Spontaneous symmetry breaking and the flat phase of crystalline membranes

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(Received 12 June 2019; revised manuscript received 26 July 2019; published 5 September 2019)

Crystalline membranes are one of the rare examples of bidimensional systems in which long-range order can stabilize an ordered phase in the thermodynamic limit. By a careful analysis of the Goldstone mode counting, we propose a symmetry-breaking mechanism associated with the generation of the flat phase, and we show how it highlights the crucial role played by the crystalline lattice in the establishment of long-range order in these objects. Comparison with other symmetry-breaking mechanisms in membrane physics is also used to unveil the links between symmetry-breaking patterns and the physical properties of the flat phase.

DOI: 10.1103/PhysRevB.100.125406

Crystals are generally defined as materials that possess an underlying periodic structure: at the microscopic scale, their atoms (or any other microscopic constituent) lie on a periodic lattice. However, this can only be true at $T = 0$, otherwise thermal agitation causes the atoms to fluctuate around their equilibrium position—these fluctuations are called phonons. For temperatures smaller than the melting temperature of the crystal, such fluctuations remain weak enough, so that each atom is still on average at its equilibrium position defined by the lattice. This is the definition of a crystalline phase at a nonzero temperature.

This picture holds for most of the observed crystalline materials; however, it depends strongly on their dimension. In the 1930s, simple arguments were found that show that crystalline order is indeed destroyed at any $T \neq 0$ for one-dimensional crystals in the thermodynamic limit [1–3]. Later, more rigorous arguments enabled the proof to be extended to two dimensions [4]. Thus, only in three dimensions can genuine crystals exist.

In this context, the observation of graphene sheets—a one-atom-thick hexagonal lattice of carbon atoms—first came as a surprise [5,6]. Since then, many more bidimensional crystals have been observed and synthesized (see [7], for example, for a review). How does this agree with previous findings about the impossibility of the existence of crystalline phases in two dimensions? It was later understood that height fluctuations (also called flexurons or flexural phonons), or more precisely their interaction with acoustic phonons, play a crucial role in maintaining the stability of such bidimensional crystal sheets: the long-range order is indeed not of a positional nature (as in the previous definition of crystals), but of an orientational one, i.e., it is not defined by the atoms being at precise positions on average, but by the vectors normal to the surface embodied by the crystal being correlated at large distances [8,9]. Hence, despite their name, the ordered phase of crystalline membranes is not a crystalline phase but a so-called flat phase (in reference to the membrane's typical configurations, although other geometries are allowed, e.g., as spherical shapes for vesicles). This can be seen quite clearly in diffraction patterns observed from graphene sheets [10].

The presence of long-range orientational order in crystalline membranes is at the origin of a number of unusual scaling behaviors, captured by an anomalous exponent η: the bending rigidity κ increases with the size L of the system as $\kappa \sim L^{\eta}$, whereas the elastic moduli such as Young's modulus $\mathcal{Y}$ get weakened as $\mathcal{Y} \sim L^{2-2\eta}$ (see [11] and references therein). Additionally, Hooke's law, which determines the system's response to an external stress, also becomes anomalous [9,12,13]. The anomalous exponent η is moreover a universal quantity that does not depend on the nature of the material under consideration.

Still, the theorem of Hohenberg-Mermin-Wagner [14,15] forbids the existence of such an ordered phase in most two-dimensional systems. As for crystalline membranes, Nelson and Peliti have shown, in an attempt to examine the possibilities of a Kosterlitz-Thouless melting of the bidimensional crystals, that interactions between the Gaussian curvature of the membrane at different locations turn out to be long-range [16]. In this paper, we show that a similar line of reasoning can be applied at the level of the interaction involving the order parameter field, which is also of long-range type, thereby providing an explanation of why the theorem of Hohenberg-Mermin-Wagner does not apply here.

In the scenario of Hohenberg-Mermin-Wagner, long-range order is destroyed by thermal fluctuations, which are dominated by the massless modes, if any, in the thermodynamic limit, because such modes do not get screened at large distances. The appearance of such modes generally requires particular symmetries to be at play in the system, which can prevent the existence of a mass term; it is then possible (at least in principle) to establish links between massless modes and the symmetries of the state in which the system lies. The Goldstone theorem [17] is one of the most useful tools that one can use to relate the symmetries of an ordered phase to the number of massless modes, but its original formulation is valid only for Lorentz-invariant systems. It has long been known that this theorem cannot be easily generalized to

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systems that do not possess this symmetry, the most well-known example being Heisenberg ferromagnets that have only one massless magnon instead of the two that would be expected from a naive application of Goldstone's theorem [18]. Despite thorough investigations (see [19] for a review), a proper counting rule for the massless modes in full generality only came out very recently [20–23].

In this paper, we use these recent findings to shed new light on the puzzling question of long-range order in crystalline membranes. In particular, by applying the Goldstone counting rule [22], we show that the previously proposed symmetry-breaking mechanisms associated with the flat phase [9,12,24] are not consistent with the well-known infrared spectrum of this phase because they forget the subtle (but nonetheless crucial) role played by the underlying crystalline lattice. Indeed, in membranes without such a lattice—also called fluid membranes—the long-range orientational order is destroyed by thermal fluctuations [25]. The corrected mechanism is then compared with other symmetry-breaking patterns from membrane physics to investigate further its relations to the theorem of Hohenberg-Mermin-Wagner. We identify two main important features of this mechanism (namely the occurrence of two different types of fluctuation modes and the presence of Goldstone modes with quadratic dispersion relation) that are, respectively, associated with the stabilization of the ordered phase in the thermodynamic limit and the generation of a nontrivial field anomalous dimension in the ordered phase, related to a strong degree of anharmonicity of the fluctuation modes, and which is a key quantity to understand the physics of the flat phase [11].

The paper is organized as follows: We first present the Goldstone counting rule. Then, we apply it to find a consistent symmetry-breaking mechanism associated with the flat phase. In the third part, we show why long-range orientational order is preserved by performing a careful analysis of the hypotheses of Hohenberg-Mermin-Wagner's theorem. Finally, we apply the same procedure to the study of the overstretched phase, where the material undergoes a strong stretching effort, thereby giving another example that helps us track down the relations between the symmetry-breaking mechanism and the peculiar properties of the flat phase of crystalline membranes, and then we conclude.

I. GOLDSTONE MODES COUNTING WITHOUT LORENTZ INVARIANCE

Different low-energy states that can be discriminated from each other must have the same energy if they are related by a symmetry of the free energy. In particular, if a ground state breaks a symmetry of the free energy, all states that can be obtained by recursively applying this symmetry to the ground state are also possible ground states. This is how the comparative study of the symmetries of the states that the system can reach, and those of the free energy, gives insight into the degeneracies of the spectrum of the theory. In the following, we shall restrict ourselves to systems simple enough that more refined arguments about Higgs mechanisms, or other subtle ways to generate gaps [26,27], are not necessary. Then the low-energy spectrum can be directly read from the spontaneous symmetry-breaking pattern.

Obviously, not all symmetries need to be broken by the ground state, i.e., there can be residual symmetries. In the case when the broken symmetries are associated with continuous transformations, it is possible to relate any pair of ground states by a continuous path of other ground states. That is, the spectrum includes massless excitations, which are the *Goldstone modes*. Because of their massless nature, Goldstone modes are not screened at large distances, and therefore they play a crucial role in the infrared physics of the system. In relativistic systems, Lorentz symmetry imposes on these modes a dispersion relation of the form $\omega = qc$; however, this no longer holds for nonrelativistic systems. For example, the ferromagnetic magnon has a low-energy dispersion relation of the form $\omega \sim q^2$ [18]. Hence for systems without Lorentz invariance, the counting rule should give access not only to the number of Goldstone modes, but also to their type of dispersion relation.

The counting of Goldstone modes can be related to the algebra of the symmetry groups of the free energy and the ground states by Goldstone's theorem [17]: let G be the symmetry group of the free energy, and let H be the group of symmetries of the ground states, namely the group of residual symmetries, with $H \subset G$ in light of the above. Goldstone's theorem (in its original version) states that the number of Goldstone modes is simply given by dim (G/H). However, we must note the following:

(i) This rule cannot be trivially extended to nonrelativistic systems. Indeed, in ferromagnets, for example, $G = \mathrm{O}(3)$ is broken into $H = \mathrm{O}(2)$ (the ground state is still invariant with respect to rotations of axis in the direction of the spontaneous magnetization). The ferromagnetic state thus breaks two generators of G, but there is only one magnon [18].

(ii) As Goldstone's theorem is intended only for relativistic systems, it does not give any information about the dispersion relations, which are crucial characteristics of the massless modes in nonrelativistic theories. It was first pointed out by Nielsen and Chadha that the infrared behavior of the Goldstone bosons has a direct influence on the number of generated massless modes: to continue with the example of magnets, ferromagnets have one magnon with a quadratic dispersion relation $\omega \sim q^2$, whereas antiferromagnets have two magnons with a linear dispersion relation $\omega \sim q$, while in both cases two rotation generators are broken by the ground state [18].

(iii) The knowledge of the algebra of broken generators is not sufficient in general. Indeed, two different broken generators can generate the same excitation on a given ground state (see Fig. 1 for a simple example), and therefore they are not associated with different Goldstone modes [28]. This is all the more important for systems that possess spacetime symmetries, which frequently generate nonindependent transformations of the ground states [29].

It can be understood in light of the above that the central quantity involved in the counting of Goldstone bosons should include more information than the mere algebra of the broken generators. Let us define the matrix ρ of the *commutators* of *independent* broken generators $\{Q_i\}$ *evaluated* in the ground state $|0\rangle$ [21]:

$$\rho_{ab} = \langle [Q_a, Q_b] \rangle . \tag{1}$$

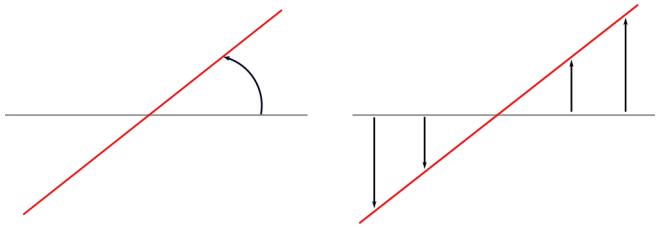

FIG. 1. Left: effect of a rotation on a line. Right: the action, on the same line, of a combination of local translations induces the exact same transformation of the line. It is one of the simplest examples in which mathematically independent transformations can induce similar excitations of a given state. This example is discussed in [28].

It is also important to discriminate the generated massless modes according to their dispersion relation: we will call type-*A* Goldstone bosons those that have a linear dispersion relation, and type-*B* those whose dispersion relation is quadratic (the rigorous definition is a little bit more subtle, but the difference is irrelevant to our purpose [22]). The numbers n_A and n_B of each type of Goldstone mode is then given by the following formulas [21]:

$$\begin{aligned} n_A &= \dim(G/H) - \mathrm{rank}(\rho), \\ n_B &= \frac{\mathrm{rank}(\rho)}{2}. \end{aligned} \tag{2}$$

Note that in the relativistic case, the existence of a nonzero expectation value of the commutator in the ground state would break Lorentz invariance [22], thus Lorentz symmetry requires $\rho = 0$, and we recover the original Goldstone theorem with dim (G/H) type-*A* massless modes.

As a less obvious example, consider an ideal crystalline solid in D dimensions. The ground state of the system is given by the periodic lattice that breaks both the translation and rotation symmetries of the free energy (given in that case by the theory of elasticity [30]). The residual symmetry group H is the discrete subgroup of symmetry of the crystalline lattice denoted $\mathcal{C}$. The spontaneous symmetry-breaking mechanism characterizing the crystalline ground state can be written as follows:

$$\underline{\textbf{Mechanism 1:}} \quad \mathrm{ISO}(D) \to \mathcal{C}, \tag{3}$$

where ISO (D) is the group of isometries.

There are D broken translation generators, as well as $D(D-1)/2$ broken rotation generators. However, the action of translations and rotations on the ground state do not generate independent excitations [31,32], and the group of *independent* broken generators reduces to the broken translations. Because translations commute with each other, $\rho = 0$ and the counting rule Eq. (2) gives D type-*A* Goldstone bosons, corresponding to the well-known D acoustic phonons of crystals, and not $D + D(D-1)/2 = D(D+1)/2$ Goldstone bosons as a naive application of Goldstone's theorem would have given.

This simple example sheds light on two properties of the counting rule Eq. (2): First, for non-Lorentz-invariant systems, the total number of Goldstone modes does not need to be equal to the total number of broken generators, even if all of them have a linear dispersion relation (the equality occurs only if we suppose further that the actions of each of the generators on the ground state are independent). Second, a nontrivial algebra of independent broken generators is required for the existence of type-*B* Goldstone bosons.

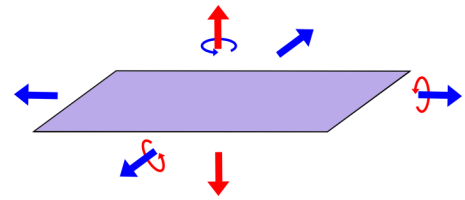

FIG. 2. Symmetries of the infinite plane in three dimensions: the plane is invariant under two translations and one rotation (in blue in the figure), but it breaks the other two rotations and the remaining translation (in red). The SO$(d-D)$ group reduces to the discrete $\mathbb{Z}_2$ group corresponding to reversing the up and down sides of the plane.

II. SPONTANEOUS SYMMETRY BREAKING IN THE FLAT PHASE

In this section, we study the application of the Goldstone counting rule Eq. (2) to the flat phase. Note that we are concerned here with the symmetries of the zero-temperature ground state of the crystalline membrane, which is still a periodic crystal. The influence of thermal fluctuations on the stability of such a state is discussed in the next section.

In addition to usual crystals, a fundamental property of crystalline membranes is their ability to fluctuate in a bigger embedding space, whose dimension will be called d. This paves the way to more complex spacetime symmetry-breaking patterns, which, as we can already anticipate, will be of paramount importance to explain the quadratic dispersion relation of flexurons (see below). The massless fluctuation spectrum of the flat phase of crystalline membranes is well-known. It includes the following [11]:

(i) D acoustic phonons, which are type-*A* Goldstone bosons, as in any crystal.

(ii) $d - D$ flexurons, which are type-*B* Goldstone bosons, and therefore much stronger than the phonons in the infrared limit.

There is no clear consensus in the literature on the spontaneous symmetry-breaking pattern associated with the flat phase, to the best of our knowledge [9,12,24]. Note that most patterns were proposed at a time when the Goldstone counting rule was not known.

In the following, we will only discuss the more widely used mechanism, which is also the one that best respects the symmetries of the flat phase (see Fig. 2), and we will argue why we think it is not correct. Consider the following symmetry-breaking pattern [12]:

$$\underline{\textbf{Mechanism 2:}} \quad \mathrm{ISO}(d) \to \mathrm{ISO}(D) \times \mathrm{SO}(d-D). \tag{4}$$

The free-energy is invariant under all isometries of the embedding space, forming the group ISO (d), and the flat phase configuration is an infinite plane, which is still invariant under the plane isometries ISO (D) as well as the rotations of SO$(d-D)$, which act only on directions of the embedding space that are all orthogonal to the flat phase plane (see Fig. 2 for a picture for the physical dimensions $D = 2$, $d = 3$).

The set of broken symmetry generators thus contains the following:

(i) The d-D translations in the directions orthogonal to the flat phase plane, denoted $\{P_\alpha\}_{\alpha\in[\![D+1;d]\!]}$. In the following, greek letters denote indices in $[\![1;d]\!]$, whereas latin ones only run into $[\![1;D]\!]$.

(ii) The $D\times(d-D)$ *mixed* rotations that bring the vector giving the ith direction inside the flat phase plane to the direction given by the αth vector of the embedding space outside the membrane's plane $\{J_{i\alpha}\}_{i\in[\![1;D]\!],\alpha\in[\![D+1;d]\!]}$.

The commutation relations between these generators are given by the iso(d) algebra of the isometry group [33]:

$$\begin{aligned}[J_{\mu\nu},J_{\alpha\beta}]&=\delta_{\nu\alpha}J_{\mu\beta}-\delta_{\mu\alpha}J_{\nu\beta}-\delta_{\nu\beta}J_{\mu\alpha}+\delta_{\mu\beta}J_{\nu\alpha},\\ [J_{\mu\nu},P_\theta]&=\delta_{\nu\theta}P_\mu-\delta_{\mu\theta}P_\nu,\\ [P_\mu,P_\nu]&=0.\end{aligned}\tag{5}$$

The next question one should answer is, which of these generators acts independently on the ground state $|0\rangle$? Consider the action of a mixed rotation on the ground state:

$$J_{i\alpha}|0\rangle=(x_iP_\alpha-x_\alpha P_i)|0\rangle=x_iP_\alpha|0\rangle.\tag{6}$$

The second equality follows from the fact that the translations inside the flat phase plane are not broken. Hence the action of a mixed rotation can always be canceled by a combination of broken translations. The set of independent broken generators thus reduces to $\{P_\alpha\}_{\alpha\in[\![D+1;d]\!]}$, and the counting rule Eq. (2) yields

$$\begin{aligned}n_A&=d-D,\\ n_B&=0,\end{aligned}\tag{7}$$

which is not consistent with the expected spectrum for the flat phase of crystalline membranes.

This computation is quite enlightening though, since it underlines the necessity of having a nontrivial algebra of broken generators to describe type-B Goldstone bosons such as the flexurons in crystalline membranes. Looking more thoroughly at Eq. (7), we notice that the number of generated Goldstone bosons is equal to the number of directions of the embedding space orthogonal to the flat phase plane, which indicates that they must be flexurons (but not with the expected dispersion relation): the mechanism in Eq. (4) misses the phonons. Thus, this mechanism seems more appropriate to describe the flat phase of fluid membranes in which only flexurons are at play. (Such an ordered phase only exists at $T=0$, however, because of Hohenberg-Mermin-Wagner's theorem). As a matter of fact, the constituents of an incompressible fluid membrane are free to diffuse on its surface, thereby forbidding the definition of any particular reference state by the position of the molecules, and therefore any kind of positional order. Such materials thus do not have acoustic phonons.

This assertion can indeed be checked for consistency with usual models of fluid membranes: from the Canham-Helfrich free energy [34,35], the flexuron's propagator $G(q)$ has the following asymptotic behavior in the infrared limit:

$$G(q)=\langle h(q)h(-q)\rangle\underset{q\to0}{\sim}\frac{1}{\sigma q^2},\tag{8}$$

where σ is the tension of the membrane. Such behavior is typical of type-A Goldstone bosons. One could argue that, according to Eq. (8), another type of behavior could be expected in tensionless fluid membranes, but this does not hold since even if not present at the microscopic scale, σ is generated by the renormalization-group flow when going toward the infrared regime [11]. Our Goldstone modes analysis leads to the following complementary argument: the tension term is not protected by the symmetries of the system, and therefore cannot be consistently enforced to be zero.

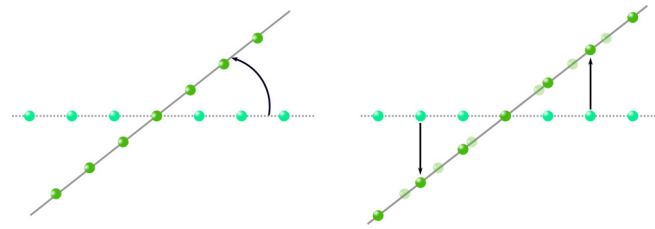

FIG. 3. Left: action of a mixed rotation on a one-dimensional lattice. Right: a linear combination of translations can align the lattice on the same line, but the state of the system is different because of the dilation that the translations induce on the lattice due to Pythagoras' theorem.

As we have seen in Eq. (3), the origin of acoustic phonons in crystals is the breaking of the isometry group by the discrete group of the crystalline lattice $\mathcal{C}$. This must also hold for crystalline membranes in which, although not well preserved by the thermal fluctuations, a crystalline lattice is still present in the flat phase. In light of these arguments, it seems reasonable to take a closer look at the following symmetry-breaking pattern:

$$\underline{\textbf{Mechanism 3:}}\quad \mathrm{ISO}(d)\to\mathcal{C}\times\mathrm{SO}(d-D).\tag{9}$$

Note that the track of the discrete symmetry group $\mathcal{C}$ is still present even in the continuum theory of crystalline membranes: in [36], for example, the transition to the flat phase is presented as the generation of a nonzero expectation value for the metric of the flat phase g^0_{ij}, which then characterizes the ground state. Because the membrane is assumed to be homogeneous and isotropic, its metric in the ground state has to be proportional to δ_{ij}. The proportionality coefficient ζ^2, called the extension parameter, characterizes the unit of length inside the membrane's plane. The presence of this fixed reference metric is one of the key differences between crystalline membranes and fluid ones.

The broken symmetry generators in the mechanism given by Eq. (9) are thus as follows:

(i) The $d-D$ external translations $\{P_\alpha\}_{\alpha\in[\![D+1;d]\!]}$.

(ii) The D internal translations $\{P_i\}_{i\in[\![1;D]\!]}$.

(iii) The $D\times(d-D)$ mixed rotations, mixing the internal and external spaces $\{J_{i\alpha}\}_{i\in[\![1;D]\!],\alpha\in[\![D+1;d]\!]}$.

(iv) The $\frac{D(D+1)}{2}$ internal rotations $\{J_{ij}\}_{(i,j)\in[\![1;D]\!]^2}$.

The action of internal translations and rotations are not independent, as in usual crystals. This time, however, the action of mixed rotations on the ground state can no longer be canceled by a carefully chosen combination of external translations (see Fig. 3 for a picture in the one-dimensional case). Indeed, whereas rotations always preserve the ground-state metric, a combination of translations bringing the system into a similar plane would induce a dilation of the system

(as an obvious application of Pythagoras' theorem shows) and therefore a flat phase state with a different extension parameter ζ. This is a direct consequence of the presence of a microscopic lattice, or equivalently of the presence of phonons in the system.

The internal space isometries can be used to relate the mixed rotations with the same external index α, but different internal indices i. All in all, the total number of independently acting broken symmetry generators is as follows: D internal translations + $(d-D)$ external translations + $(d-D)$ mixed rotations, so that finally

$$\dim(G/H) = 2d - D. \tag{10}$$

The commutation relations between these generators evaluated in a ground state are given as usual by the algebra of isometries iso(d):

$$\begin{aligned} \langle [J_{\alpha i}, J_{\beta i}] \rangle \propto \langle J_{\alpha\beta} \rangle &= 0, \\ \langle [P_\alpha, P_\beta] \rangle &= 0, \\ \langle [J_{\alpha i}, P_\gamma] \rangle &\underset{\gamma\neq i, \gamma\neq\alpha}{=} 0, \\ \langle [J_{\alpha i}, P_\alpha] \rangle &= \langle P_i \rangle \neq 0, \\ \langle [J_{\alpha i}, P_i] \rangle &= \langle P_\alpha \rangle \neq 0. \end{aligned} \tag{11}$$

It is then possible to write ρ is a basis where the determination of its rank is simple, even without knowing the precise value of the nonzero matrix coefficients, denoted "$*$" below. We choose to write in a basis in which the d translations are displayed first, and then the $d-D$ rotations:

$$\rho = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & * & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & * & \cdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & * & * & \cdots & 0 & 0 & 0 & 0 & \cdots \\ * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{12}$$

which leads to

$$\operatorname{rank}(\rho) = 2(d-D), \tag{13}$$

and finally, thanks to the counting rule Eq. (2),

$$\begin{aligned} n_A &= D, \\ n_B &= d - D. \end{aligned} \tag{14}$$

Finally, we get D type-A Goldstone modes with a linear dispersion relation, corresponding to the acoustic phonons, and $d-D$ type-B Goldstone bosons with a quadratic dispersion relation, corresponding to the flexurons; this is the exact spectrum of crystalline membranes that we recalled at the beginning of the section.

The crucial difference between the second and third proposed mechanisms—Eqs. (4) and (9), respectively—is the presence of the crystalline lattice discrete group in the latter, which allows us to keep some independent rotation generators required to get a nontrivial algebra Eq. (11) and thus a nonzero rank for ρ, which allows for the existence of the type-B flexurons. The consequences are far-reaching as among the proposed mechanisms, only the third one is associated with a stable ordered phase in two dimensions: phonons Eq. (3) or flexurons Eq. (4) alone cannot stabilize a long-range order of positional or orientational nature. Already at this stage, we can notice two main differences between mechanisms 1 and 2 on the one hand and mechanism 3 [Eq. (9)] on the other hand (although all three are built from the same groups), namely that mechanism 3 has two different types of fluctuation modes, and it has type-B Goldstone modes. The rest of this paper is dedicated to an analysis of the consequences of these two differences.

III. HOHENBERG-MERMIN-WAGNER'S THEOREM

The spontaneous symmetry-breaking pattern combined with the counting rule [Eq. (2)] gives access to the number of massless modes as well as their dispersion relation in the large-distance limit in the ordered phase predicted by mean-field theory, but it is not sufficient to know if such an ordered phase is robust to thermal fluctuations. For that last purpose, the most useful tool is the Hohenberg-Mermin-Wagner theorem [14,15]. In their original paper, Mermin and Wagner stressed an important hypothesis for their theorem to apply: the interaction needs to have a short enough range. Namely, if J is the coupling constant describing the strength of the interaction between the order-parameter fields, $J(x)x^2$ must be an integrable function in D dimensions [14]. Let us test this hypothesis in the case of crystalline membranes.

First, we need the action describing the small fluctuations around the flat phase. As stated before, crystalline membranes can be described as an elastic medium fluctuating in an embedding space, and their action thus contains both a curvature term, proportional to the bending energy κ of the membrane, and an elastic term quadratic in the strain tensor ε_{ij}:

$$S = \int_x \left[\frac{\kappa}{2} (\partial^2 \vec{r})^2 + \frac{c_{ijab}}{2} \varepsilon_{ij} \varepsilon_{ab} \right], \tag{15}$$

where $\int_x = \int d^D x$ is an integral over the internal space of the membrane, $\vec{r}$ is the position vector describing the membrane, and c_{ijab} is the elastic tensor, which can be expressed, for example, in terms of the Lamé coefficients as $\lambda \delta_{ij}\delta_{ab} + \mu(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})$. The strain tensor ε_{ij} can be expressed as half the difference between the metric in the current state of the membrane $g_{ij} = \partial_i \vec{r}.\partial_j \vec{r}$ and that of the flat phase reference state $g^0_{ij} = \zeta^2 \delta_{ij}$. To build a theory of small fluctuations around the flat phase, it is necessary to expand $\vec{r}$ around the equilibrium configuration with extension parameter ζ:

$$\vec{r} = \zeta x_i \vec{e}_i + \vec{u} + \vec{h}, \tag{16}$$

where a basis of the flat phase plane $\{\vec{e}_i\}$ has been introduced. This expansion causes the phonons $\vec{u}$ and the flexurons $\vec{h}$ to appear explicitly. Using the fact that the phonons and flexurons vibrate in orthogonal spaces, the action in terms of

the most relevant terms reads [8]

$$S = \int_x \Big[\frac{\kappa}{2} (\partial^2 \vec{h})^2 + \frac{c_{abcd}}{2} u_{ab} u_{cd} + \frac{c_{abcd}}{2} u_{ab} (\partial_c \vec{h} \cdot \partial_d \vec{h}) + \frac{c_{abcd}}{8} (\partial_a \vec{h} \cdot \partial_b \vec{h})(\partial_c \vec{h} \cdot \partial_d \vec{h}) \Big], \tag{17}$$

where the symmetric tensor $u_{ab} = (\partial_a u_b + \partial_b u_a)/2$ has been introduced.

At first glance, it seems that the action Eq. (17) contains only local interactions between phonons and flexurons, which could lead one to conclude that the long-range orientational order is broken by thermal fluctuations. But in the original argument of Mermin and Wagner, there is only one fluctuation mode.

We have seen that flexurons are the dominant modes in the infrared limit. Moreover, the action Eq. (17) is quadratic in the phonons, thus it is possible to perform an exact integration over the phonons to define an effective action with the flexurons as only fields [37]. For the sake of simplicity, we give it here in Fourier space, with implicit momentum conservation, and the shorthand notation $\vec{h}(k_i) = \vec{h}_i$:

$$S_{\text{eff}} = \int_{k_1,k_2} \frac{\kappa}{2} k_1^4 (\vec{h}_1 \cdot \vec{h}_2) + \int_{k_1,k_2,k_3,k_4} \Big[\frac{\mathcal{R}_{abcd}(q)}{4} k_1^a k_2^b k_3^c k_4^d (\vec{h}_1 \cdot \vec{h}_2)(\vec{h}_3 \cdot \vec{h}_4) \Big], \tag{18}$$

with $\int_k = \int \frac{d^D k}{(2\pi)^D}$ according to our convention for Fourier transforms, and $q = k_1 + k_2 = -k_3 - k_4$.

The price of working with only one type of field is that now the interaction vertex $\mathcal{R}$ is nonlocal. It depends only on two coupling constants, exactly like the elasticity tensor c, which can be made explicit by decomposing it onto the following set of orthogonal projectors:

$$N_{abcd}(q) = \frac{1}{D-1} P^T_{ab}(q) P^T_{cd}(q),$$
$$M_{abcd}(q) = \frac{1}{2} \big[P^T_{ac}(q) P^T_{bd}(q) + P^T_{ad}(q) P^T_{bc}(q) \big] - N_{abcd}(q), \tag{19}$$

where $P^T_{ij}(q) = \delta_{ij} - q_i q_j / q^2$ is the projector in the direction orthogonal to q. The effective interaction vertex is then [37]

$$\mathcal{R}_{abcd}(q) = \frac{\mu(D\lambda + 2\mu)}{\lambda + 2\mu} N_{abcd}(q) + \mu M_{abcd}(q). \tag{20}$$

Note that in the particular case $D = 2$, corresponding to physical membranes, the two projectors Eq. (19) are equal, and $\mathcal{R}$ depends on only one elastic constant, which turns out to be Young's modulus $\mathcal{Y}$ [16].

The first proof of the presence of long-range content of the interaction in Eq. (18) has been given by Nelson and Peliti [16]. They showed that in two dimensions, the interaction term in Eq. (18), S_{int}, can be rewritten as an interaction between the local Gaussian curvature $s(x) = \det(\partial_i \partial_j h)$:

$$S_{\text{int}} = \frac{\mathcal{Y}}{16\pi} \int_{x,y} \mathcal{G}(x-y) s(x) s(y), \tag{21}$$

where the (nonlocal) interaction vertex between the Gaussian curvature $\mathcal{G}$ behaves as $\mathcal{G}(x) \simeq x^2 \ln(x/a)$ at large distance (a being the lattice spacing), which is clearly a long-range type of interaction.

To make the connection with the original work of Mermin and Wagner [14], we must first find the equivalent to the $J(x)$ interaction term. An order parameter associated with the flat phase is given by the extension parameter ζ, which is always nonzero in the flat phase and equal to zero in a completely disordered crumpled configuration. It can also be expressed as a function of the correlation between the tangent vectors to the surface generated by the membrane [12]:

$$\zeta^2 = \frac{1}{D} \langle \partial_i \vec{r} \rangle \cdot \langle \partial_i \vec{r} \rangle. \tag{22}$$

Finally, taking into account the fact that the action in Eq. (18) is generated after an integration over the phonon fields, the analog of $J(x)$ in our model is the interaction between the ∂h terms, which turn out to be $\mathcal{R}$.

To test if the range of $\mathcal{R}$ is short enough for Hohenberg-Mermin-Wagner's theorem to apply, it must be reexpressed in direct space. We do not give here the full expression of $\mathcal{R}(x-y)$, but rather we analyze the following elementary block:

$$P^T_{ab}(q) P^T_{cd}(q) = \delta_{ab}\delta_{cd} - \delta_{ab} \frac{q_c q_d}{q^2} - \delta_{cd} \frac{q_a q_b}{q^2} + \frac{q_a q_b q_c q_d}{q^4}. \tag{23}$$

Each term can be expressed in direct space thanks to the following formula of the Fourier transform of power laws (see, for example, Ref. [38]):

$$\frac{1}{(p^2)^a} = \frac{1}{4^a \pi^{\frac{D}{2}}} \frac{\Gamma\left(\frac{D}{2} - a\right)}{\Gamma(a)} \int_x \frac{e^{ipx}}{(x^2)^{\frac{D}{2}-a}}, \tag{24}$$

which finally gives

$$\int_q \delta_{ab}\delta_{cd} e^{i q(x-y)} = \delta_{ab}\delta_{cd}\, \delta^{(D)}(x-y),$$
$$\int_q \frac{q_a q_b}{q^2} e^{i q(x-y)} = \frac{\delta_{ab}}{2\pi |x-y|^2} - \frac{(x_a - y_a)(x_b - y_b)}{\pi |x-y|^4}$$
$$\int_q \frac{q_a q_b q_c q_d}{q^4} e^{i q(x-y)} = \frac{X_{abcd}}{4\pi |x-y|^2} - \frac{Y_{abcd}(x-y)}{2\pi |x-y|^4} + \frac{2(x_a - y_a)(x_b - y_b)(x_c - y_c)(x_d - y_d)}{\pi |x-y|^6}, \tag{25}$$

with the following tensors being defined:

$$X_{\mu\nu\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho},$$
$$Y_{\mu\nu\rho\sigma}(\vec{x}) = x_\mu x_\nu \delta_{\rho\sigma} + x_\mu x_\rho \delta_{\nu\sigma} + x_\mu x_\sigma \delta_{\nu\rho} + x_\nu x_\rho \delta_{\mu\sigma} + x_\nu x_\sigma \delta_{\mu\rho} + x_\rho x_\sigma \delta_{\mu\nu}. \tag{26}$$

Among the different terms of Eq. (25), only the first one is local. The other ones are not integrable over the membrane's internal space once multiplied by $(x-y)^2$. Hence, Hohenberg-Mermin-Wagner's theorem does not apply, and the orientational order in crystalline membranes can be long-range.

In the previous argument, it is the nonlocal structure of the effective interaction vertex between flexurons that is at the origin of the stabilization of long-range order in two dimensions. In light of our analysis of the spontaneous symmetry-breaking pattern [Eq. (9)], we can add the following: in the flat phase, even if flexurons are the modes that dominate in the infrared limit, they are not the only important fluctuation modes. In particular, acoustic phonons carry a nonlocal effective interaction between flexurons at various locations in the flat phase's plane. The mechanism in Eq. (9) moreover guarantees that the phonons are Goldstone modes, i.e., thanks to the massless nature they possess by symmetry, they are not efficiently screened out at large distances. Thus, the effective interaction they carry is a true long-range one, which explains why Hohenberg-Mermin-Wagner's theorem does not apply, and the flat phase is robust against thermal fluctuations.

IV. THE ORIGIN OF LONG-RANGE ORDER IN THE FLAT PHASE

In the previous sections, we identified two main differences between the third mechanism [Eq. (9)] and the other two mechanisms [Eqs. (3) and (4)]. To refine our comprehension of the necessary ingredients to generate a stable order phase in the system, we propose to analyze a fourth mechanism.

Whenever a crystalline membrane undergoes a strong enough stress, it buckles. This buckling phenomenon can be understood as a (second-order) phase transition [9,12,13,39,40]: depending on the type of applied forces, the membrane will either become buckled if compressed—in this state, the membrane appears as a mosaic of locally flat domains, the orientation of which is random [12]—or overstretched if sufficiently dilated. Because our main concern is the origin of long-range order in two-dimensional systems, we will focus here on the overstretched phase in which a strong long-range orientational order is present.

In the overstretched phase, the external stress screens the effects of flexurons [39,41], which become energetically disfavored. As a result, the massless infrared spectrum of overstretched membranes contains only type-A Goldstone bosons.

Macroscopically, the shape of the overstretched membrane is still flat, with weaker height fluctuations compared to the flat phase. Microscopically, the stress dominates the local rearrangement of atoms, and the ancient lattice positional order is broken. Contrary to the flat phase examined in Sec. II, the ground state in the overstretched phase cannot be uniquely characterized by its metric [36]. Indeed, the local arrangement of atoms depends both on the intrinsic properties of the material (captured by ζ) and the external stress. Consequently, the previous argument that allowed us to disentangle mixed rotations and external translations does not hold anymore: the situation is analogous to that of Fig. 1 rather than Fig. 3. To avoid confusion, and to ensure the disentanglement between the action of translations and mixed rotations on the ground state, the symmetry-breaking mechanism will be written as

$$\underline{\textbf{Mechanism 4:}} \quad \mathrm{ISO}(d) \to \mathrm{SO}(d-D). \tag{27}$$

The set broken generators is the same as for the third mechanism, but now the unit of length at the surface of the membrane is determined by the applied stress rather than the sole extension parameter ζ. The set of independent broken generators hence reduces to the d translations, so that

$$\dim(G/H) = d, \quad \rho = 0, \tag{28}$$

and the counting rule Eq. (2) leads to

$$\begin{aligned} n_A &= d, \\ n_B &= 0, \end{aligned} \tag{29}$$

namely D type-A acoustic phonons and $d-D$ type-A flexurons, as expected.

Finally, mechanism 4 [Eq. (27)] provides an example in which long-range orientational order can be maintained in two dimensions without requiring the help of type-B Goldstone bosons. Indeed in the overstretched phase, the previous argument with regard to the Hohenberg-Mermin-Wagner theorem still holds: despite the fact that the flexurons are screened by the stress, they remain Goldstone bosons, and an effective theory of interacting flexurons can still be built, in which the phonons, which are also massless, carry an effective long-range interaction between flexurons.

Note, however, that in the ground state of overstretched membranes the local pseudo-ordering persists, which causes the breaking of the group isometries inside the membrane's plane, which is in strong contrast with the second mechanism [Eq. (4)], in which the microscopic constituents are free to move and no phonon is generated (and therefore no long-range interaction can occur and the ordered phase is destroyed by the thermal fluctuations).

A remaining question involves the role of the type-B Goldstone bosons. To address it, we must compare mechanisms 3 [Eq. (9)] and 4 [Eq. (27)], which differ only by the dispersion relation of the flexurons. The most striking difference between the flat phase and the overstretched phase is the presence in the former of a strong anomalous exponent $\eta \simeq 0.85$ [11,42–44], at the origin of the highly anharmonic behavior of the thermal fluctuations, which leads to many unusual effects, as well as a modified elasticity theory, which is in total contrast with the overstretched phase in which conventional elasticity is restored and $\eta = 0$ (see [39–41] and [45] for a comparative study). The role of the type-B Goldstone bosons is thus probably related to the generation of such an anharmonic behavior and unusual scaling relations.

To sum up on the Goldstone physics, we have analyzed various nontrivial symmetry-breaking patterns related to the physics of crystalline membranes, which have highlighted a number of general features of the physics of Goldstone modes in theories without Lorentz invariance. First, mechanism 1 [Eq. (3)] illustrates the fact that whenever different broken symmetry generators generate linearly dependent transformations of the ground state, the total number of associated Goldstone modes is smaller than the total number of broken generators, which is the main lesson of Ref. [28]. This feature is quite common in theories presenting spacetime symmetry

breaking, since the action of rotations and translations are rarely independent (it is much more difficult to see this if an internal symmetry group is broken). Mechanism 2 [Eq. (4)] teaches us the importance of the underlying microscopics, even in continuum theories. Indeed, taking into account the mere overall shape of the membrane leads to a spontaneous symmetry-breaking mechanism without phonons, which is much more sensitive to thermal fluctuations (as such, a system cannot sustain long-range order in two dimensions). The comparison between mechanisms 3 [Eq. (9)] and 4 [Eq. (27)] shows that the presence of two different types of interacting Goldstone modes is a sufficient condition to generate a stable ordered phase in two dimensions, as whenever one can reexpress the theory as an effective theory of a single mode, the effective interaction carried by the second Goldstone mode must be long range, because of its massless character. Having two different types of Goldstone modes, however, requires particular patterns of symmetry breaking. Finally, we also showed the necessity of having a nontrivial algebra of independently acting broken generators to generate type-*B* Goldstone bosons, which can be seen as an obvious consequence of the Goldstone counting rule [Eq. (2)], but which we have shown is not so easy to achieve in practice. We also give hints at the possible link between the presence of such type-*B* Goldstone bosons and unusual scaling behaviors in the ordered phase, related to the generation of a nontrivial field anomalous dimension η.

V. CONCLUSION

To conclude, the corrections of the symmetry-breaking mechanism at the origin of the flat phase teach us two main lessons on the physics of crystalline membranes. First, acoustic phonons cannot be overlooked, even though crystalline order is broken by thermal fluctuations, and by the fact that they are subdominant at large distances. It is all the more important that the presence of a massless effective interaction carrier is required to ensure that long-range orientational order is not destroyed by fluctuations. This is why genuine bidimensional crystals or fluid membranes, in which only phonons alone or flexurons alone survive at large distances, do not present any stable ordered phase in two dimensions, but crystalline membranes, which possess both modes, also present long-range order.

Second, the origin of the flat phase anomalous scaling laws can be traced back to a delicate geometrical interplay between the intrinsic properties of the material and its embedding in the three-dimensional space. In the presence of a strong enough external stress field—which drives the system into the overstretched phase—this subtle balance is broken, and conventional elasticity is restored, probably due to the absence of type-*B* Goldstone bosons.

The key to understanding all these results is the Goldstone counting rule Eq. (2). We hope that our work will motivate further studies in the context of condensed matter physics, in which Lorentz invariance is frequently absent, spacetime symmetries are often at play, and therefore such tools are certainly of interest.

ACKNOWLEDGMENTS

The author thanks D. Mouhanna for useful discussions and a careful reading of this manuscript.

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