Rapid Communications

Cavity quantum electrodynamics of strongly correlated electron systems: A no-go theorem for photon condensation

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In spite of decades of work it has remained unclear whether or not superradiant quantum phases, referred to here as photon condensates, can occur in equilibrium. In this Rapid Communication, we first show that when a nonrelativistic quantum many-body system is coupled to a cavity field, gauge invariance forbids photon condensation. We then present a microscopic theory of the cavity quantum electrodynamics of an extended Falicov-Kimball model, showing that, in agreement with the general theorem, its insulating ferroelectric and exciton condensate phases are not altered by the cavity and do not support photon condensation.

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Introduction. Superradiance [1–5] refers to the coherent spontaneous radiation process that occurs in a dense gas when a radiation field mode mediates long-range intermolecule interactions. Superradiance was observed first more than 40 years ago in optically pumped gases [2,3] and has recently been identified in optically pumped electron systems in a semiconductor quantum well placed in a perpendicular magnetic field [6]. In 1973 Hepp and Lieb [7] and subsequently Wang and Hioe [8] pointed out that for sufficiently strong light-matter coupling the Dicke model, often used to describe superradiance in optical cavities, has a finite-temperature second-order *equilibrium* phase transition between normal and superradiant states. To the best of our knowledge, this phase transition has never been observed [9]. In the superradiant phase the ground state contains a macroscopically large number of coherent photons, i.e., $\langle \hat{a} \rangle \neq 0$, where $\hat{a}$ ($\hat{a}^\dagger$) destroys (creates) a cavity photon. To avoid confusion with the phenomenon discussed in the original work by Dicke [1], we refer to the equilibrium superradiant phase as a photon condensate.

Theoretical work on photon condensation has an interesting and tortured history. Early on it was shown that photon condensation is robust against the addition of counter-rotating terms [10,11] neglected in Refs. [7,8]. Soon after, however, Rzażewski *et al.* [12] pointed out that addition of a neglected term related to the Thomas-Reiche-Kuhn (TRK) sum rule [13,14] and proportional to $(\hat{a} + \hat{a}^\dagger)^2$ destroys the photon condensate. These quadratic terms are naturally generated by applying minimal coupling $\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - q\boldsymbol{A}/c$ to the electron kinetic energy $\hat{\boldsymbol{p}}^2/(2m)$. More recent research has focused on ground-state properties. The quantum chaotic and entanglement properties of the Dicke model photon condensate were studied in Refs. [15,16]. The authors of Ref. [17] criticized these studies however, pointing again to the importance of the quadratic term. The no-go theorem for photon condensation was revisited in Ref. [18], where it was claimed that it can be bypassed in a circuit quantum electrodynamics (QED) system with Cooper pair boxes capacitively coupled to a resonator. Soon after, however, Ref. [19] showed that the no-go theorem for cavity QED applies to circuit QED as well. The claims of Ref. [19] were then criticized in Ref. [20]. (See also subsequent discussions [21,22] on light-matter interactions in circuit QED.) Later, it was argued [23] that the linear band dispersion of graphene provides a route to bypass the no-go theorem, and that photon condensation could occur in graphene in the integer quantum Hall regime. This claim was later countered in Refs. [24,25], where it was shown that a dynamically generated quadratic term again forbids photon condensation.

Recent experimental progress has created opportunities to study light matter interactions in new regimes in which direct electron-electron interactions play a prominent role. For example [26], two-dimensional (2D) electron systems can be embedded in cavities or exposed to the radiation field of metamaterials, making it possible to study strong light-matter interactions in the quantum Hall regime [27–32]. Other emerging possibilities include cavity QED with quasi-2D electron systems that exhibit exciton condensation, superconductivity, magnetism, or Mott insulating states. This Rapid Communication is motivated by interest in strong light-matter interactions in these new regimes and by fundamental confusion on when, if ever, photon condensation is allowed.

We present a no-go theorem for photon condensation that is valid for generic nonrelativistic interacting electrons at $T = 0$. This result generalizes to interacting systems existing no-go theorems for photon condensation in two-level [12,18,33,34] and multilevel [19] Dicke models, which are based on the TRK sum rule. We then present a theory of cavity QED of an extended Falikov-Kimball model [35], which, in the absence of the cavity, has insulating ferroelectric and exciton condensate phases. We show through explicit microscopic calculations how the theorem is satisfied in this particular strongly correlated electron model.

Gauge invariance excludes photon condensation. We consider a system of N electrons of mass m_i described by a nonrelativistic many-body Hamiltonian of the form

$$\hat{\mathcal{H}} = \sum_{i=1}^{N}\left[\frac{\hat{\boldsymbol{p}}_i^2}{2m_i} + V(\hat{\boldsymbol{r}}_i)\right] + \frac{1}{2}\sum_{i\neq j} v(\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j). \tag{1}$$

Here, $V(\boldsymbol{r})$ is a generic function of position and $v(\boldsymbol{r})$ is a generic (nonretarded) two-body interaction, which need not even be spherically symmetric. In a solid $V(\boldsymbol{r})$ is the one-body crystal potential. Below we first exclude the possibility of a continuous transition to a condensed state, and then use this insight to exclude first-order transitions. For future reference, we denote by $|\psi_m\rangle$ and E_m the *exact* eigenstates and eigenvalues of $\hat{\mathcal{H}}$ [36,37], with $|\psi_0\rangle$ and E_0 denoting the ground state and ground-state energy, respectively.

We treat the cavity electromagnetic (e.m.) field in a quantum fashion, via a uniform quantum field $\hat{\boldsymbol{A}}$ corresponding to only one mode [5,7,8,11,12,14,15,18–25,33,38,39], i.e., $\hat{\boldsymbol{A}} = A_0\boldsymbol{u}(\hat{a} + \hat{a}^\dagger)$, where $\boldsymbol{u}$ is the polarization vector, $A_0 = \sqrt{2\pi\hbar c^2/(V\omega_c\epsilon_r)}$, V is the volume of the cavity, ϵ_r is its relative dielectric constant, and the photon Hamiltonian $\hat{\mathcal{H}}_{\rm ph} = \hbar\omega_c\hat{a}^\dagger\hat{a}$, where ω_c is the cavity frequency. The full Hamiltonian, including light-matter interactions in the Coulomb gauge [33,34,40,41], is

$$\hat{\mathcal{H}}_{\boldsymbol{A}_0} = \hat{\mathcal{H}} + \hbar\omega_c\hat{a}^\dagger\hat{a} + \sum_{i=1}^{N}\frac{e}{m_i c}\hat{\boldsymbol{p}}_i\cdot\boldsymbol{A}_0(\hat{a}+\hat{a}^\dagger) + \sum_{i=1}^{N}\frac{e^2A_0^2}{2m_ic^2}(\hat{a}+\hat{a}^\dagger)^2, \tag{2}$$

where $\boldsymbol{A}_0 \equiv A_0\boldsymbol{u}$ and $-e < 0$ is the electron charge. The third and fourth terms in Eq. (2) are often referred to respectively as the paramagnetic and diamagnetic contributions to the light-matter coupling Hamiltonian. Our aim is to make general statements about the ground state $|\Psi\rangle$ of $\hat{\mathcal{H}}_{\boldsymbol{A}_0}$. For future reference we define (i) the paramagnetic (number) current operator [36,37], $\hat{\boldsymbol{j}}_{\rm p} \equiv (c/e)\delta\hat{\mathcal{H}}_{\boldsymbol{A}_0}/\delta\boldsymbol{A}_0|_{\boldsymbol{A}_0=\boldsymbol{0}} = \sum_{i=1}^{N}\hat{\boldsymbol{p}}_i/m_i$, and (ii) $\Delta \equiv \sum_{i=1}^{N} e^2A_0^2/(2m_ic^2)$.

The term proportional to Δ in Eq. (2) can be removed by performing the transformation $\hat{b} = \cosh(x)\hat{a} + \sinh(x)\hat{a}^\dagger$, where $\cosh(x) = (\lambda+1)/(2\sqrt{\lambda})$ and $\sinh(x) = (\lambda-1)/(2\sqrt{\lambda})$ with $\lambda = \sqrt{1+4\Delta/(\hbar\omega_c)}$. The Hamiltonian (2) becomes: $\hat{\mathcal{H}}_{\boldsymbol{A}_0} = \hat{\mathcal{H}} + (e/c)\hat{\boldsymbol{j}}_{\rm p}\cdot\boldsymbol{A}_0\lambda^{-1/2}(\hat{b}+\hat{b}^\dagger) + \hbar\omega_c\lambda\hat{b}^\dagger\hat{b}$. It can be shown [see Sec. I of the Supplemental Material (SM) [42]] that in the thermodynamic limit ($N\to\infty$, $V\to\infty$ limit at fixed N/V), the ground state $|\Psi\rangle$ of $\hat{\mathcal{H}}_{\boldsymbol{A}_0}$ does not contain light-matter entanglement, i.e., we can take $|\Psi\rangle = |\psi\rangle\,|\Phi\rangle$, where $|\psi\rangle$ and $|\Phi\rangle$ are matter and light wave functions. Using this property we see that in the thermodynamic limit the ground state $|\Phi\rangle$ of the effective photon Hamiltonian $\langle\psi|\,\hat{\mathcal{H}}_{\boldsymbol{A}_0}\,|\psi\rangle$ is a coherent state [43,44] $|\beta\rangle$ satisfying $\hat{b}\,|\beta\rangle = \beta\,|\beta\rangle$. The ground-state energy is therefore given by

$$E_\psi(\beta) = \langle\psi|\hat{\mathcal{H}}|\psi\rangle + \frac{e}{c}\langle\psi|\hat{\boldsymbol{j}}_{\rm p}|\psi\rangle\cdot\boldsymbol{A}_0\frac{2\,{\rm Re}[\beta]}{\sqrt{\lambda}} + \hbar\omega_c\lambda|\beta|^2. \tag{3}$$

We need to minimize $E_\psi(\beta)$ with respect to β and $|\psi\rangle$. The minimization with respect to β can be done analytically. We find that the optimal value $\bar{\beta}$ for this minimum problem is a real number given by

$$\bar{\beta} = -\frac{1}{\hbar\omega_c\lambda^{3/2}}\frac{e}{c}\langle\psi|\hat{\boldsymbol{j}}_{\rm p}|\psi\rangle\cdot\boldsymbol{A}_0. \tag{4}$$

We are therefore left with a *constrained* minimum problem for the matter degrees of freedom. Its solution must be sought among the normalized antisymmetric states $|\psi\rangle$ which yield (4). This is the typical scenario that can be handled with the stiffness theorem [37].

For photon condensation to occur we need $E_\psi(\bar{\beta}) < E_{\psi_0}(0)$ or, equivalently,

$$\hbar\omega_c\lambda\bar{\beta}^2 > \langle\psi|\hat{\mathcal{H}}|\psi\rangle - \langle\psi_0|\hat{\mathcal{H}}|\psi_0\rangle\,, \tag{5}$$

where, because of (4), $|\psi\rangle$ depends on $\bar{\beta}$. The dependence of $\langle\psi|\hat{\mathcal{H}}|\psi\rangle - \langle\psi_0|\hat{\mathcal{H}}|\psi_0\rangle$ on $\bar{\beta}$ can be calculated *exactly* up to order $\bar{\beta}^2$ by using the stiffness theorem [37]. We find $\langle\psi|\hat{\mathcal{H}}|\psi\rangle - \langle\psi_0|\hat{\mathcal{H}}|\psi_0\rangle = \alpha\bar{\beta}^2/2 + O(\bar{\beta}^3)$, where $\alpha = -1/\chi(0) > 0$ and

$$\chi(0) \equiv -\frac{2}{\hbar^2\omega_c^2\lambda^3}\frac{e^2}{c^2}\sum_{n\neq 0}\frac{|\langle\psi_n|\hat{\boldsymbol{j}}_{\rm p}\cdot\boldsymbol{A}_0|\psi_0\rangle|^2}{E_n - E_0} < 0 \tag{6}$$

is proportional to the static paramagnetic current-current response function in the Lehmann representation [36,37]. We have used that $(e/c)\,\langle\psi_0|\hat{\boldsymbol{j}}_{\rm p}|\psi_0\rangle\cdot\boldsymbol{A}_0 = 0$, as proven in Sec. II of the SM [42]. It follows that photon condensation occurs if and only if

$$4\frac{e^2}{c^2}\sum_{n\neq 0}\frac{|\langle\psi_n|\hat{\boldsymbol{j}}_{\rm p}\cdot\boldsymbol{A}_0|\psi_0\rangle|^2}{E_n - E_0} > \hbar\omega_c + 4\Delta. \tag{7}$$

However, as shown in Sec. III of the SM [42],

$$\frac{e^2}{c^2}\sum_{n\neq 0}\frac{|\langle\psi_n|\hat{\boldsymbol{j}}_{\rm p}\cdot\boldsymbol{A}_0|\psi_0\rangle|^2}{E_n - E_0} = \Delta. \tag{8}$$

Equation (8) is the TRK sum rule [13] which expresses the fact that the paramagnetic and diamagnetic contributions to the physical current-current response function cancel in the uniform static limit [36,37], as discussed more fully in Sec. III of the SM [42], i.e., it expresses gauge invariance. Using Eq. (8) we can finally rewrite Eq. (7) as $c^2 4\Delta > c^2(\hbar\omega_c + 4\Delta)$ which cannot be satisfied. We conclude that photon condensation cannot occur and that, upon minimization with respect to $|\psi\rangle$, the ground state is $|\psi_0\rangle$ and $\bar{\beta} = 0$. From this analysis it is clear that first-order transitions to states with finite photon density are also excluded, because interactions with a coherent equilibrium photon field do not lower the matter energy [45]. Gauge invariance excludes photon condensation for any Hamiltonian of the form (2). This is the first important result of this Rapid Communication.

Cavity QED of an extended Falikov-Kimball model. We now illustrate how this general conclusion applies to a specific properly gauge invariant model of strongly correlated electrons in a cavity. We consider spinless electrons in a one-dimensional (1D) inversion-symmetric crystal with N sites, each with one atom with two atomic orbitals of opposite parity (s and p). When this lattice model is augmented by the addition of on-site repulsive electron-electron interactions, it is often referred to as an extended Falikov-Kimball (EFK) model [35]. The EFK model has been used to discuss exciton condensation [46] and electronic ferroelectricity [47,48]. The coupling of cavity photons to the matter degrees of freedom of a 1D EFK model can be described [49–52] by employing a Peierls substitution in the site representation with a uniform linearly polarized vector potential of amplitude A_0, as detailed in Sec. IV of the SM [42]. We obtain

$$\hat{\mathcal{H}}_{A_0} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\rm ee} + \hbar\omega_{\rm c}\hat{a}^\dagger\hat{a} + \frac{g_0}{\sqrt{N}}\frac{\hbar}{a}\hat{j}_{\rm p}(\hat{a}+\hat{a}^\dagger) - \frac{g_0^2}{2N}\hat{\mathcal{T}}(\hat{a}+\hat{a}^\dagger)^2, \tag{9}$$

where $\hat{\mathcal{H}}_0 = \sum_{k,\alpha,\beta}\hat{c}^\dagger_{k,\alpha}H_{\alpha\beta}(k)\hat{c}_{k,\beta}$ is the band Hamiltonian,

$$H_{\alpha\beta}(k) = \begin{pmatrix} E_{\rm s} - 2t_{\rm s}\cos(ka) & 2i\tilde{t}\sin(ka) \\ -2i\tilde{t}\sin(ka) & E_{\rm p} + 2t_{\rm p}\cos(ka)\end{pmatrix}, \tag{10}$$

and the Hubbard interaction term

$$\hat{\mathcal{H}}_{\rm ee} = U\sum_{j=1}^{N}\hat{c}^\dagger_{j,{\rm s}}\hat{c}_{j,{\rm s}}\hat{c}^\dagger_{j,{\rm p}}\hat{c}_{j,{\rm p}}. \tag{11}$$

In Eq. (9), $\hat{j}_{\rm p} = \sum_{k,\alpha,\beta}\hat{c}^\dagger_{k,\alpha}j_{\alpha\beta}(k)\hat{c}_{k,\beta}$ with $j_{\alpha\beta}(k) \equiv \hbar^{-1}\partial H_{\alpha\beta}(k)/\partial k$ is the paramagnetic number current operator, and $\hat{\mathcal{T}} = \sum_{k,\alpha,\beta}\hat{c}^\dagger_{k,\alpha}\mathcal{T}_{\alpha\beta}(k)\hat{c}_{k,\beta}$ with $\mathcal{T}_{\alpha\beta}(k) \equiv -a^{-2}\partial^2 H_{\alpha\beta}(k)/\partial k^2$ is the diamagnetic operator. In Eq. (10), $E_{\rm s}$ and $E_{\rm p}$ are on-site energies for the s and p orbitals, $t_{\rm s} \in \mathbb{R}$ and $t_{\rm p} \in \mathbb{R}$ are hopping parameters, and $\tilde{t} \in \mathbb{R}$ is the interband hopping parameter. At the single-particle level (i.e., for $U = 0$), $\tilde{t}$ is the only term responsible for interband transitions due to light. All sums over the wave number k are carried out in the 1D Brillouin zone and become integrals in the thermodynamic limit with the usual rule $N^{-1}\sum_k \to a\int_{-\pi/a}^{+\pi/a}dk/(2\pi)$, where a is the lattice constant. In these equations the Greek labels take values $\alpha, \beta = {\rm s}, {\rm p}$. The momentum-space and site representations for field operators are linked by the usual relationship $\hat{c}^\dagger_{j,\alpha} = N^{-1/2}\sum_k \hat{c}^\dagger_{k,\alpha}e^{-ikja}$. The dimensionless light-matter coupling constant in Eq. (9) is defined by $g \equiv eaA_0/(\hbar c) = g_0/\sqrt{N}$, where $g_0 \equiv \sqrt{2\pi e^2/(\hbar v_0\omega_{\rm c}\epsilon_{\rm r})}$ and $v_0 = V/N$ is the cavity volume per site.

We emphasize that the operators $\hat{j}_{\rm p}$ and $\hat{\mathcal{T}}$ describing light-matter interactions are completely determined by the matrix elements $H_{\alpha\beta}(k)$ of the band Hamiltonian. This property is crucial to have a properly gauge-invariant model [53] and must be a general feature of *any* strongly correlated lattice model coupled to cavity photons.

In the limit $g_0 \to 0$, the model reduces to a 1D EFK model [35,47,48]. In the limit $ka \to 0$ and $U = 0$, Eq. (9) reduces to the Dicke model, augmented by the addition of a term proportional to $\sum_{k,\alpha,\beta}\hat{c}^\dagger_{k,\alpha}\sigma^{(z)}_{\alpha\beta}\hat{c}_{k,\beta}(\hat{a}+\hat{a}^\dagger)^2$ [38,39,54], where $\sigma^{(z)}_{\alpha\beta}$ are the matrix elements of the corresponding 2×2 Pauli matrix. For noninteracting systems, the diamagnetic term prevents photon condensation from occurring in the thermodynamic limit [12,18]. We now show that interactions do not help. $\hat{\mathcal{H}}_{A_0}$ does not support photon condensation.

To make progress in analyzing the interacting problem we treat the Hubbard term using an unrestricted Hartree-Fock (HF) approximation [37,55]. As detailed in Sec. V of the SM [42] we arrive at

$$\hat{\mathcal{H}}^{\rm (HF)}_{\rm ee} = -U\frac{\mathcal{M}}{2}\sum_k(\hat{c}^\dagger_{k,{\rm p}}\hat{c}_{k,{\rm p}} - \hat{c}^\dagger_{k,{\rm s}}\hat{c}_{k,{\rm s}}) - U\sum_k(\mathcal{I}\hat{c}^\dagger_{k,{\rm s}}\hat{c}_{k,{\rm p}} + \mathcal{I}^*\hat{c}^\dagger_{k,{\rm p}}\hat{c}_{k,{\rm s}}) + U\frac{n_0}{2}\sum_{k,\alpha}\hat{n}_{k,\alpha} + UN\left(\frac{\mathcal{M}^2 - n_0^2}{4} + |\mathcal{I}|^2\right). \tag{12}$$

In Eq. (12) we have introduced the following self-consistent fields: (i) the electronic polarization

$$\mathcal{M} \equiv \frac{1}{N}\sum_k(\langle\hat{c}^\dagger_{k,{\rm p}}\hat{c}_{k,{\rm p}}\rangle - \langle\hat{c}^\dagger_{k,{\rm s}}\hat{c}_{k,{\rm s}}\rangle), \tag{13}$$

(ii) the complex excitonic order parameter

$$\mathcal{I} \equiv \frac{1}{N}\sum_k\langle\hat{c}^\dagger_{k,{\rm p}}\hat{c}_{k,{\rm s}}\rangle, \tag{14}$$

and (iii) the number of electrons per site $n_0 \equiv N^{-1}\sum_{k,\alpha}\langle\hat{n}_{k,\alpha}\rangle$, where $\hat{n}_{k,\alpha} \equiv \hat{c}^\dagger_{k,\alpha}\hat{c}_{k,\alpha}$. The term proportional to $n_0/2$ in Eq. (12) acts as a renormalization of the chemical potential in the grand-canonical Hamiltonian and can be discarded in this study since we study the phase diagram only at half filling and $n_0 = 1$ in all phases.

In order to reduce the number of free parameters in the problem, from now on we enforce particle-hole symmetry in the bare band Hamiltonian $\hat{\mathcal{H}}_0$ by setting $E_{\rm s} \equiv -E_{\rm p} = -E_{\rm g}/2$ and $t_{\rm s} \equiv t_{\rm p} = t$ (with $|t| > E_{\rm g}/4$, see Fig. S1). In order to find the ground state of the Hamiltonian (9) with Hubbard interactions treated as in Eq. (12), we follow the same steps outlined in the proof of the no-go theorem above. We seek a ground state of the unentangled form $|\Psi\rangle = |\psi\rangle\,|\Phi\rangle$. After removing the term proportional to $(\hat{a}+\hat{a}^\dagger)^2$, one finds that $|\Phi\rangle$ must be a coherent state $|\bar{\beta}\rangle$ with $\bar{\beta} = -g_0\mathcal{J}\sqrt{N}/(\lambda^{3/2}\hbar\omega_{\rm c})$. [We remind the reader that the photon condensate order parameter is $\langle\bar{\beta}|\hat{a}|\bar{\beta}\rangle/\sqrt{N} = \langle\bar{\beta}|\cosh(x)\hat{b} - \sinh(x)\hat{b}^\dagger|\bar{\beta}\rangle/\sqrt{N} = \bar{\beta}/\sqrt{N}\lambda$. See Sec.VI of the SM [42].] Here, $\mathcal{J} \equiv \hbar\,\langle\psi|\,\hat{j}_{\rm p}\,|\psi\rangle\,/(aN)$, λ has the same expression as in the proof of the no-go theorem with $\Delta = -g_0^2\mathcal{T}/2$, and $\mathcal{T} \equiv \langle\psi|\,\hat{\mathcal{T}}\,|\psi\rangle$. Note that both $\mathcal{J}$ and $\mathcal{T}$ have units of energy and are finite in the $N \to \infty$ limit.

The resulting effective Hamiltonian for the matter degrees of freedom, i.e., $\langle\bar{\beta}|\,\hat{\mathcal{H}}_{A_0}\,|\bar{\beta}\rangle$, can be diagonalized exactly since, after the HF decoupling, it is quadratic in the fermionic operators $\hat{c}_{k,\alpha}$, $\hat{c}^\dagger_{k,\alpha}$. To this end, it is sufficient to introduce the Bogoliubov operators $\hat{\gamma}^\dagger_{k,-} = u_k\hat{c}^\dagger_{k,{\rm s}} + v_k\hat{c}^\dagger_{k,{\rm p}}$ and $\hat{\gamma}^\dagger_{k,+} = v_k^*\hat{c}^\dagger_{k,{\rm s}} - u_k^*\hat{c}^\dagger_{k,{\rm p}}$, where the quantities u_k and v_k depend on the parameters of the bare Hamiltonian $\hat{\mathcal{H}}_0$, on the Hubbard parameter U, on the light-matter coupling constant g_0, and on the quantities $\mathcal{I}$, $\mathcal{M}$, $\mathcal{J}$, and $\mathcal{T}$. The ground state

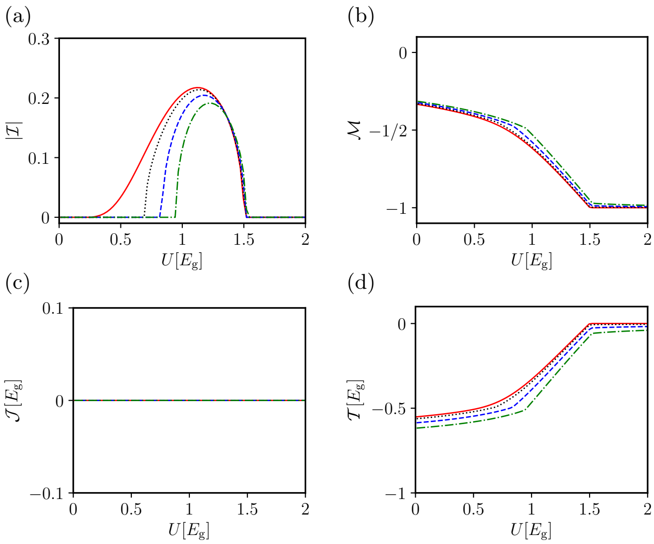

FIG. 1. (a) The excitonic order parameter $|\mathcal{I}|$ is plotted as a function of U (in units of E_g). Numerical results have been obtained by setting $t = 0.5E_g$ and $\hbar\omega_c = E_g$. Different curves correspond to different values of $\tilde{t}$. Red solid line: $\tilde{t} = 10^{-4}E_g$. Black dotted line: $\tilde{t} = 0.05E_g$. Blue dashed line: $\tilde{t} = 0.1E_g$. Green dashed-dotted line: $\tilde{t} = 0.15E_g$. Note that for $\tilde{t} \neq 0$, $|\mathcal{I}| \neq 0$ for $U_{c1} < U < U_{c2}$. (b) Same as in (a) but for the electronic polarization $\mathcal{M}$. (c) Same as in other panels but for $\mathcal{J}$. Note that $\mathcal{J} = 0$ for all values of $\tilde{t}$ and U/E_g. This implies $\bar{\beta} = 0$ and therefore no photon condensation. (d) Same as in other panels but for $\mathcal{T}$ (in units of E_g) .

$|\psi\rangle = \prod_k \hat{\gamma}^\dagger_{k,-}\,|\text{vac}\rangle$ can be written in a BCS-like fashion,

$$|\psi\rangle = \prod_k [u_k + v_k \hat{c}^\dagger_{k,\text{p}}\hat{c}_{k,\text{s}}]\,|\emptyset\rangle\,, \tag{15}$$

where $|\emptyset\rangle = \prod_k \hat{c}^\dagger_{k,\text{s}}\,|\text{vac}\rangle$ and $|\text{vac}\rangle$ is the state with no electrons. The final ingredients which are needed are the expressions for the quantities $\mathcal{M}$, $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{T}$ in terms of u_k, v_k: $\mathcal{M} = N^{-1}\sum_k(|v_k|^2 - |u_k|^2)$, $\mathcal{I} = N^{-1}\sum_k v^*_k u_k$, $\mathcal{J} = 2N^{-1}\sum_k[-t\sin(ka)(|v_k|^2 - |u_k|^2) - 2\tilde{t}\cos(ka)\text{Im}(u^*_k v_k)]$, and $\mathcal{T} = 2N^{-1}\sum_k[t\cos(ka)(|v_k|^2 - |u_k|^2) - 2\tilde{t}\sin(ka)\text{Im}(u^*_k v_k)]$. The technical details of this calculation are summarized in Sec. VI of the SM [42].

The quantities $\mathcal{I}$, $\mathcal{M}$, $\mathcal{J}$, and $\mathcal{T}$ can be determined by solving this nonlinear system of equations. A typical solution is shown in Fig. 1. We have found that all observables are independent of g_0. In other words, in the thermodynamic limit the ground state is given by Eq. (15) with u_k and v_k evaluated at $g_0 = 0$, in agreement with the general theorem proven above. The self-consistent solutions always have $\mathcal{J} = 0$ (i.e., $\bar{\beta} = 0$), as clearly seen in Fig. 1(c), and therefore display no photon condensation but may have a finite excitonic order parameter and finite polarization, as shown in Figs. 1(a) and 1(b), respectively. This is the second important result of this Rapid Communication. We have checked that the self-consistent solutions always have $\mathcal{J} = 0$ on a wide range of parameters (not shown). Also, it is easy to prove that the stability of the solutions is guaranteed by the condition $\mathcal{T} \leqslant 0$. At $\tilde{t} = 0$ the HF ground state has a single transition at $U = U_{\text{XC}}$. For $0 < U < U_{\text{XC}}$ the ground state is an exciton condensate with spontaneous coherence between the s and p bands [47,48] which are not hybridized when $U = 0$. The ordered state appears on the small U side of the transition because interactions favor orbital polarization over coherence. The value of U_{XC} can be determined analytically as detailed in Sec. VIII of the SM [42]. We find, in agreement with earlier work [56,57], that $U_{\text{XC}} = 8t^2/E_g - E_g/2$.

In the limit $\tilde{t} = 0$, $\hat{\mathcal{H}}_{A_0}$ separately conserves the number of electrons with band indices $\alpha = \text{s}, \text{p}$, and has a global $U(1)$ symmetry associated with the arbitrariness of the relative phase between s and p electrons [35]. The HF ground state breaks this symmetry. For $\tilde{t} \neq 0$ the $U(1)$ symmetry is reduced to a discrete Z_2 symmetry reflecting the invariance of the Hamiltonian under spatial inversion. This symmetry is broken for $U_{c1}(\tilde{t}) < U < U_{c2}(\tilde{t})$. Note that $\lim_{\tilde{t}\to 0} U_{c2}(\tilde{t}) = U_{\text{XC}}$. Corrections to $U_{c2}(0)$ can be found perturbatively for $\tilde{t}/t \ll 1$ and are of $O(\tilde{t}^2)$ (see Sec. VIII of the SM [42]). For $0 < U < U_{c1}(\tilde{t})$ inversion symmetry is unbroken and $\mathcal{I} = 0$. For $U > U_{c1}(\tilde{t})$ the ground state is an insulating ferroelectric that breaks the Z_2 symmetry (see Sec. IX of the SM [42]). The dependence of U_{c1} on $\tilde{t}$ is nonanalytical and can be extracted asymptotically for $\tilde{t}/t \ll 1$. We find that $U_{c1}(\tilde{t}) \to \pi(4t^2 - E_g^2/4)^{1/2}/|\ln(\tilde{t}/t)|$ (see Sec. VIII of the SM [42]).

In summary, we have presented a no-go theorem for photon condensation that applies to all quantum many-body Hamiltonians of the form (1), greatly extending previous no-go theorems for Dicke-type Hamiltonians [18,19]. Since the proof is nonperturbative in the strength of electron-electron interactions, our arguments against photon condensation apply to all lattice models of strongly correlated electron systems that can be derived from Eq. (1). We have then explained how the theorem manifests in practice, presenting a theory of cavity QED of a 1D model that supports insulating ferroeletric and exciton condensate phases. We have shown that these electronic orders are never entwined with photon condensation [58]. In the future, it will be interesting to study the role of spatially varying multimode cavity fields and their interplay with retarded interactions [59,60], or strong magnetic fields [61]. Our work emphasizes that theoretical models of interacting light-matter systems must retain precise gauge invariance, which is often lost when the matter system is projected onto a low-energy model.

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Coupling to the uniform vector potential of the cavity *must* be done via the paramagnetic current operator $\hat{j}_{\rm p} = \sum_{\ell=1}^{N} \hat{j}_{{\rm p},\ell}$ (while, at the same time, including the diamagnetic term). This is manifestly displayed by our Hamiltonian (9) at $g_0 \neq 0$. Further details on the f-sum rule can be found in Sec. VII of the SM [42].

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