Even-odd effect of a spin-S impurity coupled to a quantum critical system

Kun Chen and Yashar Komijani

Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08854, USA

(Received 15 January 2019; revised manuscript received 12 August 2019; published 9 September 2019)

We discuss an even-odd effect for an impurity with N-fold-degenerate internal states immersed in a two-dimensional superfluid–Mott-insulator quantum critical bath which is described by a spin-S XY Bose-Kondo impurity model with $N = 2S + 1$. Using a dimensional- and momentum-cutoff regularized renormalization group and unbiased large-scale Monte Carlo numerical simulations, we establish the phase diagram for the $S = 1$ impurity with all relevant terms included. We show that a threefold-degenerate $S = 1$ impurity coupled to a critical bath has a nondegenerate ground state with a quantized charge, in qualitative contrast to the spin-1/2 case where the impurity is partially screened. We then argue that all impurities with odd-2S degeneracy share the same universal physics as the spin-1/2 case and all impurities with even-2S degeneracy are the same as the spin-1 case. We validate our conjecture with unbiased Monte Carlo simulations up to $S = 2$. A physical consequence of this even-odd effect is that two spin-1/2 impurities in the critical bath form a long-range entangled state at a low temperature, which can be realized in ultracold atoms in an optical lattice.

DOI: 10.1103/PhysRevB.100.094514

I. INTRODUCTION

There has been a widespread increase of interest in impurities coupled to a bosonic bath in recent years. Impurities immersed in a weakly and strongly interacting bosonic bath are shown to exhibit many exotic emergent phenomena, e.g., the Bose polaron [1,2], the halon effect [3–6], and the trapping collapse effect. [7] Many of these impurity physics are directly accessible with modern techniques in ultracold-atom experiments [6,8,9]. Furthermore, the bosonic impurity also finds applications in a numerical method: A framework of dynamical mean-field theory was recently developed on top of a bosonic impurity model, providing a new powerful tool to attack strongly interacting bosonic systems [10–12].

It was recently demonstrated [4,6] that a sufficiently strong trapping potential in a Bose-Hubbard system, tuned to its bulk superfluid–Mott-insulator transition, leads to a boundary critical phenomenon with the XY universality class. The bulk of such a system is described by the $d = 2$ dimensional $O(2)$ ϕ^4 theory with the action

$$S_{\text{bath}} = \int d^{d+1}x\left[(\partial^\mu\vec{\phi})(\partial^\mu\vec{\phi}) + \frac{r}{2}\vec{\phi}^2 + \frac{g_0}{4!}(\vec{\phi}^2)^2\right], \quad (1)$$

where $\vec{\phi} = (\phi_x, \phi_y)$, $\mu = x_0, x_1, \ldots, x_d$, and the parameter r is tuned to the critical point, governed by the Wilson-Fisher (WF) finite-coupling fixed point [13,14]. The boundary critical point induced by the trapping potential is then described by the XY Bose-Kondo model [5,6,15]

$$H_{\text{BK}} = H_{\text{bath}}[\phi^-, \phi^+] + \gamma[\phi^-(0)S^+ + \text{H.c.}], \quad (2)$$

where $\phi^\pm \equiv \phi_x \pm i\phi_y$ and in the simplest case S^+ transforms under the fundamental spin-1/2 representation of the SU(2) group [16,17]. The bulk Hamiltonian is obtained from Eq. (1) and is given by

$$H_{\text{bath}} = \int d^dx\left\{\frac{1}{2}[\vec{\pi}^2 + (\partial^a\vec{\phi})(\partial^a\vec{\phi}) + r\vec{\phi}^2] + \frac{g_0}{4!}(\vec{\phi}^2)^2\right\}, \quad (3)$$

where $a = x_1, \ldots, x_d$ and ϕ and π obey the equal-time commutation relations

$$[\phi_\alpha(\vec{x}), \pi_\beta(\vec{x}')] = \delta_{\alpha\beta}\delta^d(\vec{x} - \vec{x}'). \quad (4)$$

The SU(2) version of this problem arises by doping an antiferromagnet at its quantum critical point [3,18–20].

In contrast to the Fermi-Kondo model [21], the Bose-Kondo interaction is linear in the Bose field, which makes γ a highly relevant perturbation. Moreover, the bulk interaction in Eq. (2) is an important feature of the model, and therefore, the interacting Bose-Kondo model cannot be reduced to an effective one-dimensional problem as in the Fermi-Kondo model. Therefore, the conformal field theory, Bethe ansatz, and numerical renormalization group approaches that have been very successfully applied to the Fermi-Kondo model are ineffective here.

The special case $S = 1/2$ of Eq. (2) is argued to feature a boundary quantum critical point (BQCP) with the $S = 1/2$ XY Bose-Kondo universality class [4–6]. The hallmark of this BQCP is the partial screening of the impurity, which leads to the so-called *halon effect*: When a small polarization field h_zS_z is used to lift the twofold degeneracy of the ground state, the half-integer projection of the pseudospin on its z axis gets delocalized into a halo of critically divergent radius $\sim 1/|h_z|^{\nu_z}$, where $\nu_z = 2.33(5)$ [4]. In other words, the integer charge carried by a trapping potential impurity gets fractionalized into two parts: a microscopic core with half-integer charge and a critically large halo carrying a complementary charge of $\pm 1/2$. This critical impurity state—referred to as a halon—describes the emergent physics of one potential scattering impurity in ultracold atoms in an optical lattice near the superfluid–Mott-insulator quantum critical point [6].

In experiments, one can create multiple potential scattering impurities in the system. Tuning the potential strength then leads to a multihalon, or, equivalently, multiple spin-1/2 XY Bose-Kondo impurities, coupled to the same critical bath. The nature of the ground state of such a system is still an open question. We find that this problem is closely related to the problem of the XY Bose-Kondo impurity problem with a generic $S > 1/2$. Indeed, in the long-wavelength limit, the problem of two spin-1/2 impurities is effectively described by a spin-1 Bose-Kondo model (Appendix A) with threefold degeneracy (up to a small perturbation). This motivates us to ask the following question: What is the ground state when a $(2S+1)$-degenerate spin impurity is coupled to a quantum critical bath as in Eq. (2)?

Whitsitt and Sachdev recently studied the boundary critical point for the spin-S XY Bose Kondo model in an $O(N)$ critical bosonic bath [5] under the assumption of the criticality. Here, we revisit the problem because even for the case of $S > 1/2$ and $N = 2$, the impurity models admit (more and more) relevant operators that need to be fine-tuned to maintain the $(2S+1)$ degeneracy of the ground state. Moreover, there also exist other stable quantum phases with partial lifting of $(2S+1)$ degeneracy, which is perhaps more relevant to the experiments.

Here, we argue that under U(1) and Z_2 symmetry (to be introduced shortly) the ground state of a spin-S impurity coupled to a quantum critical bath exhibits an even-odd effect: For all even-$2S$ impurities, the degeneracy is completely lifted, and the impurity ground state has a sharply defined charge quantum number, while for all odd-$2S$ impurities, the impurity is partially screened and has the same universal physics as the $S = 1/2$ case.

We first establish the phase diagram, as shown in Fig. 1, of the spin-1 Bose-Kondo model with all relevant terms included. The transition lines and the nontrivial critical exponents are calculated from an ϵ-expansion renormalization group (RG) approach and are cross-checked with unbiased large-scale quantum Monte Carlo (QMC) simulations. Our calculations show that the $N = 3$ degeneracy of the impurity as shown in Eq. (2) is effectively lifted by the impurity-bath interaction in the long-wavelength limit, in qualitative contrast to the previously established spin-1/2 impurity. We then use symmetry reasons to argue that all odd-$2S$ impurities share the same physics as the spin-1/2 Bose-Kondo impurity, whereas all even-$2S$ impurities are different and behave as the spin-1 impurity. This conjecture is validated with unbiased Monte Carlo simulations up to $S = 2$.

The even-odd effect implies that two spin-1/2 impurities in a critical bath form a long-range entangled state regardless of their distance d, a prediction that can be tested with ultracold atoms in an optical lattice. This scenario can be compared to the two-impurity spin-1/2 Fermi-Kondo problem [22–25]. The Ruderman-Kittel-Kasuya-Yosida interaction $J_H \vec{S}_1 \cdot \vec{S}_2$ between the impurities competes with the local Kondo screening $J_K \vec{S}_i \cdot \vec{s}_i$ by the conduction band spin density $\vec{s}_i$. When J_H is large and positive, the two spins form a singlet, and the conduction electrons are reflected with the phase shift $\delta_c = 0$. In the opposite regime of large negative J_H, the two

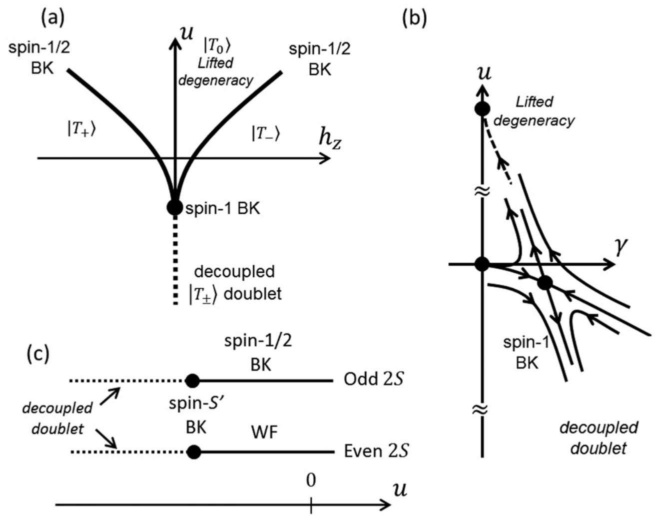

FIG. 1. (a) The phase diagram of the spin 1 with XY coupling to a critical bosonic bath is very similar to the decoupled impurity. $|T_0\rangle$ corresponds to a charge-quantized ground state, whereas spin-1 BK represents the partially screened case. The dashed line represents a level crossing of the decoupled doublet. (b) The perturbative RG flow in the presence of Z_2 symmetry $h_z = 0$ (solid line) and the speculative extrapolation (dashed line) show that fixed point requires fine-tuning u. (c) The phase diagram in the presence of Z_2 symmetry $h_z = 0$ for a general spin-S impurity.

impurities form a spin-1 impurity which is screened by two conduction electron bands with the phase shift $\delta_c = \pi$. Since the phase shift has to be either zero or π in the presence of particle-hole symmetry, there has to be a quantum phase transition at which the phase shift jumps. The (single) Bose type of coupling in Eq. (2) and the XY symmetry change this picture qualitatively.

II. SPIN-1 BOSE-KONDO MODEL

We consider the spin-1 Bose Kondo model

$$H_{\rm BK} = H_{\rm bath}[\phi, \phi^\dagger] + \gamma[\phi^-(0)S^+ + {\rm H.c.}] + u(S_z)^2. \quad (5)$$

This is the most general form of the interaction with Z_2 and U(1) invariance,

$$U(1): \quad \phi^\pm \to e^{\pm i\alpha}\phi^\pm, \quad S_\pm \to e^{\pm i\alpha}S^\pm, \quad S_z \to S_z,$$
$$Z_2: \quad \phi^\pm \to \pm i\phi_\mp, \quad S_\pm \to \pm iS_\mp, \quad S_z \to -S_z,$$

which are subgroups of the SU(2) group of the spin and are preserved under the RG. The problem without the u term was studied by Whitsitt and Sachdev [5] using the RG and ϵ expansion. In the presence of SU(2) symmetry, $\vec{S}^2$ is a conserved quantity, and therefore, no new active impurity term is generated under the RG. However, in a spin-1 system, the u term is allowed by both U(1) and Z_2 symmetries and is dynamically generated at low energies.

For the spin impurities coupled to an interacting bath, we characterize the screening based on the fluctuations of the total charge in the system $Q = \sum_i \phi_i^\dagger \phi_i + S_z$. At low energies, the residual charge fluctuations $\langle \delta Q^2 \rangle - \langle \delta Q^2 \rangle_{\rm bulk}$ determine if the impurity is fully or partially screened or it has decoupled from the bath.

The quadratic part of the bath action has the zero-temperature, equal-position, two-point function

$$D_d(\tau) \equiv \langle T_\tau \phi_\alpha(\tau)\phi_\alpha\rangle \sim \frac{1}{|\tau|^{d-1}} \tag{6}$$

for $\alpha = x, y$. At $d = 3 - \epsilon$, the bulk is weakly interacting. Additionally, the impurity-bath coupling in Eq. (5) is barely relevant. This allows us to treat the impurity problem in Eq. (5) perturbatively and then extrapolate the result to $\epsilon \to 1$.

First, we discuss the effect of the u term in lifting the internal degeneracy of the triplets in the spin-1 impurity. At large positive values $u \to \infty$, the impurity has a nondegenerate ground state $|T_0\rangle$, with $\langle S_z^2\rangle_{T\to 0} = 0$. The single-boson coupling is thus projected to zero, and the second-order perturbation theory leads to

$$H_{\rm imp} \to \frac{\gamma^2}{u}[\phi^+(0)\phi^-(0) + {\rm H.c.}], \tag{7}$$

which is a potential scattering with the strength γ^2/u. For $\gamma^2/u \ll 1$ this is irrelevant at $d = 3 - \epsilon$, including the marginal case of $d = 2$ [6]. In the opposite regime of $u \to -\infty$, $|T_+\rangle$ and $|T_-\rangle$ form a degenerate ground state with $\langle S_z^2\rangle_{T\to 0} = 1$, which acts as an effective spin 1/2, but their mixing with the bosons requires two-boson exchange:

$$H \to \frac{\gamma^2}{u}[(\phi^-)^2(S^+)^2 + {\rm H.c.}]. \tag{8}$$

This is again irrelevant for $d = 3 - \epsilon$ and $\epsilon \ll 1$ but becomes marginal at $d = 2$. Beyond tree level, for a noninteracting bath of bosons with the propagator $D_d(\tau) \sim 1/\tau^{d-1}$, we can study this term in $d = 2 - \epsilon$. Second-order perturbation theory contains $D^2_{2-\epsilon}(\tau) \sim 1/\tau^{2-2\epsilon}$ instead of $D_{3-\epsilon}(\tau) \sim 1/\tau^{2-\epsilon}$; the only modification to the β function of the spin-1/2 BK [5] is to replace $\epsilon \to 2\epsilon$. This gives a nontrivial fixed point that goes to zero as $\epsilon \to 0$. We conclude that for a noninteracting bosonic bath this term is irrelevant. In the presence of an interacting bath, we can only rely on our QMC simulations. We will see that for $u < u_*$, the system flows to a WF fixed point and a decoupled doublet. We have also confirmed that this interaction is irrelevant in an effective spin-1/2 model.

Since the u term is relevant, the boundary critical point requires fine tuning of at least one parameter, and this is expected to affect the exponents

$$\langle T_\tau S_z(\tau)S_z\rangle \sim \frac{1}{|\tau|^{\eta_z}}, \qquad \sum_\alpha \langle T_\tau S_\alpha(\tau)S_\alpha\rangle \sim \frac{1}{|\tau|^{\eta_\perp}} \tag{9}$$

at the critical point. This is plausible considering that the $O(\epsilon^2)$ exponents computed in Ref. [5], although very accurate for $S = 1/2$, become unreliable for $S = 1$; for the $O(2)$ model $\eta_\perp = \epsilon - 5.2838\epsilon^2 + \ldots$ and $\eta_z = 2\epsilon - 8.5676\epsilon^2 \ldots$, both become negative for $\epsilon > 0.234$ and cannot access the $\epsilon \to 1$ limit. Interestingly, this observation also holds for the SU(2) case, and the $O(\epsilon^2)$ results [19] become unreliable for $S > 1/2$. This indicates the breakdown of ϵ expansion for accessing the $\epsilon \to 1$ limit and is probably linked to the fact that the classical limit $S \to \infty$ is expected to behave differently [26].

To order $O(\epsilon)$ in the presence of $u \neq 0$, these exponents remain unchanged: $\eta_\perp = \epsilon$ and $\eta_z = 2\epsilon$. Moreover, we can compute the exponent corresponding to the relevant operator

$$\left\langle T_\tau S_z^2(\tau)S_z^2\right\rangle \sim \frac{1}{|\tau|^{\eta_u}}, \tag{10}$$

which we compare with the Monte Carlo simulations.

III. RG ANALYSIS

The uS_z^2 term leads to IR divergences in perturbation theory which are cut by the temperature, i.e., expanding in terms of $u\beta$ instead of u. For sufficiently small u, β can be very large, and we use the zero-temperature form (6) of the boson propagator.

Additionally, the dynamic generation of mass means that perturbation theory is plagued with UV divergences that are not cured by the dimensional regularization. Hence, in addition to dimensional regularization, we introduce a momentum cutoff μ_0 (Appendix B). Eventually, β^{-1} is replaced with the energy μ of interest, and μ_0 is sent to infinity after renormalization.

Renormalization is achieved by introducing scale-dependent operators and coupling constants [27]. In addition to $\phi \to \sqrt{Z}\tilde{\phi}$ in the bulk, we have [5]

$$S_z \to \sqrt{Z_z}\tilde{S}_z, \quad S_{x,y} \to \sqrt{Z_\perp}\tilde{S}_{x,y}, \quad S_z^2 \to Z_u\tilde{S}_z^2. \tag{11}$$

Note that S_z^2 is a distinct operator and gets its own renormalization factor. We find that the correlation functions of the renormalized operators are independent of the UV scale μ_0 and analytical in ϵ, provided that they are expressed in terms of renormalized couplings constants and we choose Z parameters to absorb the poles in ϵ. We find (Appendix C)

$$\frac{d\gamma}{d\ell} = \frac{\gamma}{2}[\epsilon - \gamma^2(1+u)], \qquad \frac{du}{d\ell} = u + (1+3u)\gamma^2, \tag{12}$$

where $d\ell \equiv -d\ln\mu$. Figure 1(b) shows the RG flow in the vicinity of the fixed point. To $O(\epsilon)$ the factors $1 + u$ in the first line and $1 + 3u$ in the second line can be neglected. These equations have a nontrivial fixed point at $(\gamma_*^2, u_*) = (\epsilon, -\epsilon)$. Since $\epsilon \to 1$, $\gamma_*^2 \to 1$, and $u_* \to -1$. At the BQCP the critical exponents of spin susceptibility $\chi_\alpha = \int d\tau \langle T_\tau S_\alpha(\tau)S_\alpha\rangle \propto T^{3-2/\nu_\alpha}$ remain unchanged. But we can compute one more exponent corresponding to $\chi_u = \int d\tau \langle T_\tau S_z^2(\tau)S_z^2\rangle$. To $O(\epsilon)$, this is given by

$$\eta_u = -\frac{d\ln Z_u^2}{d\ln\mu} = -2\frac{d\ln Z_u}{d\gamma}\beta_\gamma = 6\gamma^2 = 6\epsilon, \tag{13}$$

from which we obtain $\nu_u \equiv (1 - \eta_u/2)^{-1} = 1 + 3\epsilon \to 4$.

IV. NUMERICAL ANALYSIS

We now study this impurity problem with an effective cubic lattice model in Euclidean space-time with equal sizes of temporal and spatial directions, which allows an efficiently unbiased Monte Carlo simulation using the worm algorithm. The details and the definition of this lattice model can be found in Appendix E. Here we show the main results obtained with large-scale simulations and finite-size scaling analysis.

Figure 2 shows the charge fluctuations $\langle\delta Q^2\rangle$ as a function of the relevant term u for various system sizes. For each curve

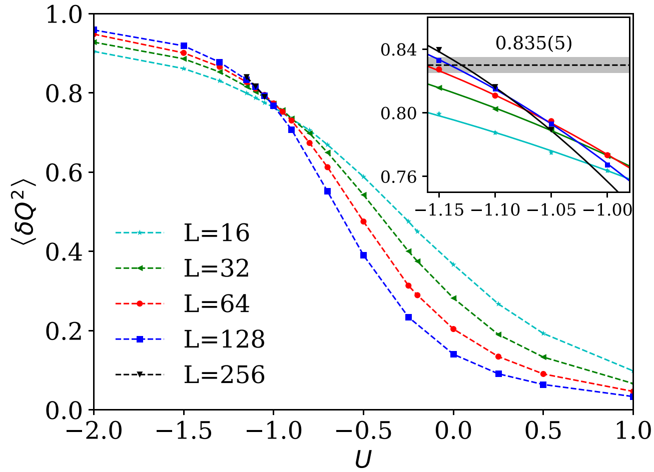

FIG. 2. The change in the total charge number fluctuation $\langle \delta Q^2 \rangle = \langle Q^2 \rangle - \langle Q \rangle^2$ (subtracting the universal bulk contribution $\langle \delta Q^2 \rangle_0 = 0.5160(6)$ [28]) as a function of the impurity interaction strength U. The inset shows that the curves near the critical strength $U_c = 1.12(2)$ can be fitted with the same scaling ansatz $q_0 + q_1(U - U_c)L^{1/\nu_u} + q_2(U - U_c)^2 L^{2/\nu_u} + c/L^{\omega}$. We extract the universal constant $q_0 = 0.835(5)$ and the boundary critical exponent $\nu_u = 3.6(3)$.

with a given system size, a bulk WF value of $\langle \delta Q^2 \rangle_0 \sim 0.516$ is subtracted from the data. A BQCP featuring the partially screened fixed point is found at $u_* = -1.12(2)$. For $u > u_*$ (including $u = 0$) the system flows to the WF fixed point, as if the system has fully screened the impurity. However, for $u < u_*$ it flows to a WF and a decouple doublet with $\langle \delta Q^2 \rangle - \langle \delta Q^2 \rangle_0 \to 1$. At $u = u_*$, a universal charge fluctuation of $\langle \delta Q^2 \rangle - \langle \delta Q^2 \rangle_0 = 0.835(2)$ and a critical exponent $\nu_u = 3.6(3)$ are extracted, in good agreement with $\nu_u \sim 4$ from RG analysis.

Figure 3 shows the $\chi_z(\tau) = \langle S_z(\tau) S_z \rangle$ correlation function as a function of τ for various system sizes. $\chi_z(\tau)$ becomes a power law with the exponent $\nu_z = 1.10(2)$, corresponding to $\eta_z = 0.18$, in marked contrast to the leading value of $\eta_z \sim 2\epsilon$ from RG analysis.

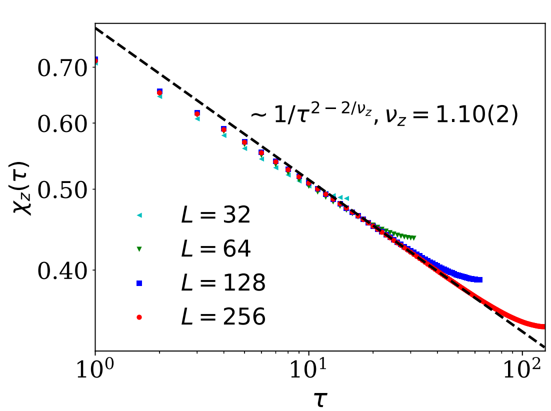

FIG. 3. The longitudinal spin-spin correlation function in the imaginary time $\chi_z(\tau) = \langle S^z(\tau) S^z(0) \rangle$ at the BQCP for different system sizes L. In the thermodynamic limit $L \to \infty$, there develops a power-law tail (the dashed line) expected from the general analysis (see the text), yielding $\nu_z = 1.10(2)$.

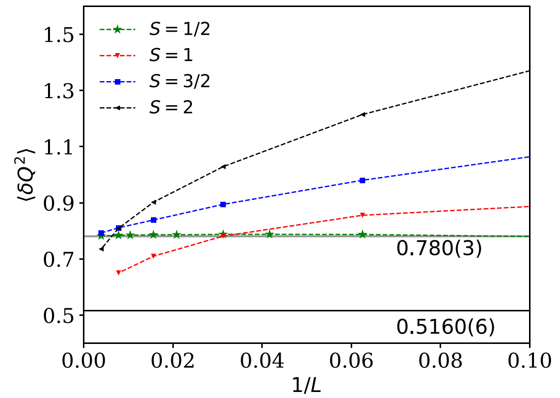

FIG. 4. The finite-size flows of the total charge number fluctuation $\langle \delta Q^2 \rangle = \langle Q^2 \rangle - \langle Q \rangle^2$ for an impurity with spin $S = 1/2, 1, 3/2,$ and 2. In the thermodynamic limit $1/L \to 0$, an impurity with even-$2S$ spin and odd-$2S$ spin demonstrates different trends. An even-$2S$ impurity flows to the bulk universal constant 0.5160(6) [28], which characterizes the Wilson-Fisher universality class as if it is fully screened. On the other hand, An odd-$2S$ impurity is only partially screened, and it flows to a different universal constant 0.780(3), which characterizes the spin-1/2 XY Bose Kondo boundary universality class.

V. EVEN-ODD EFFECT

We showed that in the spin-1 case, one needs to fine-tune the coefficient of S_z^2 in order to flow to the critical point. In impurities with higher spin representations, more independent operators appear that lift various degeneracies. However, we can argue that even $2S$ and odd $2S$ behave qualitatively differently. In the presence of the Z_2 symmetry and no fine tuning, the ground state is either unique or twofold degenerate.

In the even-$2S$ case, the ground state is generically nondegenerate or degenerate but with large spin differences, and therefore, the impurity decouples, and bulk remains at the WF fixed point. However, odd $2S$ in the presence of Z_2 symmetry is guaranteed to have an (at least doubly) degenerate ground state. Moreover, the sign of the dynamically generated u term can be shown to be positive. Integrating out the bosons to lowest order results in

$$\begin{aligned} H_{\text{mass}} &= -\frac{\gamma^2}{2}(S^+S^- + S^-S^+) \int d\Delta\tau D_d(\Delta\tau) \\ &= \gamma^2 f(0) S_z^2 + \text{const.}, \end{aligned} \tag{14}$$

where $f(0)$ is a positive constant (Appendix B) and we have used SU(2) algebra of the spins. Therefore, either the ground state is nondegenerate (as in $S = 1$), or the degenerate doublets are more than $\Delta m_z > 1$ spin apart and are incapable of coupling to the bath.

The coefficients of higher-order terms of the form $u_n(S_z)^n$, with $n > 2$, depend on the higher-order *connected* diagrams of bosonic fields and are absent in the Gaussian theory. Therefore, the ground state is dominated by positive $u_2 S_z^2$. Our numerical results indicate that this argument is valid even for the interacting theory all the way to $S \leqslant 2$, although a rigorous proof is currently unavailable.

Figure 4 shows QMC results on charge fluctuations for different spin-S impurities as a function of system size. Only

XY coupling is included in this calculation, and no fine tuning is included. For $S = 1$ and $S = 2$, the charge fluctuations approach the WF fixed point at low-energies, suggesting a full screening. However, $S = 1/2$ and $S = 3/2$ have the same IR fixed point, higher than the WF value, indicating a partially screened impurity. This is in agreement with the above argument and the phase diagram suggested before.

VI. DISCUSSION

In the multi-impurity problem, the even-odd effect indicates that an even number of spin-1/2 impurities form an entangled state with $S_z = 0$ projection at low energies. In the language of two potentials separated by d forming two halons, each halon provides 1/2 boson which is delocalized in the environment. Naturally, they merge into one shared boson via the two potentials. The shared boson induces an attractive interaction that decays as $\sim 1/d$. If the halons are mobile and heavy, this attractive interaction can potentially give rise to a bound state. Such physics can be realized in ultracold atoms in an optical lattice. At a sufficiently low temperature, the particle numbers at two impurity sites are anticorrelated against any low-energy and long-wave-length probe; if one impurity traps more charge by adiabatically increasing the laser strength, the charge around the other impurity decreases to keep the total charge a good quantum number.

In conclusion, we have studied spin-S impurities coupled via an XY coupling to a critical bosonic bath. The simplest case $S = 1$ admits relevant symmetry-preserving terms that need to be fine-tuned at the criticality. We have computed the critical exponents, which show good agreement with QMC. We have also argued that impurities with even $2S$ and odd $2S$ behave qualitatively differently and demonstrated this using QMC. We also proposed the experimental protocol to observe this effect in ultracold atoms in an optical lattice.

ACKNOWLEDGMENTS

It is a pleasure to thank B. Svistunov for multiple discussions, his critical reading, and detailed comments that helped to improve the manuscript. The authors also acknowledge valuable discussions with N. Prokof'ev and Y. Deng. Y.K. acknowledges funding from Rutgers University Materials Theory postdoctoral fellowship. K.C. acknowledges supports from the Simons Collaboration on the Many Electron Problem.

K.C. and Y.K. contributed equally to this work.

APPENDIX A: RELATION WITH THE TWO-IMPURITY PROBLEM

In the Appendixes we provide detailed calculations and proofs of statements used in the paper. This Appendix shows the connection between two spin-1/2 impurities and a spin-1 impurity, coupled to the critical bath. Appendix B contains the detailed perturbation theory calculations to first order in u and second order in γ. Appendix C provides the details of renormalization group analysis as well as the calculation of the critical exponents. Appendix D contains a short discussion of the relation between the exponent of the static susceptibilities and the correlation functions. Finally, we discuss the detailed implementation of the worm algorithm for Monte Carlo simulations in Appendix E.

The problem of two spin-1/2 impurities with $U(1)$ symmetry is described by the Hamiltonian

$$H = H_{\text{bath}} + \sum_i \gamma[\phi^+(\vec{r}_i)S_i^- + \text{H.c.}] \tag{A1}$$

$$+ J_H(S_1^+S_2^- + \text{H.c.}), \tag{A2}$$

where $|\vec{r}_2 - \vec{r}_1| = d$. At $d = 3 - \epsilon$, the γ couplings are barely relevant. At long wavelength (energies small compared to $1/d$) we have

$$\phi^\pm(\vec{r}_1) \approx \phi^\pm(\vec{r}_2) \to \phi^\pm(\bar{\vec{r}}) \tag{A3}$$

as the corrections are highly irrelevant. Here $\bar{\vec{r}} = (\vec{r}_1 + \vec{r}_2)/2$. Therefore, the effective action becomes

$$H = H_{\text{bath}} + \gamma[\phi^+(\bar{\vec{r}})(S_1^- + S_2^-) + \text{H.c.}] \tag{A4}$$

$$+ J_H(S_1^+S_2^- + \text{H.c.}). \tag{A5}$$

The two spins can be written in terms of a singlet and a triplet, and the former has decoupled from the bosons. Therefore, up to a constant the Hamiltonian is equal to the spin-1 Hamiltonian of Eq. (6), where $S_1^\pm + S_2^\pm \to S^\pm$.

APPENDIX B: PERTURBATION THEORY

In this section we calculate the correlation functions

$$\mathcal{M}_\perp(\tau) \equiv \Sigma_\alpha \langle T_\tau S_\alpha(\tau)S_\alpha(0)\rangle, \tag{B1}$$

$$\mathcal{M}_z(\tau) \equiv \langle T_\tau S_z(\tau)S_z(0)\rangle, \tag{B2}$$

$$\mathcal{M}_u(\tau) \equiv \left\langle T_\tau S_z^2(\tau)S_z^2(0)\right\rangle - \left\langle S_z^2\right\rangle^2 \tag{B3}$$

using perturbation theory in γ_0 and u_0. Since $u = O(\epsilon)$ and $\gamma = O(\sqrt{\epsilon})$, we stop at order $O(\epsilon)$. We discuss the derivation for the first correlation functions and for the other two, providing only the final result.

By expanding to second order in γ (but exact in u) we have

$$\mathcal{M}_\perp = \mathcal{M}_\perp^0 + \frac{\gamma^2}{2}\int d\tau_1 d\tau_2 D_d(\tau_1 - \tau_2)\Lambda_\perp(\tau, 0; \tau_1, \tau_2).$$

Here $D_d(\tau)$ is the zero-temperature two-point function of the bosonic bath and is given by

$$D_d(\tau) \equiv \langle T_\tau \phi_\alpha(\tau)\phi_\alpha\rangle = \int \frac{d^dk\,d\omega}{(2\pi)^{d+1}}\frac{e^{-i\omega\tau}}{k^2 + \omega^2} = \frac{\tilde{S}_{d+1}}{|\tau|^{d-1}}, \tag{B4}$$

where

$$\tilde{S}_d = \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}}. \tag{B5}$$

Using $S_\pm(\tau) = e^{\tau H}S_\pm e^{-\tau H}$, the effect of u can be calculated nonperturbatively. The zeroth order in γ is

$$\mathcal{M}_\perp^0(\tau) = \frac{e^{-u\tau} + e^{-u(\beta-\tau)}}{1 + 2e^{-\beta u}}. \tag{B6}$$

For the order γ^2 we have

$$\Lambda_\perp = \sum_{\alpha\mu=x,y}\langle T_\tau S_\alpha(\tau)S_\alpha(0)[S_\mu(\tau_1)S_\mu(\tau_2) - \langle\cdots\rangle_u]\rangle_u.$$

The spins do not obey Wick's theorem, and one has to calculate the trace over spins for each time ordering separately [5,29]. Also, note that S_α is the generator of a spin-1 representation of SU(2). Therefore, $S_\alpha^2 \neq 1$. For the case of $\mathcal{M}_\perp$ we have

$$\begin{aligned}\Lambda_\perp = &\frac{2}{1+2e^{-\beta u}}[2e^{-u(\tau+\Delta\tau)} + e^{-\beta u}e^{u(\tau+\Delta\tau)}](\theta_1+\theta_2)\\ &+\frac{2}{1+2e^{-\beta u}}[e^{-(\tau-\Delta\tau)u} + 2e^{-\beta u}e^{u(\tau-\Delta\tau)}]\theta_3\\ &+\frac{2}{1+2e^{-\beta u}}[e^{(\tau-\Delta\tau)u} + 2e^{-\beta u}e^{-u(\tau-\Delta\tau)}]\theta_4\\ &+\frac{2}{1+2e^{-\beta u}}[e^{-u\tau}e^{u(\tau_1+\tau_2)} + e^{u(\tau-\beta)}e^{-u(\tau_1+\tau_2)}]\theta_5\\ &+\frac{2}{1+2e^{-\beta u}}[e^{u\tau}e^{-u(\tau_1+\tau_2)} + e^{-\beta u}e^{-u\tau}e^{u(\tau_1+\tau_2)}]\theta_6\\ &-4\frac{e^{-u\tau}+e^{-u(\beta-\tau)}}{(1+2e^{-\beta u})^2}[e^{-u\Delta\tau} + e^{-u(\beta-\Delta\tau)}]\theta_0,\end{aligned}$$

where we defined $\Delta\tau \equiv \tau_1 - \tau_2$ and $2\bar{\tau} = \tau_1 + \tau_2$. Expansion of these expressions to zero and first order in u gives the terms $O(\gamma^2)$ and $O(u\gamma^2)$, which we are interested in here. The θ_i factors are shorthand notation for the following Heaviside functions:

$$\theta_1 \equiv \theta(\tau_1 > \tau_2 > \tau > 0), \tag{B7}$$
$$\theta_2 \equiv \theta(\tau > 0 > \tau_1 > \tau_2), \tag{B8}$$
$$\theta_3 \equiv \theta(\tau > 0 > \tau_1 > \tau_2), \tag{B9}$$
$$\theta_4 \equiv \theta(\tau_1 > \tau > 0 > \tau_2), \tag{B10}$$
$$\theta_5 \equiv \theta(\tau > \tau_1 > 0 > \tau_2), \tag{B11}$$
$$\theta_6 \equiv \theta(\tau_1 > \tau > \tau_2 > 0), \tag{B12}$$
$$\theta_0 \equiv \theta(\tau > 0)\theta(\tau_1 > \tau_2). \tag{B13}$$

The integration over these ranges appears with an integrand that is only a function of $\tau_1 - \tau_2$. Denoting

$$I_i \equiv \int d\tau_1 d\tau_2 \theta_i G(\tau_1 - \tau_2), \tag{B14}$$

we have

$$I_1 = \int_0^{\beta/2-\tau} d\Delta\tau[\beta/2 - \tau - \Delta\tau]G(\Delta\tau), \tag{B15}$$
$$I_2 = \int_0^{\beta/2} [\beta/2 - \Delta\tau]G(\Delta\tau), \tag{B16}$$
$$I_3 = \int_0^{\tau} d\Delta\tau[\tau - \Delta\tau]G(\Delta\tau), \tag{B17}$$
$$\begin{aligned}I_4 = &\int_\tau^{\beta/2} d\Delta\tau[\Delta\tau - \tau]G(\Delta\tau)\\ &+\int_{\beta/2}^{\tau+\beta/2} d\Delta\tau[\beta/2 - \tau]G(\Delta\tau)\\ &+\int_{\tau+\beta/2}^{\beta} d\Delta\tau[\beta - \Delta\tau]G(\Delta\tau).\end{aligned} \tag{B18}$$

I_5 and I_6 sometimes appear with an integrand that depends on both $\Delta\tau$ and $\bar{\tau}$. In that case,

$$\begin{aligned}I_5 = &\int_0^\tau d\Delta\tau \int_{-\Delta\tau/2}^{\Delta\tau/2} + \int_\tau^{\beta/2} d\Delta\tau/2 \int_{-\Delta\tau/2}^{\tau-\Delta\tau/2} d\bar{\tau}\\ &+\int_{\beta/2}^{\tau+\beta/2} d\Delta\tau \int_{-\beta/2+\Delta\tau/2}^{\Delta\tau/2} d\bar{\tau},\end{aligned} \tag{B19}$$
$$\begin{aligned}I_6 = &\int_0^\tau d\Delta\tau \int_{\tau-\Delta\tau/2}^{\tau+\Delta\tau/2} d\bar{\tau} + \int_\tau^{\beta/2-\tau} d\Delta\tau \int_{\Delta\tau/2}^{\tau+\Delta\tau/2} d\bar{\tau}\\ &+\int_{\beta/2-\tau}^{\beta/2} d\Delta\tau \int_{\Delta\tau/2}^{\beta/2-\Delta\tau/2} d\bar{\tau}.\end{aligned} \tag{B20}$$

If the integrand depends on only $\tau_1 - \tau_2$, we find

$$\begin{aligned}I_5 = &\int_0^\tau d\Delta\tau(\Delta\tau)G(\Delta\tau) + \tau\int_0^{\beta/2} d\Delta\tau G(\Delta\tau)\\ &+\int_{\beta/2}^{\beta/2+\tau} d\Delta\tau(\tau + \beta/2 - \Delta\tau)G(\Delta\tau),\end{aligned} \tag{B21}$$
$$\begin{aligned}I_6 = &\int_0^\tau d\Delta\tau(\Delta\tau)G(\Delta\tau) + \tau\int_\tau^{\beta/2-\tau} d\Delta\tau G(\Delta\tau)\\ &+\int_{\beta/2-\tau}^{\beta/2} d\Delta\tau(\beta/2 - \Delta\tau)G(\Delta\tau).\end{aligned} \tag{B22}$$

The $\tau = 0$ boundary in these integrals has to be replaced by the inverse UV cutoff. Moreover, since we use the expression of $D_d(\tau) \sim 1/\tau^{d-1}$ defined for $\tau \in (-\beta/2, \beta/2)$, the integrals have to be folded back to this range using the periodicity of the Green's functions. The final result is

$$\begin{aligned}\mathcal{M}_\perp(\tau) = &\frac{4}{3} + \frac{2}{9}u\beta + \gamma^2\left[\frac{1}{9}\beta f(0) - \frac{4\tilde{S}_{d+1}}{3}\frac{\tau^\epsilon}{\epsilon}\right]\\ &+\frac{\gamma^2 u}{3}\beta\left\{\left[\frac{1}{9} - \frac{2\tau}{\beta} + \frac{\tau^2}{\beta^2}\right]\beta f(0) - \frac{8\tilde{S}_{d+1}}{9}\frac{I_\perp}{\epsilon}\right\}\\ &+\frac{u^2\beta^2}{3}\left[\frac{1}{9} + \frac{\tau^2}{\beta^2} - \frac{2\tau}{\beta}\right],\end{aligned} \tag{B23}$$

where $I_\perp = (3/2)[(\beta/2)^\epsilon - \tau^\epsilon/3] = 1 + O(\epsilon)$ and we have defined $f(0) \equiv f(\mu_0^{-1})$,

$$f(\mu_0^{-1}) \equiv \left[\int_{\mu_0^{-1}}^{\beta/2} + \int_{-\beta/2}^{-\mu_0^{-1}}\right] d\tau D_{3-\epsilon}(\tau) = 2\tilde{S}_{d+1}\mu_0^{1-\epsilon}. \tag{B24}$$

That this value is independent of the IR cutoff is an artifact of using the $T = 0$ expression of $D(\tau)$. We are interested in only the UV part of this expression, and the IR dependence is not important. Similarly,

$$\begin{aligned}\mathcal{M}_z(\tau) = &\frac{2}{3} - \frac{2}{9}u\beta - \gamma^2\left[\frac{1}{9}\beta f(0) + \frac{4\tilde{S}_{d+1}}{3}\frac{\tau^\epsilon}{\epsilon}\right]\\ &+\gamma^2 u\beta\left\{-\frac{1}{27}\beta f(0) + \frac{10\tilde{S}_{d+1}}{9}\frac{I_z}{\epsilon}\right\} - \frac{1}{27}u^2\beta^2,\end{aligned} \tag{B25}$$

where $I_z = (3/5)[2(\beta/2)^\epsilon - \tau^\epsilon/3] = 1 + O(\epsilon)$, and

$$\mathcal{M}_u(\tau) = \frac{2}{9} + \frac{2}{27}u\beta + \gamma^2\left[\frac{1}{27}\beta f(0) - \frac{4\tilde{S}_{d+1}}{3}\frac{\tau^\epsilon}{\epsilon}\right] - \gamma^2 u\beta\left\{\frac{1}{27}\beta f(0) + \frac{2\tilde{S}_{d+1}}{9}\frac{\tau^\epsilon}{\epsilon}\right\} - \frac{1}{27}u^2\beta^2. \tag{B26}$$

These correlation functions diverge in two ways: One diverges as $\epsilon \to 0$ or $d \to 3^-$. And the other one diverges through the explicit UV cutoff dependence $\mu_0 \to \infty$. The goal of the next section is to renormalize the coupling constants to remove these divergences.

APPENDIX C: RENORMALIZATION

As discussed, e.g., by Whitsitt and Sachdev [5], the bulk renormalization is achieved by

$$\phi_\alpha = \sqrt{Z}\tilde{\phi}_\alpha, \quad g_0 = \frac{\mu^\epsilon Z_g}{S_{d+1}Z^2}g, \quad S_d = \frac{2}{\Gamma(d/2)(4\pi)^{d/2}}, \tag{C1}$$

where

$$Z = 1 - \frac{4}{144}\frac{g^2}{\epsilon}, \tag{C2}$$

$$Z_g = 1 + \frac{5}{3}\frac{g}{\epsilon} + \frac{25}{9}\frac{g^2}{\epsilon^2} - \frac{8}{9}\frac{g^2}{\epsilon}, \tag{C3}$$

leading to the β function [$d\ell \equiv -d\ln\mu$]

$$\beta_g \equiv \frac{dg}{d\ell} = \epsilon g - \frac{5}{3}g^2, \tag{C4}$$

which has a fixed point at $g^* = 3\epsilon/5$.

In order to renormalize the impurity problem, we define the renormalized (tilde) coupling constants as

$$\gamma = \tilde{\gamma}\mu^{\epsilon/2}A_\gamma, \quad A_\gamma = \frac{Z_\gamma}{\sqrt{\tilde{S}_{d+1}Z_\perp Z}}, \quad u = u' + Z_u\tilde{u}\mu, \tag{C5}$$

where u' is introduced to absorb the nonuniversal part of the u. We assume $\beta\mu = cte$ is a constant that can be absorbed into a redefinition of $\tilde{u}$. This implies we are comparing theories in which the temperature is equal to the energy scale of interest $T \sim \mu$. The renormalized (tilde) correlation functions are defined as

$$\mathcal{M}_\perp(\tau) = Z_\perp\tilde{\mathcal{M}}_\perp(\tau), \quad \mathcal{M}_z(\tau) = Z_z\tilde{M}_z(\tau), \tag{C6}$$

and

$$\mathcal{M}_u(\tau) = Z_u^2\tilde{\mathcal{M}}_u(\tau). \tag{C7}$$

We see that, first, the nonuniversal UV dependence can be removed by choosing

$$u' = -\frac{\gamma^2}{2}\left[f\left(\mu_0^{-1}\right) - f(\mu^{-1})\right] = \gamma^2\tilde{S}_{d+1}\left[\mu^{1-\epsilon} - \mu_0^{1-\epsilon}\right] \tag{C8}$$

and, second, the $1/\epsilon$ divergence can be eliminated by choosing

$$Z_u = 1 - \frac{3\tilde{\gamma}^2}{\epsilon}, \tag{C9}$$

$$Z_\perp = 1 - \frac{\tilde{\gamma}^2}{\epsilon} - \tilde{u}\frac{\tilde{\gamma}^2}{\epsilon}, \tag{C10}$$

$$Z_z = 1 - 2\frac{\tilde{\gamma}^2}{\epsilon} + 2\tilde{u}\frac{\tilde{\gamma}^2}{\epsilon}. \tag{C11}$$

Taking the derivative of Eqs. (C5) with respect to $d\ell = -d\ln\mu$, we find

$$\left[1 + \tilde{\gamma}\frac{d\ln A_\gamma}{d\gamma}\right]\beta_\gamma + \tilde{\gamma}\frac{d\ln A_\gamma}{du}\beta_u + \tilde{\gamma}\frac{d\ln A_\gamma}{dg}\beta_g = \frac{\epsilon}{2}\tilde{\gamma},$$

$$\tilde{u}\frac{d\ln Z_u}{d\gamma}\beta_\gamma + \left[1 + \tilde{u}\frac{d\ln Z_u}{du}\right]\beta_u + \tilde{\gamma}\frac{d\ln Z_u}{dg}\beta_g = \tilde{u} + \mu\frac{du'}{d\mu}.$$

Reference [5] also calculated vertex corrections by the bosonic interaction, leading for $S = 1$ to

$$Z_\gamma = 1 + \frac{2\pi^2}{9}\frac{g\gamma^2}{\epsilon}. \tag{C12}$$

It can be shown that Z and Z_γ are not important to $O(\epsilon)$, and we can use the simplification $A_\gamma \propto Z_\perp^{-1/2}$. Moreover, the bulk interaction g will not play a role to $O(\epsilon)$. Inverting the resulting matrix

$$\begin{pmatrix} 1 - (\tilde{\gamma}/2)\partial_\gamma \ln Z_\perp & -(\tilde{\gamma}/2)\partial_u \ln Z_\perp \\ \partial_\gamma \ln Z_u & 1 + \tilde{u}\partial_u \ln Z_u \end{pmatrix}\begin{pmatrix}\beta_\gamma \\ \beta_u\end{pmatrix} = \begin{pmatrix}(\epsilon/2)\tilde{\gamma} \\ \tilde{u} + Z_u^{-1}\partial_\mu u'\end{pmatrix}, \tag{C13}$$

we find the β functions reported in Eq. (12) of the main text.

Since $\tilde{\mathcal{M}}_\perp(\tau) \sim (\mu\tau)^{-\eta_\perp}$ and the bare correlation function $\mathcal{M}_\perp(\tau) = Z_\perp\tilde{\mathcal{M}}(\tau)$ is independent of μ, we find

$$-\eta_\perp = \frac{d\ln Z_\perp}{d\ln\mu} = \frac{d\ln Z_\perp}{d\gamma}\beta_\perp + \frac{d\ln Z_\perp}{du}\beta_u + \frac{d\ln Z_\perp}{dg}\beta_g,$$

$$-\eta_z = \frac{d\ln Z_z}{d\ln\mu} = \frac{d\ln Z_z}{d\gamma}\beta_\perp + \frac{d\ln Z_z}{du}\beta_u + \frac{d\ln Z_z}{dg}\beta_g,$$

$$-\eta_u = \frac{d\ln Z_u^2}{d\ln\mu} = \frac{d\ln Z_u^2}{d\gamma}\beta_\perp + \frac{d\ln Z_u^2}{du}\beta_u + \frac{d\ln Z_u^2}{dg}\beta_g.$$

Again, to $O(\epsilon)$ we can drop the bulk contribution in the last terms. This gives the exponents $\eta_\perp = \gamma^2$, $\eta_z = 2\gamma^2$, and $\eta_u = 6\gamma^2$ discussed in the paper.

APPENDIX D: RELATION BETWEEN η AND ν

Here we discuss the relation between the exponent in the correlation function and the free energy. For example, from

$$\langle T_\tau S_z(\tau)S_z\rangle \sim \tau^{-\eta_z}, \tag{D1}$$

the corresponding susceptibility is found to be

$$\chi_z = \int_0^\beta d\tau\langle T_\tau S_z(\tau)S_z\rangle = \beta^{1-\eta_z}. \tag{D2}$$

This susceptibility can be obtained by taking the derivative with respect to a source term h_zS_z in the action. In the presence of the source term, the scaling form of the free energy is

$$F(h_z) = b^{-1}\Phi(h_z b^{1/\nu_z}), \tag{D3}$$

where $b \sim \beta$ is a given length scale. We find

$$\chi_z = \frac{d^2F}{dh_z^2} = \beta^{2/\nu_z - 1}. \tag{D4}$$

By comparing Eqs. (D2) and (D4) we find

$$\nu_z = [1 - \eta_z/2]^{-1}. \tag{D5}$$

APPENDIX E: LATTICE MODEL

Simulations by the worm algorithm allow us to perform a comprehensive study of the universal properties of an impurity in a two-dimensional $O(2)$ quantum critical environment. As long as we are interested in only the critical properties, we are allowed to simulate the environment system with a J-current model with a spin impurity at the origin.

In the present $(2+1=3)$-dimensional case, the bulk part of this model consists of integer currents J living on the bonds of a three-dimensional $L^2 \times L_\tau$ cubic lattice, with L being the size of the spatial dimensions and L_τ being the size along the "temporal" direction (in the absence of the impurity, all three dimensions are absolutely equivalent). The currents are subject to the zero-divergence constraint,

$$\operatorname{div} J = 0, \tag{E1}$$

meaning that at each site, the algebraic—incoming minus outgoing—sum of all the currents is zero. To have a really minimalistic model, one also confines the allowed values of the bond currents to just three numbers:

$$J = 0, \pm 1. \tag{E2}$$

The Hamiltonian of the model reads

$$H_J = \frac{1}{2K}\sum_{i,\hat{e}} J_{i,\hat{e}}^2 (\hat{e} = \hat{x}, \hat{y}, \hat{\tau}). \tag{E3}$$

Here the vector $i = (x, y, \tau)$ labels the sites on the cubic lattice by three discrete coordinates: x, y, and z; $\hat{x}$, $\hat{y}$, and $\hat{\tau}$ are the lattice unit translation vectors in corresponding directions. $J_{i,\hat{e}} \equiv -J_{i+\hat{e},-\hat{e}}$ is the J current of the bond going from site i in the direction $\hat{e}$.

In terms of the mapping onto a two-dimensional system of lattice bosons (at an integer filling factor), the closed loops of currents should be understood as the world lines of $O(2)$ charge quanta, with $J_{i=(x,y,\tau),\hat{\tau}}$ having the meaning of the particle/hole charge on the site (x, y) at the imaginary-time moment τ. This model (E1)–(E3) describes the universal properties of the insulator-to-superfluid criticality; the corresponding transition takes place at the critical value $K_c = 0.3332052(20)$ of the control parameter K [4].

We now discuss the implementation of the spin impurity in the J-current models. The model used in this paper is inspired by the spin-1/2 impurity model for the halon effect [6]. We introduce a spin-1 degree of freedom at the origin by replacing the original J currents with the spin currents on the bonds going from the sites $(0, 0, \tau)$ in the direction $\hat{\tau}$. For spin-1 impurity, the spin current can take only three values,

$$S_\tau = 0, \pm 1. \tag{E4}$$

At the impurity site, the zero-divergency condition also includes the algebraic sum of the spin currents associated with this site, which guarantees the conservation of total charge: the τ independence of the total charge number Q, where

$$Q = S_\tau + \sum_{x,y} J_{(x,y,\tau),\hat{\tau}}. \tag{E5}$$

The impurity Hamiltonian contains an interaction term and a possible magnetic field term:

$$H_{\rm imp} = \frac{U}{2}\sum_\tau S_\tau^2 + h_z \sum_\tau S_\tau. \tag{E6}$$

A bare spin 1 with three degenerate states corresponds to the parameter $U = 0$. In our simulations, we turn off the magnetic field term, which explicitly breaks the Z_2 symmetry of the Hamiltonian.

The coupling between the impurity and the environment is introduced as

$$H_{\rm imp-bulk} = \frac{1}{2K_I} J_{(0,0,\tau),\hat{e}}^2 (\hat{e} = \hat{x}, \hat{y}). \tag{E7}$$

When the bulk is fine-tuned to the critical point $K = K_c$, the universal physics of the impurity model should be independent of the choice of K_I. Therefore, we fix $K_I = K$ for simplicity.

We also point out that the above lattice model with a spin-1 impurity can be easily generalized to a generic spin-S impurity by allowing the spin current to fluctuate between $-S, -S+1, \ldots, S-1, S$.

[1] E. Compagno, G. De Chiara, D. Angelakis, and G. Palma, Sci. Rep. **7**, 2355 (2017).

[2] F. Grusdt, R. Schmidt, Y. E. Shchadilova, and E. Demler, Phys. Rev. A **96**, 013607 (2017).

[3] M. Punk and S. Sachdev, Phys. Rev. A **87**, 033618 (2013).

[4] Y. Huang, K. Chen, Y. Deng, and B. Svistunov, Phys. Rev. B **94**, 220502(R) (2016).

[5] S. Whitsitt and S. Sachdev, Phys. Rev. A **96**, 053620 (2017).

[6] K. Chen, Y. Huang, Y. Deng, and B. Svistunov, Phys. Rev. B **98**, 214516 (2018).

[7] K. Chen, N. V. Prokof'ev, and B. V. Svistunov, Phys. Rev. A **98**, 041602(R) (2018).

[8] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).

[9] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinsen, R. S. Christensen, G. M. Bruun, and J. J. Arlt, Phys. Rev. Lett. **117**, 055302 (2016).

[10] W.-J. Hu and N.-H. Tong, Phys. Rev. B **80**, 245110 (2009).

[11] K. Byczuk and D. Vollhardt, Phys. Rev. B **77**, 235106 (2008).

[12] P. Anders, E. Gull, L. Pollet, M. Troyer, and P. Werner, New J. Phys. **13**, 075013 (2011).

[13] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[14] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).
[15] G. Zaránd and E. Demler, Phys. Rev. B **66**, 024427 (2002).
[16] A. M. Sengupta, Phys. Rev. B **61**, 4041 (2000).
[17] J. L. Smith and Q. Si, Europhys. Lett. **45**, 228 (1999).
[18] S. Sachdev, C. Buragohain, and M. Vojta, Science **286**, 2479 (1999).
[19] M. Vojta, C. Buragohain, and S. Sachdev, Phys. Rev. B **61**, 15152 (2000).
[20] S. Sachdev, Phys. C (Amsterdam, Neth.) **357–360**, 78 (2001).
[21] A. C. Hewson, *The Kondo Problem to Heavy Fermions*, Cambridge Studies in Magnetism (Cambridge University Press, Cambridge, 1993).
[22] C. Jayaprakash, H. R. Krishna-murthy, and J. W. Wilkins, Phys. Rev. Lett. **47**, 737 (1981).
[23] B. A. Jones, C. M. Varma, and J. W. Wilkins, Phys. Rev. Lett. **61**, 125 (1988).
[24] I. Affleck and A. W. W. Ludwig, Phys. Rev. Lett. **68**, 1046 (1992).
[25] I. Affleck, A. W. W. Ludwig, and B. A. Jones, Phys. Rev. B **52**, 9528 (1995).
[26] S. Sachdev, J. Stat. Phys. **115**, 47 (2004).
[27] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 2002).
[28] K. Chen, L. Liu, Y. Deng, L. Pollet, and N. Prokof'ev, Phys. Rev. Lett. **112**, 030402 (2014).
[29] V. Barzykin and I. Affleck, Phys. Rev. Lett. **76**, 4959 (1996).