From coupled-wire construction of quantum Hall states to wave functions and hydrodynamics

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In this paper we use a close connection between the coupled-wire construction (CWC) of Abelian quantum Hall states and the theory of composite bosons to extract the Laughlin wave function and the hydrodynamic effective theory in the bulk, including the Wen-Zee topological action, directly from the CWC. We show how rotational invariance can be recovered by fine tuning the interactions. A simple recipe is also given to construct general Abelian quantum Hall states described by the multicomponent Wen-Zee action.

I. INTRODUCTION

Topological order [1] is one of the most fundamental concepts in modern condensed matter physics. The history starts with the discovery of the $\nu = \frac{1}{3}$ fractional quantum Hall effect (FQHE) in the 1980's [2], which was essentially understood after Laughlin proposed his famous wave function [3]. Since then, theorists have proposed a variety of topologically ordered states in two and three dimensions. Examples in $2 + 1$ dimensions are Abelian hierarchical quantum Hall (QH) states [4–6], non-Abelian QH states [7,8], and various kinds of spin liquids [9–14]. Some, but far from all, of these states have strong experimental support.

Topologically ordered states are featureless and symmetric in the bulk, and as such they defy the characterization of phases by local order parameters in the manner of Landau. Nevertheless, there are field-theoretic descriptions. It is belived that the topological properties can be encoded in various types of topological field theories, the most well-known examples being the Chern-Simons theories of the QHE [15,16].

Going beyond the topological scaling limit [17], but still in the infrared region, there are hydrodynamical descriptions that supplement the topological action with higher-order derivative terms. These theories typically encode information about collective excitations. Yet another type of field theories are those based on statistical transmutation, or "flux attachment." These theories of "composite" fermions [18] or bosons [19], which are closely related to various model wave functions, are in principle microscopic, but can only be solved using various kinds of mean-field approximations.

In addition to the various field theories, there are several other ways to describe topologically ordered states in general, and quantum Hall states in particular. Examples of the latter are the approach based on the thin-torus limit [20], and the coupled-wire construction (CWC) of Kane and co-workers [21,22]. The aim of this paper is to make a rigorous connection between this last approach and the Chern-Simons field theory for composite bosons.

The CWC is quite general, and has been employed to construct various two- and three-dimensional topological states.

The list includes the original work on Abelian and non-Abelian fractional quantum Hall states [21–23], chiral spin-liquid states [24], topological insulators and superconductors in two and three dimensions [25–28], and the construction of higher-dimensional Abelian topological phases [29]. A rederivation of the periodic table of integer and fractional fermionic topological phases was given in [30]. In this paper, we shall only consider Abelian QH states, and in particular the Laughlin states.

The essence of the CWC is to build interacting fermion/boson systems by starting from a collection of parallel "wires" in a strong magnetic field, each one supporting a Luttinger liquid. The wires are then coupled by tunneling interactions, and there are also forward scattering interactions on each wire that couple right- and left-moving electrons. Keeping only the intrawire current-current interactions, the system is in the so-called sliding Luttinger liquid phase [31–35] and remains invariant under *independent* global U(1) transformations and translations on each wire.

A key observation is that interwire interactions can be used to freeze most of the above symmetry. Kane *et al.* [21], showed that, in the limit of strong interwire coupling, and at certain rational filling fractions, one retains various two-dimensional ground states whose properties depend on the the the details of the construction. In the simplest case of the Laughlin state, it is sufficient to couple neighboring wires, but in general several wires have to be coupled.

The CWC gives an intuitive description of the chiral edge states in a way resembling the occurrence of fractional spins at the ends of the Affleck-Kennedy-Lieb-Tasaki chain [36], or of Majorana modes at the ends of a Kitaev chain [37]. The basic mechanism is that the right- and left-moving electrons on neighboring chains pair, leaving a right-moving channel on one side of the sample and a left-moving one on the other. One can also show that the number of degenerate ground states are as expected, and that the kink excitations on the wires indeed are the fractionally charged anyons characteristic of FQH liquids.

It is less clear how the CWC, which, by definition, explicitly breaks rotational invariance, will describe other

characteristics of the Laughlin states, such as its wave function and collective excitations. In this paper, we reformulate the CWC in terms of gauge fields, in a way that makes it clear how to reproduce the known bulk properties. Not surprisingly, a fine tuning of a parameter is needed to get an isotropic two-dimensional liquid, but we also show that the topological properties do not depend on this. Concretely, we shall use the bosonic fields in the CWC to define a gauge field that can be identified with the statistical gauge field in the Chern-Simons-Ginzburg-Landau (CSGL) theory of the Laughlin states [19,38]. Using this, we derive the Laughlin wave function, both in the isotropic and anisotropic cases, and the hydrodynamic effective theory which contains the Wen-Zee topological action [16] as its leading term in a derivative expansion.

A comment is in order about an interesting recent paper by Fuji and Furusaki [39]. In fact, there is an overlap in the underlying idea between it and this work. Rather than directly identifying the composite-boson field in the CWC, they explicitly carry out the flux attachment on the wires, and then add an interaction term that stabilizes the superfluid phase of the composite bosons. Finally, using a coupled-wire version of the boson-vortex duality transformation [40], they arrive at an action that couples the dual (vortex) fields, describing the quasiparticles, to a hydrodynamic gauge field with a Chern-Simons action. Despite similar looking, this theory differs in some important respects from the hydrodynamic effective theory we shall derive below. We comment on this in the end of Sec. V. Also, fluctuations and the wave function, which are main subjects of this paper, are not discussed in Ref. [39].

The paper is organized as follows. We begin by briefly reviewing the CWC in Sec. II and show how to identify the physically inequivalent ground states. In Sec. III, we define a gauge field which can be interpreted as a statistical gauge field and explain the connection to the CSGL theory. Section IV contains the derivation of the Laughlin wave function starting from our effective Hamiltonian, as well as a discussion of the effects of the spatial anisotropy which is naturally present in the coupled wires. In Sec. V, we derive the hydrodynamical theory, that contains the topological Wen-Zee action, from the effective action obtained in the previous section and show that the Kohn mode is correctly reproduced. The generalization to arbitrary Abelian states is given in Sec. VI, and we end with a concluding section. Some technicalities are summarized in two appendices.

II. LAUGHLIN STATE IN COUPLED-WIRE CONSTRUCTION

The Laughlin state is the paradigmatic example of topological states of matter (see, e.g., Refs. [6,41] for reviews of the Laughlin and other quantum Hall states). The defining properties of the $\nu = 1/m$ state are encoded in the Wen-Zee topological action which is a level-m U(1) Chern-Simons gauge theory [15,18]. The corresponding edge theory is a chiral Luttinger liquid, which is a chiral boson compactified on a circle with radius $\sqrt{m}$ [42]. We now review the CWC for the Laughlin state following the original work by Kane *et al.* [21].
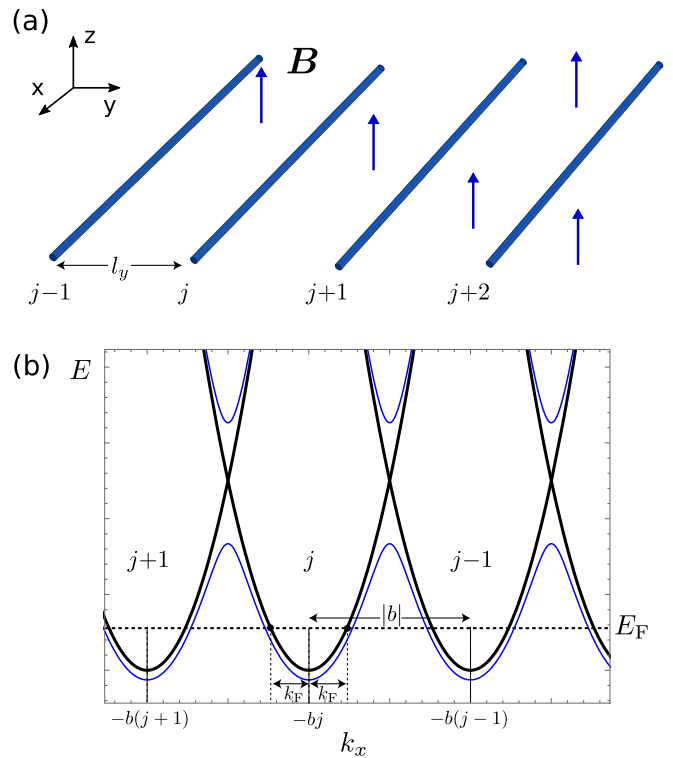

FIG. 1. (a) An array of coupled wires in a perpendicular magnetic field. (b) The energy band structure of the coupled wires. In the limit of decoupled wires, a perpendicular magnetic field in the Landau gauge shifts the quadratic dispersion relations of the individual wires (labeled by j) in a j-dependent manner: $k \to k + bj$ (thick curves). When $2k_F = |b|$ ($\nu = 1$), the single-particle hopping between the adjacent wires opens the gaps at Fermi points and the anisotropic Landau levels are formed (thin blue curves).

A. From uncoupled to coupled wires

Consider a set of parallell one-dimensional wires running along the x direction and stacked in the y direction with a distance l_y [see Fig. 1(a)]. We then apply a strong magnetic field $B (> 0)$ which is perpendicular to the xy plane and is given by the gauge potential $\mathbf{A} = (-By, 0)$ (Landau gauge). We consider nonrelativistic spinless fermions $c_j(x)$ (with mass M_e) moving on the wire j with the dispersion relation $E_j(k_x) = (k_x + bj)^2/2M_e$, where $b = eBl_y/\hbar$ [see Fig. 1(b); $e < 0$ for electrons]. When these bands are filled up to the Fermi momentum k_F [see Fig. 1(b)], the fermion density on each wire is given by $\bar{\rho}^{1D} = k_F/\pi$, and this gives the filling fraction $\nu = 2k_F/|b|$. When $\nu = 1$, the Fermi points are located exactly at the lowest crossing points and the inclusion of the single-particle hopping $c_{j+1}^\dagger(x)c_j(x) + (\text{H.c.})$ among the neighboring wires opens a band gap there, which is identified with one of the gaps separating the (quasi-one-dimensional) Landau levels as is seen in the thin curves in Fig. 1(b).

At a filling fraction $\nu < 1$, which is relative to the filled Landau levels, the single-particle hopping never opens a gap at the Fermi points and we need interactions to make the system gapped. To develop a systematic approach, we linearize the low-energy dispersion around the Fermi points $k_x^R(j) = -bj + k_F$ and $k_x^L(j) = -bj - k_F$ to obtain two Dirac fermions R_j and L_j, respectively, which are related to the

original spinless fermion $c_j(x)$ as

$$c_j(x) \approx e^{ik_x^R(j)x} R_j(x) + e^{ik_x^L(j)x} L_j(x). \quad (1)$$

The low-energy effective Hamiltonian for the jth wire is given by that of massless Dirac fermion

$$\mathcal{H}_{\text{Dirac}}^{(j)} = v_F^0 \int dx [-iR_j^\dagger \partial_x R_j + iL_j^\dagger \partial_x L_j], \quad (2)$$

where the velocity $v_F^0 = \hbar k_F / M_e$ is common to all wires. (To ease the notations, we set $\hbar = 1$ in what follows.)

By Abelian bosonization [43], the chiral spinless fermions R_j and L_j on each wire (j) are expressed, at low energies, in terms of the chiral bosons $\phi_j^{L/R}$ as

$$R_j = \frac{\kappa_j}{\sqrt{2\pi a_0}} : \exp\left(2i\phi_j^R\right) :, \quad L_j = \frac{\kappa_j}{\sqrt{2\pi a_0}} : \exp\left(2i\phi_j^L\right) :, \quad (3)$$

where $\{\kappa_j\}$ are the Klein factors necessary to guarantee the anticommutation among the spinless fermions on different wires. The symbol $: \cdots :$ denotes the normal ordering necessary to regularize the operator products; to simplify the notations, we suppress it hereafter. In terms of the bosonic fields Φ_j (compactified on a circle with radius 1) and their duals Θ_j

$$\Phi_j = \phi_j^L + \phi_j^R, \quad \Theta_j = \phi_j^L - \phi_j^R, \quad (4)$$

the low-energy effective Hamiltonian for the spinless fermion is

$$\mathcal{H}_{\text{Dirac}}^{(j)} = \frac{v_F}{2\pi} \int dx \sum_{j=1}^{N_y} \left\{ \frac{1}{K_j} : (\partial_x \Phi_j)^2 : + K_j : (\partial_x \Theta_j)^2 : \right\}, \quad (5)$$

where N_y is the number of wires in the stacking (y) direction. The bosonic fields satisfy

$$[\Phi_j(x), \Phi_{j'}(x')] = [\Theta_j(x), \Theta_{j'}(x')] = 0,$$
$$[\partial_x \Phi_j(x), \Theta_{j'}(x')] = -i\pi \delta_{jj'} \delta(x - x'), \quad (6)$$

and the Luttinger liquid parameter K_j equals to 1 for free fermions but, in general, K_j and v_F are renormalized in the presence of interactions. The field Θ_j is related to the particle density [measured from its average $\bar{\rho}^{1D} = k_F/\pi = \nu|b|/(2\pi)$] on the jth wire:

$$\rho_j^{1D}(x) = \bar{\rho}^{1D} + \delta\rho_j^{1D}(x) = \bar{\rho}^{1D} + \partial_x \Theta_j(x)/\pi. \quad (7)$$

This relation will be used frequently in the following sections.

To introduce interactions, we consider an adjacent pair, e.g., j and $(j+1)$. In order to gap out most of the degrees of freedom and leave gapless chiral Luttinger liquids only at the edges, the authors of Refs. [21,22] introduced the following, carefully designed, interwire interaction allowed by U(1) and translation symmetries:

$$\mathcal{H}_{\text{int}} = -g \sum_j : (L_j^\dagger R_j)^{\frac{m-1}{2}} L_j^\dagger (L_{j+1}^\dagger R_{j+1})^{\frac{m-1}{2}} R_{j+1} :, \quad (8)$$

where m is an odd integer for fermions.[1]

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[1] At this stage, m must be integer as the number of backscattered fermions must be integer: $(m-1)/2 \in \mathbb{Z}$. However, once the
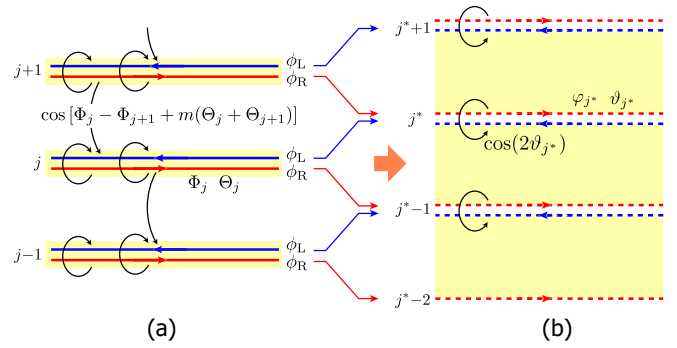

FIG. 2. (a) Original wires (located at j) on which the original bosons $\{\Phi_j, \Theta_j\}$ are defined and (b) virtual wires (with positions specified by j^*) corresponding to the on-strip fields $\{\varphi_{j^*}, \vartheta_{j^*}\}$. Dashed lines denote the new set of chiral bosons $\tilde{\phi}_{j^*}^{L/R}$, with which interwire interaction (9) looks like an ordinary backscattering $\cos(2\vartheta_{j^*})$.

This interaction is made up of two simultaneous $\frac{m-1}{2}$-pled (on-wire) backscattering processes on the adjacent wires and interwire single-particle hopping (i.e., $\mathcal{H}_{\text{int}}$ consists of m-particle processes). According to Eq. (1), the chiral fermions $R_j(x)$ and $L_j(x)$ are accompanied, respectively, by the oscillating factors $e^{ik_x^R(j)x}$ and $e^{ik_x^L(j)x}$. Therefore, if we construct the interaction $\mathcal{H}_{\text{int}}$ from the original c_j fermions, it must contain the factor $e^{i(-b+2mk_F)x}$. Requiring $\mathcal{H}_{\text{int}}$ to be nonoscillatory (i.e., translationally invariant) fixes the filling fraction $\nu = 2k_F/|b| = 1/m$.

In terms of the bosonic fields, $\mathcal{H}_{\text{int}}$ reads as[2]

$$\mathcal{H}_{\text{int}} = -g \sum_j \int dx : \cos[\Phi_j - \Phi_{j+1} + m(\Theta_j + \Theta_{j+1})] :$$
$$\equiv -g \sum_j \int dx : \cos(2\vartheta_{j^*}) :, \quad (9)$$

where ϑ_{j^*} lives on the "fictitious" wire (or strip) j^* located in the middle of the two wires j and $j+1$ (see the dashed lines in Fig. 2). Physically, the part $\Phi_{j+1} - \Phi_j$ comes from the single-particle hopping between neighboring wires, and $m(\Theta_{j+1} + \Theta_j)$ from the tailored backscattering *within* individual chains [21,22]. The new fields $\{\vartheta_{j^*}\}$ may be viewed as parts of a new set of bosons on strips (see Fig. 2)

$$2\varphi_{j^*} = \Phi_j + \Phi_{j+1} + m(\Theta_j - \Theta_{j+1}),$$
$$2\vartheta_{j^*} = \Phi_j - \Phi_{j+1} + m(\Theta_j + \Theta_{j+1}) \quad (10)$$

that, from (6), obey the following commutation relation:

$$[\varphi_{j^*}(x), \partial_x \vartheta_{j'^*}(x')] = i\pi m \delta_{j^* j'^*} \delta(x - x'). \quad (11)$$

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interaction is given in terms of bosons Φ_j and Θ_j as in Eq. (9), we can think of bosonic problem with even m.

[2] We were not precise about the ordering of the fermion operators in (8); we assume that they are correctly ordered in such a way that, when bosonized, they reproduce (9).

They may be viewed as made up of the new set of "chiral bosons"

$$\tilde{\phi}_{j*}^{\mathrm{L}} = (\Phi_j + m\Theta_j)/2, \quad \tilde{\phi}_{j*}^{\mathrm{R}} = (\Phi_{j+1} - m\Theta_{j+1})/2 \quad (12)$$

as $\varphi_{j*} = \tilde{\phi}_{j*}^{\mathrm{L}} + \tilde{\phi}_{j*}^{\mathrm{R}}$ and $\vartheta_{j*} = \tilde{\phi}_{j*}^{\mathrm{L}} - \tilde{\phi}_{j*}^{\mathrm{R}}$.

In the strong-coupling limit $g \to \infty$, the bosonic fields are pinned at one of the minima of the cosine potential,

$$2\bar{\vartheta}_{j*} = 2n_{j*}\pi \quad (j^* = 0, \ldots, N_y - 1), \quad (13)$$

where $n_j \in \mathbb{Z}$. Any semiclassical ground state is then specified by a set of N_y integers $\{n_{j*}\}$. However, all these states, labeled by $\{n_{j*}\}$, are *not* physically distinct since, as will be shown in the next section, some of them must be identified up to the periodicity of the bosonic fields Φ_j and Θ_j.

B. Ground-state degeneracy

One of the crucial signatures of a topologically ordered state in $2 + 1$ dimensions is the ground-state degeneracy on higher-genus surfaces. The simplest nontrivial example is the m-fold degeneracy in the $\nu = 1/m$ Laughlin states [44,45]. In the strong-coupling limit, the (semiclassical) CWC ground states are found by minimizing the potential $-g \sum_j :$ $\cos[2\vartheta_{j*}]$ with respect to ϑ_{j*}, and there seems to be infinitely many of them labeled by the set of integers $\{n_{j*}\}$ in (13). Reference [22] gave an argument about the ground-state degeneracy based on the limit $N_y \to 1$. Here, we count the number of inequivalent ground states by taking into account the periodic structure of the bosonic fields [46,47]. As the derivation is straightforward but slightly involved, we just give a sketch here and the details, for both fermions and bosons, are given in Appendix A.

First, we assume that the system is defined on a torus and impose periodic boundary condition both in the wire (x) direction and in the stacking direction ($N_y + 1 \equiv 1$, $\vartheta_{0*} \equiv \vartheta_{N_y^*}$). One may think that the wires are infinitely long and that the geometry of the system is cylindrical. However, the argument (in Appendix A) relies essentially on the structure of the zero modes which is peculiar to periodic systems (see, e.g., Fig. 5), and we implicitly assume the periodicity in the x direction as well.

Then, we use the expressions (3) expressing the Dirac fermions in terms of the bosons to infer that states which differ by a π shift of $\phi_j^{\mathrm{L,R}}$ must be identified:

$$\phi_j^{\mathrm{L,R}} \simeq \phi_j^{\mathrm{L,R}} + \pi\mathbb{Z}. \quad (14)$$

(Bosons have a different periodicity, as explained in Appendix A.) Due to the periodicity, most of the would-be ground states are equivalent. In fact, as proven in Appendix A, any pair of ground states $\{\bar{\vartheta}_{j*}^{(1)}\}$ and $\{\bar{\vartheta}_{j*}^{(2)}\}$ are equivalent and represent the same physical state if, and only if,

$$\frac{1}{\pi} \sum_{j^*=1}^{N_y} \delta\vartheta_{j*} \equiv \frac{1}{\pi} \sum_{j^*=1}^{N_y} \left(\bar{\vartheta}_{j*}^{(1)} - \bar{\vartheta}_{j*}^{(2)}\right) = 0 \ (\mathrm{mod}\, m). \quad (15)$$

Thus, we conclude that in the coupled-wire system with strong interwire interaction (9), there are precisely m distinct ground

states that are characterized by

$$\frac{1}{\pi} \sum_{j^*=1}^{N_y} \bar{\vartheta}_{j*} = 0, 1, \ldots, m - 1. \quad (16)$$

From the above, it follows that the eigenvalues of the Wilson loop operator

$$\mathcal{W}_y = \exp\left[i\frac{2}{m} \sum_{j^*=1}^{N_y} \vartheta_{j*}(x_0)\right] \quad (x_0 : \mathrm{fixed}) \quad (17)$$

distinguish the m different ground states. Then, by using Eq. (11), we readily see that the operator

$$\mathcal{T}_x = \exp\left[\frac{i}{m} \int_0^{L_x} dx\, \partial_x \varphi_{j_0^*}(x)\right] \quad (j_0^* : \mathrm{fixed}) \quad (18)$$

changes the eigenvalue of $\mathcal{W}_y$ by $e^{i\frac{2\pi}{m}}$ and creates a different ground state. Since, according to the arguments in Ref. [22], the operator $\mathcal{T}_x$ may be viewed as transporting a quasiparticle along the wire j_0^*, this perfectly agrees with the well-known picture that the insertion of fluxes, corresponding to quasiparticle transport around nontrivial loops, generates topologically different ground states [45].

III. FROM COUPLED WIRES TO CHERN-SIMONS THEORY

The argument in Sec. II B already suggests us to identify ϑ_{j*} with a gaugelike degree of freedom. In this section, we proceed along this line to identify the bosonic fields with the statistical gauge field.

A. Introducing gauge fields

Since x-dependent local gauge transformations of fermions $\psi_j \to \psi_j e^{i\xi_j(x)}$ change the boson field as $\Phi_j(x) \to \Phi_j(x) + \xi_j(x)$, it is natural to identify

$$a_x(x, jl_y) = -\partial_x \Phi_j(x) \quad (19)$$

as the x component of a two-dimensional gauge field. To obtain a bona fide two-dimensional gauge field, we also need the y component, and to this end, we note that ϑ_j transforms as the y component of a lattice gauge field. To see this, recall that a link variable transforms as[3]

$$U_{(x,j),(x,j+1)} \equiv e^{il_y a_y(x, j\delta)} \to e^{i\xi_j(x)} U_{(x,j),(x,j+1)} e^{-i\xi_{j+1}(x)}, \quad (20)$$

where the lattice spacing is the distance l_y between the wires. This implies

$$a_y(x, jl_y) \to a_y(x, jl_y) + \frac{1}{l_y}[\xi_j(x) - \xi_{j+1}(x)], \quad (21)$$

which suggests us to identify $2\vartheta_{j*}(x) = l_y a_y(x, jl_y)$ with a proper choice of gauge. In the continuum where we put

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[3]Note that the x dimension is continuous while the y dimension is discrete. This can be thought of as an Euledian version of the Kogut-Susskind formulation of lattice gauge theory [65].

$y = jl_y$, the y component reads as

$$a_y = -\partial_y \Phi + \frac{2m}{l_y} \Theta. \qquad (22)$$

Combining (19) and (22), we obtain the following gauge field:

$$\vec{a} \equiv (a_x, a_y) = -\vec{\nabla}\Phi + (0, 2m\Theta/l_y), \qquad (23)$$

which, by Eq. (6), satisfies the commutation relation

$$[a_x(\vec{r}_1), a_y(\vec{r}_2)] = i2\pi m \delta^2(\vec{r}_1 - \vec{r}_2). \qquad (24)$$

This can also be derived from a theory with a Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{1}{2\pi m} a_y \dot{a}_x \qquad (25)$$

in the Lagrangian, which is precisely how the statistical gauge field appears in the Chern-Simons-Ginzburg-Landau (CSGL) theory [19]. That the normalization of $\vec{a}$ is consistent with this interpretation is seen by calculating the statistical magnetic field

$$\frac{1}{2\pi} b_a = \frac{1}{2\pi} \epsilon^{ij} \partial_i a_j = \frac{m}{l_y} \delta \rho^{\text{1D}} = m \delta \rho, \qquad (26)$$

where we have used Eq. (7) and that the two-dimensional density (ρ) is related to the one-dimensional one by $\rho^{\text{1D}} = l_y \rho$. Equation (26) implies that $m\, a$ fluxes are attached to each electron, thereby confirming our interpretation.

An attentive reader should have noticed that in the above argument, the normalization of $\vec{a}$ was of central importance. For instance, renormalizing the relation (23) by a factor m would have given a coefficient $m/2\pi$ in the CS Lagrangian (25), and in that case it would have been tempting to identify a with the hydrodynamical field in the Wen-Zee theory of quantum Hall liquids [16]. However, in the next subsection we show that our identification of $\vec{a}$ as a statistical gauge field is indeed the correct one.

A remark on the strong-coupling limit is in order. In the usual treatment of the CWC, the strong-coupling limit is used to simply gap out all the bulk degrees of freedom leaving the gapless modes only at the boundaries. Here, as we shall see, the strong-coupling limit enters via the consistency with the continuum limit. In Ref. [26], an infinitely strong interwire interaction locks an *external* electromagnetic field to ϑ. For the external gauge field to be completely canceled by the statistical gauge field, which is the essence of the mean-field approximation in the CSGL theory, the coupling must be strong. In the CSGL theory, the large energy scale is the cyclotron energy in the uncancelled magnetic field, while in our case it is the interwire coupling g.

B. Relation to the CSGL theory

So far, we have considered only the extreme strong-coupling limit $g \to \infty$ where a_y is strictly pinned to the minimum of the cosine potential. Relaxing this condition, and also retaining the terms from the Luttinger Hamiltonian (5), we get the effective Hamiltonian for the a field

$$H_{\text{eff}} = \int d^2 r \left\{ \frac{\omega_c}{4\pi m} \left(\frac{1}{K} a_x^2 + 2m\beta a_y^2 \right) + \frac{K\omega_c \pi}{4m} l_y^2 (\delta\rho)^2 \right\}, \qquad (27)$$

where we used the mean-field relation $2\pi m\bar{\rho} = eB$, with B the external magnetic field and $\bar{\rho}$ the mean *area* density, to rewrite the Fermi velocity as $v_F = \pi \bar{\rho}^{\text{1D}}/M_e = \omega_c l_y v/2 = \omega_c l_y/(2m)$. Here, $\omega_c = eB/M_e$ is the cyclotron energy (M_e is the electron band mass), and we introduced the dimensionless coupling strength β by

$$g = \frac{\omega_c}{\pi l_y} \beta. \qquad (28)$$

In the last term in (27), which comes from the term $\sim (\partial_x \Theta)^2$ in the Luttinger Hamiltonian (5), $\delta\rho$ denotes the deviation from the average area density $\bar{\rho} = eB/(2\pi m)$.

So far, K and β are free parameters but we shall now impose the condition $\beta K = 1/(2m)$ which ensures rotational invariance in (27). It should be no surprise that fine tuning is needed to retain rotational invariance; it is in fact more surprising that rotational invariance can at all be recovered in the CWC. In order for the continuum approximation to make sense [for $\beta = 1/(2mK) = $ finite], we must take l_y to be small and thus g must be large $g \sim 1/l_y$, albeit not infinite. The rotationally invariant effective Lagrangian now reads as

$$\mathcal{L}_{\text{CWC}} = \frac{1}{4\pi m} \epsilon^{ij} a_i \dot{a}_j - \frac{\beta}{2\pi} \omega_c (a_x^2 + a_y^2) - \frac{K\omega_c \pi}{4m} l_y^2 (\delta\rho)^2. \qquad (29)$$

This is *not* a topological field theory since the leading term (a_i^2) in a derivative expansion does depend on the metric. Thus, we could not have interpreted a as the hydrodynamical field even if we had changed the normalization to get the "correct" CS term.

It is illuminating to compare this with the standard CSGL theory given by the Lagrangian [19,38]

$$\mathcal{L}_{\text{CSGL}} = \phi^*(i\partial_0 - \alpha_0)\phi - \frac{1}{2M_e}|(-i\vec{\nabla} - e\vec{A} - \vec{\alpha})\phi|^2$$

$$- \frac{1}{2} \int d\vec{y}\, |\phi(\vec{x})|^2 V(\vec{x} - \vec{y})|\phi(\vec{y})|^2$$

$$+ \frac{1}{4\pi m} \epsilon^{\mu\nu\lambda} \alpha_\mu \partial_\nu \alpha_\lambda, \qquad (30)$$

where ϕ is a composite boson minimally coupled to the external electromagnetic gauge field $\vec{A}$, $(\alpha_0, \vec{\alpha})$ is a statistical gauge field, and $V(\vec{r})$ is a repulsive two-body repulsive potential.

We proceed by first parametrizing the bosonic field ϕ as

$$\phi(\vec{x}) = \sqrt{\rho(\vec{x})} e^{i\theta(\vec{x})}, \qquad (31)$$

and then integrate out the nondynamical field α_0 to get the constraint $\rho = 2\pi m b_\alpha$, where $b_\alpha = \epsilon^{ij} \partial_i \alpha_j$ is the flux of the statistical gauge field $\vec{\alpha}$, and the chemical potential is such that V has a minimum at $\rho = \bar{\rho}$. Neglecting derivatives of the density, making the mean-field approximation $\vec{a} = e\vec{A} + \vec{\alpha}$, and assuming a local potential $V(\vec{r}) = V\delta(\vec{r})$, we get

$$\mathcal{L}_{\text{CSGL}} = \frac{1}{4\pi m} \epsilon^{ij} \alpha_i \dot{\alpha}_j - \frac{\bar{\rho}}{2M_e} \vec{\alpha} \cdot \vec{\alpha} - \frac{V}{2}(\delta\rho)^2 \qquad (32)$$

with $2\pi m\bar{\rho} = eB$. To derive this result, we absorbed the gradient of the phase $\vec{\nabla}\theta$ in the vector potential $\vec{\alpha}$, which amounts to picking a unitary gauge.

Recalling that $\bar{\rho}/M_e = \omega_c/(2\pi m)$, (29) and (32) become identical if we take $\beta = 1/(2m)$ (i.e., $K = 1$), identify $\vec{a}$ in (29) as the fluctuation in the statistical gauge field, and pick V in the CSGL theory (32) to match the coefficient in front of $(\delta\rho)^2$ in the CWC expression. In what follows, we fix $K = 1$ and $\beta = 1/(2m)$ except in Sec. VI. This equivalence is a main result of this paper, and it shows that, after introducing interactions, the bosons on the wires become bona fide composite bosons.

Having shown the equivalence between the CWC and the CSGL theory, it is reasonable to expect that the results from the latter can be derived also in the present context. In the next two sections, we shall show that this is indeed the case. Before doing so, however, we shall make some comment about the edge modes.

The presence of chiral gapless edge modes is at the heart of the CWC. The way the right- and left-moving modes on the adjacent wires are coupled is designed in such a way that the correct chiral modes are left at the wires at the edges of the system. It is clearly of importance that our continuum description in terms of gauge fields is able to describe these modes as well. In fact, as we already advertised, we will eventually derive the Wen-Zee theory, which is known to give a correct description of the edge [48]. Turning to the intermediate theory $\mathcal{L}_{CWC}$ given by (29), one might naively expect that there are no gapless modes because of the quadratic terms.

IV. LAUGHLIN WAVE FUNCTION FROM COUPLED WIRES

In this section, we derive the Laughlin wave function from the CWC. We do it first for the (fine-tuned) rotationally invariant theory (29), and then for the general anisotropic case.

A. Isotropic case

In the standard derivation [38,49] of the wave function from the CSLG theory, one uses the Coulomb gauge and rewrites the Hamiltonian as a collection of harmonic oscillators in the variables $\delta\rho$ and θ. One then finds the wave function for the composite bosons, by using the density representation of the wave functional. Finally, the full lowest-Landau-level Laughlin wave function is regained by reintroducing the phase factor that was taken out in the statistical transmutation from the original electrons to the composite bosons. In the present context, the derivation of the norm of the wave functions only differs from the standard CSGL procedure by some technicalities, while that for the phase requires a careful treatment, and provides a nontrivial consistency check of our calculations.

Instead of the Coulomb gauge, we are implicitly using the unitary gauge, where the phase fluctuations are absorbed in the gauge field [see, e.g., Eq. (32)], and our treatment here also differs from that in the standard CSLG treatment in that the relation between $\vec{a}$ and the density is anisotropic as is seen in Eq. (26). With this in mind, we proceed by making the following decomposition for $\vec{a}$:

$$a_i = \epsilon_{ij}\partial_j\chi + \partial_i\eta \tag{33}$$

so that $b_a = -\nabla^2\chi$. Substituting (33) in the Lagrangian density (29) gives the Lagrangian

$$L_{CWC} = \frac{1}{2\pi m}\int d^2r\, \eta\dot{b}_a - \frac{\omega_c}{4\pi m}\int d^2r\,(-\eta\nabla^2\eta - \chi\nabla^2\chi)$$
$$- \frac{\omega_c\pi}{4m}l_y^2\int d^2r\,(\delta\rho)^2, \tag{34}$$

where we have set $\beta = 1/(2m)$ and used that, by partial integration, the cross terms between η and χ vanish in the Hamiltonian part while the diagonal terms vanish in the kinetic term. Using $b_a = 2\pi m\delta\rho$, we see that $\delta\rho$ and η are conjugate variables satisfying $[\delta\rho(\vec{r}_1), \eta(\vec{r}_2)] = i\delta^2(\vec{r}_1 - \vec{r}_2)$. The Hamiltonian becomes

$$H = \frac{\omega_c}{4\pi m}\int d^2r\left\{-\eta\nabla^2\eta - \delta\rho\frac{(2\pi m)^2}{\nabla^2}\delta\rho\right\}, \tag{35}$$

where we used $\beta = 1/(2m)$ and only kept the leading terms in a derivative expansion. In the density ($\delta\rho$) representation, the wave functional is that of an assembly of harmonic oscillators labeled by r. Using the explicit nonrelativistic form of the density operator, $\delta\rho(\vec{r}) = \rho(\vec{r}) - \bar{\rho} = \sum_i \delta^2(\vec{r} - \vec{r}_i) - \bar{\rho}$ with $\vec{r}_i$ being the position of the ith electron, and the mean density $\bar{\rho}$ which is related to the magnetic length ℓ_B by $2\pi m\bar{\rho} = eB = 1/\ell_B^2$, the wave function reads as

$$\Psi_m(\{\vec{r}_i\}) \sim \prod_{i<j}|z_i - z_j|^m e^{-\frac{1}{4\ell_B^2}\sum_i |z_i|^2}. \tag{36}$$

In the above, we have used the complex notaion $z_i = x_i + iy_i$, and we refer to Refs. [38,49] for details of the derivation.

The Laughlin wave function differs from (36) by the phase factor

$$\prod_{i<j}(z_i - z_j)^{\frac{m}{2}} \Big/ \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{-\frac{m}{2}}, \tag{37}$$

in which exchanging two electrons yields a phase angle $m\pi$, while transporting one around another yields $2m\pi$. We now show that we can reconstruct the additional phase factor (37) within the CWC by carefully calculating the Berry phase factor acquired when transporting an electron along the path shown in Fig. 3. In doing so, we first calculate the phase using Eq. (37). When moving along the paths I and III, the transported electron exchanges its position with other electrons sitting on the wires 0 and M thereby acquiring the phase $m\pi$ each time (in the $\nu = 1/m$ Laughlin state). On the segments II and IV, on the other hand, there are no exchanges. Nevertheless, the full loop encircles all the electrons sitting on the wires 1 to $M - 1$ and picks up the phase $2m\pi$ from each electron. If we define $\theta_j(x)$ to denote the ground-state expectation value of the operator $\Theta_j(x)$, then, by Eq. (7), the number of electrons in the interval $x \in [0, a]$ of the wire j is given by $[\theta_j(a) - \theta_j(0)]/\pi$. Combining all this, we expect the total phase acquired by the electron:

$$\gamma_{loop} = m\{\theta_0(a) - \theta_0(0) + \theta_M(0) - \theta_M(a)\}$$
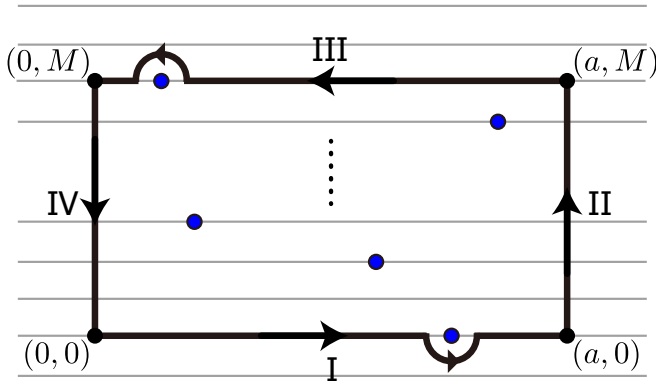$$+ 2m\sum_{j=1}^{M-1}\{\theta_j(a) - \theta_j(0)\}. \tag{38}$$

FIG. 3. Loop used to calculate the phase of the Laughlin wave function. Particles on wires 1 to $M - 1$ are fully encircled and each gives the phase $2m\pi$. Exchanging the positions with particles on wires 1 or M amounts to replacing the straight lines with half-circles, each one giving rise to a phase $m\pi$.

To calculate the Berry phase related to transporting a right-moving electron (we could equally well have considered a left mover) within the CWC, we first define the states

$$|\vec{r}\rangle = e^{i\Phi_j(x) - i\Theta_j(x)}|0\rangle \equiv e^{i\varphi(\vec{r})}|0\rangle, \quad (39)$$

where $\vec{r} = (x, j)$ and $|0\rangle$ is the ground state. We have also introduced the notation $\varphi(\vec{r}) = 2\phi_j^R(x)$ to avoid clutter. The Berry connection is defined by

$$\vec{\mathcal{A}}(\vec{r}) = \langle\vec{r}|i\nabla_{\vec{r}}|\vec{r}\rangle = \langle 0|e^{-i\varphi(\vec{r})}i\nabla_{\vec{r}} e^{i\varphi(\vec{r})}|0\rangle$$
$$= \langle 0|\partial_x\Theta_j(x)|0\rangle, \quad (40)$$

where we used that, because of reflection symmetry when $x \to -x$, there is no x component of the current in the ground state: $\langle 0|\partial_x\Phi_j(x)|0\rangle = 0$, and properly regularized the product $e^{-i\varphi(\vec{r})}e^{i\varphi(\vec{r})}$. It is now easy to calculate the Berry phase corresponding to the segment I:

$$\Upsilon(\mathrm{I}) = \int_0^a dx \, \mathcal{A}_x(x, 0) = -[\theta_0(0) - \theta_0(a)]. \quad (41)$$

In the same way, we get for the segment III

$$\Upsilon(\mathrm{III}) = \theta_M(0) - \theta_M(a). \quad (42)$$

Along to the segment II, we shall return to the original discretized formulation and define the the Berry phase factor as we do in lattice field theories

$$e^{i\Upsilon(\mathrm{II})} = \langle(a, M)|(a, M-1)\rangle\langle(a, M-1)|(a, M-2)\rangle$$
$$\ldots \langle(a, 1)|(a, 0)\rangle \quad (43)$$

with

$$\langle(x, j+1)|(x, j)\rangle = \langle 0|e^{-i[\varphi(x, j+1) - \varphi(x, j)]}|0\rangle, \quad (44)$$

where we used that $\varphi(x, j)$ on different wires commute. The next, and crucial, step is to rewrite the exponent in (44) as

$$\varphi(j+1) - \varphi(j)$$
$$= \Phi_{j+1} - \Phi_j - m(\Theta_{j+1} + \Theta_j)$$
$$+ (m-1)\Theta_{j+1} + (m+1)\Theta_j$$
$$= -2\vartheta_{j^*} + (m+1)\Theta_j + (m-1)\Theta_{j+1}, \quad (45)$$

where we have suppressed the x dependence. We now recall, from Sec. II A, that ϑ_{j^*} is pinned to a constant $\bar{\vartheta}_{j^*}$, which we take to be zero (as $2\bar{\vartheta}_{j^*} \equiv 0 \mod 2\pi$, we arrive at the same phase factor for any other choices; the important thing is that it does not depend on the geometry of the loop). Adding the contributions from all the wires ($j = 0, \ldots, M - 1$) together and taking the expectation value, we are left with the phase

$$\Upsilon(\mathrm{II}) = -\{(m+1)\theta_0(a) + 2m[\theta_1(a)$$
$$+ \cdots + \theta_{M-1}(a)] + (m-1)\theta_M(a)\}. \quad (46)$$

The segment IV gives a similar expression, now evaluated at $x = 0$, and with an additional minus sign. Putting everything together, and taking into account that the Berry phase differs by a sign from the exchange phase calculated from the wave function [50], we obtain precisely the phase (38) that was derived from (36). Note that, only after carefully taking into account both the phases $m\pi$ from the particle exchange *on* the loop and $2m\pi$ from the encircled charges *inside*, we obtained the correct result.

Because of the close analogy to the CSGL theory, it is natural to expect that the above derivation can also be modified to give the wave functions of the Laughlin holes. We shall not elaborate on the details, but just make two observations. First, adding a kink $\Theta_{j_\eta}^{\mathrm{kink}}(x) = \pi \, \mathrm{sgn}(x - \eta_x)/(2m)$ on the wire j_η in the loop in Fig. 3, will, by exactly the same argument as above, give the Berry phase (38) without m thereby leading to an extra phase factor

$$\prod_i (z_i - \eta)^{\frac{1}{2}} \Big/ \prod_i (\bar{z}_i - \bar{\eta})^{-\frac{1}{2}},$$

where $\eta = \eta_x + ij_\eta l_y$. Since it will also give a contribution $\delta^2(\vec{r} - \vec{\eta})/m$ [$\vec{\eta} = (\eta_x, j_\eta l_y)$] to the density operator $\rho(\vec{r})$, it is easy to see that it will add the correct modulus $\prod_i |z_i - \eta|$ which, together with the above phase factor, reproduces the correct Laughlin hole factor $\prod_i(z_i - \eta)$.

The quasielectrons are harder to deal with. Although it is straightforward to calculate the fractional charges and the statistical phase factors for states with antikinks on the wires, it is nontrivial to extract wave functions. This asymmetry in the description of quasiholes and quasielectrons is well known both in the CSGL theory, and in approaches based on conformal field theory [41].

B. Anisotropic case

If we do not make the (fine-tuned) choice $\beta K = 1/(2m)$, that is necessary to get the isotropic theory (29), the Hamiltonian will depend on the combination

$$a_x^2 + \alpha^2 a_y^2$$

with $\alpha^2 = 2m\beta K$. Since $a_x = -\partial_x\Phi$ and $a_y = -\partial_y\Phi + \frac{2m}{\ell}\Theta$, a rescaling $(x, y) \to (x', y') = (\alpha x, y)$ implies

$$a_x(x, y) \to a_x'(x', y') = \alpha^{-1}a_x(x, y),$$
$$a_y(x, y) \to a_y'(x', y') = a_y(x, y). \quad (47)$$

Alternatively, we could rescale $(x, y) \to (x', y') = (x, y/\alpha)$ and $l_y \to l_y' = l_y/\alpha$. In both cases, the a^2 terms change as

$$dx \, dy \, (a_x^2 + \alpha^2 a_y^2) \to \alpha \, dx' \, dy' \, \{(a_x')^2 + (a_y')^2\}, \quad (48)$$

while $dx\,dy\,a_x\dot{a}_y$ remains invariant. Thus, if we use the primed coordinates, the previous derivation of the Laughlin wave function will go through without any changes, while the complex coordinate will be redefined as $z = x + iy/\alpha$. In terms of the wave functions in the lowest Landau level, this amounts to having coherent states which are not circular, but deformed into ellipses. As has been stressed by Haldane [51] and others [41,52], the Laughlin state is more general in nature than its usual incarnation as a holomorphic polynomial $\prod_{i<j}(z_i - z_j)^m$ in $z = x + iy$.

V. TOPOLOGICAL AND HYDRODYNAMIC ACTIONS

Here, we derive the effective hydrodynamical theory which contains the Wen-Zee topological field theory as the leading term in the infrared. From this, we extract the collective Kohn mode and its dispersion. We also show how to couple the hydrodynamic theory to an external perturbing electromagnetic field.

A. Topological action

When doing the gauge transformation

$$a_i \rightarrow a_i + \partial_i \xi, \tag{49}$$

the CS Lagrangian (25) picks up an extra piece $\sim \epsilon^{ij}a_i\partial_j\dot{\xi}$, and, using the notation $a_0 = \dot{\xi}$, we can recast it into the relativistic form

$$\mathcal{L}_{CS}(a) = \frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{50}$$

If we introduce a new gauge field b, the partition function for this CS theory can be rewritten as follows:

$$Z = \int \mathcal{D}[a]e^{iS_{CS}[a]} = \int \mathcal{D}[a]\mathcal{D}[b]e^{iS_{eff}[a,b]}, \tag{51}$$

where $S_{CS}[a] = \int d^3r\,\mathcal{L}_{CS}(a)$ and $S_{eff}[a,b] = \int d^3r\,\mathcal{L}_{eff}(a,b)$ with

$$\mathcal{L}_{eff}(a,b) = -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda. \tag{52}$$

Integrating out the statistical gauge field a in (52) yields the constraint $\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda = 0$ (remember that the background charge has been subtracted already) leaving only the first term which is precisely the Wen-Zee topological action for the $\nu = 1/m$ Laughlin state.

The Wen-Zee action is trivial on an infinite plane, but codes for the m^g-fold ground-state degeneracy on higher-genus (g) surfaces, as well as giving the kinetic term for the chiral edge modes [48]. The ground-state degeneracy is consistent with the counting done in Sec. II B, and the presence of chiral edge modes in the direction of the wires was a starting point of the CWC [21].

The fact that the Wen-Zee action follows from the CWC strongly suggests that there will be gapless edge modes also in the perpendicular direction in the case of finite length wires with open boundary conditions. However, we do not claim to have proven this since the derivation of the topological action did not incorporate open boundary conditions in the x direction. Here, a comment on the edge Hamiltonian is in order. From the CWC perspective, the edge modes parallel

to the wires are nothing but the decoupled chiral components at the outermost wires. As such, they have a Hamiltonian given explicitly by the chiral Luttinger theory which is at the basis of the construction. In particular, the velocity depends explicitly on the topological number m. There is no reason to believe that this would give a good description of a *real* QH edge, where the edge velocity is known to depend on the edge potential. Thus, the edge velocity should be considered as an extra phenomenological parameter, and the same holds for the multicomponent case discussed in Sec. VI.

B. Hydrodynamical action and the Kohn mode

In the previous section, we only retained the topological part (25) of the full action (29). We now show that, by including the Hamiltonian part, we can complement the Wen-Zee topological term with higher derivative contributions that describe collective modes. To this end, we begin with the low-energy effective Lagrangian (29):

$$\mathcal{L}_{CWC} = \frac{1}{4\pi m}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - a_0\delta\rho$$
$$- \frac{\beta}{2\pi}\omega_c(a_x^2 + a_y^2) - \frac{\omega_c\pi}{4m}l_y^2(\delta\rho)^2, \tag{53}$$

where we have added the term $a_0\delta\rho$ to retain the correct constraint. Following the same logic as in the previous section, but with $\mathcal{L}_{CS}(a)$ replaced by the above expression, we first integrate out the density fluctuations $\delta\rho$ to get a term $\sim a_0^2$ in the action

$$\mathcal{L} = -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda$$
$$- \frac{\omega_c}{4\pi m}\vec{a}\cdot\vec{a} + \frac{m\omega_c}{\pi\gamma}a_0^2, \tag{54}$$

where we defined the dimensionless parameter $\gamma = K\omega_c^2 l_y^2$ and used the relation $\beta = 1/(2m)$.

As above, we then proceed to integrate the a field to get the desired effective hydrodynamic theory:

$$\mathcal{L}_{hydro} = -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{m}{4\pi}\frac{1}{\omega_c}\vec{E}_b^2 - \frac{\gamma}{16\pi m}\frac{1}{\omega_c}B_b^2, \tag{55}$$

where $\vec{E}_b = -\vec{\nabla}b_0 - \dot{\vec{b}}$ and $B_b = \epsilon_{ij}\partial_i b_j$ are the field strengths related to b.

It is now straightforward to extract the dispersion relation for the collective mode described by the dynamics of this effective theory,

$$\omega(q) = \omega_c + \frac{\gamma}{8m^2}\frac{1}{\omega_c}q^2. \tag{56}$$

This should be compared with the result from the CSGL theory (30) [38,53]:

$$\omega(q) = \omega_c + \frac{\bar{\rho}}{2B}V(q)q^2, \tag{57}$$

where $V(q)$ is the Fourier transform of the two-body potential $V(\vec{r})$. We see, as expected from (29) and (32), that the result from the coupled-wire construction corresponds to having a delta function potential.

C. Coupling to an external electromagnetic field

So far, we did not include the coupling to electromagnetism, except for the constant B field that determines the ground-state density $\bar{\rho}$. Minimal coupling of electromagnetism to a phase field is implemented by the standard substitution

$$\partial_i \Phi \to \partial_i \Phi - eA_i, \qquad (58)$$

and recalling the definition (23), this implies that $\vec{A}$ is incorporated by the substitution $\vec{a} \to \vec{a} + e\vec{A}$ in the Hamiltonian, and A_0 is coupled by adding the term $eA_0\rho$ to (53). With this in mind, the Lagrangian (54) generalizes to

$$\mathcal{L} = -\frac{m}{4\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda$$
$$- \frac{\omega_c}{4\pi m}(\vec{a}+e\vec{A})\cdot(\vec{a}+e\vec{A}) + \frac{m\omega_c}{\pi\gamma}(a_0+eA_0)^2. \quad (59)$$

[Note that the substitution (58), which amounts to a minimal coupling, should be done *only* in the Hamiltonian; the CS action, which encodes the proper commutation relations, is not to be changed.] By a shift, $a \to \tilde{a} - eA$, the integration over $\tilde{a}$ can be performed as in the previous section, and we regain the result (55) with the extra term

$$\mathcal{L}_A = -\frac{e}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu b_\lambda, \qquad (60)$$

which is the desired coupling of the electromagnetic potential to the conserved current $j^\mu = \frac{e}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu b_\lambda$.

At this point, it behooves us to clarify the relation to the approach by Fuji and Furusaki in Ref. [39]. Our hydrodynamic Lagrangian (55), supplemented by the electromagnetic coupling (60), should be compared with their Eq. (52). They use the notation α for our hydrodynamic gauge field b and also consider couplings to a bosonic quasiparticle described by Φ^{VCB}. As they point out, their final expression contains a term that may be obtained by discretizing the topological part of (55). However, it crucially differs from (55) in that the field α does not have any dynamics, and that Eq. (52) has other terms that may eventually generate more relevant contributions (e.g., α^2) in a derivative expansion.

VI. GENERAL ABELIAN QH STATES

Having understood how a topological quantum field theory emerges from the CWC in the simplest case of the Laughlin states, we now ask what in the previous analysis will carry over to general Abelian QH states which are described by the Wen-Zee Lagrangian [16]

$$\mathcal{L} = -\frac{1}{4\pi}\sum_{I,J=1}^{n_K}\mathcal{K}_{IJ}\epsilon^{\mu\nu\rho}b_\mu^{(I)}\partial_\nu b_\rho^{(J)} - \frac{e}{2\pi}\sum_{I=1}^{n}q_I\epsilon^{\mu\nu\rho}b_\mu^{(I)}\partial_\nu A_\rho. \qquad (61)$$

In the above, $\{b_\mu^{(I)}\}$ are n_K Chern-Simons gauge fields, and the $n_K \times n_K$ "K matrix" $\mathcal{K}$ is symmetric and integer valued. The integer-valued "charge vector" $\mathbf{q} = (q_1, \ldots, q_n)$ determines how the Ith component of the U(1) charge $\frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\mu b_\nu^{(I)}$ couples to the external electromagnetic field A_μ [48]. The topological field theories of the form (61) are known to describe not only genuine topological phases [16], but also
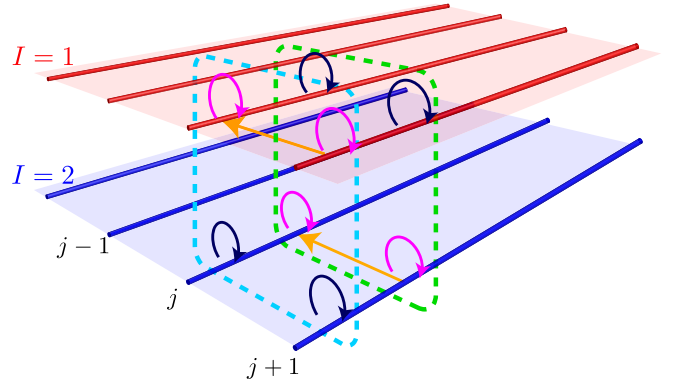

FIG. 4. Multilayer construction that reproduces the Abelian Chern-Simons gauge theory (61).

symmetry-protected ones [54]. In Ref. [22], Teo and Kane gave a construction for the simple case of a 2×2 K matrix. Their method generalizes quite straightforwardly to a general K matrix, and so does the extraction of the topological field theory (61).

To closely follow the steps given in Sec. II A for the single-component case, we prepare $N_{\mathrm{layer}} = n_K$ layers of coupled wires and define the following bosons as in Eq. (12):

$$\vec{\phi}_{j^*}^{\mathrm{L}} \equiv \tfrac{1}{2}(\vec{\Phi}_j + \mathcal{K}\vec{\Theta}_j), \qquad \vec{\phi}_{j^*}^{\mathrm{R}} \equiv \tfrac{1}{2}(\vec{\Phi}_{j+1} - \mathcal{K}\vec{\Theta}_{j+1}), \quad (62)$$

where we have introduced the vectorial notations, e.g.,

$$\vec{\Phi}_j \equiv \big(\Phi_j^{(1)}, \ldots, \Phi_j^{(N_{\mathrm{layer}})}\big)^{\mathrm{T}}, \qquad \vec{\Theta}_j \equiv \big(\Theta_j^{(1)}, \ldots, \Theta_j^{(N_{\mathrm{layer}})}\big)^{\mathrm{T}}. \qquad (63)$$

Each pair of bosons $(\Phi_j^{(I)}, \Theta_j^{(I)})$ obey the commutation relations (6) and describe a Luttinger liquid on the wire j of the Ith layer (see Fig. 4) with the Luttinger parameter K. From these fields, we define the following bosons living on the strip j^*:

$$\vec{\vartheta}_{j^*} \equiv \vec{\phi}_{j^*}^{\mathrm{L}} - \vec{\phi}_{j^*}^{\mathrm{R}} = \tfrac{1}{2}(\vec{\Phi}_j - \vec{\Phi}_{j+1}) + \tfrac{1}{2}\mathcal{K}(\vec{\Theta}_j + \vec{\Theta}_{j+1}). \quad (64)$$

Note that the on-strip fields $\vartheta_{j^*}^{(I)}$ are not local in that they contain $\Theta^{(J)}$ on *different* layers as well as $\Phi^{(I)}$ on the same layer.

Again in analogy with the single-component case, it is clear that to reproduce the Wen-Zee action (61) we need the following N_{layer} interactions between the adjacent wires:

$$\mathcal{V} = \sum_{I=1}^{N_{\mathrm{layer}}}\sum_{j=1}^{N_y} g_I \cos\big(2\vartheta_{j^*}^{(I)}\big). \qquad (65)$$

Following the procedure given in Ref. [22], we find that, in the strong-coupling limit of $\mathcal{V}$, there are N_{layer} gapless modes at the lower edge $j^* = 0$ described by the following edge Hamiltonian:

$$\mathcal{H}_{\mathrm{edge}}(j^* = 0)$$
$$= \frac{v_{\mathrm{F}}^0}{2\pi}\int dx \sum_{I,J=1}^{N_{\mathrm{layer}}}\{\mathbf{1} + (\mathcal{K}^{-1})^2\}_{IJ}\big(\partial_x\widetilde{\phi}_0^{\mathrm{R},(I)}\big)\big(\partial_x\widetilde{\phi}_0^{\mathrm{R},(J)}\big) \quad (66)$$

and similarly for the upper edge $j^* = N_y$. If we identify the nonuniversal velocity matrix **V** as

$$\mathbf{V} = 2v_{\mathrm{F}}^0 \{\mathbf{1} + (\mathcal{K}^{-1})^2\}, \tag{67}$$

the edge Hamiltonian (66) coincides with that predicted by the hydrodynamic Chern-Simons action (61) [48]. As has been remarked in Sec. V A, that the K matrix, which is of purely topological origin, appears in the edge Hamiltonian (66) through the nonuniversal velocity **V** is just an artifact of using the special interwire interaction (65) to construct the QH state.

Then, by properly identifying the statistical gauge fields $\{a^{(I)}\}$, we can derive the Wen-Zee action (61) with $A = 0$. Since we follow almost the same steps as before, we just sketch the derivation in Appendix B 2. Also, it is not hard to use the method in Sec. V B to calculate the higher-derivative corrections to the topological action (61) but we do not give the details here. The construction here uses the multilayer setting. However, as is described in Appendix B 3, we can always recast the multilayer system into a single-layer one with *longer-range* interactions as in Ref. [22].

It is straightforward to rewrite (65) in terms of suitably ordered interactions among fermions or bosons that generalize (8). In fact, as is shown in Appendix B 1, in order for the above interactions to be written only in terms of local operators of the original fermionic/bosonic theories, the $N_{\mathrm{layer}} \times N_{\mathrm{layer}}$ matrix K must satisfy

$$\mathcal{K}_{II} = \text{odd}, \quad \mathcal{K}_{IJ} = \text{even} \ (I \neq J) \quad (\text{for fermions}),$$
$$\mathcal{K}_{IJ} = \text{even} \ \forall \, I, J \quad (\text{for bosons}). \tag{68}$$

Recovering all the Klein factors (in the fermion case) and combining (68) with the relation $e^{i\vartheta_{j^*}^{(I)}} e^{i\vartheta_{j^*-1}^{(I)}} = (-1)^{\mathcal{K}_{IJ}} e^{i\vartheta_{j^*-1}^{(J)}} e^{i\vartheta_{j^*}^{(I)}}$ derived from the commutation relations, we see that, for both fermions and bosons, all the terms in (65) are commuting and can be minimized simultaneously.

By requiring translational invariance of the interwire interactions, we can determine the filling at which these interactions are allowed. The magnetic field b and the particle density of each layer may be recovered by the following substitution for all layers I and wires j:

$$\Phi_j^{(I)}(x) \to \Phi_j^{(I)}(x) + bjx, \quad \Theta_j^{(I)}(x) \to \Theta_j^{(I)}(x) - \pi \bar{\rho}_{(I)}^{\mathrm{1D}} x. \tag{69}$$

Then, the interwire interactions (65) acquire additional x dependence $(2\pi \sum_J \mathcal{K}_{IJ} \bar{\rho}_{(I)}^{\mathrm{1D}} + b)x$ in the cosines. The translational invariance requires that these oscillating factors should vanish:

$$\bar{\rho}_{(I)}^{\mathrm{1D}} = \frac{|b|}{2\pi} \sum_J (\mathcal{K}^{-1})_{IJ}. \tag{70}$$

From this, we can read off the corresponding filling fraction as

$$\nu = \frac{2\pi}{|b|} \sum_I \rho_0^{(I)} \bar{\rho}_{(I)}^{\mathrm{1D}} = \sum_{I,J} (\mathcal{K}^{-1})_{IJ}. \tag{71}$$

Comparing this with the general expression from the Chern-Simons theory $\nu = \mathbf{q}^{\mathrm{T}} \mathcal{K}^{-1} \mathbf{q}$, we see that our construction corresponds to the symmetric basis $\mathbf{q} = (1, 1, \ldots, 1)$ [48].

Although it is rather straightforward, we will not derive the ground-state degeneracy along the lines in Sec. II B[4] since the result also follows directly from the effective Wen-Zee theory (61) to be derived in Appendix B 2 [55,56].

Regarding the wave functions, there is an important distinction between states formed by multicomponent states where the particles are distinguishable by spin or some "layer" index [57], and where they are indistinguishable, as in the hierarchy of spin-polarized states in the lowest Landau level. In the first case, which includes the Halperin (m, m, n) states [58], one can derive the wave functions using the same technique as in Sec. IV. The hierarchy wave functions pose a much more difficult problem. Here, the electrons in the layers (or effective Landau levels in the language of composite fermions) are distinguished by their orbital spin. This is a topological quantity that is most directly revealed by geometrical response since it couples to external curvature [59], but it is also related to the Hall viscosity which is a transport coefficient [60]. It is a challenge to to incorporate the orbital spin in the CWC.

VII. SUMMARY AND OUTLOOK

In this paper we have considered the bulk properties in the CWC, i.e., the (topological) ground-state degeneracy, the wave function, the bulk effective theory, and the low-energy excitations. In the limit of sufficiently strong interwire interactions, the ground states are found by minimizing the interaction energy. We gave, for the Laughlin states, the precise condition to identify physically inequivalent ground states and showed that the CWC correctly reproduces the number of degenerate ground states on a torus.

In order to describe the low-energy properties in the bulk, we first identified the bosons on the wires as composite bosons and found the expressions for the statistical gauge field in the CSGL theory. These enabled us to obtain the low-energy effective action for the statistical (Chern-Simons) gauge field, from which we constructed the bulk Laughlin wave function within the framework of the CWC. By integrating out the statistical gauge field, we then derived the hydrodynamic effective action (the Maxwell-Chern-Simons theory) in which the leading term is the topological Wen-Zee action. We also discussed the effects of anisotropy, which is inherent in the CWC, on the bulk properties. The methods developed for the simplest Laughlin states were readily generalized to give a simple recipe for constructing general Abelian quantum Hall states, characterized by a K matrix, and the corresponding multicomponent Wen-Zee action.

Our work points at several directions that might be fruitful to explore. In the CWC, it is by its nature quite easy to

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[4]Now, the equivalence relation corresponding to (15) is

$$\frac{1}{\pi} \sum_{j^*=1}^{N_y} \delta \vec{\vartheta}_{j^*} = \sum_{I=1}^{N_{\mathrm{layer}}} n_I e^{(I)} \quad (n_I \in \mathbb{Z})$$

with a set of N_{layer} vectors defined by $(e^{(I)})_J = K_{JI} = K_{IJ}$. That is, if the difference $\{\delta \vec{\vartheta}_{j^*}\}$ between any given pair of ground states is on the lattice spanned by $\{e^{(I)}\}$, they are equivalent. Then, the volume of a unit cell of this lattice $|\det K|$ gives the ground-state degeneracy.

find the gapless edge modes at the boundaries parallel to the wires. Although we derived the bulk Wen-Zee action that does encode the chiral edge states at *any* boundaries, it is not at all clear how this will work microscopically at the boundaries in the direction perpendicular to the wires (i.e., y direction). To investigate this, one would have to impose box boundary conditions on the wires, but we postpone this for future studies.

We already mentioned that one can derive the Laughlin quasihole wave functions in the CWC framework, but that it is a harder problem to find the wave function for states with quasielectrons, and even harder to find even the ground-state wave functions for the hierarchical states. In particular, it is a challenge to understand how the orbital spin, which couples to the curvature of the manifold on which the QH liquid is defined, could emerge from a CWC.

Another related and difficult, but very interesting, question is whether one could use methods similar to those developed in this paper to extract topological field theories for non-Abelian states. The obvious first try would be the $\nu = 1$ bosonic Moore-Read state for which Teo and Kane found a reasonably simple CWC [22].

Note added. Recently, we noticed another very recent related preprint [61], in which the effective theory similar to ours has been derived directly without evoking composite bosons. We, however, disagree on how the gauge constraint is handled in this paper.

APPENDIX A: GROUND-STATE DEGENERACIES

In Sec. II, we have claimed that the coupled-wire system with the fine-tuned interaction (9) has exactly m different ground states labeled by the value

$$\sum_{j^*=1}^{N_y} \bar{\vartheta}_{j^*}/\pi \ (\text{mod } m)$$

($\vartheta_0 = \vartheta_{N_y}$ by periodic boundary condition) in the bulk. In this Appendix we prove this proposition.

The key is that states that are equal up to the periodicity of the bosonic fields must be identified [46,47]. The situation is different for fermionic and bosonic cases as the two cases have different periodic structures (see Fig. 5). The periodicity in the fermionic case is defined with respect to the chiral bosons [the periodicity of (Φ_j, Θ_j) differs in the fermionic and bosonic sectors]:

$$\phi_j^L \sim \phi_j^L + \pi, \quad \phi_j^R \sim \phi_j^R + \pi. \tag{A1}$$
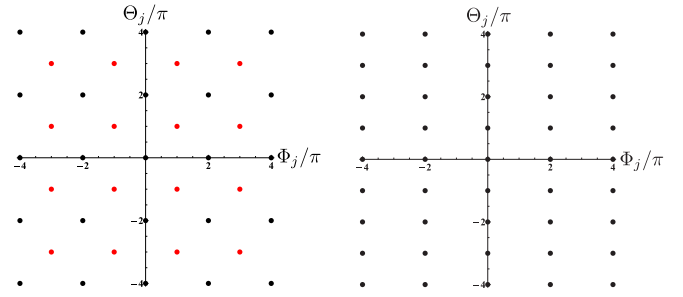

FIG. 5. Points $(\Phi_j/\pi, \Theta_j/\pi)$ identified by redundancy for fermionic (left) and bosonic (right) cases. In the fermionic case, the entire lattice splits into two sublattices corresponding to fermionic (red) and bosonic (black) sectors.

The "lattice points" (Φ_j, Θ_j) identified with $(0,0)$ up to this periodicity are shown in the left panel of Fig. 5. In the bosonic cases, on the other hand, the periodicity is defined as

$$\Phi_j \sim \Phi_j + 2\pi, \quad \Theta_j \sim \Theta_j + \pi \tag{A2}$$

and we have a different lattice (see the right panel of Fig. 5).

Let us begin with the fermion case. We take a pair of ground states $\{\bar{\vartheta}_{j^*}^{(1)}\}$ and $\{\bar{\vartheta}_{j^*}^{(2)}\}$ and denote the difference of the $\bar{\vartheta}_{j^*}^{(1)} - \bar{\vartheta}_{j^*}^{(2)}$, $\phi_j^{(1),L/R} - \phi_j^{(2),L/R}$, etc., in the two ground states by $\delta\vartheta_{j^*}$, $\delta\phi_j^{L/R}$, etc. As $\delta\vartheta_{j^*} \equiv 0 \ (\text{mod } \pi)$, it is convenient to specify the difference of the two ground states $\{\delta\vartheta_{j^*}\}$ by a set of integers $n_{j^*} \equiv \delta\vartheta_{j^*}/\pi$. Using the definitions

$$\vartheta_{j^*} = \frac{1}{2}(\Phi_j - \Phi_{j+1}) + \frac{m}{2}(\Theta_j + \Theta_{j+1}) \ (j = 1, \ldots, N_y - 1),$$

$$\vartheta_{N_y^*} = \frac{1}{2}(\Phi_{N_y} - \Phi_1) + \frac{m}{2}(\Theta_{N_y} + \Theta_1) = \vartheta_{0^*} \ (\text{periodic}),$$

$$\Phi_j = \phi_j^L + \phi_j^R, \quad \Theta_j = \phi_j^L - \phi_j^R,$$

we readily see that

$$\sum_{j^*=1}^{N_y} \delta\vartheta_{j^*}/\pi = \sum_{j^*=1}^{N_y} n_{j^*} = m \sum_{j=1}^{N_y} \left(\delta\phi_j^L - \delta\phi_j^R\right)/\pi. \tag{A3}$$

If the two ground states are equivalent, $\delta\phi_j^L/\pi, \delta\phi_j^R/\pi \equiv 0$ (mod 1), and hence $\sum_{j^*=1}^{N_y} n_{j^*} \equiv 0 \ (\text{mod } m)$.

Now let us prove the converse, i.e., that if a given pair of ground states satisfy $\sum_{j^*=1}^{N_y} n_{j^*} \equiv 0 \ (\text{mod } m)$, then they are equivalent. To this end, first we show that, by applying a series of transformations to one of the two ground states, we can reduce an arbitrary set $\{n_{1^*}, n_{2^*}, \ldots n_{N_y^*}\}$ to a simpler one: $\{0, 0, \ldots 0, M\}$ with $M = \sum_{i^*=1}^{N_y} n_{i^*}$. If we perform the transformation $\Phi_2 \to \Phi_2 + 2\pi n_{1^*}$ to the ground state "1," $\delta\vartheta_{1^*} \to 0$ in the transformed state, leaving us with the new configuration $\{0, n_{1^*} + n_{2^*}, n_{3^*}, \ldots n_{N_y^*}\}$. In the next step, we make the transformation $\Phi_3 \to \Phi_3 - 2\pi(n_{1^*} + n_{2^*})$ to obtain $\{0, 0, n_{1^*} + n_{2^*} + n_{3^*}, \ldots n_{N_y^*}\}$, and so on. After $N_y - 1$ steps, we have got the configuration $\{0, 0, \ldots 0, M\}$, where $M = \sum_{i^*=1}^{N_y} n_{i^*} \equiv mk + r$ with $k \in \mathbb{Z}$ and $r = 0, 1, \ldots m - 1$. Since $\Phi_j \to \Phi_j + 2\pi\mathbb{Z}$ is realizable in both fermionic and bosonic cases, the above procedure is applicable to both cases alike.

Now suppose that $M \equiv 0 \ (\text{mod } m)$, i.e., we have $\{0, 0, \ldots, mk\}$ after the steps described above. Then, the

question is whether we can find a transformation that eliminates mk in the last component or not. In fact, the transformation depends on the parity of m. Consider the following two transformations allowed for fermionic systems [see Eq. (A1)]: $(\phi_{N_y}^{\mathrm{L}}, \phi_{N_y}^{\mathrm{R}}) \to (\phi_{N_y}^{\mathrm{L}} - \pi, \phi_{N_y}^{\mathrm{R}})$ $[(\Phi_{N_y}, \Theta_{N_y}) \to (\Phi_{N_y} - \pi, \Theta_{N_y})]$ and $(\phi_{N_y}^{\mathrm{L}}, \phi_{N_y}^{\mathrm{R}}) \to (\phi_{N_y}^{\mathrm{L}}, \phi_{N_y}^{\mathrm{R}} - \pi)$ $[(\Phi_{N_y}, \Theta_{N_y}) \to (\Phi_{N_y} - \pi, \Theta_{N_y} + \pi)]$. They, respectively, change $\{n_1, n_2, \dots n_{N_y}\}$ by

$$\{0, 0, \dots 0, -(m-1)/2, -(m+1)/2\}$$
$$\text{and} \quad \{0, 0, \dots 0, (m+1)/2, (m-1)/2\}. \quad \text{(A4)}$$

When m is odd, the above transformations correctly shift the set $\{n_j\}$ by integers. It is clear that if we repeat the first and second transformation $(m+1)k/2$ and $(m-1)k/2$ times, respectively, we can reduce $\{0, 0, \dots, mk\} \to \{0, 0, \dots, 0\}$. Note that the above procedure is allowed only when m is odd, i.e., only for the fermionic Laughlin states. Thus, we have proved that a given pair of ground states are equivalent if and only if $\sum_{j^*=1}^{N_y} \delta\vartheta_{j^*}/\pi \equiv 0 \pmod{m}$.

In the bosonic case, the periodicity is defined by (A2). Since we now have

$$\sum_{j^*=1}^{N_y} \delta\vartheta_{j^*}/\pi = \sum_{j^*=1}^{N_y} n_{j^*} = m \sum_{j=1}^{N_y} \delta\Theta_j/\pi \quad \text{(A5)}$$

instead of (A3), any pair of equivalent ground states ($\delta\Theta_j \equiv 0 \bmod \pi$) must satisfy the same relation: $\sum_{j^*=1}^{N_y} n_{j^*} \equiv 0 \pmod{m}$. When this relation holds, we can again deform the initial $\{n_{1^*}, n_{2^*}, \dots n_{N_y^*}\}$ into $\{0, 0, \dots, mk\}$ by the series of transformations. Now, we consider the two transformations $(\Phi_{N_y}, \Theta_{N_y}) \to (\Phi_{N_y} - 2\pi, \Theta_{N_y})$ and $(\Phi_{N_y}, \Theta_{N_y}) \to (\Phi_{N_y}, \Theta_{N_y} - \pi)$, which shift $\{n_{1^*}, n_{2^*}, \dots n_{N_y^*}\}$ by $\{0, 0, \dots 0, 1, -1\}$ and $\{0, 0, \dots 0, -m/2, -m/2\}$, respectively. Then, we can eliminate mk by repeating the first and second transformation $mk/2$ and k times, respectively. Clearly, this construction requires $m = \text{even}$ ($m/2$ must be integer), which is expected also from the property of the bosonic Laughlin states. Therefore, when m is odd (even), the coupled fermionic (bosonic) wires with the interwire interaction (9) exhibit precisely m degenerate ground states. A similar but different approach to the ground-state counting based on the edge states has been presented in Ref. [62].

APPENDIX B: CWC FOR GENERAL ABELIAN QH STATES

In this Appendix we provide details of the CWC for a general K matrix discussed in Sec. VI, and the derivation of the associated Wen-Zee topological field theory.

1. Constraints on K

So far, we have not assumed any particular statistics of particles on the individual wires. However, in order for the above interactions to be written as products of *local* operators (say, R_j and L_j) on the wires, the elements of the K matrix are constrained. Any local operators of the wire (J, j) $(J = 1, \dots, N_{\mathrm{layer}}, j = 1, \dots, N_y)$ can be written as vertex operators of the form

$$V_j^{(J)}\big(M_j^{(J)}, M_j^{(J)}\big) = \exp\big\{iM_j^{(J)}\Phi_j^{(J)} + i2N_j^{(J)}\Theta_j^{(J)}\big\}, \quad \text{(B1)}$$

where the possible values of $(M_j^{(J)}, N_j^{(J)})$ are restricted as [63,64]

$$M_j^{(J)} \in \mathbb{Z}, \quad 2N_j^{(J)} \in \mathbb{Z}, \quad (-1)^{M_j^{(J)}+2N_j^{(J)}} = 1 \quad \text{for bosons,}$$
$$M_j^{(J)} \in \mathbb{Z}, \quad N_j^{(J)} \in \mathbb{Z} \quad \text{for bosons.} \quad \text{(B2)}$$

The most general expression of the (nearest-neighbor) interwire coupling is

$$\prod_{J=1}^{N_{\mathrm{layer}}} V_j^{(J)}\big(M_j^{(J)}(I), M_j^{(J)}(I)\big) V_{j+1}^{(J)}\big(M_{j+1}^{(J)}(I), M_{j+1}^{(J)}(I)\big)$$

$$= \exp\left\{ i \sum_{J=1}^{N_{\mathrm{layer}}} \big(M_j^{(J)}(I)\Phi_j^{(J)} + M_{j+1}^{(J)}(I)\Phi_j^{(J)}\big) \right.$$
$$\left. + i \sum_{J=1}^{N_{\mathrm{layer}}} \big(2N_j^{(J)}(I)\Theta_j^{(J)} + 2N_{j+1}^{(J)}(I)\Theta_j^{(J)}\big) \right\}. \quad \text{(B3)}$$

Comparing this with the Ith interwire interaction $\cos(2\vartheta_{j^*}^{(I)})$, we obtain

$$M_j^{(J)}(I) = -M_{j+1}^{(J)}(I) = \delta_{IJ},$$
$$2N_j^{(J)}(I) = 2N_j^{(J)}(I) = \mathcal{K}_{IJ} \quad \text{(B4)}$$

with the integers $\{M_j^{(J)}(I), N_j^{(J)}(I)\}$ satisfying the condition (B2). Clearly, for bosonic wires, all the elements $\mathcal{K}_{IJ}$ must be even. On the other hand, for systems consisting of fermion wires,

$$\mathcal{K}_{II} = \text{odd}, \quad \mathcal{K}_{IJ} = \text{even} \ (I \neq J). \quad \text{(B5)}$$

The above approach was was based on having many layers and nearest-neighbor interactions between the wires. Alternatively, as in Ref. [22], we can use a single layer at the expense of having interactions with longer range. An example of this is given in Appendix B 3 below.

One can ask if the CWC scheme allows for more general possibilities. A natural extension of the above construction is to allow for more than one interwire scattering. This can be thought of as forming FQH states of composites of electrons, and we provide some details in Appendix B 4.

2. Derivation of multilayer topological action

We now define a multicomponent statistical gauge field by

$$a_x^{(I)} = -\partial_x \Phi_j^{(I)},$$
$$a_y^{(I)} = -\frac{2}{l_y}\vartheta_{j^*}^{(I)} \quad \text{(B6)}$$

and using the commutation relations for the fields $\vec{\Theta}_j$ and $\vec{\Phi}_j$ we get

$$\big[a_x^{(I)}(x, j), a_y^{(J)}(x', k)\big] = -2\pi i K_{IJ}\left(\frac{1}{l_y}\delta_{jk}\right)\delta(x - x')$$
$$\to -2\pi i K_{IJ}\delta(\vec{x} - \vec{x}'). \quad \text{(B7)}$$

These commutation relations amounts to having an action with the kinetic term

$$S = \frac{1}{2\pi} \int d^3x \sum_{I,J} \mathcal{K}_{IJ}^{-1} \epsilon^{ij} a_i^{(I)}(x) \dot{a}_j^{(J)}(x). \tag{B8}$$

As in Sec. III B, the corresponding Hamiltonian is given by

$$\mathcal{H} = \sum_I \left[\beta_x^I (a_x^{(I)})^2 + \beta_y^I (a_y^{(I)})^2 \right], \tag{B9}$$

where the coefficients β_i^I are easily extracted from the Luttinger Hamiltonians for the individual wires, and the expansion of the interwire interactions (65). With (B8) and (B9) in hand, we can step by step follow the derivation in Sec. V to arrive at the Wen-Zee Lagrangian (61) with a charge vector $\mathbf{q} = (1, 1, \ldots 1)$ corresponding to the symmetric basis of the K matrix using the terminology of Wen [48]. Using the above charge vector $\mathbf{q}$, we can write the filling fraction ν compactly as $\nu = \mathbf{q}^{\mathrm{T}} \mathcal{K}^{-1} \mathbf{q}$ reproducing the well-known formula.

Integrating the statistical gauge fields a_i^I in the presence of the Hamiltonian (B9) will generate higher-derivative corrections to to the topological action (61) and the resulting hydrodynamical theory can be used to study collective modes, just as in the single-component case.

3. Single-layer description

Above we gave the CWC for generic Abelian topological states using the multilayer scheme, where we used $N_{\mathrm{layer}} = n$ sheets of coupled wires to realize topological states characterized by an n-dimensional K matrix. However, we can easily transform the multilayer scheme to a single-layer one proposed in, e.g., Ref. [22] by "crushing" the stack of layers. The idea is to first relabel the jth wire on the Ith layer (I, j) $(I = 1, \ldots, n, j = 1, \ldots, N_y)$ as the j'th one $[j' = n(j-1) + I]$ on a *single* layer. Now, the interaction between the wires (I, j) and $(J, j+1)$ is transformed to a long-range one [with the range $(n + J - I)$] between the wires $n(j-1) + I$ and $nj + J$. This way, we formally rewrite the original n-layer system in terms of a *single*-layer system including at most range-$(2n - 1)$ interactions. However, this is not the end of the story. In fact, there is a freedom of changing the boson fields while preserving the commutation relations.

Let us demonstrate how the procedure described above works in the Haldane-Halperin $\nu = \frac{2}{5}$ state [4,5] characterized by the following K matrix (in the symmetric basis):

$$\mathcal{K} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \tag{B10}$$

According to the procedure described in Sec. VI [see Eq. (62)], the modified chiral bosons are defined as

$$\tilde{\phi}_j^{\mathrm{L}(1)} = (\Phi_{2j-1} + 3\Theta_{2j-1} + 2\Theta_{2j})/2,$$

$$\tilde{\phi}_j^{\mathrm{R}(1)} = (\Phi_{2j-1} - 3\Theta_{2j-1} - 2\Theta_{2j})/2, \tag{B11a}$$

$$\tilde{\phi}_j^{\mathrm{L}(2)} = (\Phi_{2j} + 2\Theta_{2j-1} + 3\Theta_{2j})/2,$$

$$\tilde{\phi}_j^{\mathrm{R}(2)} = (\Phi_{2j} - 2\Theta_{2j-1} - 3\Theta_{2j})/2, \tag{B11b}$$

where we have made the replacement $(\Phi_j^{(1)}, \Theta_j^{(1)}, \Phi_j^{(2)}, \Theta_j^{(2)})$ $\rightarrow (\Phi_{2j-1}, \Theta_{2j-1}, \Phi_{2j}, \Theta_{2j})$ $(j = 1, \ldots, N_y)$ on the

right-hand side. In terms of the new set of variables, the original interwire interactions read as

$$\cos[2\vartheta_{j*}^{(1)}] = \cos[2(\tilde{\phi}_j^{\mathrm{L}(1)} - \tilde{\phi}_{j+1}^{\mathrm{R}(1)})]$$
$$= \cos\{(\Phi_{2j-1} - \Phi_{2j+1}) + 3\Theta_{2j-1} + 2\Theta_{2j}$$
$$+ 3\Theta_{2j+1} + 2\Theta_{2j+2}\},$$

$$\cos[2\vartheta_{j*}^{(2)}] = \cos[2(\tilde{\phi}_j^{\mathrm{L}(2)} - \tilde{\phi}_{j+1}^{\mathrm{R}(2)})]$$
$$= \cos\{(\Phi_{2j-1} - \Phi_{2j+1}) + 2\Theta_{2j-1} + 3\Theta_{2j}$$
$$+ 2\Theta_{2j+1} + 3\Theta_{2j+2}\}, \tag{B12}$$

which include backscattering processes on four wires $(2j - 1, 2j, 2j + 1, 2j + 2)$ as well as single-particle hoppings between second-neighbor wires. Therefore, if we squeeze the double-layer systems to a single-layer one, interactions involving four wires are introduced.

Now, we show that we can reduce the number of wires involved in the interwire interactions by the redefinition of the chiral bosons in Eqs. (B11a) and (B11b). In fact, we can readily check that all the commutation relations among $\{\tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(I)}\}$ are preserved even after we redefine the chiral bosons as

$$\tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(1)} \rightarrow \tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(1)} + \Theta_j^{(2)}, \quad \tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(2)} \rightarrow \tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(2)} - \Theta_j^{(1)}. \tag{B13}$$

Then, it is clear that all the arguments on the underlying topological properties in Appendix B 2 carry over and that the same topological phase is obtained for the new system as well. With the new strip variables defined by $\vartheta_{j*}^{(I)} = \tilde{\phi}_j^{\mathrm{L}/\mathrm{R}(I)} - \tilde{\phi}_{j+1}^{\mathrm{L}/\mathrm{R}(I)}$, the two interwire interactions now read as

$$\cos[2\vartheta_{j*}^{(1)}]$$
$$= \cos\{(\Phi_j^{(1)} - \Phi_{j+1}^{(1)}) + 3(\Theta_j^{(1)} + \Theta_{j+1}^{(1)}) + 4\Theta_j^{(2)}\},$$

$$\cos[2\vartheta_{j*}^{(2)}]$$
$$= \cos\{(\Phi_j^{(2)} - \Phi_{j+1}^{(2)}) + 4\Theta_{j+1}^{(1)} + 3(\Theta_j^{(2)} + \Theta_{j+1}^{(2)})\}. \tag{B14}$$

After relabeling the wires as before, we obtain the interactions proposed in Ref. [22] containing only three-wire couplings.

4. FQHE of composite particles

So far, we have been discussing the case only with single-particle (interwire) hopping where the coefficients of the Φ fields appearing in the interwire interactions are always ± 1. However, we may think of the situations where multiparticle hopping occurs or, more specifically, when we have the following interwire interactions:

$$\sum_{I=1}^{N_{\mathrm{layer}}} \sum_{j=1}^{N_y} g_I \cos\left\{n(\Phi_j^{(I)} - \Phi_{j+1}^{(I)}) + \sum_J \widetilde{\mathcal{K}}_{IJ}(\Theta_j^{(J)} + \Theta_{j+1}^{(J)})\right\}.$$

In the above, $\widetilde{\mathcal{K}}$ is an invertible, symmetric integer-valued matrix which is *not* necessarily the K matrix of the underlying Chern-Simons theory. As in the previous cases, we can formally introduce the following chiral bosons:

$$\vec{\tilde{\phi}}_j^{\mathrm{L}} \equiv \frac{1}{2}(n\vec{\Phi}_j + \widetilde{\mathcal{K}}\vec{\Theta}_j), \quad \vec{\tilde{\phi}}_j^{\mathrm{R}} \equiv \frac{1}{2}(n\vec{\Phi}_j - \widetilde{\mathcal{K}}\vec{\Theta}_j). \tag{B15}$$

Then, the on-strip fields defined by

$$
\vec{\varphi}_{j*} \equiv \vec{\phi}_j^{\,\mathrm{L}} + \vec{\phi}_{j+1}^{\,\mathrm{R}} = \frac{n}{2}(\vec{\Phi}_j + \vec{\Phi}_{j+1}) + \frac{1}{2}\widetilde{\mathcal{K}}(\vec{\Theta}_j - \vec{\Theta}_{j+1}),
$$

$$
\vec{\vartheta}_{j*} \equiv \vec{\phi}_j^{\,\mathrm{L}} - \vec{\phi}_{j+1}^{\,\mathrm{R}} = \frac{n}{2}(\vec{\Phi}_j - \vec{\Phi}_{j+1}) + \frac{1}{2}\widetilde{\mathcal{K}}(\vec{\Theta}_j + \vec{\Theta}_{j+1})
$$

$$(\mathrm{B}16)$$

enable us to rewrite the above interwire interactions as

$$
\sum_{I=1}^{N_{\mathrm{layer}}} \sum_{j=1}^{N_y} g_I \cos\left(2\vartheta_{j*}^{(I)}\right). \tag{B17}
$$

The hidden symmetry now reads as

$$
\begin{pmatrix} \vec{\phi}_j^{\,\mathrm{L}} \\ \vec{\phi}_{j+1}^{\,\mathrm{R}} \end{pmatrix} \rightarrow \begin{pmatrix} \vec{\phi}_j^{\,\mathrm{L}} \\ \vec{\phi}_{j+1}^{\,\mathrm{R}} \end{pmatrix} + \frac{n}{2}\begin{pmatrix} \widetilde{\mathcal{K}}\vec{\chi}_{j*} \\ \widetilde{\mathcal{K}}\vec{\chi}_{j*} \end{pmatrix}, \tag{B18}
$$

with the N_{layer}-dimensional vector $\vec{\chi}_{j*}$ parametrizing the residual U(1)$^{N_{\mathrm{layer}}}$ symmetry:

$$
a_x^{(I)}(x, j) \equiv \sum_J (\widetilde{\mathcal{K}}^{-1})_{IJ} \partial_x \Phi_j^{(J)}(x),
$$

$$
a_y^{(I)}(x, j) \equiv \frac{2}{n}\frac{1}{l_y}\Theta_j^{(I)}(x). \tag{B19}
$$

Plugging these expressions into the Berry-phase part of the Luttinger liquid action, we obtain

$$
-\int d^3x \frac{1}{4\pi} \sum_{I,J} (n\widetilde{\mathcal{K}}_{IJ})\epsilon^{ij} \dot{a}_i^{(I)}(x) a_j^{(J)}(x), \tag{B20}
$$

which immediately implies that the underlying topological field theory is the Wen-Zee action with the K matrix given by $n\widetilde{\mathcal{K}}$:

$$
\tilde{\rho}_{j*}^{(I)}(x) = \frac{1}{\pi} \sum_{J=1}^{N_{\mathrm{layer}}} (\widetilde{\mathcal{K}}^{-1})_{IJ} \partial_x \vartheta_{j*}^{(J)}(x)
$$

$$
\simeq \frac{n}{2\pi} l_y \left(\partial_x a_y^{(I)} - \partial_y a_x^{(I)}\right). \tag{B21}
$$

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