load} \to 1/2$.

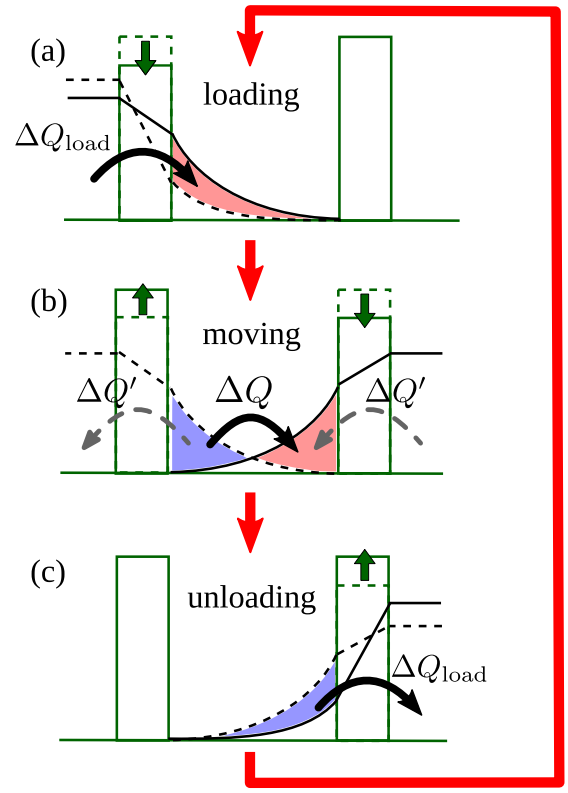

FIG. 2. Cartoon of the three steps corresponding to 1a, 1b, and 1c, described in Sec. III A, for a system without a Lamb shift ($\lambda = 0$). The central region represents the quantum dot, separated from the reservoirs by barriers, whose height we can vary to change the tunnel coupling between the dot and reservoirs. Although the dot is a single site, it helps our intuition to show the dot's hybridization with reservoir L (R) as a decaying wave function penetrating the dot from the left (right). Pink represents the average occupation of the state increasing with time, while blue represents it decreasing. The arrows indicate the average charge flow; the arrows in (b) indicate the charge $\Delta Q_{\rm load}$ is split in two, with $\Delta Q' = \Delta Q_{\rm load} - \Delta Q$ going back into L, being replaced by a charge $\Delta Q'$ from R, see Sec. V.

B. Seeing the topology

The adiabatic charge pumped per cycle can be said to be topological when it is the same for all adiabatic pumping cycles of the gate voltages that have the same topology. We will show this is the case for the cycle of $V_{\rm L}$ and $V_{\rm R}$ outlined above under certain conditions, and up to exponentially small corrections; so we call it "adiabatic almost-topological" pumping.

To see what this means, one must write the charge pumped into R as the surface integral,

$$\Delta Q_{\rm R} = e \int_{I(C)} dV_{\rm L} dV_{\rm R}\, \Pi_{\rm R}[V_{\rm L}, V_{\rm R}], \tag{8}$$

where $I(C)$ is the surface in the $V_{\rm L}$-$V_{\rm R}$ plane enclosed by the pumping cycle C.

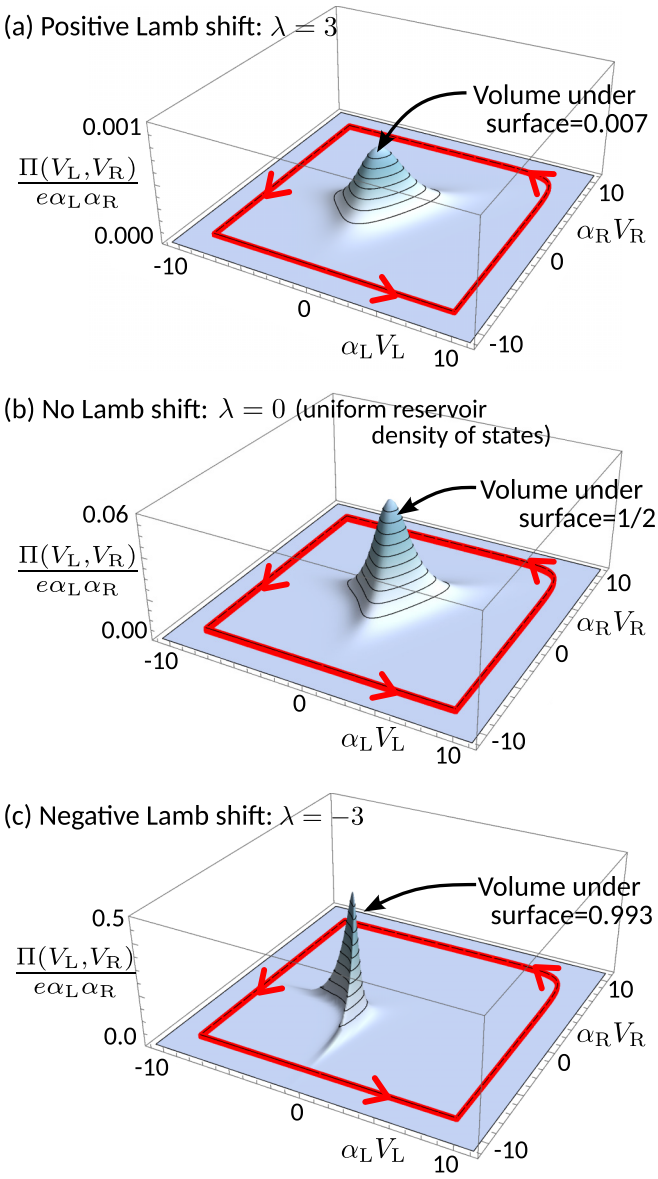

FIG. 3. Plots of the Berry curvature, $\Pi_R(V_L, V_R)$, for the dot-reservoir coupling in Eq. (7). This is given by Eq. (17) for $\lambda = 0$ and by Eq. (40) for arbitrary λ. It is always a sharp peak, but the volume under the peak is highly λ-dependent, and given by Eq. (11). Contour 1 from Fig. 1(b) is also shown.

Then one calculates $\Pi_R[V_L, V_R]$, which is known as the Berry curvature, for the pumping. If one finds that this Berry curvature is a Dirac δ function, then the pumping is entirely topological; the adiabatically pumped charge only depends on how many times the pumping contour winds around the δ function. Here, our central result, Eq. (40), is that the Berry curvature is not a δ function, but it is strongly peaked with an exponential decay away from the peak, see Fig. 3. Then we call the pumping *almost topological*, because it depends only on the contour's topology (how many times it winds around the peak) if the contour stays away from the peak, and if we neglect the exponentially small corrections coming from the tail of the peak. Thus contours 1 and 2 in Fig. 1(b) pump the same charge (up to exponentially small corrections) because they both have the same topology—each winds once around the peak.

Figure 3(b) shows the peak for reservoirs with uniform density of states. The integral over this peak is 1/2, so the contours in Fig. 3(b) will thus pump the charge

$$\Delta Q_{\text{quantized}} = e/2. \tag{9}$$

In the limit of thick tunnel barriers, $L \to \infty$, one sees that α_i in Eq. (7) also goes to infinity. Then the Berry curvature peak becomes a Dirac δ function in the V_L-V_R plane. This means that the adiabatic pumping will becomes *entirely topological*. However, for $L \to \infty$, the tunnel coupling is exponentially small, so we require exponentially small temperatures, so $\epsilon_D - \mu$ can be as small as the couplings, to ensure we can make X_L and X_R of order one, so the pumping contour can enclose the δ-function peak.

C. Different fractions of an electron per cycle

Let us now consider reservoirs with a nonuniform density of states, so $\rho(\omega)$ depends on ω. In this case, the Lamb shift in Eq. (5) is nonzero; this means that the dot-reservoir coupling does not only broaden the dot level into a resonance, it also causes the center of that resonance to be shifted in energy. Section VII will use Keldysh theory to show that the adiabatic almost-topological pumping is quantized at a fraction of an electron (between 0 and 1), which is given by the ratio of the Lamb shift to the level broadening. We define λ as the following dimensionless measure of this ratio at $\omega = \mu$:

$$\lambda = 2\Lambda(\mu, t)/\Gamma(\mu, t), \tag{10}$$

where the factor of 2 is to make our results compact. We will show that the almost topological charge that is pumped by the cycle described in Sec. III A above is

$$\Delta Q_{\text{quantized}} = \frac{e}{\pi}\left[\frac{\pi}{2} - \arctan(\lambda) - \frac{\lambda}{1+\lambda^2}\right]. \tag{11}$$

Hence, for this pumping cycle, $\Delta Q_{\text{quantized}}$ is a monotonically decaying function of λ, and it take values between e and 0. More precisely, $\Delta Q_{\text{quantized}}$ equals $[1 - 2/(3\pi\lambda^2)]e$ for $\lambda \ll -1$, equals $e/2$ at $\lambda = 0$, and equals $2e/(3\pi\lambda^2)$ for $\lambda \gg 1$.

It is surprising that the exact result for pumping at low temperatures only depends on the ratio of the Lamb shift *at the electrochemical potential* to the level broadening *at the electrochemical potential*, when many other observables depend on these quantities integrated over all energies [see, e.g., $n(\boldsymbol{K})$ is Sec. V]. It is not easy to explain how this quantity emerges in the exact calculation, but we believe it is because we are at very low temperature and zero bias, so all charge flow between reservoirs occurs at energies at (or extremely close to) the electrochemical potential. Hence the pumped charge also only depends on the physics of the Lamb shift and level broadening at the electrochemical potential.

Crucially, λ is entirely determined by the reservoir band structure since Eqs. (4), (5), and (10) mean that

$$\lambda = \frac{2}{\rho(\mu)} \fint d\varepsilon \, \frac{\rho(\varepsilon)}{\mu - \varepsilon}, \tag{12}$$

so it is independent of K_L, K_R and t. Hence, any given reservoir band structure will have a given λ, and hence a

given quantized fraction of an electron pumped per cycle. By choosing a suitable reservoir band structure, one can choose that fraction.

D. Requirements for experimental observation

There are four requirements for observing this quantized pumping of a fraction of an electron per cycle.

The first requirement is a quantum dot that mimics the Hamiltonian in Eq. (1) which neglects electron-electron interactions on the dot. The simplest experimental implementation of Eq. (1) is an interacting quantum dot (described by an Anderson impurity Hamiltonian) in a large enough magnetic field that the dot's spin state with higher energy is always empty, which makes the on-dot interaction term negligible.

The second requirement is that k_BT is much smaller than $(\epsilon_d - \mu)$, larger temperatures will destroy the quantization. At the same time $(\epsilon_d - \mu)$ should be small enough that we can make the dot-reservoir coupling $K \gg (\epsilon_d - \mu)/\rho(\mu)$. Thus we require that $k_BT \ll K_{\max}\rho(\mu)$, which means the required value of T depends on how strongly the dot can be coupled to the reservoirs.

The third requirement is related to the fact that the charge pumping is probabilistic, with only the *average* charge being quantized. This probabilistic nature of the pumping is typical whenever there is part of the pumping cycle in which the dot is coupled to both reservoirs at the same time (segment 1b of the cycle). Thus in any given cycle $n = 0, \pm1, \pm2, \ldots$ electrons might flow. Central limit theorem tells us that averaging over many cycles will give an answer that will converge to the quantized fraction that we predict.

The fourth requirement is due to our assumption that ϵ_d is time-independent during the pumping cycle. Unfortunately, in practice, the electrostatic gates that vary K_L and K_R, will also have a capacitive coupling to the dot level, causing ϵ_d to vary. Gate M in Fig. 1(a) will minimize this capacitive coupling, by partially screening the dot from gates L and R. Any remaining capacitive coupling to gates L and R will act much like the Lamb shift. However, this coupling goes linearly in V_L and V_R, while the level broadening and Lamb shift (if present) go exponentially, as above Eq. (7). Hence any effect of the capacitive coupling on ϵ_d will become negligible compared to the broadening at large α_iV_i.

IV. SCATTERING THEORY

The central calculation in this work uses the Keldysh technique, however as a warm up exercise, we can use the scattering theory of quantum pumping [6] for the special case where the reservoirs have uniform density of states. Readers interested in the Keldysh calculation of the general case can skip this section.

The scattering matrix of a single-level dot (see, e.g., Refs. [33,34]) at energy μ is

$$\begin{pmatrix} S_{LL} & S_{RL} \\ S_{LR} & S_{RR} \end{pmatrix} = \boldsymbol{I} - \frac{i}{\mu - \epsilon_d + i\frac{1}{2}\Gamma} \begin{pmatrix} \Gamma_L & \sqrt{\Gamma_L\Gamma_R} \\ \sqrt{\Gamma_L\Gamma_R} & \Gamma_R \end{pmatrix}, \tag{13}$$

where $\boldsymbol{I}$ is a 2-by-2 unit matrix. The scattering theory [6] for pumping of K_L and K_R around the contour C, gives the charge pumped per cycle into reservoir R as the integral over the surface enclosed by C,

$$\Delta Q_R = e \int_C dK_L dK_R \; \Pi_R(\boldsymbol{K}), \tag{14}$$

where the Berry curvature $\Pi_R(\boldsymbol{K})$ for our system is

$$\Pi_R(\boldsymbol{K}) = \frac{1}{\pi} \mathrm{Im}\left[\frac{\partial S^*_{RL}}{\partial K_L} \frac{\partial S_{RL}}{\partial K_R} + \frac{\partial S^*_{RR}}{\partial K_L} \frac{\partial S_{RR}}{\partial K_R} \right]. \tag{15}$$

Substituting in Eq. (13) and using Eq. (6), one find that the zero-temperature result for pumped charge per cycle (in units of e) is given by the dimensionless integral

$$\frac{\Delta Q_R}{e} = \frac{2}{\pi} \int_{I(C')} dX_L dX_R \; \frac{X}{[1 + X^2]^2}, \tag{16}$$

where $X = X_L + X_R$. The surface of integration $I(C')$ is that enclosed by the contour C in Eq. (14) rescaled using Eq. (6). One can show [6] that $\Delta Q_L = -\Delta Q_R$.

Now we cast this result in terms of the gate voltages that control the couplings. Using Eq. (7), ΔQ_R is given by Eq. (8) with the Berry curvature

$$\frac{\Pi_R[V_L, V_R]}{e} = \frac{2\alpha_L\alpha_R}{\pi} \frac{e^{\alpha_L V_L} e^{\alpha_R V_R} (e^{\alpha_L V_L} + e^{\alpha_R V_R})}{[1 + (e^{\alpha_L V_L} + e^{\alpha_R V_R})^2]^2}. \tag{17}$$

This is shown in Fig. 3(b). The crucial feature is that this is highly peaked at small $|\alpha_iV_i|$ and decays exponentially with increasing $|\alpha_iV_i|$ (for both $i = L$ and $i = R$). Hence it fulfils the conditions for adiabatic almost-topological pumping discussed in Sec. III B.

To find the charge pumped by a contour that encloses the above peak once, we take contour 1 in Fig. 1(b), whose segment 1b is chosen such that $\exp[\alpha_L V_L] + \exp[\alpha_R V_R]$ is constant. We then go back to Eq. (16), for which this contour maps via Eq. (7) to the triangular contour shown in Fig. 1(c). The contour C' is the triangle defined by (X_L, X_R) going from $(0, 0) \to (X_{\max}, 0) \to (0, X_{\max}) \to (0.0)$, where $X_{\max} = \rho K_{\max}/[2(\epsilon_d - \mu)]$. We write

$$\int_{I(C')} dX_L dX_R \, (\cdots) \; = \; \tfrac{1}{2} \int_0^{X_{\max}} dX \int_{-X}^{X} dY \, (\cdots), \tag{18}$$

where $Y = X_L - X_R$. Then

$$\frac{\Delta Q_R}{e} = \frac{1}{\pi} \left[\arctan[X_{\max}] - \frac{X_{\max}}{1 + X^2_{\max}} \right] \tag{19}$$

for the above triangular contour. This goes to $1/2$ for large $X_{\max}$, which corresponds to encircling the peak in Eq. (17). Hence, for uniform reservoir density of states, the pumping is quantized at half an electron per cycle.

We do not know of a scattering theory for nonuniform reservoir density of states, so we use the Keldysh formalism to treat such cases in Secs. VI and VII.

V. COMPARISON WITH DOT OCCUPATION

One might naively guess that the pump is simply due to filling the dot state from L in the "loading" part of the cycle, and then emptying it into R in the "unloading" part of the cycle. Then the charge transferred from L to R would equal the charge loaded into the dot, ΔQ_{load}. We show here that this

is not the case; there is no simple relation between the pumped charge and $\Delta Q_{\rm load}$.

We are pumping adiabatically slowly, so electrons are continuously tunneling in and out of the dot from L and R (and tunneling though the dot from L to R) during the "moving" part of the cycle. They have too little energy to remain in the dot, but the uncertainty principle means they can be there for a time of order $\hbar/(\epsilon_d - \mu)$. So there is no reason to assume the pumped charge is related to the dot occupation. Indeed, the occupation of the dot at low temperatures, see, e.g., Refs. [35,36], is

$$n(\boldsymbol{K}) = \int_{-\infty}^{\mu} \frac{d\omega}{2\pi} \frac{\Gamma(\omega;\boldsymbol{K})}{[\omega - \epsilon_d - \Lambda(\omega;\boldsymbol{K})]^2 + \left[\frac{1}{2}\Gamma(\omega;\boldsymbol{K})\right]^2}.$$

For a uniform density of states $\Lambda(\omega) = 0$, the integrand is a Lorentzian, and so $n(\boldsymbol{K}) = \arctan[X]/\pi$. Then

$$\Delta Q_{\rm load} = e[n(\boldsymbol{K}_{\rm max}) - n(0)] = \frac{e \arctan[X_{\rm max}]}{\pi}. \tag{20}$$

From Eq. (19), we see the pumped charge is smaller than $\Delta Q_{\rm load}$ by a factor of $\Delta Q' = eX_{\rm max}/[\pi(1 + X_{\rm max}^2)]$, which vanishes when $X_{\rm max} \to \infty$. This means that the "moving" part of the pumping cycle in Sec. III A involves a small flow, $\Delta Q'$, from the R to L through the dot [the dashed arrows in the Fig. 2(b)].

For nonuniform density of states, $\Delta Q_{\rm load}$ depends on the ω dependence of $\Gamma(\omega;\boldsymbol{K})$ and $\Lambda(\omega;\boldsymbol{K})$ for all $\omega \leqslant \mu$. In contrast, the pumped charge in Eq. (11) depends *only* on their values at $\omega = \mu$. Thus in general ΔQ and $\Delta Q_{\rm load}$ will not be related in any way, although both will be between 0 and e. Either can be larger, so $\Delta Q'$ can be of either sign. Indeed, two different setups can have the same ΔQ and different $\Delta Q_{\rm load}$ or vice versa.

VI. KELDYSH FORMALISM

The dot's occupation and current into reservoir i at time t are [37–41]

$$n(t) = \left\langle d^\dagger(t) d(t) \right\rangle = -iG^<(t,t), \tag{21}$$

$$\begin{aligned} J_i(t) &= -e\frac{d}{dt}\sum_k \langle c_{ik}^\dagger(t) c_{ik}(t)\rangle \\ &= e\int dt_1 [G^{\rm R}(t,t_1)\Sigma_i^<(t_1,t) + G^<(t,t_1)\Sigma_i^A(t_1,t) \\ &\quad - \Sigma_i^{\rm R}(t,t_1)G^<(t_1,t) - \Sigma_i^<(t,t_1)G^A(t_1,t)]. \end{aligned} \tag{22}$$

respectively, in terms of the Keldysh Green's functions in Appendix A. We will derive the pumped charge for a large driving contour C, by summing the contributions from all infinitesimal circular contours inside it $\{C_n\}$ (see Fig. 4), as was done in scattering theory by Ref. [6].

The infinitesimal contour C_n satisfies

$$\boldsymbol{K}(t) \equiv \begin{pmatrix} K_{\rm L}(t) \\ K_{\rm R}(t) \end{pmatrix} = \boldsymbol{K}_{n,0} + \delta K \begin{pmatrix} \cos[\Omega t] \\ \sin[\Omega t] \end{pmatrix}, \tag{23}$$

where Ω is a pumping frequency, δK is an infinitesimally small amplitude of driving around the time-independent point $\boldsymbol{K}_{n,0}$. Under this infinitesimal driving, the time-dependent

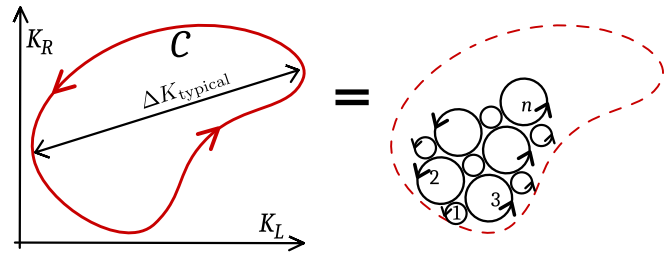

FIG. 4. A sketch of the subdivision of contour C into infinitesimal circular contours $\{C_n\}$, with the diameter of each infinitesimal contour chosen to ensure C is filled densely.

charge current into reservoir i is

$$J_i(t) = \mathcal{G}_i^{\rm L}(\Omega;\boldsymbol{K}_{n,0})\,\delta K\cos(\Omega t) + \mathcal{G}_i^{\rm R}(\Omega;\boldsymbol{K}_{n,0})\,\delta K\sin(\Omega t). \tag{24}$$

where $\mathcal{G}_i^{\rm L}(\Omega;\boldsymbol{K}_{n,0})$ is the Fourier transform of the dynamic conductance for the infinitesimal contour C_n;

$$\mathcal{G}_i^j(t,t_1;\boldsymbol{K}_{n,0}) = \left.\frac{\delta J_i(t)}{\delta K_j(t_1)}\right|_{\delta K=0}. \tag{25}$$

This is given in terms of Keldysh Green's functions in Appendix A, and it only depends on the time differences $(t - t_1)$ because it is evaluated for $\delta K = 0$. We assume the condition for adiabatic driving;

$$\Omega \ll \tau^{-1}, \tag{26}$$

where τ is the typical time for electrons in the dot to relax. Then it is sufficient to take the dynamic conductance at leading order in the pumping frequency Ω; $\mathcal{G}_i^{i'}(\Omega;\boldsymbol{K}) = A_i^{i'}(\boldsymbol{K})\ \Omega\ +\ \mathcal{O}[\Omega^2]$. Substituting this into Eq. (24), and integrating Ωt from 0 to 2π, we find the charge pumped per cycle on the infinitesimal contour C_n is $\delta Q_{i,n} = \int_{C_n}[A_i^{\rm L}(\boldsymbol{K})dK_{\rm L} + A_i^{\rm R}(\boldsymbol{K})dK_{\rm R}]$.

Summing all infinitesimal contours inside the large contour C, gives charge pumped per cycle around C as

$$\Delta Q_i = \oint_C d\boldsymbol{K}\cdot\boldsymbol{A}_i(\boldsymbol{K}), \tag{27}$$

where we define the *Berry connection* as the vector $\boldsymbol{A}_i(\boldsymbol{K}) = (A_i^{\rm L}(\boldsymbol{K}), A_i^{\rm R}(\boldsymbol{K}))$. Rewriting this in terms of a surface integral using Stokes theorem, we get

$$\Delta Q_i = \int_{I(C)} d\boldsymbol{S}\cdot\boldsymbol{\Pi}_i(\boldsymbol{K}), \tag{28}$$

where $\boldsymbol{\Pi}_i(\boldsymbol{K}) = \nabla_{\boldsymbol{K}}\times\boldsymbol{A}_i(\boldsymbol{K})$ is the *Berry curvature*. This integral is over the surface $I(C)$ which is enclosed by the pumping contour C. As this surface is the K_x-K_y plane, only the component of $\boldsymbol{\Pi}_i(\boldsymbol{K})$ perpendicular to this plane contributes; we call this component

$$\Pi_i(\boldsymbol{K}) = \frac{d}{dK_{\rm L}}\left[A_i^{\rm R}(K)\right] - \frac{d}{dK_{\rm R}}\left[A_i^{\rm L}(K)\right]. \tag{29}$$

we will calculate this for our model in the next section.

We end this derivation with an adiabaticity condition for the large contour C. Given Eq. (26) for the infinitesimal

circular contours, adiabaticity on C requires

$$|d\boldsymbol{K}/dt| \ll \Delta K_{\text{typical}}/\tau, \tag{30}$$

where $\Delta K_{\text{typical}}$ is the typical scale of the contour (see Fig. 4), and τ is the relaxation time of the dot state. The magnitude of $1/\tau$ is discussed in Sec. VIII.

VII. RESULTS FOR OUR MODEL

For the Hamiltonian in Eq. (1), we find that the Berry connection in Eq. (27) contains two terms

$$\boldsymbol{A}_i(\boldsymbol{K}) = \boldsymbol{A}_i^{\text{broad}}(\boldsymbol{K}) + \boldsymbol{A}_i^{\text{shift}}(\boldsymbol{K}) \tag{31}$$

because $\boldsymbol{A}_i(\boldsymbol{K})$ involves a derivative with respect to $\boldsymbol{K}$, and that derivative can act on the level broadening (giving $\boldsymbol{A}^{\text{broad}}$) or the Lamb shift (giving $\boldsymbol{A}^{\text{shift}}$). If there is no Lamb shift then $\boldsymbol{A}_i^{\text{shift}}(\boldsymbol{K}) = 0$, while if the Lamb shift is much greater than the level broadening, then Eq. (31) is dominated by $\boldsymbol{A}_i^{\text{shift}}(\boldsymbol{K})$. The Keldysh calculations outlined in Appendix give

$$\left[\boldsymbol{A}_i^{\text{broad}}(\boldsymbol{K})\right]_j = \int \frac{d\omega}{2\pi}\left[\left(\mathcal{B}^2 - \tfrac{1}{4}\mathcal{A}^2\right) f\Lambda_i - \tfrac{1}{2}\mathcal{A}\mathcal{B} f\Gamma_i - \delta_{i,j}\mathcal{B}' f\right]\frac{\partial\Gamma}{\partial K_j}, \tag{32}$$

$$\left[\boldsymbol{A}_i^{\text{shift}}(\boldsymbol{K})\right]_j = \int \frac{d\omega}{2\pi}\left[2\mathcal{B}\mathcal{A} f\Lambda_i + \tfrac{1}{4}\mathcal{A}^2(f'\Gamma_i - f\Gamma_i') + \mathcal{B}^2(f\Gamma_i)' - \delta_{i,j}(\mathcal{A}f)'\right]\frac{\partial\Lambda}{\partial K_j}, \tag{33}$$

where i and j are L or R, and $f = [1 + e^{(\omega-\mu)/T}]^{-1}$ is the Fermi function. The primed denotes the partial derivative with respect to ω. The quantities Λ_i and Γ_i are given in Eqs. (2)–(5), while $\mathcal{A} = 2\text{Im}[G^A(\omega)]$ and $\mathcal{B} = \text{Re}[G^A(\omega)]$, with $G^A(\omega)$ given in Eq. (A6).

Turning to the Berry curvature in Eq. (29), we see that it contains two derivatives (with respect to K_j), because $[\boldsymbol{A}_i(\boldsymbol{K})]_j$ contained one. Hence $\Pi_i(\boldsymbol{K})$ contains three terms; a "broad-broad" term due to both derivatives acting on the broadening, a "shift-shift" term due to both derivatives acting the Lamb shift, and a "shift-broad" term with one derivative on each of them. The "shift-shift" term turns out to be zero, showing that the Lamb shift alone is not enough to do pumping. Intuitively, this can be understood as the Lamb shift only moving the dot level, which is not enough to do pumping. Hence

$$\Pi_{\text{R}}(\boldsymbol{K}) = \Pi_{\text{R}}^{\text{broad-broad}}(\boldsymbol{K}) + \Pi_{\text{R}}^{\text{shift-broad}}(\boldsymbol{K}), \tag{34}$$

and $\Pi_{\text{L}}(\boldsymbol{K}) = -\Pi_{\text{R}}(\boldsymbol{K})$, with

$$\Pi_{\text{R}}^{\text{broad-broad}}(\boldsymbol{K}) = \int \frac{d\omega}{4\pi}\, f'\mathcal{A}\mathcal{B}\,\frac{\Gamma^2(\omega,\boldsymbol{K})}{K^2}, \tag{35}$$

$$\Pi_{\text{R}}^{\text{shift-broad}}(\boldsymbol{K}) = \int \frac{d\omega}{4\pi}\, f'\mathcal{A}^2\,\frac{\Gamma(\omega,\boldsymbol{K})\Lambda(\omega,\boldsymbol{K})}{K^2}, \tag{36}$$

where we have used the fact that Λ and Γ are proportional to $K = K_{\text{L}} + K_{\text{R}}$. A bit more algebra gives

$$\Pi_{\text{R}}(\boldsymbol{K}) = \frac{e}{2}\int \frac{d\omega}{2\pi}\,\frac{(\omega-\epsilon_d)\,\rho^2(\omega)\,\Gamma(\omega;\boldsymbol{K})\,(\partial f/\partial\omega)}{\left[[\omega-\epsilon_d-\Lambda(\omega;\boldsymbol{K})]^2 + \left[\tfrac{1}{2}\Gamma(\omega;\boldsymbol{K})\right]^2\right]^2}. \tag{37}$$

This depends on the sum of the couplings, $K = (K_{\text{L}} + K_{\text{R}})$, but not the difference $(K_{\text{L}} - K_{\text{R}})$.

A. Low-temperature pumping

In the limit of small temperature, we can make the approximation $(\partial f/\partial\omega) = -\delta(\omega-\mu)$ in Eq. (37). To justify this approximation, one needs the other terms in the integrand of Eq. (37) to vary little over the window of ω given by $\mu \pm k_{\text{B}}T$. Then, the Berry curvature is

$$\Pi_{\text{R}}(\boldsymbol{K}) = \frac{e}{4\pi}\,\frac{(\epsilon_d-\mu)\,\rho^2(\mu)\,\Gamma(\mu;\boldsymbol{K})}{\left[[\mu-\epsilon_d-\Lambda(\mu;\boldsymbol{K})]^2 + \left[\tfrac{1}{2}\Gamma(\mu;\boldsymbol{K})\right]^2\right]^2}. \tag{38}$$

Writing this in terms of λ in Eq. (10), the low-temperature result for pumped charge per cycle (in units of e) is given by the dimensionless integral

$$\frac{\Delta Q_{\text{R}}}{e} = \frac{2}{\pi}\int_{I(C')} dX_{\text{L}}dX_{\text{R}}\,\frac{X}{[(1+\lambda X)^2 + X^2]^2}, \tag{39}$$

where X_i is defined in Eq. (6) with ρ being $\rho(\mu)$, and $X = X_{\text{L}} + X_{\text{R}}$. The surface of integration $I(C')$ is that enclosed by the contour C in Eq. (14) rescaled using Eq. (6).

As explained in Sec. II, we control gate voltages V_i, in experiments. By substituting Eq. (7) into Eq. (39), we find the Berry curvature in the $(V_{\text{L}}, V_{\text{R}})$ plane

$$\frac{\Pi_{\text{R}}(V_{\text{L}},V_{\text{R}})}{e} = \frac{2}{\pi}\,\frac{\alpha_{\text{L}}\alpha_{\text{R}}\,X\,e^{\alpha_{\text{L}}V_{\text{L}}}e^{\alpha_{\text{R}}V_{\text{R}}}}{[(1+\lambda X)^2+X^2]^2}\bigg|_{X=\frac{e^{\alpha_{\text{L}}V_{\text{L}}}}{+e^{\alpha_{\text{R}}V_{\text{R}}}}}, \tag{40}$$

shown in Fig. 3. This is our *central result*, because the fractional and topological nature of the adiabatic pumping both follow from it, as we now show.

Equation (40) has a peak at small $|\alpha_i V_i|$ and decays exponentially as $|V_i|$ grows. Hence, any pumping contour that encloses the peak without encroaching on it will give the same pumped charge per cycle (up to exponentially small corrections), ensuring quantized pumping.

To calculate the charge pumped by such a cycle, we return to Eq. (39) and consider a triangular contour C' explained above Eq. (18). Equation (7) means that for large $X_{\max}$ this triangular contour corresponds to contour 1 in Fig. 1(b), that encloses the peak in $\Pi_{\text{R}}(V_{\text{L}}, V_{\text{R}})$. We transform to X and Y as in Eq. (18), then

$$\frac{\Delta Q_{\text{R}}}{e} = \frac{1}{\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{1+\lambda X_{\max}}{X_{\max}}\right) - \frac{X_{\max}(1+\lambda X_{\max})}{1+X_{\max}^2+\lambda X_{\max}(2+\lambda X_{\max})}\right], \tag{41}$$

see Fig. 5. It reduces to Eq. (19) for $\lambda = 0$, since $\arctan(x) + \arctan(1/x) = \text{sgn}[x]\pi/2$. We take $X_{\max} \to \infty$ to get the pumping for a contour that corresponds to one enclosing the peak of Eq. (40); this gives Eq. (11).

This analysis has given us our main results; the adiabatic pumping is almost topological, and pumps a fraction of an electron (between 0 and 1) given by the value of λ, which is determined purely by the reservoir's band structure.

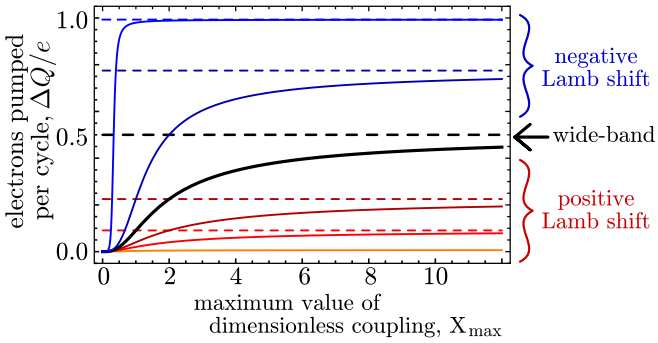

FIG. 5. The solid curves are the charge pumped per cycle on the triangular pumping cycle, given by Eq. (41). From top to bottom we have $\lambda = -3,\ -0.5,\ 0,\ 0.5,$ and $1, 3$. The horizontal dashed lines show the large X_{max} limit given by Eq. (11).

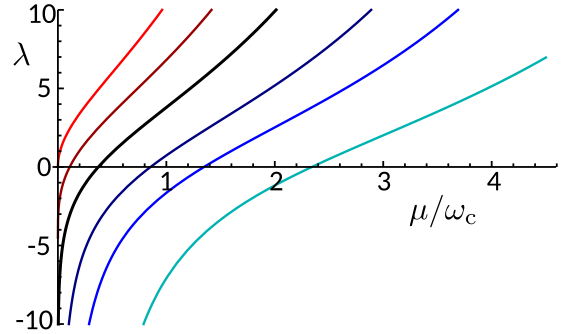

FIG. 6. The parameter λ for a reservoir density of states given by Eq. (42) as a function of μ/ω_c, for $s = -0.5,\ -0.3,\ 0,\ 0.5,\ 1,$ and 2 (from top to bottom). One can access almost any λ by a suitable choice of s and μ/ω_c.

To generalize to a voltage dependence of the form below Eq. (7), we substitute it into Eq. (39). Then Eq. (40) changes, but it remains strongly peaked with exponentially small tails. This ensures that there is still adiabatic almost-topological pumping. Further more, the faction pumped per cycle is the same for any voltage dependence, since it was calculated directly from Eq. (39).

VIII. ADIABATICITY AND BAND GAPS

Up to now this work has only discussed pumping in the adiabatic limit. However, from the argument in Sec. VI, it is clear that a large but finite pumping period $T_{\rm period} \sim \Delta K_{\rm typical}/(d\boldsymbol{K}/dt) \gg \tau$, will induce a nonadiabatic correction of order $(\tau/T_{\rm period})$. This nonadiabatic correction is much larger than that in many proposals for topological pumping, in which the topology makes the nonadiabatic corrections exponentially small at large $T_{\rm period}$. Thus to observe the topological pumping in our system it is crucial to estimate τ, and then choose the pumping to be slow enough (large $T_{\rm period}$) to make corrections of order $(\tau/T_{\rm period})$ negligible.

It is simple to estimate τ when the reservoirs have uniform density of states, since there the dot state decays at the rate given by the level broadening in Sec. II; $1/\tau = \Gamma = (K_{\rm L} + K_{\rm R})\rho$. For systems with nonuniform density of states, we can place a lower bound on $1/\tau$ by saying that $1/\tau \gtrsim K\rho_{\rm min}$, where $\rho_{\rm min}$ is the minimal value of the density of states.

However, this poses a problem for reservoirs with band gaps, as the density of states vanishes in the band gap, so the above lower-bound does not allow us to say when the pumping is slow enough to be considered adiabatic. To investigate this further we consider the case where the electrochemical potential is near a band edge in the reservoir, so the reservoir's density of states is

$$\rho(\omega) = \begin{cases} \rho_0\,(\omega/\omega_c)^s\, e^{-\omega/\omega_c} & \omega > 0 \\ 0 & \omega < 0 \end{cases}, \qquad (42)$$

where, without loss of generality, we measure energy ω from the band edge. Then the level broadening is $\Gamma_i = K_i\rho(\omega)$, and the Lamb shift is (see, e.g., Refs. [36,42]),

$$\lambda = -2\,\Gamma(1+s)\,\mathrm{Re}[(-1)^s\,\Gamma(-s, -\mu/\omega_c)]. \qquad (43)$$

Figure 6 plots this and shows that a suitable choice of s and μ/ω_c will give almost any desired value of λ.

It has long been known that this model exhibits an infinite-lifetime bound state [43–47], see Refs. [48,49] for reviews. Electron dynamics in various time-dependent versions of this model have been studied; particularly the decay of an initially prepared dot state [42,50–54], the response to switching on a bias [55,56], or the response to periodic driving [57,58]. For $s > 0$, this bound state appears when the coupling exceeds a critical value [35,36,42,50–57,59] $K_c = \epsilon/\Gamma[s]$. This state has $\tau = \infty$, so pumping never satisfies the adiabaticity condition in Eq. (30) when $K > K_c$. Intriguingly, the Berry curvature in Eq. (37) does not contain K_c; it is a smooth function across this line of critical coupling $K = (K_{\rm L} + K_{\rm R}) = K_c$. However, the Berry curvature in Eq. (37) ceases to have a physical meaning when one crosses the line of critical coupling, because nonadiabatic contributions dominate beyond this line ($K > K_c$), no matter how slow the pumping is.

For $K < K_c$, it is difficult to determine the dot's decay rate, $1/\tau$, because it contains terms with an oscillatory power-law decay, for which there is no unique way to define $1/\tau$. Fig. 7

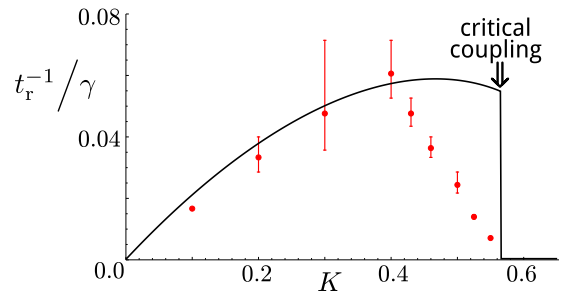

FIG. 7. Estimate of the dot relaxation rate versus coupling K, for reservoirs with a band gap; Eq. (42) with $s = 1/2$ and $\omega_c = 10\epsilon_d$. The points estimate the rate via Eq. (44). The solid curve estimates it from the exponential part of the decay, neglecting the oscillatory or power-law components. At small K, the decay is almost exponential, and the two protocols coincide. However, as $K \to K_c$, the power law completely dominates, and the solid curve fails to capture the slowing of the decay. The infinite-lifetime bound state emerges at K_c; so $1/t_r = 0$ for $K \geqslant K_c$. Note the error bars are not error bars in the usual sense, see Sec. VIII.

is an attempt to give a feeling for how $1/\tau$ depends on the coupling. The red data points are the inverse time for the dot occupation to decay to threshold (using the method reviewed in Refs. [35,36]) that we set at 2% of its initial value, i.e., we plot the $1/t_r$ that satisfies

$$\frac{|n(t_r) - n_0|}{|n_\infty - n_0|} = 2\%. \tag{44}$$

The solid curve is the time taken to reach this threshold, if one approximates the decay to an exponential at the rate given by the imaginary part of the resonance's energy (i.e., neglecting all power-law or oscillatory components in the decay). This approximation captures much of the true decay, but misses the sharp drop in $1/t_r$ as $K \to K_c$. This sharp drop shown by the data points indicates that the timescale to decay diverges as K approaches K_c. Hence it is increasingly difficult to pump slowly enough to be adiabatic as K gets closer to K_c.

The "error bars" on the data in Fig. 7 are *not* numerical uncertainties in t_r (such uncertainties are about the size of the red dots). They indicate the period of the oscillations in the decay, which is maximal for $K \simeq K_c/2$. A small change in the system parameters (e.g., a change of ω_c or μ) would shift the phase of the oscillations, thereby shifting where the oscillating decay crosses the threshold to a different place on the vertical "error bar." Hence, we can expect a change in system parameters to induce a large change in $1/t_r$ when $K \simeq K_c/2$, while the change will be modest for $K \ll K_c$ and $K \sim K_c$.

IX. CONCLUSIONS

We show that a system without exotic physics (a noninteracting single-level quantum dot at low temperature) can exhibit an adiabatic almost-topological pumping of a fraction of an electron per cycle, when averaged over many cycles. We call it "almost" topological because the pumped charge depending only on the number of times the contour winds around the peak in the Berry curvature, shown in Fig. 3, under the conditions that (i) the contour does not touch the peak and (ii) we neglect the exponentially small corrections coming from the tail of the peak. Section III B mentions a specific limit in which the adiabatic pumping is entirely topological. The fraction pumped (between zero and one electron) is determined by the ratio of the Lamb shift to the level broadening. This ratio is imposed by the reservoir band structure, which can be chosen to give the desired fraction. A uniform reservoir density of states gives the quantized pumping of half an electron per cycle. We emphasize that it is the *average* charge pumped per cycle that is (almost) topological and fractional. Each cycle has a finite probability that n electrons are pumped for $n = 0, \pm1, \pm2, \ldots$; the quantized fraction is only revealed by averaging over many cycles.

Hence, if one wants to prove the existence of fractionally charged particles in some system, one would need more evidence than just adiabatic pumping of fractional average charge. This evidence could be that nonadiabatic corrections decay exponentially when the period of the pumping cycle is made large, since these corrections only decay like one over this period in our model.

ACKNOWLEDGMENTS

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APPENDIX: KELDYSH GREEN'S FUNCTIONS

The quantum dot's Green's functions are defined as [37–39]

$$G^A(t_1,t_2) = i\Theta(t_2-t_1)\langle[d(t_1), d^\dagger(t_2)]_+\rangle, \tag{A1}$$

$$G^R(t_1,t_2) = -i\Theta(t_1-t_2)\langle[d(t_1), d^\dagger(t_2)]_+\rangle, \tag{A2}$$

$$G^<(t_1,t_2) = i\langle d^\dagger(t_2)d(t_1)\rangle, \tag{A3}$$

where $\Theta(t)$ is a Heaviside function and $[\cdot,\cdot]_+$ is an anticommutator. Their algebraic form is given by Dyson's equations

$$G^\kappa(t_1,t_2) = g^\kappa(t_1,t_2) + \int dt_3 dt_4\, g^\kappa(t_1,t_3)\Sigma^\kappa(t_3,t_4)G^\kappa(t_4,t_2),$$

for $\kappa = R, A$, and

$$G^<(t_1,t_2) = \int dt_3 dt_4\, G^R(t_1,t_3)\Sigma^<(t_3,t_4)G^A(t_4,t_2).$$

Here, $g^{A/R}(t_1,t_2)$ are Green's function of electron of the isolated quantum dot, and $\Sigma^\kappa(t_1,t_2)$ are the one-particle-irreducible self-energy for $\kappa = A, R, <$,

$$\Sigma_i^\kappa(t_1,t_2) = \sum_k \gamma_i(t_1) g_{ik}^\kappa(t_1,t_2)\gamma_i(t_2), \tag{A4}$$

with $\Sigma^\kappa = \Sigma_L^\kappa + \Sigma_R^\kappa$. Here $g_{ik}^\kappa(t_1,t_2)$ is a Green's function of electrons in the isolated electron reservoirs,

$$\begin{aligned} g_{ik}^A(t_1,t_2) &= i\Theta(t_2-t_1)\langle[c_{ik}(t_1), c_{ik}^\dagger(t_2)]_+\rangle_{\gamma_i=0},\\ g_{ik}^R(t_1,t_2) &= -i\Theta(t_1-t_2)\langle[c_{ik}(t_1), c_{ik}^\dagger(t_2)]_+\rangle_{\gamma_i=0},\\ g_{ik}^R(t_1,t_2) &= i\langle c_{ik}^\dagger(t_2)c_{ik}(t_1)\rangle_{\gamma_i=0}. \end{aligned} \tag{A5}$$

The dynamic conductance in Eq. (25) is

$$\begin{aligned} \mathcal{G}_i^j(t,t_1) = \frac{e}{2}\big\{ & G^R\big|\Sigma_j^R G^R\Sigma_i^< + G^R\Sigma_j^R\big|G^R\Sigma_i^<\\ & - \Sigma_i^< G^A\big|\Sigma_j^A G^A + \Sigma_i^< G^A\Sigma_j^A\big|G^A\\ & + G^R\big|\Sigma_j^R G^<\Sigma_i^A + G^R\Sigma_j^R\big|G^<\Sigma_i^A + G^R\big|\Sigma_j^< G^A\Sigma_i^A\\ & + G^R\Sigma_j^<\big|G^A\Sigma_i^A + G^<\big|\Sigma_j^A G^A\Sigma_i^A + G^<\Sigma_j^A\big|G^A\Sigma_i^A\\ & - \Sigma_i^R G^R\big|\Sigma_j^R G^< - \Sigma_i^R G^R\Sigma_j^R\big|G^< - \Sigma_i^R G^R\big|\Sigma_j^< G^A\\ & - \Sigma_i^R G^R\Sigma_j^<\big|G^A - \Sigma_i^R G^<\big|\Sigma_j^A G^A - \Sigma_i^R G^<\Sigma_j^A\big|G^A\\ & + \delta_{i,j}(G^R\big|\Sigma_i^< + G^<\big|\Sigma_i^A - \Sigma_i^R\big|G^< - \Sigma_i^<\big|G^A)\big\}, \end{aligned}$$

where we define $AB|CD = [AB](t,t_1)[CD](t_1,t)$ with $[AB](t_1,t_2) = \int dt_3\ A(t_1,t_3)B(t_3,t_2)$. One can Fourier transform $G^{A/R}(t_1,t_2)$ to get

$$G^A(\omega) = [G^R(\omega)]^* = \left(\omega - \epsilon_d - \Lambda(\omega) + i\tfrac{1}{2}\Gamma\right)^{-1}. \tag{A6}$$

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