Kramers-Kronig relations for magnetoinductive waves

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Kramers-Kronig relations for propagating modes are a fundamental property of many but not all structures supporting wave propagation. While Kramers-Kronig relations for a transfer function of a causal system are always satisfied, it was discovered in the past that Kramers-Kronig relations for individual modes of a system may fail, e.g., in leaky structures. Our aim in this paper is to scrutinize whether magnetoinductive waves propagating in discrete metamaterial structures by virtue of interelement coupling satisfy Kramers-Kronig relations. Starting with the dispersion relations of magnetoinductive waves in the nearest-neighbor approximation we investigated the real and imaginary parts of two complex functions: the propagation coefficient and the transfer function. In order to show that the real and imaginary parts of these functions are related to each other by the Kramers-Kronig relations we had to modify the functions by deducting from them the values of the functions at infinite frequencies. It was shown that these modified functions and their Kramers-Kronig pairs perfectly coincided. The much more complicated case of magnetoinductive wave propagation with long-range coupling assumed was also investigated. The mode structure was derived for interactions up to the third neighbor. It was shown that the Kramers-Kronig relations are not applicable to the individual modes, but are valid for the transfer functions of the waveguide after solving the excitation problem and taking all the modes into account. Our method can be applied to practically relevant cases enabling rapid evaluation of transfer functions, e.g., in metamaterials used for wireless power transfer.

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I. INTRODUCTION

It all started with Kramers and Kronig. They worked independently although they knew each other. Their respective papers were published close to a hundred years ago, in 1926 and 1927. The chronology is actually in reverse order of the names: Kronig [1] published his paper a year before Kramers [2]. Kronig was Dutch, Kramers was German.

Kronig investigated both the propagation and the attenuation of x rays due to electrons treated by a semiclassical theory [1], and derived a relationship between them. Kramers [2] used similar arguments but he could also justify the integral relationship between absorption and propagation by relying on Cauchy's theory of the functions of complex variables. The interesting thing is that neither author mentions causality (that the effect cannot precede the cause) which nowadays is the usual starting point in deriving the Kramers-Kronig (KK) relations. The causality condition was, in fact, treated later by Kronig [3]. He showed that the dispersion relation is both a necessary and sufficient condition for causality to be satisfied. For a more rigorous treatment of causality see Toll [4]. A somewhat different derivation is given in [5]. An application to ferrites is offered in [6].

Since the 1920s there has been a large amount of work concerned with one or another aspect of the KK relations. Those interested were both experimenters and theoreticians. If experimenters managed to measure either propagation or refraction as a function of frequency, they acquired the means of predicting the absorption characteristics as a function of frequency, and vice versa. [see, e. g., [7–9]]. The subject of acoustic waves was often revisited [10–12]. A topic of interest was also the actual evaluation of the two integral expressions. As far as we know there was only one case, that of the Debye relaxation equations [13], when the residue theorem could be applied and analytical solutions could be obtained. This means that numerical solutions abound and the question is what approach to use. A combination of the finite element method with the Runge-Kutta technique was used by Montagna *et al.* [14], the truncation of the integrals was discussed by Waters *et al.* [15]. A Fourier series method was given by Johnson [16]. The various numerical methods were compared by Ohta and Ishida [17]. A fast-Fourier transform technique was introduced by Lucas *et al.* [18].

To what physical phenomena could the KK relations apply? They must be causal but that is obvious. No engineer would ever consider a situation when the effect comes before the cause. A further condition that no wave can propagate faster than the velocity of light in vacuum is not very burdensome either. Krylov [12] states a more relevant restriction that KK is generally invalid when energy leaks from an acoustic waveguide. In contrast, it is claimed by Haakestaed *et al.* [5]

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that for weakly guiding dielectric waveguides, when the index contrast is small, the KK relations are applicable. We should also mention a further, not widely known, criterion that needs to be satisfied for the KK relations to be valid not only to real and imaginary parts of a transfer function but also to its phase and logarithm of its modulus: the equivalent network has to be a minimum phase network [19]. Such network satisfies the condition that it blocks signal propagation when it is cut at any point.

The range of topics to which the KK relations are applicable is quite wide. A list would contain nonlinear optics [14,20,21], hollow waveguides with infinitely conducting walls [5,12], slightly lossy coaxial waveguides [18], different kind of acoustic waveguides [12], transfer functions of any physically realizable linear circuits and systems, interferometers [22], dielectric waveguides, index guiding waveguides, Bragg reflection waveguides [5], relations between reflectance and phase [23], and extraction of metamaterial parameters [24]. It has been shown by Kirby *et al.* [25] that attenuation can be reversed even for the case of "slow" or "stopped light." Going further it was proved in Refs. [26,27] that the KK relations can be applied to negative refractive index materials. In particular, Wuestner *et al.* [27] showed their relevance to amplifying light in a left-handed material known as a fishnet structure.

In fact, whenever a new structure capable of supporting any kind of wave is discovered, then it is legitimate to ask the question whether the KK relations are applicable, and then supply the proof. That brings us to magnetoinductive waves, relatively recently discovered, for which we shall derive the relevant relations. Magnetoinductive waves supported by 1D, 2D, and 3D structures of magnetically coupled split ring resonators were first reported in 2002 [28,29]. They were born as part of the research on metamaterials. We shall introduce them in Sec. II. Similar waves on coupled resonant elements have also been reported: on optical cavities [30], on electrically coupled metamaterial elements [31], and on arrays of spherical nanoparticles [32]. We shall show in Secs. III and IV that the KK relations are valid in the nearest-neighbor approximation even for large ohmic losses. Then, in Sec. V we consider the KK relations for long-range coupling proving they are valid for a general excitation problem rather than for individual modes. Conclusions are drawn in Sec. VI.

II. MAGNETOINDUCTIVE WAVES IN THE NEAREST-NEIGHBOR APPROXIMATION

The physics of the operation of magnetoinductive waves is quite simple. They can propagate on a discrete set of current carrying resonant elements, like coils or split ring resonators. In the one-dimensional case there are two distinct configurations, axial and planar, as shown in Figs. 1(a) and 1(b). The current I_n flowing in element n is then coupled via its magnetic field to elements $n-1$ and $n+1$ when we rely on nearest-neighbor interaction. The equivalent circuit is shown in Fig. 1(c). We can attempt the solution in the form

$$I_n = I_0 e^{j(\omega t - kan)}, \quad n = -\infty, \dots \infty, \tag{1}$$

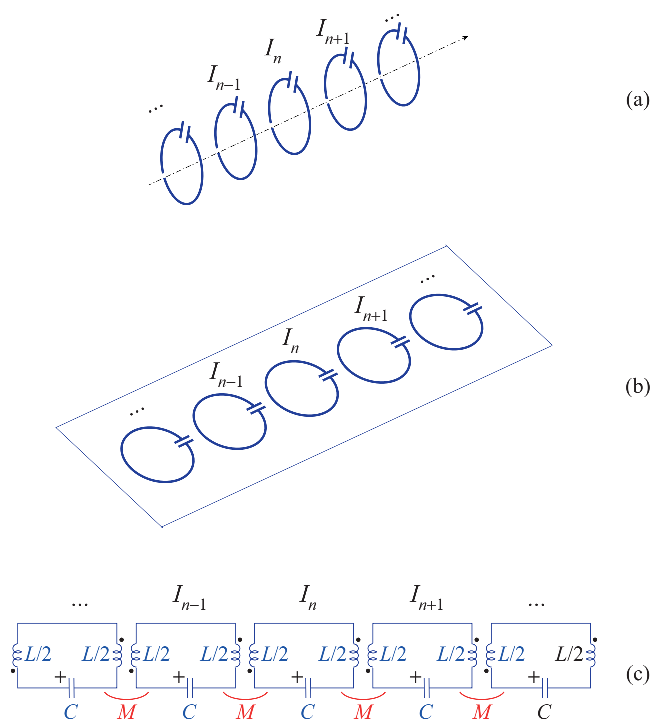

FIG. 1. (a) Axial, (b) planar configuration of magnetoinductive waveguides, and (c) equivalent circuit.

where ω is the frequency, t is time, k is the complex propagation coefficient

$$k = k_r + jk_i, \tag{2}$$

k_r and k_i are the wave number and attenuation constant respectively, a is the separation between the elements, and n are integers. Writing down the relevant Kirchhoff's equations for three neighboring elements

$$ZI_n + j\omega M(I_{n-1} + I_{n+1}) = 0, \tag{3}$$

where

$$Z = j\omega L\left(1 - \frac{\omega_0^2}{\omega^2} - j\frac{\omega_0}{\omega Q}\right) \tag{4}$$

is the self-impedance, L is the self-inductance, Q is the quality factor, ω_0 is the resonant frequency of the ring, and M is the mutual inductance between neighboring elements, we find the dispersion equation as

$$\zeta + \kappa\cos(ka) = 0, \tag{5}$$

where $\zeta = Z/j\omega L$ is the normalized self-impedance and $\kappa = 2M/L$ is the coupling constant between neighboring elements. The coupling constant κ is real but can be positive or negative. When we derived (5) we assumed that it was positive for axial, and negative for planar configurations [see Figs. 1(a) and 1(b)].

The solution of (5) for ω or k is straightforward and gives two waves traveling in opposite directions:

$$ak_{1,2} = \pm\cos^{-1}(y), \tag{6}$$

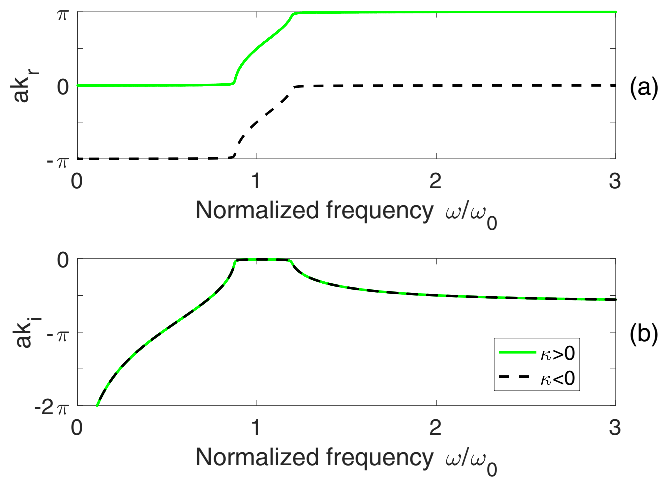

FIG. 2. Dispersion curves of magnetoinductive waves for positive (solid green curves) and negative (dashed black curves) coupling constants in the case of positive group velocities: (a) The real part and (b) the imaginary part of the complex propagation coefficient against normalized frequency.

with

$$y = -\frac{\zeta}{\kappa} = \frac{1}{\kappa}\left[s^{-2} + \frac{j}{Qs} - 1\right], \quad s = \frac{\omega}{\omega_0}. \tag{7}$$

We should be careful in selecting the correct sign for a particular wave because some of the waves, those which grow in the direction of propagation, are not physically realizable with passive structures. For the wave carrying a signal (power) in the direction of positive n and having $V_g > 0$ (V_g is the group velocity), we should select

$$ak = \begin{cases} ak_1 = \cos^{-1}(y), & \text{if} \quad \kappa > 0, \\ ak_2 = -\cos^{-1}(y), & \text{if} \quad \kappa < 0. \end{cases} \tag{8}$$

This ensures that $ak_i < 0$ and the magnetoinductive wave decays while propagating away from the source as dictated by causality. When $V_g < 0$ (meaning that power propagates in the negative n direction), the selection of the signs should be opposite, giving $ak_i > 0$ and attenuation of the power along the propagation direction toward negative n to comply again with causality. The positive sign of κ corresponds to a forward wave, and the negative sign to a backward wave. The former occurs in the axial configuration, and the latter in the planar configuration. The dispersion characteristics for these two cases are shown in Fig. 2 for $V_g > 0$, $\kappa = \pm 0.3$, $Q = 100$ and for positive frequencies. One can see that the signs of V_g and the phase velocity are opposite indeed if $\kappa < 0$. The attenuation of magnetoinductive waves is minimum within the passband. It would be exactly zero in the passband from $s = 1/\sqrt{1+|\kappa|}$ to $s = 1/\sqrt{1-|\kappa|}$ in the absence of ohmic losses. In contrast to microwave waveguides which have lossless propagation only above a certain cut-off frequency, lossless Magnetoinductive waves exist only in a passband. In the stop bands k_r quickly approaches either zero or $\pm\pi$.

III. KRAMERS-KRONIG RELATIONS FOR THE TRANSFER FUNCTION OF MAGNETOINDUCTIVE WAVES IN THE NEAREST-NEIGHBOR APPROXIMATION

The system under consideration is physically realizable and obeys the causality principle. There are all reasons to expect that the transfer function for the currents or voltages for a single element of the periodic structure $T(\omega) = T_r + jT_i$ satisfies the KK relations

$$T_r(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{T_i(\omega')}{\omega' - \omega} d\omega', \tag{9}$$

$$T_i(\omega) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{T_r(\omega')}{\omega' - \omega} d\omega', \tag{10}$$

where P stands for principal value. Note that there is another equivalent form of the KK relations where the limits of the integrals are between 0 and ∞. We found from experience that there is better convergence for the symmetric limits when the integrals are calculated numerically. Besides, this form of the KK relations is better suited for calculations using FFT. Note further that the signs in front of the principal values of the integrals in (9) and (10) differ from those often used in publications by physicists (see [33], for instance). The reason is that those authors assume that the Fourier transform relating $T(\omega)$ to the system impulse response has the plus sign in the exponent $(j\omega t)$ while many of the engineering textbooks use the minus sign [19,34]. Since we use the time factor $e^{j\omega t}$ in (1) we should use the definition of Fourier transform adopted in those textbooks. This implies that the KK relations can be derived from Cauchy's theorem by closing the integration contour not over the upper half of the complex ω plane as it is done in [33] but over the lower half-plane.

In order to check the applicability of (9) and (10) to magnetoinductive waves we need to find the transfer function T for a single period of the waveguiding structure. It is easy to do it bearing in mind (1), but again we should exercise some care in considering the direction of the signal propagation for different waves. For the waves with $V_g > 0$, the signal propagates in the positive direction of n, and

$$T = e^{-jka} = e^{-jk_r a} e^{k_i a}, \tag{11}$$

so the real and imaginary parts of the transfer function are

$$T_r = \cos(k_r a) e^{k_i a} \quad \text{and} \quad T_i = -\sin(k_r a) e^{k_i a}. \tag{12}$$

For the waves with $V_g < 0$, the input and the output for the single element of the periodic structure should be swapped, and the sign in the exponent in (11) should also be changed. This ensures $|T(\omega)| \leqslant 1$ in all cases. There is however a further problem. It follows from (1) and (11) that $T(\omega) \neq 0$ when the frequency tends to infinity. It can be found as

$$T(\infty) = \pm\frac{\sqrt{1-\kappa^2} - 1}{|\kappa|}, \tag{13}$$

where + should be taken for $\kappa > 0$ and − should be taken for $\kappa < 0$.

The fact that $T(\omega)$ does not vanish as $\omega \to \infty$ invalidates the proof of the KK relations because it can no longer be claimed that the integral in the complex plane over a semicircle with an infinitely large radius equals zero. The

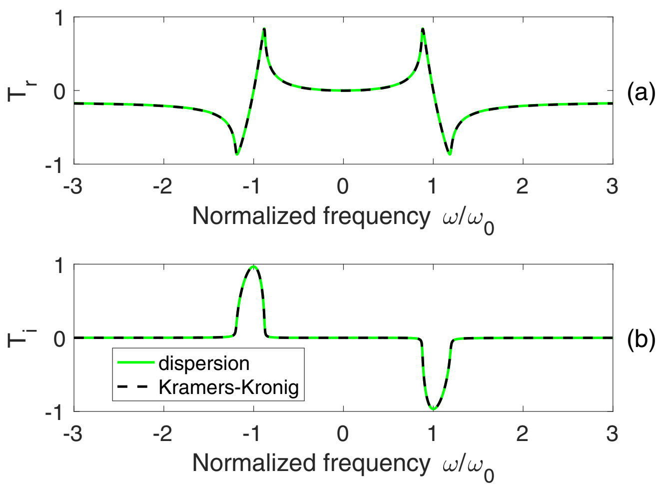

FIG. 3. (a) The real part and (b) the imaginary part of the transfer function against normalized frequency for $\kappa = 0.3$. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

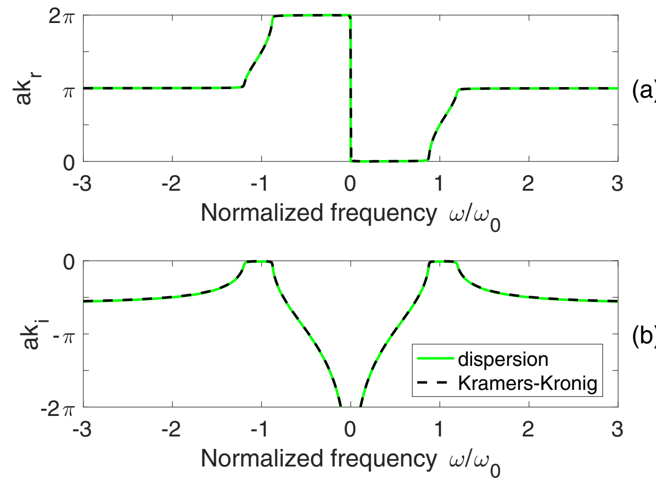

FIG. 5. (a) The real part and (b) the imaginary part of the complex propagation coefficient against normalized frequency for $\kappa = 0.3$. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

difficulty can be overcome if we apply (9) and (10) to the function $T(\omega) - T(\infty)$. We may now determine (9) and (10) by calculating numerically the Hilbert transform. We shall use the convolution theorem and the fast Fourier transform as proposed in [18] in the form

$$T_r(\omega) = \text{FFT}^{-1}\{j\,\text{sgn}(t)\,\text{FFT}[T_i(\omega)]\} + T(\infty), \quad (14)$$

$$T_i(\omega) = \text{FFT}^{-1}\{-j\,\text{sgn}(t)\,\text{FFT}[T_r(\omega) - T(\infty)]\}. \quad (15)$$

To be able to use (14) and (15), we need to extend the transfer function (11) and (12) to negative frequencies bearing in mind that $T_r(\omega)$ is an even function and $T_i(\omega)$ is an odd function to ensure a real impulse response. We use MATLAB® to calculate (14) and (15) with the number of frequency points $N = 2^{15}$ and the step $\Delta s = 0.0005$. In Figs. 3 and 4 the solid green curves are calculated from the dispersion relations (12), whereas the dashed black curves are calculated from the KK relations as given by (14) and (15) for $\kappa > 0$ and $\kappa < 0$, respectively. As may be seen, the solid green and the dashed black curves perfectly coincide proving the validity of the KK relations to magnetoinductive waves. It maybe worth noting that the integration limits $s = \pm 2^{14}\Delta s = \pm 8.192$ are sufficient in approximating the infinite limits.

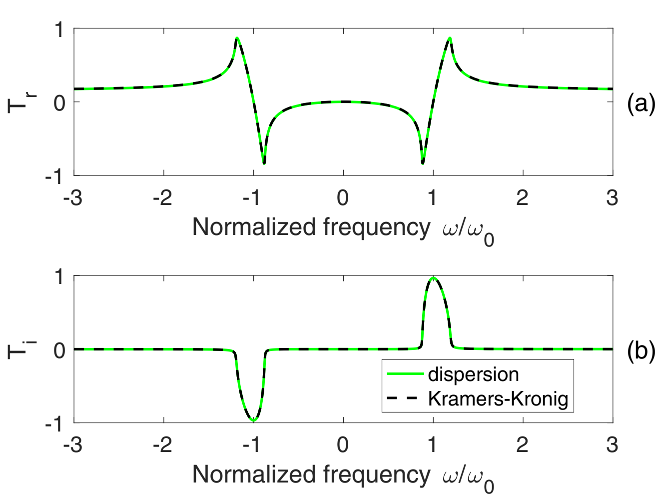

FIG. 4. (a) The real part and (b) the imaginary part of the transfer function against normalized frequency for $\kappa = -0.3$. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

IV. KRAMERS-KRONIG RELATIONS FOR THE MAGNETOINDUCTIVE PROPAGATION CONSTANT IN THE NEAREST-NEIGHBOR APPROXIMATION

The situation might be quite different for the KK relations for the propagation constant. We know from circuit theory [19,33] that KK relations work for the $\ln(|T(\omega)|) = ak_i$ and $\arg[T(\omega)] = -ak_r$ only in the case of minimum phase circuits. Can the 1D periodic structure consisting of coupled SRRs supporting the magnetoinductive waves be viewed as a minimum phase network? In the case of nearest-neighbor coupling, its equivalent circuit looks like a chain of transformers with capacitors connected to each other [Fig. 1(c)]. This means that a single break in any location of the structure would fully block the propagating signal. This property indicates that the structure is a minimum phase circuit indeed [19]. So we could expect the KK relations to be applicable to the propagation constant of magnetoinductive waves as well. Let us check it. A problem is how to take the values of ak_r and ak_i for negative frequencies. We know that for a KK pair one of them should be an even function and the other one an odd function. But which is which? As follows from (12), $ak_i(\omega)$ should be even and $ak_r(\omega)$ should be odd since T_r is even and T_i is odd. They are plotted as solid green curves against normalized frequency in Figs. 5 and 6 in the range $-3 < s < 3$ for $\kappa > 0$ and $\kappa < 0$, respectively, for the waves with $V_g > 0$ using (8).

Here we should make an important comment regarding the $\kappa > 0$ case. As one can see from the upper graph in Fig. 5, $ak_r(\omega)$ does not actually look like an odd function. This is because we have added 2π to it at $\omega < 0$. We can do it

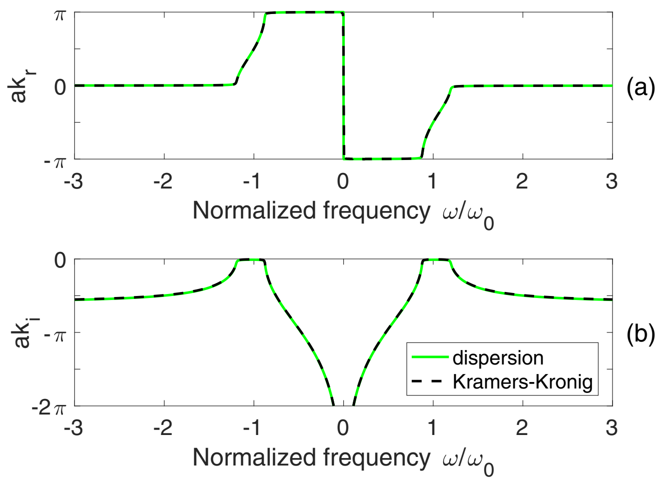

FIG. 6. (a) The real part and (b) the imaginary part of the complex propagation coefficient against normalized frequency for $\kappa = -0.3$. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

due to the discrete nature of the medium where the wave propagates. As a result, we obtain a dispersion characteristic fully equivalent to the original odd function but it has an important advantage: $k_r(-\infty) = k_r(\infty)$. This allows us to apply the same approach to the propagation constant as the one we applied to the transfer function to make the KK relations work: subtract $k(\infty)$ from $k(\omega)$. As a result, they can be written as follows:

$$k_r(\omega) = \mathrm{FFT}^{-1}\{j\,\mathrm{sgn}(t)\,\mathrm{FFT}[k_i(\omega) - k_i(\infty)]\} + k_r(\infty), \tag{16}$$

$$k_i(\omega) = \mathrm{FFT}^{-1}\{-j\,\mathrm{sgn}(t)\,\mathrm{FFT}[k_r(\omega) - k_r(\infty)]\} + k_i(\infty), \tag{17}$$

where

$$ak_r(\infty) = \pi \quad \text{if} \quad \kappa > 0 \quad \text{and} \quad ak_r(\infty) = 0 \quad \text{if} \quad \kappa < 0, \tag{18}$$

and $ak_i(\infty) = \ln[|T(\infty)|]$.

The solid green curves (Figs. 5 and 6) for the real and imaginary parts of the propagation coefficient are calculated from the dispersion relations (8) for $\kappa = 0.3$ and -0.3, respectively. The dashed black curves are determined from the KK relations (16) and (17). Again, the solid green and the dashed black curves may be seen to coincide.

All the above curves in Figs. 3–6 are calculated for $Q = 100$. Practically the same accuracy of matching between the original curves and those calculated using KK relations is achieved for a significantly lower value of $Q = 10$ as well. We shall not show the curves because they do not offer any new information but they are significant in the sense that the KK relations are still applicable for quite large ohmic losses.

V. KRAMERS-KRONIG RELATIONS: LONG-RANGE COUPLING, MODAL APPROACH

In the case of nearest-neighbor coupling, there is only one mode that is supported by the 1D chain of coils or split ring resonators. In this case, the transfer function for this mode (11) is also the transfer function for the actual physical signal propagating in the structure that obeys the causality principle. Most magnetoinductive lines can be analyzed by relying on nearest-neighbor interactions but not all of them. Sometimes higher order interactions do matter as discussed in Refs. [35,36]. Let us see what happens in this situation.

In the general case, one could take into account N_c interactions, i.e., interactions up to N_c neighbors in both directions. According to [29], the dispersion equation for the magnetoinductive waves in the long-range coupling case can be derived from Kirchhoff's equations for each of the $2N_c + 1$ elements between $m - N_c$ and $m + N_c$:

$$\zeta + \sum_{n=1}^{N_c} \kappa_n \cos(nka) = 0, \tag{19}$$

where κ_n is the coupling constant between the mth element and $(m \pm n)$th element. Let us take $N_c = 3$ as an example. To solve the dispersion equation for k as a function of ω we introduce a new unknown $y = \cos(ka)$ and rewrite (19) as a polynomial in terms of y:

$$4\kappa_3 y^3 + 2\kappa_2 y^2 + (\kappa_1 - 3\kappa_3)y - s^{-2} - \frac{j}{sQ} + 1 - \kappa_2 = 0. \tag{20}$$

Obviously the order of the polynomial is determined by the order of the long-range coupling N_c. The number of the roots y_m of the polynomial is bound to be equal to N_c, three in our case. If $Q \gg 1$ then the roots are either almost real or almost complex conjugate. Each of them gives rise to a solution for the propagation constant k_m similar to (6),

$$ak_m = \pm\cos^{-1}(y_m), \quad m = 1, \ldots, N_c. \tag{21}$$

Here we assume that the coupling constants are positive and take their values from Ref. [29]: $\kappa_1 = 0.3$, $\kappa_2 = 0.086$, $\kappa_3 = 0.035$. The dispersion curves for the three modes are shown in Fig. 7 for $Q = 100$. The solid red curve (k_1) corresponds to the fundamental mode, analogous to that of the

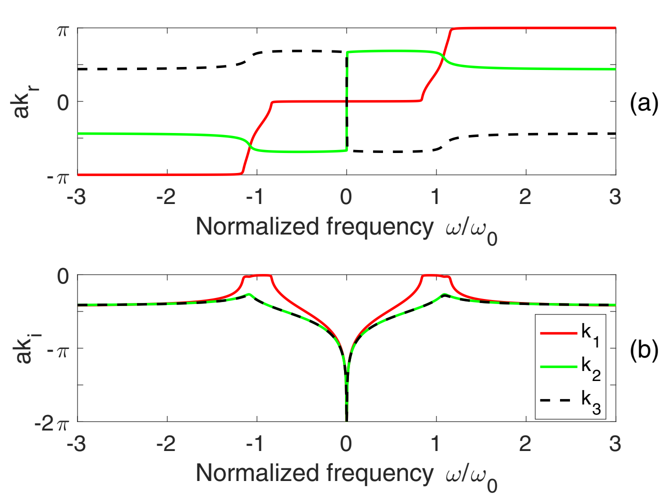

FIG. 7. (a) The real part and (b) the imaginary part of the propagation coefficients of the fundamental mode k_1 (solid red curves) and the higher-order modes k_2 (solid green curves) and k_3 (dashed black curves).

magnetoinductive wave in the nearest-neighbor approximation, the solid green curve (k_2) and the dashed black curve (k_3) correspond to the higher-order modes with significantly higher decay rate in the passband. It is important to note that, unlike in the nearest-neighbor coupling case, these modes cannot be excited separately in a causal way, hence the KK relations are not satisfied for the real and imaginary parts of k_m (this situation was discussed in [5]). We have run the numerical program implementing (16) and (17) and have indeed found that the KK relations fail in all three cases.

Clearly what we need is to find the full solution of an excitation problem taking all three modes jointly into account. As an example, we shall consider a voltage source of amplitude V connected in series to the 0th element of a semi-infinite magnetoinductive waveguide as an input signal source. The current in the mth element will then be assumed in the form

$$I_m = \begin{cases} I_{01}e^{-jk_1ma} + I_{02}e^{-jk_2ma} + I_{03}e^{-jk_3ma}, & m \geqslant 1, \\ I_0, & m = 0, \end{cases} \tag{22}$$

where I_{0i} $(i = 1, 2, 3)$ are the modal amplitudes. The propagation constants in (22) should be selected in such a way that $k_{1,2,3\ \mathrm{i}} = \mathrm{Im}(k_{1,2,3}) < 0$ (as shown in Fig. 7) in order to satisfy the condition that the mode amplitudes decline in the direction going away from the signal source. We have four unknowns hence we need four equations to find them. These will be Kirchhoff's equations applied to the first four elements as follows:

$$\zeta I_0 + \frac{\kappa_1}{2}I_1 + \frac{\kappa_2}{2}I_2 + \frac{\kappa_3}{2}I_3 = \frac{V}{j\omega L}, \quad \text{for } m = 0, \tag{23}$$

$$\zeta I_1 + \frac{\kappa_1}{2}(I_0 + I_2) + \frac{\kappa_2}{2}I_3 + \frac{\kappa_3}{2}I_4 = 0, \quad \text{for } m = 1, \tag{24}$$

$$\zeta I_2 + \frac{\kappa_1}{2}(I_1 + I_3) + \frac{\kappa_2}{2}(I_0 + I_4) + \frac{\kappa_3}{2}I_5 = 0, \quad \text{for } m = 2, \tag{25}$$

$$\zeta I_3 + \frac{\kappa_1}{2}(I_2 + I_4) + \frac{\kappa_2}{2}(I_1 + I_5) + \frac{\kappa_3}{2}(I_0 + I_6) = 0, \quad \text{for } m = 3. \tag{26}$$

Note however that apart from I_0 the unknowns we wish to find are I_{0i} $(i = 1, 2, 3)$. Thus, what we finally need are four linear equations with variables I_0 and I_{0i}. This is a rather long derivation; we shall therefore show the derivation for only one element in the 4×4 matrix in any detail. Let us choose Eq. (24) for further manipulations. We can substitute the values of I_m from (22) into (24). Note that each of the terms I_m will have a component I_{01}. Next, we shall find the coefficients of I_{01}. They may be obtained from (24) as

$$\zeta e^{-jk_1a} + \frac{\kappa_1}{2}e^{-2jk_1a} + \frac{\kappa_2}{2}e^{-3jk_1a} + \frac{\kappa_3}{2}e^{-4jk_1a} = e^{-jk_1a}\left(\zeta + \sum_{n=1}^{3}\frac{\kappa_n}{2}e^{-jnk_1a}\right). \tag{27}$$

Let us now rewrite Eq. (19) for $k = k_1$ that is one of the roots, in the form

$$\zeta + \sum_{n=1}^{3}\frac{\kappa_n}{2}e^{-jnk_1a} = -\sum_{n=1}^{3}\frac{\kappa_n}{2}e^{jnk_1a} \tag{28}$$

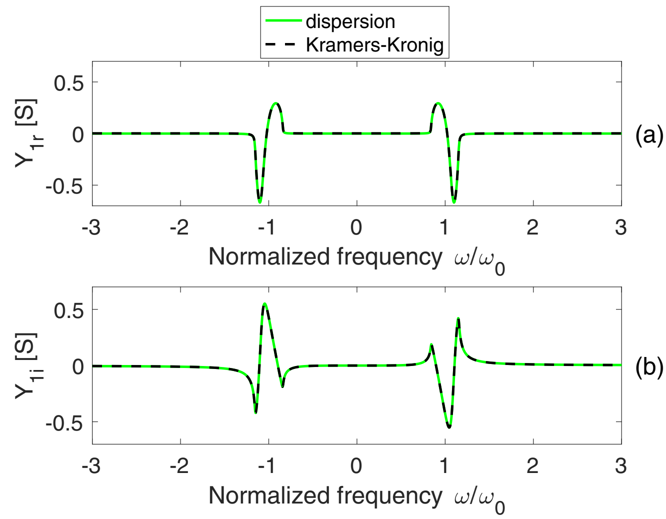

FIG. 8. (a) The real part and (b) the imaginary part of the transadmittance of the first element. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

that will give us the final form for the coefficient of I_{01},

$$X_{01} = -\sum_{n=1}^{3}\frac{\kappa_n}{2}e^{j(n-1)k_1a}. \tag{29}$$

Using analogous derivations for all the coefficients of I_0 and I_{0i} we end up with the linear equations as follows:

$$\begin{bmatrix} Z_n & X_{01} & X_{02} & X_{03} \\ \frac{\kappa_1}{2} & X_{11} & X_{12} & X_{13} \\ \frac{\kappa_2}{2} & X_{21} & X_{22} & X_{23} \\ \frac{\kappa_3}{2} & -\frac{\kappa_3}{2} & -\frac{\kappa_3}{2} & \frac{\kappa_3}{2} \end{bmatrix} \begin{bmatrix} I_0 \\ I_{01} \\ I_{02} \\ I_{03} \end{bmatrix} = \begin{bmatrix} \frac{V}{j\omega_0 Ls} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

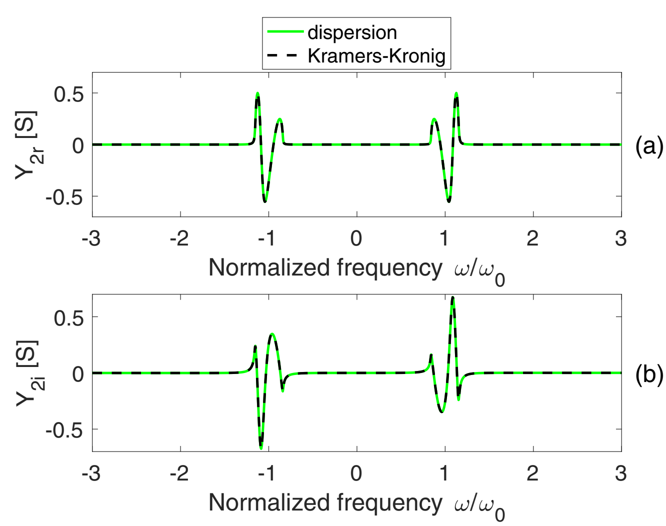

FIG. 9. (a) The real part and (b) the imaginary part of the transadmittance of the second element. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

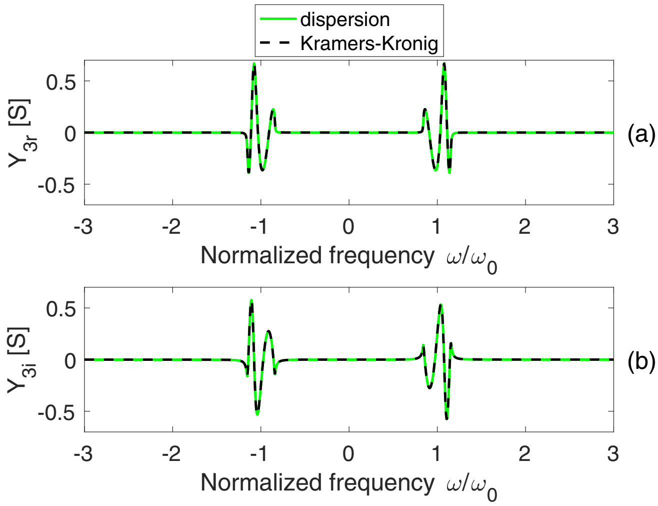

FIG. 10. (a) The real part and (b) the imaginary part of the transadmittance of the third element. Solid green curves: Dispersion. Dashed black curves: Kramers-Kronig pairs.

where

$$X_{0i} = \sum_{n=1}^{3} \frac{\kappa_n}{2} e^{-jk_i na}, \quad (31)$$

$$X_{1i} = -\sum_{n=1}^{3} \frac{\kappa_n}{2} e^{jk_i(n-1)a}, \quad (32)$$

$$X_{2i} = -\sum_{n=2}^{3} \frac{\kappa_n}{2} e^{jk_i(n-2)a}. \quad (33)$$

We are now in a position to apply the KK relations. We found the relations not to be satisfied for any of the modes on their own. The reason is that the modes are not causal in the sense that they cannot be independently excited by a physical signal source located in one of the waveguiding elements. On the other hand, we can define a transfer function, say the transadmittance for the mth element, in terms of the modal amplitudes as

$$Y_m = \frac{I_m}{V} = \frac{1}{V} \sum_{i=1}^{3} I_{0i} e^{-jmk_i a}. \quad (34)$$

Then its real and imaginary parts can be expected to satisfy the KK relations. This is indeed the case as shown in Figs. 8–10. The solid green curves show the actual functions. The dashed black curves are obtained by using the KK relations similar to (14)–(17). The agreement may be seen to be perfect.

We have been pleased to see that such complicated functions of frequency can be reproduced by the KK relations. They also work for any other transfer function of the magnetoinductive waveguide defined for the voltage V or the current I_0 as an input signal.

Finally, we wish to note that we have also calculated the KK pairs for an infinite line excited in the middle by all three modes taken together. The resulting curves look quite similar to those in Figs. 8–10 although there are some differences. Again, for all the transfer functions taking into account all three modes, we found the KK relations satisfied with the same accuracy as for the semi-infinite case.

VI. CONCLUSIONS

The applicability of the Kramers-Kronig relations to magnetoinductive waves has been investigated. Starting with the nearest-neighbor approximation, we looked at both the real and imaginary parts of the complex propagation coefficient, and also at the real and imaginary parts of the transfer function for a single period of the waveguiding structure. We were successful in proving numerically the KK relations but had to introduce some modifications, because neither function vanished as the frequency tended to infinity, a condition for the validity of the KK relations. The difficulties were overcome by deducting the functional value at infinity from the actual function. We could then prove that these modified functions calculated from the dispersion equation coincided with those determined with the aid of the KK relations.

We also looked at the applicability of the KK relations to magnetoinductive waves with long-range coupling. We found that the relations were not applicable to the individual modes supported by the 1D structure. However, they were shown to be valid for the transfer functions of the waveguide defined for the voltage or the current of the input signal source after solving the excitation problem and taking all the modes into account.

The results of this work would enable rapid evaluation, e.g., of the transfer function for signal and power transfer via magnetoinductive waves. It is the transfer function that plays the practical role in real-world applications of magnetoinductive waves including medical imaging [37] or wireless power transfer [38]. We plan to investigate the applicability of the Kramers-Kronig relations to realistic scenarios that may find applications, e.g., in designing on-the-go wireless charging for electric cars along the road. We believe our technique of proving the validity of the Kramers-Kronig relations for magnetoinductive waves may be further generalized to deal with other types of slow waves such as surface plasmons or waves of electric or magnetic coupling in nanostructured metamaterials like those described in Refs. [30–32].

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