Transport and spectral signatures of transient fluctuating superfluids in the absence of long-range order

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Results are presented for the quench dynamics of a clean and interacting electron system, where the quench involves varying the strength of the attractive interaction along arbitrary quench trajectories. The initial state before the quench is assumed to be a normal electron gas, and the dynamics is studied in a regime where long-range order is absent, but nonequilibrium superconducting fluctuations need to be accounted for. A quantum kinetic equation using a two-particle irreducible formalism is derived. Conservation of energy, particle number, and momentum emerge naturally, with the conserved currents depending on both the electron Green's functions and the Green's functions for the superconducting fluctuations. The quantum kinetic equation is employed to derive a kinetic equation for the current, and the transient optical conductivity relevant to pump-probe spectroscopy is studied. The general structure of the kinetic equation for the current is also justified by a phenomenological approach that assumes model F in the Halperin-Hohenberg classification, corresponding to a nonconserved order parameter coupled to a conserved density. Linear response conductivity and the diffusion coefficients in thermal equilibrium are also derived, and connections with Aslamazov-Larkin fluctuation corrections are highlighted. Results are also presented for the time evolution of the local density of states. It is shown that Andreev scattering processes result in an enhanced density of states at low frequencies. For a quench trajectory corresponding to a sudden quench to the critical point, the density of states is shown to grow in a manner where the time after the quench plays the role of the inverse detuning from the critical point.

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I. INTRODUCTION

Recent years have seen impressive advances in ultrafast measurements of strongly correlated systems [1–8]. In these experiments, a pump field strongly perturbs the system, while a weak probe field studies the eventual time evolution over timescales that can range from femtoseconds to nanoseconds. Thus, one can study the full nonequilibrium dynamics of the system from short times to its eventual thermalization at long times. Moreover, the access to probe fields ranging from x rays to midinfrared allows one to probe dynamics from short length and timescales to longer length and timescales associated with collective modes.

Among the various pump-probe studies, notable examples are experiments that show the appearance of highly conducting, superconducting-like states that can persist from a few to hundreds of picoseconds [9–12]. The microscopic origin of this physics is not fully understood. Explanations range from destabilization by the pump laser of a competing order in favor of superconductivity [13,14] to selective pumping of phonon modes that can engineer the effective electronic degrees of freedom so as to enhance the attractive Hubbard U, and hence T_c [15–19]. There are also theoretical studies that argue that the metastable highly conducting state observed in experiments may not have a superconducting origin [20].

The resulting superconducting-like states are transient in nature, where traditional measurements of superconductivity, such as the Meissner effect [21], cannot be performed. On the other hand, time-resolved transport and angle-resolved photoemission spectroscopy (tr-ARPES) are more relevant probes for such transient states. Thus, a microscopic treatment that makes predictions for transient conductivity and spectral features of a highly nonequilibrium system, accounting for dynamics of collective modes, is needed. This is the goal of the paper.

In what follows, we model the pump field as a quantum quench [22,23] in that it perturbs a microscopic parameter of the system, such as the strength of the attractive Hubbard U. We assume that, as in the experiments [10], the electronic system is initially in the normal phase. The quench involves tuning the magnitude of the attractive Hubbard U along different trajectories. How superconducting fluctuations develop in time and how they affect transient transport and spectral properties are studied. Denoting the superconducting order parameter as Δ, we assume that all throughout the dynamics, the system never develops true long-range order, $\langle\Delta(t)\rangle = 0$. Thus, our approach is complementary to other studies where the starting point is usually a fully ordered superconducting state, and the mean-field dynamics of the order parameter $\langle\Delta(t)\rangle$ is probed [24–29]. In our case, the nontrivial dynamics appears in the fluctuations $\langle\Delta(t)\Delta(t')\rangle$ which we study using a two-particle irreducible (2PI) formalism. We justify the selection of diagrams using a $1/N$ approach, where N denotes an orbital degree of freedom. The resulting choice of diagrams is equivalent to a self-consistent random-phase approximation (RPA) in the particle-particle channel. The physical meaning of the RPA is that the superconducting fluctuations are weakly interacting with each other.

While in this paper we study a clean system, a complementary study of the transient optical conductivity of a disordered system appears in Ref. [30], where a Kubo formalism approach was employed, and the quench dynamics arising from fluctuation corrections of Aslamazov-Larkin (AL) [31,32] and Maki-Thompson (MT) [33–35] types was explored. We avoid a Kubo-formalism approach in this paper because conserving approximations are harder to make, as compared to a quantum kinetic equation approach. This is an observation well known from other studies for transport in thermal equilibrium performed, for example, in the particle-hole channel [36]. Since we are in a disorder-free system, our quantum kinetic equations conserve momentum. Thus, to obtain any nontrivial current dynamics, we have to break Galilean invariance. We do this through the underlying lattice dispersion, while neglecting Umpklapp processes.

Complementary studies of transient conductivity involve phenomenological time-dependent Ginzburg-Landau (TDGL) theories [37,38] and microscopic approaches based on a Bardeen-Cooper-Schrieffer (BCS) mean-field treatment of the order-parameter dynamics [39]. A microscopic mean field approach has also been used to study transient spectral densities [40], while nonequilibrium dynamical mean field theory has been used to study transient spectral properties of a BCS superconductor [41].

The outline of the paper is as follows. In Sec. II, we present the model. In this section, we also outline a qualitative analysis for the transient conductivity employing a classical model-F [42] scenario for the fluctuation dynamics. In Sec. III, we derive the quantum kinetic equations using a two-particle irreducible (2PI) formalism and outline how the transient dynamics respects all the conservation laws (particle, energy, momentum) of the system. In Sec. IV, we simplify the 2PI quantum kinetic equation to that of an effective classical equation for the decay of the current employing a separation of timescales. In the process, we justify the model-F scenario discussed in Sec. II. In Sec. V, we assume thermal equilibrium and derive the conductivity and diffusion constants. We discuss these in the context of Aslamazov-Larkin (AL) and Maki-Thompson (MT) corrections [43]. In Sec. VI, we present results for the transient conductivity for some representative quench profiles. In Sec. VII, we present results for the transient local density of states, and finally in Sec. VIII, we present our conclusions. Many intermediate steps of the derivations are relegated to the Appendixes.

II. MODEL

The Hamiltonian describes fermions with spin and an additional orbital degree on a regular lattice in d spatial dimensions. Fermions are created by the operator $c^\dagger_{\sigma n}(r_i)$, where $\sigma = \pm 1$ labels spin, $n = 1 \ldots N$ labels the orbital index, and r_i gives the lattice site. These have the Fourier transform $\tilde{c}_{\sigma n}(k) = \sum_{r_i} e^{i\vec{k}\cdot\vec{r}_i} c_{\sigma n}(r_i)$. The dynamics of the fermions is governed by the Hamiltonian,

$$H(t) \equiv H_0 + H_{\rm int}(t), \tag{1}$$

$$H_0 \equiv \int_{\rm BZ} \frac{d^d k}{(2\pi)^d} \sum_{\sigma,n} (\epsilon_k - \mu)\tilde{c}^\dagger_{\sigma n}(k)\tilde{c}_{\sigma n}(k), \tag{2}$$

$$H_{\rm int} \equiv \frac{U(t)}{N} \sum_{r_i,n,m,\sigma} c^\dagger_{\sigma,n}(r_i) c^\dagger_{-\sigma,n}(r_i) c_{-\sigma,m}(r_i) c_{\sigma,m}(r_i), \tag{3}$$

where ϵ_k is the dispersion and μ is the chemical potential. We will assume for simplicity that at this chemical potential there is only a single Fermi surface. We explicitly allow for the interaction $U(t)$ to vary with time.

In addition, to probe current dynamics, we introduce an electric field $\vec{E} = -\partial_t \vec{A}$ by minimal coupling $\vec{k} \to \vec{k} - q\vec{A}$. Accounting for the charge of the electron $q = -e$, this appears in the Hamiltonian via $\vec{k} \to \vec{k} + e\vec{A}$, or equivalently, $\vec{\nabla}/i \to \vec{\nabla}/i + e\vec{A}$. In the microscopic derivation, we will set $\hbar = 1$.

In thermal equilibrium at a temperature T and for spatial dimension $d > 2$, the above system becomes superconducting below a critical temperature $T < T_C$. At $d = 2$, there is a Berezenskii-Kosterlitz-Thouless (BKT) transition at $T < T_{\rm BKT}$ where the system shows only quasi-long-range order. For $T > T_{\rm BKT}$, although the system has no long-range order, superconducting fluctuations play a key role in transport. As the spatial dimensions decrease, the role of fluctuations becomes more important. In particular, for spatial dimension $d = 2$, well-known results for the optical conductivity exist for a strongly disordered system [43]. While our derivation is valid in any spatial dimension, we will present results for spatial dimensions $d = 2$ where fluctuation effects are most pronounced. The experiments also study either a layered superconducting system or three-dimensional systems where the pump field is effectively a surface perturbation. This makes the spatial dimension $d = 2$ also experimentally relevant.

Before we go into detail, we note that our microscopic treatment leads to the following expression (reported in Ref. [44]) for the dynamics of the current j^i, generated by an applied electric field E^i,

$$\partial_t j^i(t) - E^i(t) = -\frac{1}{\tau_r}\bigg[A(t)j^i(t) - \alpha \int_0^t dt'\{B(t,t')j^i(t') + C(t,t')E^i(t')\}\bigg]. \tag{4}$$

Above τ_r, A, B, C are kinetic coefficients. We have set e=1, $\rho/\bar{m} = 1$, where ρ is the electron density and $\bar{m}$ is the electron effective mass. When $B = C = 0$, the above equation simply implies a time-dependent Drude scattering rate $\tau_r^{-1}A(t)$, which in our model arises because electrons scatter off superconducting fluctuations whose density is changing with time due to the quench. The appearance of memory terms B, C is nontrivial, and before deriving them, we will justify the appearance of these memory terms through a phenomenological approach in the next subsection.

A. Model F dynamics

In a finite-temperature phase transition, the dynamics of the fluctuations may be understood by treating them as classical stochastic fields. This will be seen by direct calculation later in the paper. However, for now, we seek to write a Langevin equation for the (classical) field Δ. In the theory of dynamical critical phenomena, we must account not only for the fluctuating mode, Δ, but also for any conserved quantities which are coupled to Δ. The pairing field Δ couples directly to the density ρ, as given by the commutator $[\Delta, \rho] = 2\Delta$. This can be translated into classical terms by the usual prescription $[\cdot,\cdot]/i\hbar \to \{\cdot,\cdot\}$ where $\{\cdot,\cdot\}$ is the classical Poisson bracket.

In the framework of dynamical critical phenomena, we specify the static free energy [see Ref. [42], Eq. (6.3)]

$$\mathcal{F}[\Delta,\rho] \equiv \int d^d r\bigg[e\Phi\delta\rho + \frac{1}{2C}(\delta\rho)^2 + r(t)|\Delta|^2 + \frac{1}{2M}|(i\vec{\nabla} - 2e\vec{A})\Delta|^2 + u|\Delta|^4 + \gamma_0(\delta\rho)|\Delta|^2\bigg]. \tag{5}$$

Here, $\delta\rho$ is density minus the equilibrium density, Φ is the scalar potential, C is the compressibility, $r(t)$ is the time-dependent detuning from the critical point, $\vec{A}(t)$ is an external vector potential, and u, γ_0 are phenomenological parameters. Note that model E [42] has $\gamma_0 = 0$, while model F has $\gamma_0 \neq 0$.

The field Δ then has the equation of motion,

$$\left(\frac{\partial\Delta}{\partial t} - \{\Delta,\rho\}\frac{\partial\mathcal{F}}{\partial\rho}\right) = -\Gamma_\Delta\frac{\partial\mathcal{F}}{\partial\Delta^*} + \xi_\Delta. \tag{6}$$

Using Eq. (5), we obtain,

$$\begin{aligned}\left[\frac{\partial\Delta}{\partial t} + 2i\Delta\left(e\Phi + \frac{\delta\rho}{C} + \gamma_0|\Delta|^2\right)\right] &= -\Gamma_\Delta\bigg[r(t) + \gamma_0\delta\rho + 2u|\Delta|^2 \\ &\quad + \frac{1}{2M}(i\nabla^i - 2eA^i)^2\bigg]\Delta + \xi_\Delta,\end{aligned} \tag{7}$$

where ξ_Δ is a Gaussian white noise obeying $\langle\xi_\Delta(r,t)\xi_{\Delta^*}(r',t')\rangle = 2T\Gamma_\Delta\delta(r-r')\delta(t-t')$, and Γ_Δ is a dimensionless phenomenological damping.

We will denote the total current by a dissipative component $j_{\rm dis}$ and a superfluid component $j_{\rm sc}$, $j^i = j^i_{\rm sc} + j^i_{\rm dis}$. The equation of motion for the density ρ is given by

$$\frac{\partial\delta\rho}{\partial t} - \{\rho,\Delta\}\frac{\partial\mathcal{F}}{\partial\Delta} - \{\rho,\Delta^*\}\frac{\partial\mathcal{F}}{\partial\Delta^*} = \vec{\nabla}\cdot\vec{j}_{\rm dis}, \tag{8}$$

where the right-hand side (rhs) can be interpreted as a definition of $j_{\rm dis}$. Under assumptions that the current may be written as a gradient of a scalar and that it reaches a well-defined steady state in the presence of a DC electric field,

$$\vec{j}_{\rm dis} = -\Gamma_{j_{\rm dis}}\left(\vec{\nabla}\frac{\partial\mathcal{F}}{\partial\rho} + e\frac{\partial\vec{A}}{\partial t}\right) + \vec{\xi}_{j_{\rm dis}}. \tag{9}$$

Above, $\vec{\xi}_{j_{\rm dis}}$ is a Gaussian white noise, whose strength is now controlled by $T\Gamma_{j_{\rm dis}}$, where $\Gamma_{j_{\rm dis}}$ is another phenomenological dissipation. Using the fact that $\partial_t\vec{A} = -\vec{E}$, the steady-state dissipative current for a spatially homogeneous system is (setting $e = 1$)

$$j^i_{\rm dis} = \Gamma_{j_{\rm dis}}E^i. \tag{10}$$

Now using

$$\begin{aligned}&\{\rho,\Delta\}\frac{\partial\mathcal{F}}{\partial\Delta} + \{\rho,\Delta^*\}\frac{\partial\mathcal{F}}{\partial\Delta^*} \\ &= \frac{i}{M}[\Delta(i\nabla^i + 2eA^i)^2\Delta^* - \Delta^*(i\nabla^i - 2eA^i)^2\Delta] = \vec{\nabla}\cdot\vec{j}_{\rm sc},\end{aligned} \tag{11}$$

we find that the definition of the superfluid current naturally emerges,

$$\vec{j}_{\rm sc} = \frac{2}{M}{\rm Im}[\Delta(\vec{\nabla} - 2ie\vec{A})\Delta^*]. \tag{12}$$

Substituting Eq. (11) in Eq. (8), we have

$$\frac{\partial\delta\rho}{\partial t} = \vec{\nabla}\cdot\vec{j}_{\rm sc} + \vec{\nabla}\cdot\vec{j}_{\rm dis}. \tag{13}$$

We now relate these equations to our equations of motion Eq. (4). The expectation value for the supercurrent is

$$\begin{aligned}\vec{j}_{\rm sc} &= \frac{2}{M}{\rm Im}\langle\Delta(x)\vec{\nabla}\Delta^*(x)\rangle \\ &= \frac{2}{M}\int\frac{d^dq}{(2\pi)^d}\vec{q}\langle\Delta_q\Delta^*_q\rangle \\ &= 2\int\frac{d^dq}{(2\pi)^d}\frac{\vec{q}}{M}F(q,t),\end{aligned} \tag{14}$$

where $F(q)$ is the equal time $\Delta_q\Delta^*_q$ correlator at momentum q. Now, we show that the integral or memory term proportional to α in Eq. (4) comes from a term of the form

$$\propto \int d^dq\,\frac{\vec{q}}{M}\partial_t F(q,t) \tag{15}$$

and that this term is precisely $\partial_t j_{\rm sc}$. The remainder can therefore be identified with the dissipative current giving an equation

$$\partial_t j^i_{\rm dis} - E^i(t) = -\tau_r^{-1}A(t)j^i_{\rm dis}. \tag{16}$$

This should be compared to the expectation value for the dissipative current Eq. (10). (Note that the damping coefficient A is not to be confused with the vector potential A^i.) Thus, in the DC limit of the kinetic equation, we identify $\Gamma_{j_{\rm dis}} = \tau_r A^{-1}$. Note that the model F dynamics are less general in the sense that they assume this DC limit and therefore that all perturbations are slow compared with $\Gamma_{j_{\rm dis}}$. The kinetic equation does not make this assumption.

Now let us evaluate $\partial_t j_{\rm sc}$. We simplify the generalized time-dependent Ginzburg-Landau theory in Eq. (7) by going into Fourier space and dropping all nonlinear terms,

$$\frac{\partial\Delta_q}{\partial t} = -\Gamma_\Delta\left\{r(t) + \frac{1}{2M}[q^i + 2eA^i(t)]^2 + \gamma_0\delta\rho_q(t)\right\}\Delta_q + \xi_\Delta. \tag{17}$$

Above, we have encoded the effect of the electric field in two places, one in the minimal coupling $\vec{q} \to \vec{q} + 2e\vec{A}$ and second in the change in the electron density at momentum q, where by symmetry

$$\delta\rho_q(t) = c\,j^i(t)q^i, \tag{18}$$

where c is a phenomenological constant. Going forward, we absorb this constant into a redefinition of γ_0 appearing in the combination $\gamma_0\delta\rho_q(t)$.

If we define

$$\lambda_q(t) = \Gamma_\Delta\left[r(t) + \frac{q^2}{2M}\right] = \Gamma_\Delta\epsilon_q(t), \tag{19}$$

in the absence of an electric field, the solution is

$$\Delta_q(t) = \int_0^t dt' \xi_\Delta(t') e^{-\int_{t'}^t dt'' \lambda_q(t'')}. \tag{20}$$

Above, we have adopted boundary conditions where the superconducting order parameter and fluctuations are zero at $t = 0$. This is because we are interested in quenches where there are initially $t \leqslant 0$ no attractive interactions, with these being switched on at some arbitrary rate from $t > 0$.

The fluctuations involve averaging over noise, giving

$$F(q, E = 0, t) = \langle \Delta_q \Delta_q^* \rangle = 2\Gamma_\Delta T \int_0^t dt' e^{-2\int_{t'}^t dt'' \lambda_q(t'')}. \tag{21}$$

In the presence of an electric field, and to leading order in it, λ_q changes to $\lambda_q + \delta\lambda_q$, with

$$\delta\lambda_q(s) = \frac{\Gamma_\Delta}{M}(2e)\vec{q} \cdot \int_s^t dt' \vec{E}(t') + \Gamma_\Delta \gamma_0 q^i j^i(s), \tag{22}$$

where the first term is due to the order parameter being charged and the second term is due to the coupling of the order parameter to the normal electrons whose density is perturbed by the electric field. We have also used that the electric field is $\vec{E} = -\partial_t \vec{A}$, so that $\vec{A}(t) = -\int_s^t dt' \vec{E}(t')$.

Expanding Eq. (21) in $\delta\lambda_q$, which is equivalent to expanding in the electric field,

$$F(q, t) \simeq 2\Gamma_\Delta T \int_0^t dt' e^{-2\int_{t'}^t dt'' \lambda_q(t'')} \left[1 - 2\int_{t'}^t du \delta\lambda_q(u)\right].$$

Splitting the integral in the exponent, $\int_{t'}^t dt'' = \int_u^t dt'' + \int_{t'}^u dt''$, and using Eq. (21), we arrive at the following expression for the superconducting fluctuations at momentum q, to leading nonzero order in the applied electric field, for an arbitrary time-dependent detuning $r(t)$,

$$\begin{aligned} F(q, t) &= F(q, E = 0, t) + \delta F(q, t), \\ \delta F(q, t) &= -2\int_0^t du \delta\lambda_q(u) e^{-2\int_u^t dt'' \lambda_q(t'')} F(q, E = 0, u). \end{aligned} \tag{23}$$

1. Aslamazov-Larkin (AL)

We now briefly show how one may recover the familiar AL conductivity in thermal equilibrium. For this, it suffices to revert to model E by setting $\gamma_0 = 0$. For a static electric field and a system in thermal equilibrium, all couplings are time independent. Thus, Eq. (23) becomes

$$\delta F(q, t) = -2F(q) \int_0^t du \delta\lambda_q(u) e^{-2(t-u)\lambda_q}. \tag{24}$$

If we use Eq. (22) for $\gamma_0 = 0$, $\delta\lambda_q(u) = \frac{\Gamma_\Delta}{M}(2e)(t - u)\vec{q} \cdot \vec{E}$. When we take the long time $t \to \infty$ limit and employ the equilibrium expression for $F(q) = T/\epsilon_q$, where $\epsilon_q = r + q^2/2M$, the change in the density of superconducting fluctuations due to the applied electric field is

$$\delta F_q = \Gamma_\Delta^{-1} \frac{T}{\epsilon_q^3} \frac{\vec{q}}{M} \cdot e\vec{E}. \tag{25}$$

The current is

$$j^i = 2e \sum_q \frac{q^i}{M} \delta F(q) = \Gamma_\Delta^{-1} 2e^2 \sum_q \frac{T}{\epsilon_q^3} \frac{q^i q^j}{M^2} E^j, \tag{26}$$

leading to the AL conductivity in d dimensions,

$$\sigma_{\rm AL} = 2e^2 \Gamma_\Delta^{-1} T \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{dM^2} \frac{1}{(r + q^2/2M)^3}. \tag{27}$$

In $d = 2$, this reduces to

$$\sigma_{\rm AL}^{d=2} = e^2 \frac{1}{2\pi\Gamma_\Delta} \frac{T}{r}. \tag{28}$$

A microscopic treatment involving electrons in a disordered potential shows that [43] $\Gamma_\Delta^{-1} = \pi/8$, giving the well-known expression, $\sigma_{\rm AL}^{d=2} = \frac{e^2}{16}\frac{T}{r}$. In our disorder-free model, Γ_Δ takes a different value.

2. Kinetic equation for the current

We now return to the derivation of the dynamics of the current in model F. For the current dynamics, we need to differentiate Eq. (23) with time,

$$\begin{aligned} &\partial_t F(q, E, t) \\ &= \partial_t F(q, E = 0, t) - 2\delta\lambda_q(t) F(q, E = 0, t) \\ &\quad + 4\lambda_q(t) \int_0^t du \delta\lambda_q(u) e^{-2\int_u^t dt'' \lambda_q(t'')} F(q, E = 0, u). \end{aligned} \tag{29}$$

The first term on the rhs, being symmetric in momentum space, does not contribute to the current. The second term on the rhs simply provides a time-dependent Drude scattering rate. It is the last term, namely the memory term proportional to the electric field, that is unique to having long-lived superconducting fluctuations. We focus only on this term, and using Eq. (22), write it as

$$\begin{aligned} \partial_t j_{\rm sc}^i(t) &= 2\int \frac{d^d q}{(2\pi)^d} \frac{q^i}{M} \partial_t F(q, t) \\ &= \frac{8\Gamma_\Delta}{M} \int \frac{d^d q}{(2\pi)^d} q^i q^j \lambda_q(t) \int_0^t du \bigg[\gamma_0 j^j(u) \\ &\quad + \frac{2e}{M} \int_u^t dt' E^j(t')\bigg] e^{-2\int_u^t dt'' \lambda_q(t'')} F(q, u). \end{aligned} \tag{30}$$

Above, we have dropped the $E = 0$ label in F. In the second term, we will find it convenient to replace the time integrals as follows: $\int_0^t du \int_u^t dt' \to \int_0^t dt' \int_0^{t'} du$.

Comparing Eqs. (4) and (30), we identify

$$\tau_r^{-1} \alpha B(t, t') = \frac{8\Gamma_\Delta \gamma_0}{dM} \int \frac{d^d q}{(2\pi)^d} q^2 \lambda_q(t) e^{-2\int_{t'}^t dt'' \lambda_q(t'')} F(q, t'). \tag{31}$$

The second memory term may be identified as

$$\begin{aligned} \tau_r^{-1} \alpha C(t, t') = &\frac{8\Gamma_\Delta}{dM^2}(2e) \int \frac{d^d q}{(2\pi)^d} q^2 \lambda_q(t) \\ &\times \int_0^{t'} du e^{-2\int_u^t dt'' \lambda_q(t'')} F(q, u). \end{aligned} \tag{32}$$

Comparing Eqs. (31) and (32), we obtain

$$C(t,t') = \frac{2e}{M\gamma_0}\int_0^{t'} ds B(t,s). \tag{33}$$

The microscopic treatment that follows not only justifies model F but also provides an explicit calculation for the parameters $\tau_r^{-1}, A(t), \alpha, \Gamma_\Delta, \Gamma_{j_{\rm dis}}, M, r, \gamma_0$.

III. THE 2PI EQUATIONS OF MOTION

A. Properties of 2PI formalism

We briefly recapitulate the 2PI formalism here. A detailed explanation can be found, for example, in Refs. [45–48]. The 2PI formalism begins with the Keldysh action S for the Hamiltonian H. This is written in terms of Grassmann fields $\psi_{\sigma,n,\pm}(r,t)$, $\bar{\psi}_{\sigma,n,\pm}(r,t)$, where σ labels the spin, n labels the orbital quantum number, and $\pm$ label the two branches of the Keldysh contour [49]. In terms of these, the Keldysh action is given as

$$\begin{aligned} S_K[\psi,\bar{\psi}] \equiv \int dt \Bigg\{ & i\sum_{\sigma=\uparrow,\downarrow;n,r} \bar{\psi}_{\sigma n+}(r,t)\frac{\partial}{\partial t}\psi_{\sigma n+}(r,t) \\ & - H[\bar{\psi}_+(t),\psi_+(t)] \\ & - i\sum_{\sigma=\uparrow,\downarrow;n,r} \bar{\psi}_{\sigma n-}(r,t)\frac{\partial}{\partial t}\psi_{\sigma n-}(r,t) \\ & + H[\bar{\psi}_-(t),\psi_-(t)]\Bigg\}, \end{aligned} \tag{34}$$

where $H[\bar{\psi}_\pm(t),\psi_\pm(t)]$ indicates the substitution of $\psi_\pm(t)$ for $c(t)$ in the Hamiltonian, Eq. (3), in the obvious way. Now, we consider a classical field h that couples to a general bilinear of the Grassmann field and is therefore given by $h_{\sigma n p}^{\sigma' n' p'}(r,t,r',t')$, where p, p' run over $\pm$. In order to simplify notation, we combine the five indices (spin, orbital, Keldysh, position, and time) into a single vector index, so that the Grassman field becomes a vector ψ_i and the source field a matrix h_{ij}.

From the action and the source field, we form a generating functional $\mathcal{I}[h]$:

$$\mathcal{I}[h] = \int \mathcal{D}[\bar{\psi},\psi]\exp\left[iS_K[\bar{\psi},\psi] + i\sum_{ij}\bar{\psi}_i h_{ij}\psi_j\right]. \tag{35}$$

The first derivative of $\mathcal{I}$ is precisely the Green's function in the presence of the source field h

$$G_{ij}[h] = \frac{\delta\mathcal{I}}{\delta h_{ji}} = i\int \mathcal{D}[\bar{\psi},\psi](\bar{\psi}_j\psi_i)e^{iS_K + i\bar{\psi}\cdot h\cdot\psi}. \tag{36}$$

In order to work with the physical Green's function rather than the unphysical source field h, we perform a Lagrange inversion. First, we invert Eq. (36), which gives G as a function of h, to implicitly define h as a function G. Then, we construct the Lagrange transformed functional $\mathcal{W}$,

$$\mathcal{W}[G] = \mathcal{I}[h(G)] - h(G)G. \tag{37}$$

This has the property that

$$\frac{\delta\mathcal{W}}{\delta G} = -h(G). \tag{38}$$

In particular, in the physical situation when $h=0$, we have that $\delta\mathcal{W}/\delta G = 0$. Thus, given the functional $\mathcal{W}$, we have reduced the problem of calculating G to a minimization problem.

The functional $\mathcal{W}$ can be calculated in terms of a perturbation series in U, giving

$$\mathcal{W}[G] = \frac{i}{2}\mathrm{Tr}\ln(G^{-1}) + \frac{i}{2}\mathrm{Tr}[g^{-1}G] + \Gamma_2[G], \tag{39}$$

where the Tr operates on the entire combined index space and g^{-1} is the bare electron Green's function. The functional $\Gamma_2[G]$ is constructed as follows. First, the Feynman rules for the Hamiltonian H are written down. In the present case, this is a single fermionic line and a four-fermion interaction. Then, all two-particle irreducible bubble diagrams are drawn. A diagram is two-particle irreducible (2PI) if it cannot be disconnected by deleting up to two fermionic lines. The functional $\Gamma_2[G]$ is given by the sum of all the diagrams. Lastly, the diagrams are interpreted as in usual diagrammatic perturbation theory except that the fermionic line is not the bare Green's function g, but instead the full Green's function G.

The end result is the minimization condition

$$\frac{\delta\mathcal{W}}{\delta G} = -\frac{i}{2}G^{-1} + \frac{i}{2}g^{-1} + \frac{\delta\Gamma_2}{\delta G} = 0, \tag{40}$$

$$\to G^{-1} = g^{-1} - \Sigma,\ \Sigma[G] = 2i\delta\Gamma_2/\delta G. \tag{41}$$

This is the Dyson equation for the Green's function G where the self-energy $\Sigma[G] \equiv 2i\delta\Gamma_2/\delta G$ is a self-consistent function of G. As no approximations have been made, Eq. (41) is a restatement of the original many-body problem and clearly cannot be solved. The advantage of the formalism is that we may replace Γ_2 with an approximate functional Γ_2' and still be guaranteed to preserve conservation laws and causality, which is not the case if one directly approximates Σ.

We assume that all symmetries are maintained by the solution to Eq. (41). Therefore, the Green's function can be written in terms of the original spin, orbital, Keldysh, space, and time indices as

$$\begin{aligned} & G_{\sigma_1,n_1,p_1;\sigma_2,n_2,p_2}(r_1,t_1;r_2,t_2) \\ & \quad = \delta^{\sigma_1\sigma_2}\delta^{n_1,n_2}G_{p_1p_2}(r_1-r_2;t_1,t_2), \end{aligned} \tag{42}$$

so that only the Keldysh indices p_1, p_2 will be explicitly noted.

B. RPA approximation to the 2PI generating functional

The approximation we make is the random-phase approximation (RPA) in the particle-particle channel or equivalently the $N\to\infty$, where N is the number of orbital degrees of freedom. That is, we approximate Γ_2 by Γ_2', where the latter include the set of all closed ladder diagrams,

$$\Gamma_2' \equiv -\frac{i}{2}\Pi\cdot D, \quad D \equiv (U^{-1} - \Pi)^{-1}, \tag{43}$$

$$\Pi_{p_1p_2}(r_1,r_2;t_1,t_2) \equiv \frac{i}{2}\big[G_{p_1p_2}(r_1-r_2;t_1,t_2)\big]^2, \tag{44}$$

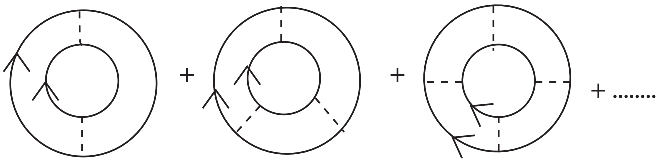

FIG. 1. 2PI Ladder diagrams for Γ_2'. Solid lines are electron Green's functions, while the dashed lines are the interaction U.

shown diagrammatically in Fig. 1, which gives the equation,

$$\Sigma_{p_1p_2}(r_1,t_1;r_2,t_2) = iD_{p_1p_2}(r_1,t_1;r_2,t_2)G_{p_2p_1}(r_2,t_2;r_1,t_1). \tag{45}$$

Above, we have used that $\Sigma(1,2) = 2i\delta\Gamma_2/\delta G(2,1)$. The equations may be equivalently derived from the functional for two Green's functions G and D given by

$$\mathcal{W}'[G,D] = \frac{i}{2}\mathrm{Tr}[\ln(G^{-1}) + g^{-1}G + \ln(D^{-1}) + U^{-1}D] + \Gamma_2'[G,D], \tag{46}$$

$$\Gamma_2'[G,D] \equiv -\frac{i}{2}\Pi\cdot D, \tag{47}$$

$$\frac{\delta\mathcal{W}'}{\delta D} = -\frac{i}{2}D^{-1} + \frac{i}{2}U^{-1} - \frac{i}{2}\Pi = 0, \tag{48}$$

$$\frac{\delta\mathcal{W}'}{\delta G} = -\frac{i}{2}G^{-1} + \frac{i}{2}g^{-1} - \frac{i}{2}\Sigma = 0. \tag{49}$$

This may be interpreted as the functional for electrons interacting with a fluctuating pairing field whose Green's function is D. The Γ_2' is the minimal diagram which has interaction between the fermions and the fluctuating pairing field.

C. RPA equations of motion

We now proceed to analyze the RPA equations of motion. As is customary, we perform a unitary rotation of the Keldysh space, defining $G_{K,A,R}$ by [49]

$$G_K \equiv (G_{++} + G_{+-} + G_{-+} + G_{--})/2, \tag{50}$$

$$G_R \equiv (G_{++} - G_{+-} + G_{-+} - G_{--})/2, \tag{51}$$

$$G_A \equiv (G_{++} + G_{+-} - G_{-+} - G_{--})/2, \tag{52}$$

$$0 = G_{++} - G_{+-} - G_{-+} + G_{--}, \tag{53}$$

the last equality holding by causality. We likewise define $\Sigma_{K,A,R}$, $D_{K,A,R}$, $\Pi_{K,A,R}$. In this basis, we may write the equations of motion as (setting $e = 1$)

$$[i\partial_{t_1} - \epsilon(i\vec{\nabla}_1 - \vec{A}(r_1,t_1))]\, g_R(r_1,t_1;r_2,t_2) = \delta(t_1 - t_2), \tag{54a}$$

$$G_R^{-1} = g_R^{-1} - \Sigma_R, \tag{54b}$$

$$G_K = G_R\cdot\Sigma_K\cdot G_A, \tag{54c}$$

$$D_R^{-1} = U^{-1} - \Pi_R, \tag{54d}$$

$$D_K = D_R\cdot\Pi_K\cdot D_A, \tag{54e}$$

and where ϵ denotes the single particle dispersion, the symbol $\cdot$ convolution, and Σ, Π are given by

$$\Sigma_R(x,y) = \frac{i}{2}[D_R(x,y)G_K(y,x) + D_K(x,y)G_A(y,x)], \tag{54f}$$

$$\Sigma_K(x,y) = \frac{i}{2}[D_K(x,y)G_K(y,x) + D_R(x,y)G_A(y,x) + D_A(x,y)G_R(y,x)], \tag{54g}$$

$$\Pi_R(x,y) = \frac{i}{2}G_K(x,y)G_R(x,y), \tag{54h}$$

$$\Pi_K(x,y) = \frac{i}{4}[G_K(x,y)^2 + G_R(x,y)^2 + G_A(x,y)^2], \tag{54i}$$

and where x and y stand for combined space and time indices. The function G_A is given by $G_A = G_R^\dagger$, where $O(x,y)^\dagger = O(y,x)^*$, and likewise for D_A, Σ_A, Π_A. We note that we also have the relationship $G_K^\dagger = -G_K$.

We now convert the definition of G_K into a more useful form by evaluating $G_R^{-1}\cdot G_K - G_K\cdot G_A^{-1}$. On the one hand, substituting in the definition of $G_{R,A}^{-1}$, this is

$$G_R^{-1}\cdot G_K - G_K\cdot G_A^{-1} = g_R^{-1}\cdot G_K - G_K\cdot g_A^{-1} - \Sigma_R\cdot G_K + G_K\cdot\Sigma_A; \tag{55}$$

on the other hand, substituting in $G_K = G_R\cdot\Sigma_K\cdot G_A$, we obtain

$$G_R^{-1}\cdot G_K - G_K\cdot G_A^{-1} = \Sigma_K\cdot G_A - G_R\cdot\Sigma_K. \tag{56}$$

Therefore, we obtain the fundamental equation

$$g_R^{-1}\cdot G_K - G_K\cdot g_A^{-1} = \Sigma_K\cdot G_A - G_R\cdot\Sigma_K + \Sigma_R\cdot G_K - G_K\cdot\Sigma_A. \tag{57}$$

Setting the two times equal and using the form of g_R, we obtain

$$\begin{aligned}&\partial_t iG_K(r_1,t;r_2,t) - [\epsilon(i\vec{\nabla}_{r_1} - \vec{A}(r_1,t)) \\ &\quad - \epsilon(-i\vec{\nabla}_{r_2} - \vec{A}(r_2,t))]G_K(r_1,t;r_2,t) = S(r_1t, r_2t), \\ &S(r_1t, r_2t') \equiv \int dy\ \{\Sigma_K(r_1,t;y)G_A(y,r_2,t') \\ &\quad + \Sigma_R(r_1,t;y)G_K(y;r_2,t') \\ &\quad - G_R(r_1,t;y)\Sigma_K(y;r_2,t') \\ &\quad - G_K(r_1,t;y)\Sigma_A(y;r_2,t')\}.\end{aligned} \tag{58}$$

Above y includes both space and time coordinates, and we have given the collision term S on the rhs for unequal times as we will need it to prove different conservation laws in the next section.

D. Conservation laws

We emphasize that the true Green's functions are found by minimizing with respect to the full 2PI potential Γ_2, whereas the functions G, D defined above are only approximations, found by minimizing with respect to the approximate potential Γ_2'. Therefore, it is not *a priori* clear that there are any conserved quantities that correspond to the conserved quantities of the true Hamiltonian. However, the existence of such quantities is guaranteed by the 2PI formalism. Here, we show that the naive expression for the conserved density

$$n(r,t) \equiv iG_K(r,t;r,t) \tag{59}$$

is correct. If we set $r_1 = r_2$ in Eq. (58),

$$\partial_t iG_K(r,t;r,t) - [\epsilon(i\vec{\nabla}_{r_1} - \vec{A}(r_1,t)) - \epsilon(-i\vec{\nabla}_{r_2} - \vec{A}(r_2,t))]G_K(r_1,t;r_2,t)|_{r_1=r_2=r} = S(rt,rt). \tag{60}$$

Inspecting the second term on the left-hand side (lhs), we see that it is a total derivative and the equation can be written as

$$\partial_t n(r,t) - \vec{\nabla}\cdot\vec{j}(r,t) = S(rt,rt),$$
$$\vec{j} = \vec{v}(i\vec{\nabla}_1 - \vec{A}(r_1,t), i\vec{\nabla}_2 + \vec{A}(r_2,t))iG_K(r_1,t;r_2,t)|_{r_1=r_2=r}, \tag{61}$$

where the velocity $\vec{v}$ generalized for an arbitrary dispersion ϵ is defined as

$$(\vec{x}+\vec{y})\cdot\vec{v}(\vec{x},\vec{y}) = \epsilon(\vec{x}) - \epsilon(-\vec{y}). \tag{62}$$

This has the form of a continuity equation for the conserved quantity n because

$$S(rt,rt) = 0. \tag{63}$$

We demonstrate this in Appendix A 1. Thus, the particle number $n(r,t) = iG_K(rt;rt)$ is a conserved quantity.

Conservation of momentum follows similarly by taking the definition of the momentum

$$p^\alpha(r,t) = \left[\frac{i}{2}\left(\nabla^\alpha_{r_1} - \nabla^\alpha_{r_2}\right) - A^\alpha(r,t)\right] iG_K(r_1,t;r_2,t)\Big|_{r_1=r_2=r}. \tag{64}$$

Evaluating the time derivative of p^α and using Eq. (58), one obtains

$$\partial_t p^\alpha(r,t) - n(r,t)E^\alpha(r,t) + \epsilon^{\alpha\beta\gamma} j^\beta(r,t)B^\gamma(r,t) - \nabla^\beta_r T^{\alpha\beta}(r,t) = 0, \tag{65}$$

$$T^{\alpha\beta} = \left[\frac{i}{2}\left(\nabla^\alpha_{r_1} - \nabla^\alpha_{r_2}\right) - A^\alpha(r,t)\right] v^\beta(i\nabla_1 - A(r_1,t), i\nabla_2 + A(r_2,t)) iG_K(r_1t, r_2t)\Big|_{r_1=r_2=r} + \delta^{\alpha\beta}U^{-1}(t)iD_K(rt,rt). \tag{66}$$

Above, the electric field is $E^\alpha = -\partial_t A^\alpha$ and the magnetic field is $B^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma}(\nabla^\beta A^\gamma - \nabla^\gamma A^\beta)$, and $T^{\alpha\beta}$ is the momentum tensor. Above, we have used the relation

$$\frac{i}{2}\left[\nabla^\alpha_{r_1} - \nabla^\alpha_{r_2}\right]S(r_1t, r_2t)\big|_{r_1=r_2=r} = iU^{-1}(t)\nabla^\alpha_r D_K(rt,rt), \tag{67}$$

which is proved in Appendix A 2. We also note that since $S(rt,rt) = 0$, $A^\alpha(r,t)S(rt,rt) = 0$. Thus, the momentum tensor has a contribution from the full electron Green's function G and also from the full superconducting fluctuations D.

Finally, conservation of energy is obtained by operating on the kinetic equation $\frac{i}{2}(\partial_{t_1} - \partial_{t_2})$ and setting $r_1 = r_2$, $t_1 = t_2$. This yields

$$\partial_t\mathcal{E}(r,t) - \vec{\nabla}\cdot\vec{j}_\mathcal{E} = \vec{E}\cdot\vec{j} + Q_{\rm ext}(r,t), \tag{68}$$

$$\mathcal{E}(r,t) = \frac{i}{2}(\partial_{t_1} - \partial_{t_2})iG_K(r,t_1;r,t_2)|_{t_1=t_2=t} - iU^{-1}(t)D_K(r,t), \tag{69}$$

$$j^\beta_\mathcal{E} = \frac{i}{2}\left(\partial_{t_1} - \partial_{t_2}\right)v^\beta(i\nabla_1 - A(r_1,t_1), i\nabla_2 + A(r_2,t_2)) \times iG_K(r_1,t_1,r_2,t_2)|_{\substack{r_1=r_2=r,\\ t_1=t_2=t}} \tag{70}$$

$$Q_{\rm ext} = -iD_K(r,t)\partial_t U^{-1}(t). \tag{71}$$

The necessary relation for S needed to derive the above is

$$\frac{i}{2}\left(\partial_{t_1} - \partial_{t_2}\right)S(rt_1, rt_2)|_{t_1=t_2=t} = i\partial_t(U^{-1}D_K(r,t;r,t)) - iD_K(r,t;r,t)\partial_t U^{-1}(t). \tag{72}$$

The proof of the above is in Appendix A 3. The above shows that the energy is not conserved due to Joule heating $\vec{E}\cdot\vec{j}$, from an applied electric field and also when the parameters of the Hamiltonian are explicitly time dependent, $\partial_t U(t) \neq 0$.

IV. REDUCTION TO A CLASSICAL KINETIC EQUATION

In this section, we project the full quantum kinetic equation onto the dynamics of the current. This can be done by noting some important separation of energy scales associated with critical slowing down of the superconducting fluctuations.

A. Critical slowing down

Although the RPA approximation produces a closed set of equations of motion, Eq. (54), these are a set of nonlinear coupled integral-differential equations and therefore cannot be solved without further reduction. The key to reducing the equations further is the phenomenon of critical slowing down [42]. This may be demonstrated by considering the particle-particle polarization in equilibrium. As the equilibrium system is time and space translation invariant, we consider the Fourier transform

$$\Pi_R(q,\omega) = \int d^d r dt\, \Pi_R(r,t,r',t')e^{-i\vec{q}\cdot(\vec{r}-\vec{r}')+i\omega(t-t')} = \frac{i}{2}\int d^d k d\eta\, G_R\Big(k+\frac{q}{2}, \eta+\frac{\omega}{2}\Big) \times G_K\Big(-k+\frac{q}{2}, -\eta+\frac{\omega}{2}\Big). \tag{73}$$

Assuming for the moment that G_R has a Fermi-liquid form

$$G_R(k,\omega) = \frac{Z}{\omega - \epsilon_k + i/2\tau} + G_{\rm inc}, \tag{74}$$

where Z is the quasiparticle residue, τ^{-1} is the decay rate and is $\propto \max(\omega^2, T^2)$, and $G_{\rm inc}$ is a smooth incoherent part whose effect is negligible at low energies. We also assume quasiequilibrium where G_K is related to G_R by the fluctuation dissipation theorem $G_K(k,\omega) = 2{\rm Im}[G_R(k,\omega)]\tanh(\omega/2T)$. We expect that $\Pi_R(r,t;r,t')$ decays exponentially on the scale T^{-1}. Thus, from the perspective of any dynamics that is slower than T^{-1}, the polarization is effectively short ranged in time. Thus, in terms of the Fourier expansion, we only need the leading behavior in ω/T. Note that the short time dynamics showing the onset of quasiequilibrium of the electron distribution function to a temperature T can also be studied by the numerical time evolution of the RPA Dyson equations [50].

Now the Green's function D_R is precisely the linear response to a superconducting fluctuation. Therefore, if $U < U_c$, where U_c denotes the critical value of the coupling, the system is in the disordered phase and D_R should decay exponentially with time. If $U > U_c$, the system is supercritical and superconducting fluctuations should grow exponentially in time. In terms of the Fourier transform $D_R(q,\omega)$, this means that there is a pole in the complex ω plane which crosses the real axis at precisely the phase transition, $U_c(T)$. Therefore, we may write

$$\begin{aligned} D_R^{-1}(q,\omega) &= U^{-1} - \Pi_R(q,\omega) \\ &= Z(q[\lambda(q,U) + i\omega/T] + \mathcal{O}\left(\frac{\omega^2}{T^2}\right), \end{aligned} \tag{75}$$

where the function $\lambda(q=0, U_c(T)) = 0$. There is a critical regime in q and $U - U_c$, where $\lambda(q,U) \ll T$. We further define a thermal wavelength q_T by the expression $\lambda(q_T, U_c) = T$, which we estimate as $q_T \sim T/v_F$, where v_F is the Fermi velocity.

On the assumption, to be explicitly shown later, that the transport is dominated by processes that are much slower than T^{-1}, we therefore replace

$$\Pi_R(q;t,t') = U^{-1}(t) - Z(q)\delta(t-t')[\lambda(q,U(t)) + \partial_t], \tag{76}$$

where we have explicitly included the possibility that $U(t)$ is time dependent. Therefore, the equations of motion for D are

$$[\partial_t + \lambda_q(t)]D_R(q,t,t') = T\nu^{-1}\delta(t-t'), \tag{77}$$

$$\lambda_q(t) = r(t) + q^2/2M, \tag{78}$$

where $r(t)$ is an effective detuning arising from the the time dependence of the interaction $U(t)$. Once D_R is known then, D_K follows from Eq. (54e) and the fact that $\Pi_K(t,t') = 4i\nu\delta(t-t')$.

We consider two cases below. One where $\lambda_q(t) = \theta(t)\lambda_q + \theta(-t)r_i$, with $r_i/T = \mathcal{O}(1)$. This represents a rapid quench, where the detuning changes from an initial value to a final value at a rate that is faster than or of the order of the temperature. The second situation we consider is an arbitrary trajectory for the detuning $r(t)$.

The solution for bosonic propagators for the former, namely the rapid quench, at times $t, t' \geqslant 0$ then become

$$\Rightarrow D_R(q,t,t') = \theta(t-t')T\nu^{-1}e^{-\lambda_q(t-t')}, \tag{79}$$

$$iD_K(q,t,t') = 2T\nu^{-1}\frac{T}{\lambda_q}(e^{-\lambda_q|t-t'|} - e^{-\lambda_q(t+t')}), \tag{80}$$

$$= 2[D_R(q,t,t')F(q,t') + F(q,t)D_A(q,t,t')], \tag{81}$$

$$F(q,t) \equiv \frac{T}{\lambda_q}(1 - e^{-2\lambda_q t}). \tag{82}$$

Note the function $F(q,t)$ changes with rate λ_q, which for small q is $\ll T$. Also note that at $t = t' = 0$, the density of superconducting fluctuations as measured by $D_K(t = t' = 0)$ or $F(t=0)$ is zero, consistent with an initial condition where the initial detuning is large, positive, and hence far from the critical point.

For an arbitrary trajectory $r(t)$ and hence $\lambda_q(t)$, the solutions for the bosonic propagators, generalize as follows:

$$D_R(q,t,t') = \theta(t-t')T\nu^{-1}e^{-\int_{t'}^{t}dt''\lambda_q(t'')}, \tag{83}$$

$$iD_K(q,t,t') = 2[D_R(q,t,t')F(q,t') + F(q,t)D_A(q,t,t')], \tag{84}$$

$$F(q,t) \equiv 2T\int_0^t dx e^{-2\int_x^t dy \lambda_q(y)}. \tag{85}$$

We show in Appendix B that the above equations for $D_{R,A,K}$ imply an effective model for a bosonic field obeying Langevin dynamics, where the Langevin noise is δ correlated with a strength proportional to the temperature T.

B. Linear response to electric field

The previous derivation of the kinetic equations Eq. (58) is correct for arbitrary electric fields strengths and therefore includes nonlinear effects. We now expand the Dyson equation to linear order in $E(t)$ in order to evaluate the linear response. Note that since $g_R(k,t,t') = e^{-i\int_{t'}^{t}ds\epsilon(k^i+A^i(s))}$, it is the combination, $g_R(\vec{k} - \vec{A}(t)) = e^{-i\int_{t'}^{t}ds\epsilon(k^i+A^i(s)-A^i(t))}$ which is gauge invariant. The leading change in the electron Green's functions at the gauge invariant momentum is

$$\begin{aligned} g_R(t,t') &= -i\theta(t-t')e^{-i\epsilon_k(t-t')} \\ &\quad \times \left(1 - i\int_{t'}^{t} ds\vec{v}_k\cdot[\vec{A}(s) - \vec{A}(t)]\right), \end{aligned} \tag{86}$$

$$n_k = n_k^{\rm eq} + \delta n_k(t), \tag{87}$$

where $n_k = \int d^d(r_1 - r_2)e^{-ik(r_1-r_2)}iG_K(r_1,r_2)$ and δn is assumed to be of order E.

We now write the kinetic equation (58) after Fourier transforming with respect to the spatial coordinates $r_1 - r_2$. The change in density at the gauge-invariant momentum is

$$\frac{d}{dt}n_{k-A(t)} = \partial_t n_k(t) - \partial_t\vec{A}\cdot\vec{\nabla}_k n_k^{\rm eq} = I_C + \vec{E}\cdot\vec{\nabla}_k n_k^{\rm eq}, \tag{88}$$

where I_C is the rhs of the kinetic equation (58) evaluated at the gauge-invariant momentum $\vec{k} - \vec{A}(t)$. Substituting for I_C and shifting the internal momentum q by $q - 2A(t)$, we obtain

$$\begin{aligned} &\partial_t n_k(t) - \vec{E}\cdot\vec{\nabla}_k n_k^{\rm eq} \\ &= 4\sum_q\int dt'{\rm Im}[iD_K(\vec{q} - 2\vec{A}(t),t,t')\delta\Pi'_A(\vec{k} - \vec{A}(t),\vec{q} \\ &\quad - 2\vec{A}(t),t',t) + iD_R(\vec{q} - 2\vec{A}(t),t,t')\delta\Pi'_K(\vec{k} - \vec{A}(t),\vec{q} \\ &\quad - 2\vec{A}(t),t',t) + \delta iD_K(\vec{q} - 2\vec{A}(t),t,t')\Pi'_A(\vec{k} - \vec{A}(t),\vec{q} \\ &\quad - 2\vec{A}(t),t',t) + i\delta D_R(\vec{q} - 2\vec{A}(t),t,t')\Pi'_K(\vec{k} - \vec{A}(t),\vec{q} \\ &\quad - 2\vec{A}(t),t,t')], \end{aligned} \tag{89}$$

where the δ in Eq. (89) indicates the first-order variation with respect to the electric field. As expected, the conservation laws continue to hold after this approximation. Above, the quantity $\Pi'(k,q,t,t')$ is defined as

$$\Pi'_{R,A}(\vec{k}-\vec{A}(t),\vec{q}-2\vec{A}(t),t,t') = \frac{i}{2}G_{R,A}(\vec{k}-\vec{A}(t),t,t')G_K(-\vec{k}+\vec{q}-\vec{A}(t),t,t'), \quad (90)$$

$$\begin{aligned}\Pi'_K(\vec{k}-\vec{A}(t),\vec{q}-2\vec{A}(t),t,t') &= \frac{i}{4}[G_K(\vec{k}-\vec{A}(t),t,t')G_K(-\vec{k}+\vec{q}-\vec{A}(t),t,t')\\ &+ G_R(\vec{k}-\vec{A}(t),t,t')G_R(-\vec{k}+\vec{q}-\vec{A}(t),t,t')\\ &+ G_A(\vec{k}-\vec{A}(t),t,t')G_A(-\vec{k}+\vec{q}-\vec{A}(t),t,t')], \quad (91)\end{aligned}$$

$$\Pi_{R,A,K}(q,t,t') = \sum_k \Pi'_{R,A,K}(k,q,t,t'). \quad (92)$$

Note that all the momenta appear in the gauge-invariant combination $\vec{k}-\vec{A}(t)$. Above, the last line highlights the relation between Π' and the polarization Π.

The fluctuations are modified by the electric field through their dependence on Π_R:

$$\delta D_R = D_R\cdot\delta\Pi_R\cdot D_R, \quad (93)$$

$$\begin{aligned}\delta D_K &= \delta D_R\cdot\Pi_K\cdot D_A + D_R\cdot\delta\Pi_K\cdot D_A + D_R\cdot\Pi_K\cdot\delta D_A\\ &= D_R\cdot\delta\Pi_R\cdot D_K + D_R\cdot\delta\Pi_K\cdot D_A + D_K\cdot\delta\Pi_A\cdot\delta D_A.\end{aligned} \quad (94)$$

In what follows, since we are interested in the response of the current, we multiply the linearized kinetic equation (89) by v_k and integrate over k.

C. Large fluctuations limit

We make one further simplification, which is based on the fact that in the critical regime $D_K \gg D_R$ by a factor T/λ_q. Thus, we will only keep the terms that are highest order in D_K.

D. Projections of equation to finite number of modes

The kinetic equation is an integral-differential equation in which the unknowns $\delta n_k(t)$ appear linearly. Conceptually, therefore, it may be solved by standard methods. However, even though it is linear, it is nonlocal in time and not time translation invariant. This makes direct analytical and numerical solution difficult.

We therefore make an additional simplification that instead of considering the full space of solutions, we instead project entirely onto the current mode. This means that we would like to fix

$$\delta n_k(t) \stackrel{?}{=} \frac{\bar{m}}{\rho}J^i(t)\nabla^i_k n^{\rm eq}_k, \quad (95)$$

$$m^{-1}_{ij} = \frac{\partial^2\epsilon(k)}{\partial k^i\partial k^j} = \partial_{k^j}v^i_k; \quad \bar{m}^{-1}\delta_{ij} = -\frac{1}{\rho}\sum_k m^{-1}_{ij}n^{\rm eq}_k, \quad (96)$$

where $\rho = -\sum_k n^{\rm eq}_k$ is the density of fermions, $\bar{m}$ is the effective mass, and $J^i(t)$ is some function to be determined. The ? in Eq. (95) is to question the correctness of this equation. This is because conservation of momentum $P^i = \sum_k k^i n_k$ causes Eq. (95) to fail qualitatively. As $\partial_t P^i = \rho E^i$, an electric field pulse, say, for simplicity taken to be a δ function in time, will generate a net momentum. Moreover, by conservation of momentum, the initial perturbation

$$\delta n_k = V^i k^i \partial n^{\rm eq}/\partial\epsilon, \quad (97)$$

with V^i arbitrary, will never decay. Therefore, instead of Eq. (95), we decompose the occupation number as

$$\delta n_k(t) = \frac{\bar{m}}{\rho}\big[J^i_r(t)\big(v^i_k - \gamma k^i/\bar{m}\big) + \gamma k^i P^i(t)/\bar{m}^2\big]\partial_\epsilon n^{\rm eq}_k, \quad (98)$$

$$\gamma \equiv \frac{\bar{m}\sum_k \vec{v}_k\cdot\vec{k}\partial_\epsilon n}{\sum_k \vec{k}\cdot\vec{k}\partial_\epsilon n}, \quad (99)$$

$$J^i_{\rm tot} \equiv \sum_k v^i_k \delta n_k = (1-\gamma)J^i_r + \gamma P^i/\bar{m}. \quad (100)$$

The parameter γ gives the amount of current that is carried by the momentum mode, which does not relax. In the limit $\gamma \to 0$, there is no overlap between the modes; the momentum mode carries no current and is thus neglected. On the other hand, when $\gamma \to 1$, as in a Galilean invariant system, the current and momentum are proportional and there is no relaxing current.

Multiplying the kinetic equation by $v^i_k - \gamma k^i/\bar{m}$ and k^i and summing over k, we obtain

$$\begin{aligned}\partial_t J^i_r(t) - \frac{\rho}{\bar{m}}E^i = \sum_{k,q}\frac{v^i_k - \gamma k^i/m}{1-\gamma}4\int dt'{\rm Im}\\ \times [iD_K(q,t,t')\delta\Pi'_A(k,q,t',t)\\ + iD_R(q,t,t')\delta\Pi'_K(k,q,t',t)\\ + \delta iD_K(q,t,t')\Pi'_A(k,q,t',t)\\ + i\delta D_R(q,t,t')\Pi'_K(k,q,t,t')], \quad (101)\end{aligned}$$

$$\partial_t P^i(t) - \rho E^i(t) = 0. \quad (102)$$

Above, in the first equation, we have used that $\sum_k(v_k - \gamma k/\bar{m})\delta n_k = (1-\gamma)J^i_r$ as follows from Eq. (100). The second equation above just follows from conservation of momentum. We also do not write the explicit dependence of the momentum labels on the vector potential as it is understood that it is always the gauge-invariant combination that appears.

As the momentum mode has trivial dependence due to momentum conservation, we will simply drop it and set $\gamma = 0$ in Eq. (101). In this limit,

$$\delta n_k(t) = \frac{\bar{m}}{\rho}J^i_r(t)\nabla^i_k n^{\rm eq}_k. \quad (103)$$

These are now a closed set of equations in terms of the single unknown function $J_r(t)$, which we will denote simply as $J(t)$. The remaining step is to evaluate the various terms. We note that $D_K(t,t')$ is generally larger than D_R by the factor T/λ_q. Thus, in what follows, we will only retain the first and third terms on the right-hand side of Eq. (101).

E. Time-dependent kinetic coefficients

We begin by evaluating the first term, proportional to $\delta\Pi^i_R(q,t,t')$ in Eq. (101). There are two contributions, one from varying n_k, and we denote it by K_J. The second is from varying g_R, and we denote it by K_E:

$$4\mathrm{Im}\int dt'\sum_{k,q} v^i_k iD_K(q,t,t')\delta\Pi'_A(k,q,t',t) = K^i_J(t) + K^i_E(t). \tag{104}$$

In Appendix C 1, we show that the term coming from varying n_k is

$$K^i_J(t) \simeq \sum_q F_q(t)J^j(t)\tilde{Q}^{ij}_J(t),$$
$$\tilde{Q}_{ij}(t) \approx \frac{\bar{m}\pi q^r q^l}{4\rho}\langle m^{-1}_{jr}m^{-1}_{il}\rangle_{\mathrm{FS}}. \tag{105}$$

To estimate the magnitude of the above term, we neglect factors of order ‘, write $\rho \sim \nu k_F^2/m$, and obtain that this term is $\sim(q/k_F)^2/\nu$.

For the term coming from varying g_R, we show in Appendix C 2 that

$$K^i_E(t) \simeq \sum_q F_q(t)\frac{\rho E^j(t)}{\bar{m}}\tilde{Q}^{ij}_E(t),$$
$$\tilde{Q}^{ij}_E(t) \approx \frac{28\pi\bar{m}q^r q^l}{16T\rho}\zeta'(-2)\langle m^{-1}_{jr}m^{-1}_{il}\rangle_{\mathrm{FS}}. \tag{106}$$

The resulting term has the form $\propto (q/k_F)^2F_q(t)\rho E(t)/\nu T\bar{m}$ and is thus a parametrically small correction to the drift term $\rho E/\bar{m}$. Therefore, we neglect it for the remainder.

We now turn to the third term, $\sim\delta D_K\Pi'_A$ in Eq. (101). We begin by considering the change in δD_K. This in turn depends on δD_R, which from Eq. (77) is given by

$$[\partial_t + \lambda_q]\delta D_R = -\delta\lambda_q D_R. \tag{107}$$

There are two contributions to $\delta\lambda_q$. One is from varying δn via

$$\lambda_q = U^{-1} - \mathrm{Re}[\Pi_R(q)] = U^{-1} + \frac{1}{4}\sum_k \frac{n_{k+q/2}+n_{-k+q/2}}{\epsilon_{k+q/2}+\epsilon_{-k+q/2}}. \tag{108}$$

Writing the above as a small q expansion, we define M such that

$$\lambda_q = r + \frac{q^2}{2M}, \tag{109}$$

where we find that, on expanding,

$$\sum_k \frac{n_{k+q/2}+n_{-k+q/2}}{2\epsilon_k} = \sum_k \frac{2n_k}{2\epsilon_k} + q^iq^j\sum_k \frac{\nabla^i_k\nabla^j_k n_k}{8\epsilon_k}$$
$$= \sum_k \frac{2n_k}{2\epsilon_k} + \frac{q^2}{d}\sum_k\frac{\nabla^2_k n_k}{8\epsilon_k},$$
$$\Rightarrow \frac{1}{M} = \partial^2_q\lambda_q = \frac{1}{4d}\sum_k \frac{\nabla^2_k n_k}{4\epsilon_k}. \tag{110}$$

where d is the spatial dimension. The above equation also shows $M \sim T/v_F^2$

Thus, the change in λ_q coming from the change in the electron distribution δn_k may be evaluated as follows (below we also use $\delta n_k = -\delta n_{-k}$):

$$\delta\lambda_q = \frac{1}{4}\sum_k\frac{\delta n_{k+q/2}+\delta n_{-k+q/2}}{\epsilon_{k+q/2}+\epsilon_{-k+q/2}}$$
$$= \frac{1}{4}\sum_k\frac{\delta n_{k+q/2}-\delta n_{k-q/2}}{\epsilon_{k+q/2}+\epsilon_{-k+q/2}}$$
$$\simeq \frac{q^i}{4}\sum_k\frac{\nabla^i_k\delta n_k}{2\epsilon_k} \quad (q\to 0). \tag{111}$$

Using the relation between δn_k and the current in Eq. (103) and the expression for M in Eq. (110), we find

$$\delta\lambda_q = \frac{q^i\bar{m}}{4\rho}J^j\sum_k\frac{\nabla^i_k\nabla^j_k n^{\mathrm{eq}}_k}{2\epsilon_k}$$
$$= \frac{q^i\bar{m}}{4\rho}J^i\frac{1}{d}\sum_k\frac{\nabla^2_k n^{\mathrm{eq}}_k}{2\epsilon_k}$$
$$= 2\frac{\bar{m}}{\rho M}q^iJ^i. \tag{112}$$

The second reason for the change in λ_q is due to the direct coupling to the electric field, where, defining $\delta A^i(s) = A^i(s) - A^i(t)$,

$$\lambda_q(s) = [q^i + 2\delta A^i(s)]^2/2M,$$
$$\Rightarrow \delta\lambda_q(s) = \frac{2q^i\delta A^i(s)}{M} = \frac{2q^i}{M}[A^i(s)-A^i(t)]$$
$$= -\frac{2q^i}{M}\int_s^t dt'\partial_{t'}A^i(t'),$$
$$\Rightarrow \delta\lambda_q(s) = \frac{2q^i}{M}\int_s^t dt'E^i(t'). \tag{113}$$

Thus, the total change in λ_q is

$$\delta\lambda_q(s) = \frac{2\bar{m}q^iJ^i(s)}{M\rho} + \frac{2q^i}{M}\int_s^t dt'E^i(t')$$
$$= \frac{2\bar{m}q^i}{\rho M}\left[J^i(s) + \int_s^t dt'\frac{\rho E^i(t')}{\bar{m}}\right]. \tag{114}$$

$\delta\lambda_q$ changes both D_R and D_K, where the change in the former is

$$\delta D_R(t,t') = e^{-\lambda_q(t-t')}\left[-\int_{t'}^t ds\delta\lambda_q(s)\right]. \tag{115}$$

Note that $\delta D_K = (\delta D_R)\Pi_K D_A + D_R(\delta\Pi_K)D_A + D_R\Pi_K(\delta D_A)$. Since the variation $\delta\Pi_K$ produces a term that is smaller by a factor of F_q, we neglect it. Using the zeroth-order calculation that $i\Pi_K \propto \nu\delta(t-t')$, we have

$$i\delta D_K(t,t') = \nu\int ds\delta D_R(t,s)D_A(s,t')$$
$$+ \nu\int ds D_R(t,s)\delta D_A(s,t'). \tag{116}$$

Substituting in for the result for δD_R, using $\delta D_R(t,t') = \delta D_A(t',t)^*$, we obtain (see Appendix C 3 for details)

$$4\int_0^t dt' \sum_{k,q} \mathrm{Im}[v_k^i i\delta D_K(q,t,t')\Pi_A'(k,q,t',t)]$$
$$= \sum_q \alpha q^i q^j \lambda_q \int_0^t du e^{-2(t-u)\lambda_q} F_q(u)$$
$$\times \left[J^j(u) + \int_u^t dt' \frac{\rho E^j(t')}{\bar{m}}\right]. \quad (117)$$

The coefficient α is given by

$$\alpha \equiv \frac{T\bar{m}}{M\nu\rho}\sum_k m_{ii}^{-1}\frac{n^{\mathrm{eq}}(\epsilon_k)}{4\epsilon_k^2+\lambda_q^2}$$
$$= \frac{T\bar{m}}{M\nu\rho}\int d\epsilon\, \kappa(\epsilon)\frac{n^{\mathrm{eq}}}{4\epsilon^2+\lambda^2},$$
$$\kappa(\epsilon) \equiv \sum_k m_{ii}^{-1}(k)\delta(\epsilon-\epsilon_k). \quad (118)$$

The above shows that $\alpha q^2 \sim (q/k_F)^2/\nu$, which is comparable to the local terms. Moreover, the sign of α for generic band structures is such as to oppose the local in time term coming from $iD_K\delta\Pi_A$.

The function $\kappa(\epsilon)$ may be Taylor expanded to give $\kappa(\epsilon) \approx \frac{\nu}{\bar{m}}(1 + b\epsilon/E_F + \cdots)$, where the parameter b is a material-dependent parameter of order 1. The sign of b is not fixed in general. However, for a "simple" band structure, the parameter b is negative. Thus,

$$\alpha \approx \frac{T}{M\rho}\int d\epsilon \left(1 + b\frac{\epsilon}{E_F}\right)\frac{n^{\mathrm{eq}}(\epsilon)}{4\epsilon^2+\lambda^2}$$
$$= \frac{bT}{M\rho E_F}\int d\epsilon \frac{\epsilon n^{\mathrm{eq}}(\epsilon)}{4\epsilon^2}$$
$$\approx \frac{bT\log(E_F/T)}{M\rho E_F} \approx \frac{bv_F^2 T\log(E_F/T)}{T\nu E_F^2}$$
$$\approx \frac{1}{k_F^2\nu} b\log(E_F/T). \quad (119)$$

For generic quench profiles, the only change that is needed is the replacement

$$e^{-2\lambda_q t} \to e^{-2\int_0^t ds \lambda_q(s)}. \quad (120)$$

In what follows, we set $\rho/\bar{m} = 1$. In these units, the ratio J/E gives a Drude scattering time and also equals the conductivity. The kinetic equation for the current for general quench trajectories becomes Eq. (4), and for convenience we rewrite it below:

$$\partial_t J^i(t) - E(t) = -\frac{1}{\tau_r}\left[A(t)J^i(t) - \alpha\int_0^t dt'\left\{B(t,t')J^i(t')\right.\right.$$
$$\left.\left. + C(t,t')E^i(t')\right\}\right]. \quad (121)$$

We define the dimensionless quantity,

$$A(t) \propto \sum_q q^2 F(q,t). \quad (122)$$

The memory terms are

$$B(t,s) \propto \sum_q \left\{q^2\lambda_q e^{-2\int_s^t du\lambda_q(u)}F_q(s)\right\}, \quad (123)$$
$$C(t,s) = \int_0^s ds' B(t,s'). \quad (124)$$

We now simplify the expressions for A, B, C using the derived expressions for the time-dependent superconducting fluctuations:

$$A(t) = \frac{1}{T^{d/2}}\int_0^t ds \frac{e^{-2\int_s^t du\, r(u)}}{(t-s)^{d/2+1}}. \quad (125)$$

Similarly, the memory term is

$$B(t,s) \propto \sum_q \left\{q^2\lambda_q e^{-2\int_s^t du\lambda_q(u)}F_q(s)\right\} \quad (126)$$
$$= \frac{1}{T^{d/2}}\int_0^s dt'\, e^{-2\int_{t'}^t du\, r(u)}$$
$$\times \left[\frac{(1+d/2)+2r(t)(t-t')}{(t-t')^{d/2+2}}\right]. \quad (127)$$

For the case of the sudden quench, since $r(t) = \theta(t)r + r_i$, the above expressions for A, B, C simplify considerably. For $r_i/T = \mathcal{O}(1)$ so that at $t = 0$, $F(q,t=0) = 0$, and for a quench to the critical point $r = 0$, Eqs. (125), (127), and (124) give

$$A_{\mathrm{cr}}(t) = \left[1 - \frac{2/d}{(1+Tt)^{d/2}}\right], \quad (128)$$
$$B_{\mathrm{cr}}(t,s) = T\left\{\frac{1}{[T(t-s)+1]^{1+d/2}} - \frac{1}{[Tt+1]^{1+d/2}}\right\}, \quad (129)$$
$$C_{\mathrm{cr}}(t,s) = \frac{2}{d}\left\{\frac{1}{[T(t-s)+1]^{d/2}} - \frac{1+(d/2)\frac{Ts}{(Tt+1)}}{[Tt+1]^{d/2}}\right\}. \quad (130)$$

Above we have regularized the integrals such that a short time cutoff of T^{-1} has been introduced. It is also helpful to study the system at nonzero detuning. For a sudden quench to a distance r from the critical point, Eqs. (125), (127), and (124) give

$$A(r,t) = 1 - \left(\frac{2r}{T}\right)^{d/2}\Gamma\left(-\frac{d}{2}, 2rt\right), \quad (131)$$
$$B(r,t,s) = T\frac{e^{-2r(t-s)}}{[T(t-s)]^{1+d/2}} - T\frac{e^{-2rt}}{(Tt)^{1+d/2}}, \quad (132)$$
$$C(r,t,s) = \left(\frac{2r}{T}\right)^{d/2}\left[\Gamma\left(-\frac{d}{2}, 2r(t-s)\right) - \Gamma\left(-\frac{d}{2}, 2rt\right)\right]$$
$$- \frac{Tse^{-2rt}}{(Tt)^{1+d/2}}. \quad (133)$$

F. Charge diffusion

We now adapt the kinetic equation to the case where the current is being driven by a density gradient. From Eq. (58), Fourier transforming with respect to the difference in position coordinates $r_1 - r_2$ and performing a gradient expansion with respect to the center-of-mass coordinate $r = \frac{r_1+r_2}{2}$ gives us

$$\partial_t n_k(r,t) + \vec{v}_k \cdot \vec{\nabla} n_k(r,t) = S(r,k,t), \tag{134}$$

where $S(k) = \int d^d(r_1 - r_2) e^{-ik(r_1-r_2)} S(r_1 t, r_2 t)$. A spatial density gradient leads to a current $J^i = \sum_k v_k^i n_k$. Multiplying Eq. (134) with v_k, taking a sum on k, and using the fact that $v_k S(k)$ in the presence of the current can be written as a contribution coming from small changes to the superconducting propagator, and small changes to the polarization bubble as summarized on the rhs of Eq. (89). The rhs can be simplified as before, leading to the rhs of Eq. (121). This leads to

$$\begin{aligned}\partial_t J^i &+ \sum_k v_k^i v_k^j \nabla^j n_k(r,t) \\ &= -\frac{1}{\tau_r}\left[A(t)J^i(t) - \alpha \int_0^t dt' \left\{B(t,t')J^i(t') \right.\right. \\ &\quad \left.\left. + \int dt' C(t,t') E^i(t')\right\}\right].\end{aligned} \tag{135}$$

The coefficients A, B are given in Eqs. (125) and (127) for a general quench profile and in Eqs. (128) and (129) for a rapid quench to the critical point. Although there is no external electric field, the density gradient induces a current, and the electric field on the rhs is a linear response to this current.

V. CONDUCTIVITY AND DIFFUSION COEFFICIENT IN THERMAL EQUILIBRIUM

In thermal equilibrium, one may simply take the long time limit of the expressions in Eqs. (128), (129), and (130). Then, at zero detuning $r = 0$,

$$\begin{aligned} A_{\rm eq}(r=0) &= 1, \\ B_{\rm eq}(r=0, t-s) &= \frac{T}{[T(t-s)+1]^{1+d/2}}, \\ C_{\rm eq}(r=0, t-s) &= \frac{(2/d)}{[T(t-s)+1]^{d/2}}. \end{aligned} \tag{136}$$

It is also useful to analyze these coefficients when the system has equilibrated at a nonzero detuning r away from the critical point. In this case, for $d = 2$ we have

$$\begin{aligned} A_{\rm eq}(r) &= A_{\rm eq}(r=0)\left[1 + \frac{2r}{T}\ln(2r/T)\right], \\ B_{\rm eq}(r, t-s) &= T\frac{e^{-2r(t-s)}}{[1+T(t-s)]^2}, \\ C_{\rm eq}(r, t-s)|_{r(t-s)\gg 1} &= \frac{T}{2r}\frac{e^{-2r(t-s)}}{[1+T(t-s)]^2}. \end{aligned} \tag{137}$$

Above, due to the separation of timescales discussed in the previous section, $|r|/T \ll 1$. We note that in Fourier space

$$\tilde{B}_{\rm eq}(r, \omega=0) = 1 + \frac{2r}{T}[\gamma + \ln(2r/T)], \tag{138}$$

$$\tilde{C}_{\rm eq}(r, \omega=0) = \frac{1}{2r}\left\{1 + \frac{2r}{T}\left[\gamma + \ln(2r/T)\right]\right\}. \tag{139}$$

Above γ is Euler γ.

We now discuss the linear response conductivity in thermal equilibrium. In this case, all the coefficients A, B, C are time-translation invariant. Fourier transforming Eq. (121), we obtain

$$J(\omega) = \frac{1 + \tau_r^{-1}\alpha\tilde{C}_{\rm eq}(\omega)}{i\omega + \tau_r^{-1}[1 - \alpha\tilde{B}_{\rm eq}(\omega)]}E(\omega). \tag{140}$$

We write

$$J = J_{\rm diss} + J_{\rm fl}, \tag{141}$$

where the first is a dissipative current arising due to Drude scattering and the second is a current arising due to the superconducting fluctuations. Then, if we take the DC limit of Eq. (140),

$$J_{\rm diss} = \tau_r E = \sigma_{\rm diss} E, \tag{142}$$

while the current from the superconducting fluctuations (neglecting B which gives small correction to the Drude scattering rate) is

$$J_{\rm fl} = \alpha\tilde{C}_{\rm eq} E = \sigma_{\rm fl} E. \tag{143}$$

The above implies that the fluctuation conductivity $\sigma_{\rm fl}$ gives a correction to the dissipative conductivity τ_r by an amount $= \tau_r^{-1}\alpha\tilde{C}_{\rm eq}(r, \omega=0)$. If we note that $\tau_r \sim T^{-1}$ and use Eq. (139), this implies a fluctuation conductivity correction that goes as $\propto \alpha T/r$. While this is qualitatively the same as the fluctuation AL conductivity [31,32], we note that for the ultraclean case considered here, material-dependent parameters such as α, which determine the strength of Galilean invariance breaking lattice effects, are unavoidable and give a nonuniversal material-dependent prefactor.

The MT correction is also discussed in the context of a disordered system [33–35,43] and is well defined only as long as the mean free path is shorter than the inelastic scattering time [51,52]. Thus, the MT conductivity does not emerge naturally in the clean limit we consider here.

Now, we discuss the diffusion constant D defined as $J^i = -D\nabla^i n$. Because of time-translation invariance in equilibrium, we write Eq. (135) in Fourier space. We also note that the electric field generated as a response to the current is given by $J = \tau_r E$, up to fluctuation corrections. This gives the diffusion constant,

$$D(\omega) = \frac{v_F^2/2}{i\omega + \tau_r^{-1}\left\{1 - \alpha\left(\tilde{B}_{\rm eq}(\omega) + \frac{\tilde{C}_{\rm eq}(\omega)}{\tau_r}\right)\right\}}. \tag{144}$$

Close to the critical point, C dominates over B. Moreover, Taylor expanding in α, the DC limit of the diffusion constant is

$$D(\omega=0) \approx \frac{v_F^2\tau_r}{2}\left[1 + \frac{\alpha}{2\tau_r r}\right]. \tag{145}$$

Thus, the fluctuation correction for the diffusion constant has qualitatively the same form as that for the conductivity in being $\propto \alpha T/r$, where we have used that $\tau_r \sim T^{-1}$.

VI. SOLVING THE KINETIC EQUATION FOR THE CONDUCTIVITY

The solution of the kinetic equation for the current was presented in much detail in Ref. [44]. For completeness, we summarize some of the findings. Two kinds of quench trajectories were studied. One was a rapid quench from deep in the disordered phase to a distance $r \geqslant 0$ away from the critical point. The second was a smooth quench trajectory which started from the disordered phase, approached the critical point $r = 0$, and returned back to the disordered phase. Regardless of the details of the trajectory, the current showed slow dynamics because of slowly relaxing superconducting fluctuations. In the kinetic equation Eq. (121), this physics is encoded in the memory terms (B, C).

In addition, the dynamics of the current for a critical quench to $r = 0$ was found to show universal behavior, with a power law aging in the conductivity $\sigma(t, t')$, a result also found for a disordered system [30]. Moreover, the dynamics at nonzero detuning r was shown to obey scaling collapse. However, it should be noted that, for the clean system studied here, the exponents entering the scaling behavior were nonuniversal in that they depend on [44] $T\tau_r$.

For a smooth quench, it was shown that transiently enhanced superconducting fluctuations create transient a low resistance current carrying channel, whose signature is a suppression of the Drude scattering rate at low frequencies. Note that the Drude scattering rate in the absence of time-translation invariance is defined as

$$\tau_{\rm Dr}(t, \omega) = -\frac{\mathrm{Im}[\sigma(\omega, t)]}{\omega \mathrm{Re}[\sigma(\omega, t)]}, \tag{146}$$

with

$$\sigma(\omega, t) = \int d\tau e^{i\omega\tau}\sigma(t + \tau, t). \tag{147}$$

In this section, we revisit the smooth quench, defined by the trajectory

$$r(t)/T = 1 - \theta(t)(1 + \epsilon)(tT/30)e^{1-tT/30}. \tag{148}$$

Above, r/T starts out being 1, smoothly approaches $-\epsilon$ at a time $Tt_* = 30$, and then smoothly returns back to its initial value of $r/T = 1$. We consider two cases, one where $\epsilon = 0$ (see Fig. 2) and one where the quench is supercritical in that $\epsilon = 0.05$. For the latter case, for a finite time, the parameters of the Hamiltonian correspond to that of an ordered phase since $r < 0$ (see Fig. 3). Despite this, for such a transient regime, the order parameter is still zero, although superconducting fluctuations are significantly more enhanced than if $r \geqslant 0$. In particular, critical slowing down prevents true long-range order to develop over timescales over which the microscopic parameters are varying.

Note that in the numerical simulation, we apply an electric field pulse which is a δ function in time. In particular, for an electric field pulse of unit strength centered at t', the conductivity equals the current, $\sigma(t, t') = J(t)$. Fourier transforming with respect to $t - t'$, we can obtain the conductivity for arbitrary ω. However, in an actual experiment, the finite temporal width of the electric field pulse places a limit on the lowest frequency accessible. Nevertheless, the physics of a suppressed Drude scattering at low frequencies is visible over a sufficiently broad range of frequencies, as shown in Figs. 2 and 3, for this to be a feasible experimental observation.

Top panels of Figs. 2 and 3 show how the superconducting fluctuations, at several different wavelengths, evolve for a critical and a supercritical quench respectively. The lower panels of the same figures show how the corresponding Drude

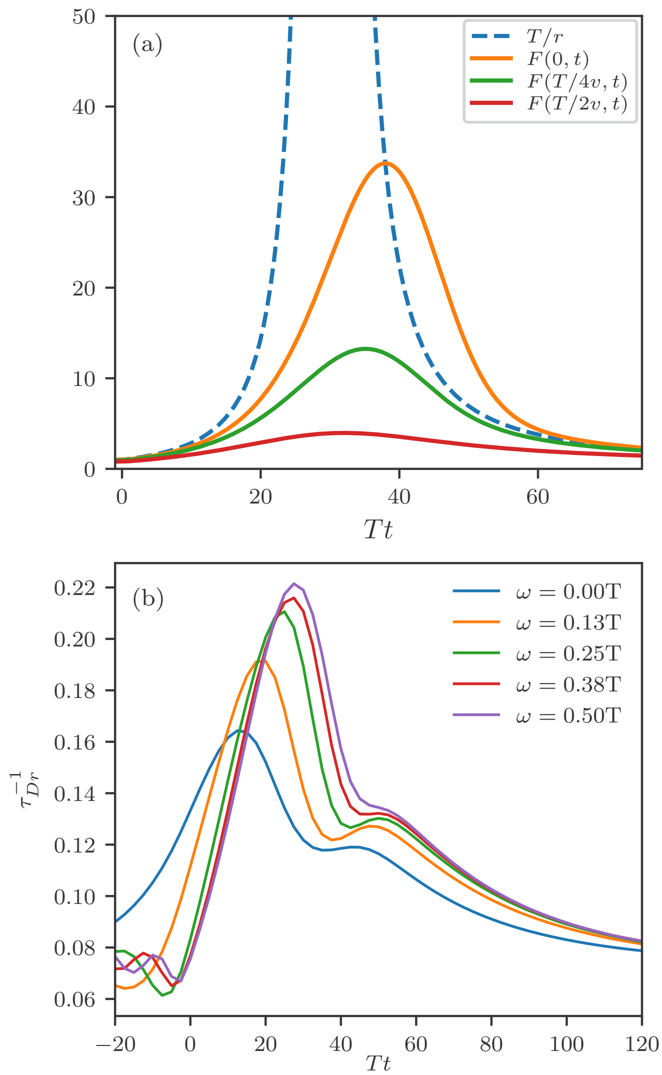

FIG. 2. Critical quench. (a) Fluctuations $F(q, t)$ for three different q and for a critical quench where the detuning varies smoothly as $r(t)/T = 1.0 - \theta(t)(Tt/30)e^{1-Tt/30}$. In thermal equilibrium, $F(q = 0) = T/r$. As a reference, $T/r(t)$ is plotted. (b) Drude scattering rate for the critical quench $r(t)/T = 1.0 - \theta(t)(Tt/30)e^{1-Tt/30}$. The high-frequency Drude scattering rate follows the profile of the fluctuations (top figure) by smoothly increasing and decreasing in time. On the other hand, the low-frequency Drude scattering rate first increases and then is suppressed approximately when the superconducting fluctuations peak.

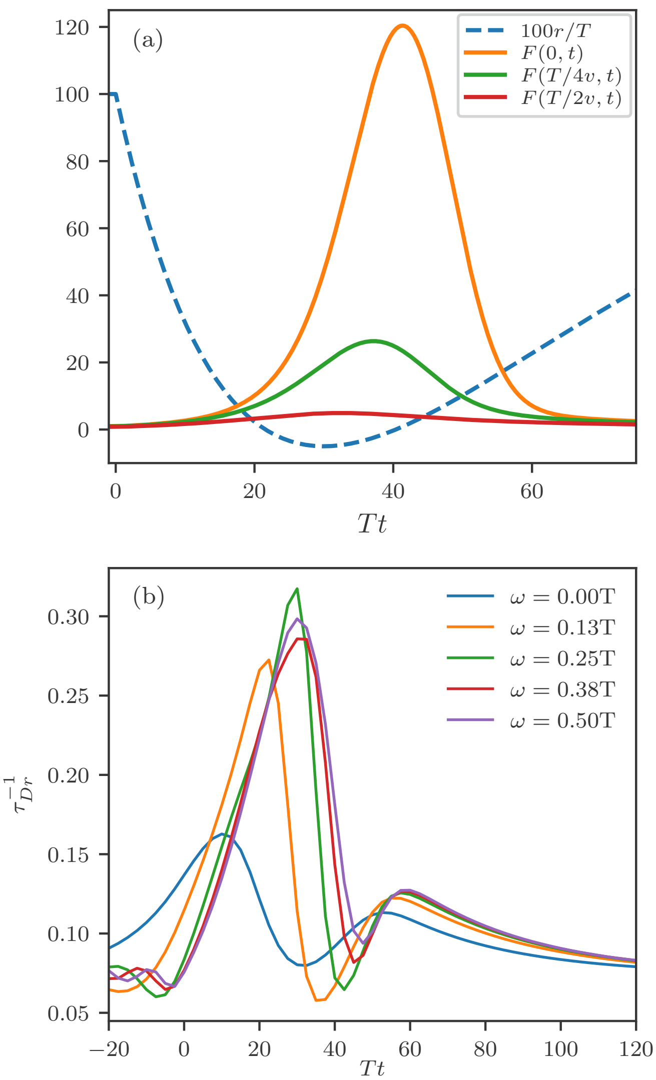

FIG. 3. Supercritical quench. (a) Fluctuations $F(q,t)$ for three different q and for a supercritical quench where the detuning varies smoothly as $r(t)/T = 1.0 - \theta(t)(1+0.05)(Tt/30)e^{1-Tt/30}$, becoming negative for a certain length of time. In thermal equilibrium, $F(q=0) = T/r$. As a reference, $100r(t)/T$ is plotted. (b) Drude conductivity for a supercritical quench where $r(t)/T = 1.0 - \theta(t)(1+0.05)(Tt/30)e^{1-Tt/30}$. Qualitatively similar behavior as in Fig. 2, but all the effects are more amplified. As the density of superconducting fluctuations $F(q,t)$ (top figure) grow in time, the high-frequency Drude scattering rate increases, while the low-frequency Drude scattering rate is suppressed at approximately when the superconducting fluctuations peak. The behavior in frequency and time is also more dispersive for the supercritical quench than the critical quench.

scattering rate, for different frequencies, evolve in time. When the density of superconducting fluctuations peak, the Drude scattering rate at low frequencies $\omega < T$ dips, with the effect being more enhanced for the supercritical quench. Moreover, the Drude scattering rate is strongly dispersive in that different frequency components of the Drude scattering rate peak at different times.

Recall that when the system returns to the normal phase, since we are in the clean limit, the steady-state Drude scattering rate is zero. A slow power-law approach to thermal equilibrium is also visible in the long time tails of the Drude scattering rate that are found to persist long after the detuning has returned back to its initial value in the normal phase. The slow dynamics highlighted above provide signatures in time-resolved optical conductivity, where, although the system lives for too short a time for true long-range order to develop, the transient dynamics can still show clear signatures of superconducting fluctuations.

VII. SPECTRAL PROPERTIES

We now present results for the time evolution of the electron spectral properties. Results for the electron lifetime obtained from the imaginary part of the electron self-energy appear in Ref. [53], where it was shown that the behavior is not Fermi liquid like due to enhanced Andreev scattering of the electrons at the Fermi energy. In this section, we discuss how this scattering affects the local density of states.

For simplicity, we consider a rapid quench from deep in the normal phase where $F(q) = 0$ initially, to a final arbitrarily small detuning $r \geqslant 0$ away from the critical point. For this quench, the superconducting fluctuations obey the equation

$$\left[\frac{1}{T}\partial_t + 2\left(\frac{v^2}{T^2}q^2 + r\right)\right]F(q^2,t) = 2, \tag{149}$$

where the solution of the above is identical to Eq. (82) with $\lambda_q/T = r + v^2q^2/T^2$. We have assumed that F depends only isotropically on q and follow a convention where r is dimensionless. We define the dimensionless variable $x = v^2q^2/T^2$. The electron self-energy using Eq. (54f) and in the limit of $D_K \gg D_R$, and in two spatial dimensions, is

$$\Sigma_{R,A}(k,t_1,t_2) = \frac{i}{2}\int \frac{d^2q}{(2\pi)^2} D_K(q,t_1,t_2) G_{A,R}(-k+q,t_2,t_1). \tag{150}$$

Going into Wigner coordinates, which involves Fourier transforming with respect to relative coordinates $t_1 - t_2$ while keeping the mean time $t = (t_1+t_2)/2$ dependence in D_K [53], and performing an angular integral, we obtain

$$\frac{\Sigma_R(\omega,\varepsilon)}{T} = +\frac{\text{Gi}}{4\pi}\int_0^\infty dx F(x,t) \times \left[\left(\frac{\omega+\varepsilon+\Sigma_A(-\omega,\varepsilon)}{T}\right)^2 - x\right]^{-1/2}, \tag{151}$$

$$\frac{\Sigma_A(-\omega,\varepsilon)}{T} = +\frac{\text{Gi}}{4\pi}\int_0^\infty dx F(x,t) \times \left[\left(\frac{-\omega+\varepsilon+\Sigma_R(\omega,\varepsilon)}{T}\right)^2 - x\right]^{-1/2}, \tag{152}$$

$$\text{where Gi} \equiv \frac{8T}{\pi\nu v^2}. \tag{153}$$

We start in the perturbative limit by setting $\Sigma_A = i0^+$, $\Sigma_R = -i0^+$. First, let us study the equilibrium case. Here, $F = 1/(x+r)$. Thus, we have

$$\frac{\Sigma_R(\omega,\varepsilon)}{T} = \frac{\text{Gi}}{8\pi}\int_0^\infty \frac{dx}{x+r} \times \left[\left(\frac{\omega+\varepsilon+\Sigma_A(-\omega,\varepsilon)}{T}\right)^2 - x\right]^{-1/2}. \quad (154)$$

$$\text{Define } \bar{x} \equiv \frac{x}{r}; \quad z_+^0 \equiv \frac{\omega+\varepsilon+i0^+}{T}\sqrt{\frac{1}{r}}, \quad (155)$$

$$\frac{\Sigma_R(\omega,\varepsilon)}{T} = \frac{\text{Gi}}{8\pi}\sqrt{\frac{1}{r}}\int_0^\infty \frac{d\bar{x}}{1+\bar{x}}\left[(z_+^0)^2 - \bar{x}\right]^{-1/2} = \frac{\text{Gi}}{8\pi\sqrt{r}} f(z_0^+), \quad (156)$$

$$\text{where } \text{Re}[f(z)] \equiv \frac{1}{\sqrt{z^2+1}}\log\frac{z+\sqrt{1+z^2}}{-z+\sqrt{1+z^2}}. \quad (157)$$

The density of states is

$$\nu(\omega) = -\frac{\nu_0}{\pi}\int d\varepsilon \Im[G^R(\varepsilon,\omega)]. \quad (158)$$

To evaluate this integral, we need to locate the pole in ε complex plane: $\omega - \varepsilon^*(\omega) - \Sigma^R(\varepsilon^*(\omega),\omega) = 0$. Then, the value of the integral follows from the residue

$$\nu(\omega) = \frac{\nu_0}{\pi}\Im\left[\frac{\pi i}{1+\frac{\partial\Sigma^R}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon^*}}\right]. \quad (159)$$

Thus, the perturbative change in the density of states is obtained from Taylor expanding

$$\frac{\delta\nu(\omega)}{\nu} = -\text{Re}[\partial_\varepsilon \Sigma_R(\omega,\varepsilon)]|_{\varepsilon=\omega} = -\text{Re}\sqrt{\frac{1}{r}}\frac{\text{Gi}}{8\pi}\sqrt{\frac{1}{r}}\partial_z f(z)|_{z=\frac{2\omega}{T\sqrt{r}}} = \frac{\text{Gi}}{8\pi r} g\left(\frac{2\omega}{T\sqrt{r}}\right), \quad (160)$$

$$\text{where} \quad g(z) = -\text{Re}[\partial_z f(z)]. \quad (161)$$

Using Eq. (157), one obtains the following asymptotic behavior:

$$\frac{\delta\nu(\omega)}{\nu} = \frac{\text{Gi}}{4\pi r}\begin{cases} -1 & \omega \ll T\sqrt{r} \\ \frac{rT^2}{8\omega^2}\log\left(\frac{4\omega^2}{rT^2}\right) & \omega \gg T\sqrt{r} \end{cases}. \quad (162)$$

Note that this shift is perturbative for all ω, that is, $\delta\nu/\nu \ll 1$, when $\text{Gi}/4\pi r \ll 1$.

Now we turn to the critical quench, where r is $\mathcal{O}(1)$ for $t<0$ and $r=0$ for $t>0$. In this case, the solution of Eq. (149) gives

$$F(x,t) = \frac{1-e^{-2xTt}}{x}. \quad (163)$$

The above implies

$$\frac{\Sigma_R(\omega,\varepsilon)}{T} = \frac{\text{Gi}}{8\pi}\int_0^\infty dx \frac{1-e^{-2xTt}}{x} \times \left[\left(\frac{\omega+\varepsilon+i0^+}{T}\right)^2 - x\right]^{-1/2}; \quad (164)$$

$$\text{define } \bar{x} = 2xTt; \quad z_+^0 \equiv \frac{\omega+\varepsilon+i0^+}{T}\sqrt{2tT}, \quad (165)$$

$$\frac{\Sigma_R(\omega,\varepsilon)}{T} = \frac{\text{Gi}}{4\pi}\sqrt{\frac{Tt}{2}}\int_0^\infty d\bar{x}\,\frac{1-e^{-\bar{x}}}{\bar{x}}[(z_+^0)^2-\bar{x}]^{-1/2} = \frac{\text{Gi}}{8\pi}(\sqrt{2Tt})h(z_0^+). \quad (166)$$

In Appendix D, we show that h has the following asymptotic form:

$$\text{Re}[h(z)] = \begin{cases} 2z & |z| \ll 1 \\ z^{-1}\left(\log(z^2) + \Gamma\left(\frac{1}{2}\right)\right) & |z| \gg 1 \end{cases}. \quad (167)$$

Proceeding to the change in the density of states, we have

$$\frac{\delta\nu(\omega)}{\nu} = -\frac{\text{Gi}}{4\pi\sqrt{2}}\sqrt{Tt}\,\text{Re}[T\partial_\varepsilon h(z_+^0)]|_{\varepsilon=\omega} = -\frac{\text{Gi}}{8\pi}(2Tt)\,\text{Re}[\partial_z h(z)]|_{z=2\sqrt{2Tt}\frac{\omega}{T}} \quad (168)$$

$$= \frac{\text{Gi}}{4\pi}\begin{cases} -2Tt & \omega \ll \sqrt{T/t} \\ \frac{1}{8}\frac{T^2}{\omega^2}\log(\omega^2 t/T) & \omega \gg \sqrt{T/t} \end{cases}. \quad (169)$$

Note that the asymptotic behaviors of the critical and equilibrium cases agree if we set $r = \frac{1}{2Tt}$. This follows from the fact that the fluctuations $F(x,t)$ behave in a comparable manner in these two cases.

A. Self-consistent regime (results for $\omega = 0$)

We now turn to the full solution that does not require $\delta\nu \ll \nu$. We will first work out the equilibrium case. The critical quench case will then follow from the substitution of $r = \frac{1}{2Tt}$.

By adding $\pm\omega+\varepsilon$, the self-consistent equation for the self-energy in Eqs. (151) and (152) may be written as

$$z_+ = z_+^0 + \gamma f(z_-), \quad (170)$$

$$z_- = z_-^0 + \gamma f(z_+), \quad (171)$$

where

$$z_+ \equiv \frac{\omega+\varepsilon+\Sigma_A(-\omega,\varepsilon)}{T}\sqrt{\frac{1}{r}}, \quad (172)$$

$$z_- \equiv \frac{-\omega+\varepsilon+\Sigma_R(\omega,\varepsilon)}{T}\sqrt{\frac{1}{r}}, \quad (173)$$

$$\gamma = \frac{\text{Gi}}{8\pi r}. \quad (174)$$

The above self-consistent equations can be solved in certain limits (see Appendix E), giving the following results for the density of states at zero frequency:

$$\nu(\omega=0) \propto \frac{\nu_0}{\log(\text{Gi}/8\pi r)}. \quad (175)$$

The critical quench case then follows by the substitution $r = \frac{1}{2Tt}$:

$$\nu(\omega = 0, t) \propto \frac{\nu_0}{\log(\mathrm{Gi}Tt/4\pi)}. \tag{176}$$

Thus, we find that there is no true steady state of the zero-frequency density of states because the solution continues to evolve logarithmically in time. This time evolution, although slow, happens because there is no non-zero-temperature critical point in $d = 2$. To obtain the correct thermalized steady state, vortices need to be accounted for in our treatment. However, for the transient regime relevant for experiments, where no superconducting gap has developed yet and the dynamics is governed by weakly interacting superconducting fluctuations, our results are still valid.

VIII. CONCLUSIONS

We have studied quench dynamics of an interacting electron system along general quench trajectories using a 2PI approach. To leading nontrivial order in $1/N$, our treatment reduced to RPA in the particle-particle channel. While the quantum kinetic equations are exact, leading to proper definitions of the conserved densities and currents, progress was made in solving them by exploiting a separation of timescales. This could be seen by noting that thermalization of the electron distribution function sets in on a timescale of T^{-1}, while the collective modes relax at a rate given by the detuning to the superconducting critical point, which can be arbitrarily slow. This is the phenomena of critical slowing down, and we derived an effective classical equation for the decay of the current in this regime. The dynamics of the current was also justified using a phenomenological mapping to model F in the Halperin-Hohenberg classification scheme. Going forward, we envision that such mappings may be helpful in other contexts when understanding time-resolved experiments that aim to probe collective modes.

Results were also presented for the local density of states, which was found to be enhanced at low frequencies due to Andreev scattering processes. For a critical quench, the self-consistent equations showed an equivalence between the time after the quench and the inverse detuning.

Future directions involve studying the coarsening regime, in particular, how true long-range order develops. The reverse quench where the initial state is ordered also needs to be explored, going beyond mean field. Experiments involving ultrafast manipulation of spin and charge order also require generalizing our study to quench dynamics in the particle-hole channel.

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APPENDIX A: PROVING CONSERVATION LAWS

1. Proof of $S(rt, rt) = 0$

Using Eq. (54),

$$\begin{aligned} S(rt, rt) = &\frac{i}{2}\int dy\{G_A(y;r,t)G_A(y;r,t)D_R(r,t;y) + G_K(y;r,t)G_A(y;r,t)D_K(r,t;y)\} \\ &+ \frac{i}{2}\int dy\{G_K(y;r,t)G_K(y;r,t)D_R(r,t;y) + G_A(y;r,t)G_K(y;r,t)D_K(r,t;y)\} \\ &- \frac{i}{2}\int dy\{G_R(r,t;y)G_R(r,t;y)D_A(y;r,t) + G_R(r,t;y)G_K(r,t;y)D_K(y;r,t)\} \\ &- \frac{i}{2}\int dy\{G_K(r,t;y)G_R(r,t;y)D_K(y;r,t) + G_K(r,t;y)G_K(r,t;y)D_A(y;r,t)\}. \end{aligned} \tag{A1}$$

The above terms can be rearranged to give

$$S(rt, rt) = 2\int d\{\Pi_K(y;r,t)D_R(r,t;y) + \Pi_A(y;r,t)D_K(r,t;y)\} - 2\int dy\{\Pi_K(r,t;y)D_A(y;r,t) + \Pi_R(r,t;y)D_K(y;r,t)\}. \tag{A2}$$

Using Eq. (54e), the second and fourth terms above are rewritten to give

$$\begin{aligned} S(rt, rt) = &2\int dy\{D_R(r,t;y)\Pi_K(y;r,t) + (D_R\Pi_K D_A)_{r,t;y}\Pi_A(y;r,t)\} \\ &- 2\int dy\{\Pi_K(r,t;y)D_A(y;r,t) + \Pi_R(r,t;y)(D_R\Pi_K D_A)_{y;r,t}\}. \end{aligned} \tag{A3}$$

Using Eq. (54d) and writing $\Pi_{A,R}D_{A,R} = U^{-1}D_{A,R} - 1$, or equivalently $D_{A,R}\Pi_{A,R} = D_{A,R}U^{-1} - 1$, gives

$$S(rt, rt) = 2U^{-1}(t)[D_R\Pi_K D_A]_{r,t;r,t} - 2U^{-1}(t)\mathrm{Tr}[D_R\Pi_K D_A]_{r,t;r,t} = 0. \tag{A4}$$

2. Proof of Eq. (67)

It is convenient to write S for unequal positions and times and to express the self-energy Σ in terms of the G, D propagators. Doing this, we obtain

$$S(r_1t_1, r_2t_2) = \frac{1}{2}\int_y [i\{D_K(r_1t_1, y)G_K(y, r_1t_1) + D_R(r_1t_1, y)G_A(y, r_1t_1) + D_A(r_1t_1, y)G_R(y, r_1t_1)\}G_A(y, r_2t_2)$$
$$+ i\Big\{D_R(r_1t_1, y)G_K(y, r_1t_1) + D_K(r_1t_1, y)G_A(y, r_1t_1)\}G_K(y, r_2t_2)]$$
$$- \frac{1}{2}\int_y [iG_R(r_1t_1, y)\{D_K(y, r_2t_2)G_K(r_2t_2, y) + D_R(y, r_2t_2)G_A(r_2t_2, y) + D_A(y, r_2t_2)G_R(r_2t_2, y)\}$$
$$+ iG_K(r_1t_1, y)\{D_A(y, r_2t_2)G_K(r_2t_2, y) + D_K(y, r_2t_2)G_R(r_2t_2, y)\}]. \tag{A5}$$

In order to prove momentum conservation, we need to study the action of the spatial gradients on S,

$$\frac{i}{2}\big[\nabla^\alpha_{r_1} - \nabla^\alpha_{r_2}\big]S(r_1t_1, r_2t_2)|_{1=2} = \int_{r't'} \big[\big\{i\nabla^\alpha_{r_1}D_K(r_1t_1, r't')\big\}\Pi^A(r't', r_1t_1) + \big\{i\nabla^\alpha_{r_1}D_R(r_1t_1, r't')\big\}\Pi^K(r't', r_1t_1)$$
$$+ \int_{r't'} \big[\Pi^R(r_1t_1, r't')\big\{i\nabla^\alpha_{r_1}D_K(r't', r_1t_1)\big\} + \Pi^K(r_1t_1, r't')\big\{i\nabla^\alpha_{r_1}D_A(r't', r_1t_1)\big\}\big], \tag{A6}$$

where we have used that $(\partial_{r_1} - \partial_{r_2})[G_a(1', 1)G_b(1', 2) + G_b(1', 1)G_a(1', 2)]\big|_{1=2} = 0$. Now, denoting by dots position and time variables that are being integrated over, and using that

$$\int_{r't'} \big\{i\nabla^\alpha_{r_1}D_K(r_1t_1, r't')\big\}\Pi^A(r't', r_1t_1) = \int_{r't',.,..} i\nabla^\alpha_{r_1}D_R(r_1t_1, .)\Pi_K(., ..)D_A(.., r't')\Pi^A(r't', r_2t_1)\big|_{r_2=r_1}$$
$$= \int_{r't',.,..} i\nabla^\alpha_{r_1}D_R(r_1t_1, .)\Pi_K(., ..)\big[D_A(.., r_2t_1)U^{-1}(t_1)\big|_{r_2=r_1} - \delta_{..,1}\big]$$
$$= i\nabla^\alpha_{r_1}D_K(r_1t_1, r_2t_1)U^{-1}(t_1)\big|_{r_1=r_2} - \int_{r't'} \big\{i\nabla^\alpha_{r_1}D_R(r_1t_1, r't')\big\}\Pi_K(r't', r_1t_1),$$

and performing a similar analysis for second last term in Eq. (A6), we obtain

$$\frac{i}{2}\big[\nabla^\alpha_{r_1} - \nabla^\alpha_{r_2}\big]S(r_1t, r_2t)\big|_{r_1=r_2=r} = iU^{-1}(t)\nabla^\alpha_{r_1}D_K(r_1t, r_2t)\big|_{r_1=r_2=r} + iU^{-1}(t)\nabla^\alpha_{r_2}D_K(r_1t, r_2t)\big|_{r_1=r_2=r}$$
$$= iU^{-1}(t)\nabla^\alpha_r D_K(rt, rt). \tag{A7}$$

3. Proof of Eq. (72)

The manipulations involved are

$$\frac{i}{2}\big(\partial_{t_1} - \partial_{t_2}\big)S(rt_1, rt_2)|_{t_1=t_2=t} = \int_{r't'} \big[\big\{i\partial_{t_1}D_K(r_1t_1, r't')\big\}\Pi^A(r't', r_1t_1) + \big\{i\partial_{t_1}D_R(r_1t_1, r't')\big\}\Pi^K(r't', r_1t_1)$$
$$+ \int_{r't'} \big[\Pi^R(r_1t_1, r't')\big\{i\partial_{t_1}D_K(r't', r_1t_1)\big\} + \Pi^K(r_1t_1, r't')\big\{i\partial_{t_1}D_A(r't', r_1t_1)\big\}\big]$$
$$= i\partial_{t_1}D_K(r, t_1; r, t_2)U^{-1}(t_2)|_{t_1=t_2=t} + i\partial_{t_2}U^{-1}(t_1)D_K(r, t_1; r, t_2)|_{t_1=t_2=t}$$
$$= iU^{-1}(t)\big[\partial_{t_1}D_K(r, t_1; r, t_2) + \partial_{t_2}D_K(r, t_1; r, t_2)\big]|_{t_1=t_2=t}$$
$$= iU^{-1}\partial_t D_K(r, t; r, t)$$
$$= i\partial_t(U^{-1}D_K(r, t; r, t)) - iD_K(r, t; r, t)\partial_t U^{-1}(t). \tag{A8}$$

APPENDIX B: EQUIVALENCE TO LANGEVIN DYNAMICS

Using the notation of Ref. [49], the bosonic quantum (Δ_q) and classical (Δ_c) fields defined on the Keldysh contour have the following correlators:

$$D_R(1, 2) = -i\langle\Delta_c(1)\Delta^*_q(2)\rangle,$$
$$D_A(1, 2) = -i\langle\Delta_q(1)\Delta^*_c(2)\rangle, \tag{B1}$$
$$D_K(1, 2) = -i\langle\Delta_c(1)\Delta^*_c(2)\rangle,$$

where 1,2 denote both spatial and temporal indices. If $D_K = D_R \cdot \Pi_K \cdot D_A$, then the above correlators can equivalently be written as a path integral over the bosonic fields as follows:

$$Z_K = \int \mathcal{D}[\Delta_{c,q}, \Delta^*_{c,q}]e^{iS_K}, \tag{B2}$$

$$S_K = \int d1 \int d2(\Delta^*_c \quad \Delta^*_q)_1 \begin{pmatrix} 0 & D_A^{-1} \\ D_R^{-1} & \Pi_K \end{pmatrix}_{1,2} \begin{pmatrix} \Delta_c \\ \Delta_q \end{pmatrix}_2. \tag{B3}$$

According to Eq. (77) and the line below,

$$D_R^{-1}(q,t,t') = \frac{\nu}{T}(\partial_t + \lambda_q(t))\delta(t-t');$$
$$\Pi_K(t,t') = 4i\nu\delta(t-t'). \tag{B4}$$

One may introduce [49] an auxiliary field ξ that decouples the $|\Delta_q|^2$ term in the action (for notational convenience, we only highlight the time coordinate)

$$e^{-4\nu\int dt\Delta_q\Delta_q^*} = \int \mathcal{D}[\xi,\xi^*]e^{-i\int dt[\xi\Delta_q^*+\xi^*\Delta_q]}e^{-\frac{1}{4\nu}\int dt\xi\xi^*}. \tag{B5}$$

If we write the action in terms of this auxiliary field,

$$S_K = \int d1\int d2\Big[\Delta_q^* D_R^{-1}\Delta_c + \Delta_c^* D_A^{-1}\Delta_q - \xi\Delta_q^* - \xi^*\Delta_q + \frac{i}{4\nu}\xi\xi^*\Big]. \tag{B6}$$

Requiring $\delta S/\delta\Delta_q^* = 0$ gives

$$D_R^{-1}\Delta_c = \xi, \tag{B7}$$

where $\langle\xi^*(t)\xi(t')\rangle = 4\nu\delta(t-t')$. If we give the explicit expression for D_R and restore the momentum label,

$$\frac{\nu}{T}(\partial_t + \lambda_q(t))\Delta_c(q,t) = \xi_q(t). \tag{B8}$$

It is convenient to rescale $\Delta_c \to \sqrt{\frac{\nu}{T}}\Delta_c$, and then the Langevin equation is

$$(\partial_t + \lambda_q(t))\Delta_c(q,t) = \sqrt{\frac{T}{\nu}}\xi_q(t), \tag{B9}$$

implying that Δ_c obeys Langevin dynamics where the noise is δ correlated with a strength equal to the temperature T.

APPENDIX C: DERIVATION OF KINETIC COEFFICIENTS

In this section, we derive the kinetic coefficients assuming a rapid quench profile $\lambda_q(t) = \theta(t)\lambda_q + \theta(-t)r_i$, where $r_i/T = O(1)$. This condition simply ensures that the density of superconducting fluctuations is zero at $t = 0$. The generalization to general quench profiles is formally straightforward and summarized in the main text.

1. Derivation of Eq. (105)

The term obtained from varying n_k is

$$K_J^i(t) \equiv \frac{1}{2}\mathrm{Im}\int dt'\sum_{k,q} iD_K(q,t,t') \times i[\delta n_{k+q/2}(t') + \delta n_{-k+q/2}(t')] \times \big[v^i_{k+q/2} + v^i_{-k+q/2}\big]e^{i(\epsilon_{k+q/2}+\epsilon_{-k+q/2})(t-t')}. \tag{C1}$$

Substituting Eq. (81) for D_K and Eq. (103) for δn_k, we see that the above can be rewritten as

$$K_J^i(t) = \int_0^t dt'\sum_q Q_J^{ij}(t,t')F_q(t')J^j(t'),$$
$$Q_J^{ij}(t,t') \equiv \frac{\bar{m}T}{\rho\nu}\mathrm{Im}\sum_k\big[\nabla_k^j n^{\rm eq}_{k+q/2} + \nabla_k^j n^{\rm eq}_{-k+q/2}\big] \times i\big(v^i_{k+q/2} + v^i_{-k+q/2}\big)e^{i(\epsilon_{k+q/2}+\epsilon_{-k+q/2}+i\lambda_q)(t-t')}. \tag{C2}$$

Following the discussion in the main text, the term Q_J^{ij} decays exponentially with rate T, whereas J and F change at a slower rate. Therefore, Q_J appears like a δ function when integrated against $F(t)J(t)$, and we may write

$$K_J^i(t) \approx \sum_q F_q(t)J^j(t)\tilde{Q}_J^{ij}(t),$$
$$\begin{aligned}\tilde{Q}_J^{ij}(t) &= \int_0^t dt' Q_J^{ij}(t,t')\\ &\approx -\frac{\bar{m}T}{\rho\nu} \times \mathrm{Im}\sum_k \frac{\big(\nabla_k^j n^{\rm eq}_{k+q/2}+\nabla_k^j n^{\rm eq}_{-k+q/2}\big)\big(v^i_{k+q/2}+v^i_{-k+q/2}\big)}{\epsilon_{k+q/2}+\epsilon_{-k+q/2}+i\lambda_q}\\ &\approx_{v_F q\ll T}\Big(-\frac{\bar{m}T}{\rho\nu}\Big)\mathrm{Im}\sum_k\frac{q^r q^l m_{il}^{-1}\nabla_k^r\nabla_k^j n_k^{\rm eq}}{2\epsilon_k + i\lambda_q}\\ &\approx_{\lambda_q\ll T}\frac{\pi\bar{m}T}{\rho\nu}\sum_k\big(q^r q^l m_{il}^{-1}\nabla_k^r\nabla_k^j n_k^{\rm eq}\big)\delta(\epsilon_k+\epsilon_{-k})\\ &= \frac{\pi\bar{m}T}{2\rho\nu}\sum_k\bigg(q^r q^l m_{il}^{-1}\frac{\partial^2\epsilon_k}{\partial_k^r\partial_k^j}\partial_{\epsilon_k}n_k^{\rm eq}\bigg)\delta(\epsilon_k)\\ &\approx \frac{\bar{m}\pi q^r q^l}{4\rho}\langle m_{jr}^{-1}m_{il}^{-1}\rangle_{\rm FS}.\end{aligned} \tag{C3}$$

In the second line above, we assume $t^{-1} \gg T$ so we are not too close to the quench. In the third line, we take $q \ll T/v_F$ and we also assume $\lambda_q/T \ll 1$.

2. Derivation of Eq. (106)

Using Eq. (81) for D_K and accounting for the change in g_R due to the electric field in Eq. (86), we find

$$K_E^i(t) = \int_0^t dt'\sum_q Q_E^{ij}(t,t')F_q(t')\frac{\rho}{\bar{m}}E^j(t'),$$
$$Q_E^{ij}(t,t')E^j(t') \equiv -\mathrm{Im}\frac{\bar{m}T}{\rho\nu}\sum_k\big(n^{\rm eq}_{k+q/2} + n^{\rm eq}_{-k+q/2}\big) \times \big(v^i_{k+q/2} + v^i_{-k+q/2}\big)\int_{t'}^t ds\, v_k^j(A^j(t) - A^j(s))e^{i(\epsilon_{k+q/2}+\epsilon_{-k+q/2}+i\lambda_q)(t-t')}. \tag{C4}$$

Assume that the frequency of the electric field $\omega \ll T$; in this case, we may approximate the integral

$$\int_{t'}^t ds(A^j(t) - A^j(s)) \approx -E^j(t)(t-t')^2/2. \tag{C5}$$

Following the discussion in the main text, the term Q_E^{ij} decays exponentially with rate T, whereas J and F change at a slower rate. Therefore, Q_E appears like a δ function when integrated against $F(t)J(t)$, and we may write, as we did for Q_J^{ij},

$$\begin{aligned}
K_E^i(t) &\approx \sum_q F_q(t)\frac{\rho E^j(t)}{\bar{m}}\tilde{Q}_E^{ij}(t),\\
\tilde{Q}_E^{ij}(t) &= \int_0^t dt' Q_E^{ij}(t,t')\\
&\approx \frac{\bar{m}T}{2\rho\nu}\text{Im}\sum_k \left(n^{\text{eq}}_{k+q/2}+n^{\text{eq}}_{-k+q/2}\right)i\frac{\left(v^i_{k+q/2}+v^i_{-k+q/2}\right)\left(v^j_{k+q/2}+v^j_{-k+q/2}\right)}{(\epsilon_{k+q/2}+\epsilon_{-k+q/2}+i\lambda_q)^3}\\
&\approx \frac{\bar{m}T}{\rho\nu}\text{Im}\sum_k i\frac{q^l q^r m_{il}^{-1}m_{jr}^{-1}n_k^{\text{eq}}}{(2\epsilon_k+i\lambda_q)^3}\\
&\approx \frac{\pi\bar{m}q^r q^l}{16\rho}\left\langle m_{jr}^{-1}m_{il}^{-1}\right\rangle_{\text{FS}}\int dx x^{-1}\frac{d^2}{dx^2}\tanh(x/2)\\
&\approx \frac{28\pi\bar{m}q^r q^l}{16\rho T}\zeta'(-2)\left\langle m_{jr}^{-1}m_{il}^{-1}\right\rangle_{\text{FS}}.
\end{aligned} \tag{C6}$$

3. Derivation of Eq (117)

Substituting Eq. (114) into Eq. (115) and assuming $t > t'$, we obtain

$$i\delta D_K(t,t') = -\nu\int_0^{t'} ds D_R(t,s)D_A(s,t')\left[\int_s^t du\delta\lambda_q(u)+\int_s^{t'} du\delta\lambda_q(u)\right] \tag{C7}$$

$$= -2\nu\int_0^{t'} du\delta\lambda_q(u)\int_0^u ds D_R(t,s)D_A(s,t') - \nu\int_{t'}^t du\delta\lambda_q(u)\int_0^{t'} ds D_R(t,s)D_A(s,t'). \tag{C8}$$

Using the definition of $D_K = D_R\cdot\Pi_K\cdot D_A$ with $\Pi_K(t,t') = 4\nu\delta(t-t')$, the above becomes

$$i\delta D_K(t,t') = -\frac{1}{2}\int_0^{t'} du e^{-\lambda_q(t'-u)}\delta\lambda_q(u)iD_K(t,u) - \frac{1}{4}\int_{t'}^t du\delta\lambda_q(u)iD_K(t,t'). \tag{C9}$$

Then, we use Eq. (81) to write

$$\begin{aligned}
i\delta D_K(t>t') &= -\frac{2T}{2\nu}e^{-\lambda_q(t-t')}\int_0^{t'} du\, e^{-2\lambda_q(t'-u)}\delta\lambda_q(u)F_q(u) - \frac{T}{2\nu}e^{-\lambda_q(t-t')}\int_{t'}^t du\delta\lambda_q(u)F_q(t')\\
&= -\frac{2T}{2\nu}e^{-\lambda_q(t-t')}\int_0^t du\, e^{-2\lambda_q(t'-u)}\delta\lambda_q(u)F_q(u) - \frac{T}{2\nu}e^{-\lambda_q(t-t')}\int_{t'}^t du\delta\lambda_q(u)[F_q(t') - 2e^{-2\lambda_q(t'-u)}F_q(u)]\\
&= -\frac{2T}{2\nu}e^{\lambda_q(t-t')}\int_0^t du\, e^{-2\lambda_q(t-u)}\delta\lambda_q(u)F_q(u) - \frac{T}{2\nu}e^{-\lambda_q(t-t')}\int_{t'}^t du\delta\lambda_q(u)[F_q(t') - 2e^{-2\lambda_q(t'-u)}F_q(u)]\\
&\approx -\frac{2T}{2\nu}e^{\lambda_q(t-t')}\int_0^t du\, e^{-2\lambda_q(t-u)}\delta\lambda_q(u)F_q(u) + \frac{T}{2\nu}e^{-\lambda_q(t-t')}(t-t')\delta\lambda_q(t)F_q(t),
\end{aligned} \tag{C10}$$

where in the last line we used the fact that $t - t'$ is of order T^{-1} as iD_K appears along with Π_A. We discuss this last term further below and show it to be parametrically smaller than the local terms calculated earlier.

We insert the second part into the collision integral, giving

$$\begin{aligned}
&4\int_0^t dt'\text{Im}\Pi_A'(t',t)\left[e^{-\lambda_q(t-t')}\frac{T}{\nu}(t-t')\delta\lambda_q(t)F_q(t)\right]\\
&\propto \delta\lambda_q(t)F_q(t)\text{Im}\sum_k i\frac{n^{\text{eq}}_{k+q/2}+n^{\text{eq}}_{-k+q/2}}{(\epsilon_{k+q/2}+\epsilon_{-k+q/2}-i\lambda_q)^2}\\
&\approx_{q\to 0} \delta\lambda_q(t)F_q(t)\text{Im}\sum_k i\frac{2n^{\text{eq}}(\epsilon_k)}{(2\epsilon_k-i\lambda_q)^2} \approx \delta\lambda_q(t)F_q(t)\text{Im}i\int d\epsilon\,\nu\frac{2n^{\text{eq}}(\epsilon)}{(2\epsilon-i\lambda_q)^2}\\
&\approx_{\lambda_q\to 0} \delta\lambda_q(t)F_q(t)\text{Re}\int d\epsilon\,\nu\mathcal{P}\frac{\partial_\epsilon n^{\text{eq}}(\epsilon)}{\epsilon} = 0.
\end{aligned} \tag{C11}$$

Thus, this term is parametrically smaller than the local term already calculated.

Now, we consider the full term

$$
\begin{aligned}
4\int_0^t dt'\mathrm{Im}\big[v_k^i i\delta D_K(t,t')\Pi_A'(t',t)\big] &= \frac{1}{2}\left[\mathrm{Im}\int_0^t dt' \sum_{k,q} i\big(v^i_{k+q/2}+v^i_{-k+q/2}\big) i\delta D_K(t',t)\big(n^{\mathrm{eq}}_{k+q/2}+n^{\mathrm{eq}}_{-k+q/2}\big)e^{i(\epsilon_{k+q/2}+\epsilon_{-k+q/2})(t-t')}\right]\\
&\approx \mathrm{Im}\left[\frac{T}{2\nu}\sum_{k,q} i\int_0^t dt'\big(v^i_{k+q/2}+v^i_{-k+q/2}\big)\big(n^{\mathrm{eq}}_{k+q/2}+n^{\mathrm{eq}}_{-k+q/2}\big)e^{i(\epsilon_{k+q/2}+\epsilon_{-k+q/2}-i\lambda_q)(t-t')}\right.\\
&\quad\left.\times\int_0^t du\left(-e^{-2(t-u)\lambda_q}\delta\lambda_q(u)F_q(u)\right)\right]\\
&\approx \mathrm{Im}\left[\frac{T}{2\nu}\sum_{k,q} q^j m^{-1}_{ij}\frac{n^{\mathrm{eq}}_{k+q/2}+n^{\mathrm{eq}}_{-k+q/2}}{\epsilon_{k+q/2}+\epsilon_{-k+q/2}-i\lambda_q}\int_0^t du\, e^{-2(t-u)\lambda_q}\delta\lambda_q(u)F_q(u)\right]\\
&\approx \sum_q \frac{q^l T}{2\nu}\sum_k m^{-1}_{il}\frac{n^{\mathrm{eq}}(\epsilon_k)\lambda_q}{4\epsilon_k^2+\lambda_q^2}\int_0^t du e^{-2(t-u)\lambda_q}\delta\lambda_q(u)F_q(u)\\
&= \sum_q \frac{Tq^l}{2\nu}\left(\sum_k m^{-1}_{il}\frac{n^{\mathrm{eq}}(\epsilon_k)\lambda_q}{4\epsilon_k^2+\lambda_q^2}\right)\int_0^t du e^{-2(t-u)\lambda_q}F_q(u)\frac{2q^j\bar{m}}{\rho M}\left[J^j(u)+\int_u^t dt'\frac{\rho E^j(t')}{\bar{m}}\right]\\
&= \sum_q \alpha q^i q^j\lambda_q\int_0^t du e^{-2(t-u)\lambda_q}F_q(u)\left[J^j(u)+\int_u^t dt'\frac{\rho E^j(t')}{\bar{m}}\right].
\end{aligned}
\tag{C12}
$$

Above, in the second last line, we used the expression for $\delta\lambda_q$ in Eq. (114).

APPENDIX D: DERIVATION OF EQ. (167)

Let us now discuss the limiting behaviors of $h(z_0)$. If $z_0 \ll 1$, the integral is restricted to the region $\bar{x} \ll 1$ and we may approximate $(1-e^{-\bar{x}})/\bar{x} \approx 1$. Therefore, in this limit, we have $\mathrm{Re}[h] \approx 2z_+^0$. If $|z_+^0| \gg 1$, then we split the integral at $\bar{x}=1$. The lower part

$$
\begin{aligned}
&\int_0^1 d\bar{x}\,\frac{1-e^{-\bar{x}}}{\bar{x}}\big[(z_+^0)^2-\bar{x}\big]^{-1/2}\\
&\sim \int_0^1 d\bar{x}\,\frac{1-e^{-\bar{x}}}{\bar{x}}\frac{1}{z_+^0}\left(1+\frac{\bar{x}}{2(z_+^0)^2}+\cdots\right)\\
&\propto \frac{1}{z_+^0}+\mathcal{O}((z_+^0)^{-2}).
\end{aligned}
\tag{D1}
$$

Whereas in the part from 1 to ∞, the exponential may be neglected, leaving the leading part

$$
\int_1^\infty d\bar{x}\,\frac{1}{\bar{x}}[(z_+^0)^2-\bar{x}]^{-1/2}\sim\frac{\log{(z_+^0)^2}}{z_+^0}. \tag{D2}
$$

Thus, we have Eq. (167).

APPENDIX E: DERIVATION OF EQ. (175)

We record here some facts about $f(z)$ as a function in the complex plane. It may be written as

$$
f(z)=\frac{1}{\sqrt{z^2+1}}\left[\log\left(\frac{\sqrt{z^2+1}+z}{\sqrt{z^2+1}-z}\right)-\pi i\,\mathrm{sgn}\Im z\right]. \tag{E1}
$$

It is analytic in the upper and lower half planes separately with a branch cut along the real axis. The points $z=\pm i$ which appear singular are in fact smooth. One may also convince oneself that $|f(z)|\leqslant\pi$ for all z, reaching this limit only when $z=0\pm i\delta$. It is also an injective function of z. In addition, we have that as $z\to\infty$,

$$
f(z)\sim 2\frac{\log z}{z}, \tag{E2}
$$

and that f is pure imaginary on the imaginary axis. Causality requires that $z_\pm$ have no zeros or branch cuts in the upper half complex ω plane, but there will be analytic structure as a function of ε.

Let us first test the validity of the perturbative approximation. To zeroth order, we set $z_\pm = z_\pm^0$. To first order, we substitute this into the rhs of Eqs. (170) and (171) and obtain

$$
z_\pm = z_\pm^0+\gamma f(z_\mp^0) \tag{E3}
$$

and to second order

$$
\begin{aligned}
z_\pm &= z_\pm^0+\gamma f(z_\mp^0+\gamma f(z_\pm^0))\\
&\approx z_\pm^0+\gamma f(z_\mp^0)+\gamma^2 f'(z_\mp^0)f(z_\pm^0).
\end{aligned}
\tag{E4}
$$

Comparing the first and second corrections, we see that the latter is smaller when

$$
\gamma \ll \left|\frac{f(z_\mp^0)}{f'(z_\mp^0)}\frac{1}{f(z_\pm^0)}\right|. \tag{E5}
$$

Considering the various limits, we see that the rhs is always at least $O(1)$ so the perturbative treatment is always valid if $\gamma\ll 1$. Making the very coarse approximation

$|f| \sim \min(1, |z^{-1}|)$, we get roughly

$$\gamma \ll \min(1, |z_+^0|) \cdot \min(1, |z_-^0|). \tag{E6}$$

This may still be satisfied even if $\gamma \geqslant 1$. If $\omega/T \gg 1/\text{Gi}$, the perturbative limit will hold for all ε. With the weaker condition $\omega^2/T^2 \gg \text{Gi}$, the perturbative limit holds except in the narrow region $|\varepsilon \pm \omega|/T \ll \sqrt{\epsilon}$, which gives only a small contribution to $\delta\nu(\omega)$.

Now let us consider the behavior in the simplest case when $z_\pm^0 = 0$. In this case, we have a self-consistent solution when $z_\pm = \mp iy$, y real, $y > 0$, since this is equivalent to finding a solution to

$$-iy(\gamma) = \gamma f(iy(\gamma)), \tag{E7}$$

which is possible since f is imaginary on the imaginary axis. In the limiting case of $\gamma \gg 1$, we get $y^2 \sim \gamma \log y^2$, which gives to logarithmic accuracy,

$$y \sim \sqrt{\gamma \log \gamma}. \tag{E8}$$

In the limit $\gamma \ll 1$, using the expansion of $f(z)$ for small argument, we obtain $z_\pm \approx \mp i\gamma\pi + \mathcal{O}(\gamma^2)$.

Now let us advance to the case $z_\pm^0 = c \in \mathbb{R}$. Recall $\omega = 0$. This has a self-consistent solution when $z_+ = z = z_-^*$,

$$z - c = \gamma f(z^*), \tag{E9}$$

since $f(z)^* = f(z^*)$. The perturbative condition in this case is

$$\gamma \ll 1/|f'(c)| \sim c^2/\log|c^2|. \tag{E10}$$

Thus, the perturbative regime begins when $c \gg y(\gamma)$. In this limit, we obtain the usual perturbative result by setting $z \approx c$.

In the opposite limit, let us assume $|z| \gg 1, |c|$. On these assumptions, writing $z = -i|z|e^{-i\phi}$, and comparing real and imaginary parts of

$$(z - c)z^* = \gamma \log(-(z^*)^2)$$
$$\Rightarrow |z|^2 - ic|z|e^{i\phi} = 2\gamma[\log|z| + i\phi],$$

we obtain

$$|z|^2 + c|z|\sin\phi = \gamma \log|z|^2, \tag{E11}$$

$$\text{and} \quad -c|z|\cos\phi = 2\gamma\phi. \tag{E12}$$

On the assumption $|z| \gg |c|$, the first line above gives $|z| = y(\gamma)$. The second line can be rearranged to give

$$\frac{\phi}{\cos\phi} = -c\frac{y(\gamma)}{2\gamma}. \tag{E13}$$

Thus, ϕ goes from $\pi/2$ to $-\pi/2$ as c goes from $-\infty$ to ∞. We may split this into two limits. When $c \ll \gamma/y(\gamma) = y(\gamma)/\log y(\gamma)$, we have that $\phi \ll 1$ and so the cos may be set to 1 giving $\phi = -cy(\gamma)/2\gamma$, and thus

$$z = -i|z|(1 - i\phi) = -iy(\gamma) + c\log(\gamma)/2. \tag{E14}$$

In the other limit, $cy(\gamma)/\gamma \gg 1$ is large so ϕ is close to $\pm\pi/2$. Let us take $c > 0$, and then

$$\phi = -\frac{\pi}{2} + \frac{\pi\gamma}{cy(\gamma)}, \tag{E15}$$

so that z is given by

$$z = -ie^{i\pi/2}|z|\left(1 - i\frac{\pi\gamma}{cy(\gamma)}\right) = y(\gamma) - i\frac{\pi\gamma}{c}. \tag{E16}$$

This is almost identical to the perturbative calculation with a slight correction to the real part.

We can collect all of these limits,

$$z = \begin{cases} -iy(\gamma) + c\log(\gamma)/2 & c \ll y(\gamma)/\log(\gamma) \\ y(\gamma) - i\frac{\pi\gamma}{c} & y(\gamma)/\log(\gamma) \ll c \ll y(\gamma), \\ c + \frac{\gamma}{c}[\log c^2 - \pi i] & y(\gamma) \ll c \end{cases} \tag{E17}$$

where in the last case, we have used the perturbative expansion in Eq. (E4).

We are now in a position to estimate the density of states for $\omega = 0$, $\gamma \gg 1$, and $c = \varepsilon/T\sqrt{r}$. We start from

$$\frac{\nu(\omega)}{\nu_0} = \frac{1}{\pi}\Im\int_{-\infty}^{\infty} d\varepsilon \frac{1}{\sqrt{r}Tz(\omega = 0, \varepsilon)}. \tag{E18}$$

We combine the positive and negative integrals over ε or c,

$$\frac{\nu(\omega)}{\nu_0} = \frac{1}{\pi}\Im\int_0^{\infty} dc\left[\frac{1}{z(c)} + \frac{1}{z(-c)}\right]. \tag{E19}$$

Next, we use the fact that at $\omega = 0$, $z(c) = -z(-c)^*$, which follows from the above calculation. This can also be seen from noting that $f(-z) = -f(z)$. Then, we use $z_+(c) = c + i\delta + \gamma f(z_-(c)) = c + i\delta + \gamma f(z_+^*(c))$. We take $c \to -c$ and conjugate so that this becomes $z_+^*(-c) = -c - i\delta + \gamma f(z_+(-c)) = -c - i\delta - \gamma f(-z_+(-c))$. Thus, $z_+(-c)^* = -z_+(c)$ is a solution of the above equation as this simultaneously requires $f(-z_+(-c)) = f(z_+^*(c))$. Using this gives

$$\frac{\nu(\omega = 0)}{\nu_0} = \frac{2}{\pi}\int_0^{\infty} dc\left[\frac{\Im z(c)}{[\Re z(c)]^2 + [\Im z(c)]^2}\right]. \tag{E20}$$

Now we split this integral into three regions, according to the asymptotics above,

$$\int_0^{\infty} dc\left[\frac{\Im z(c)}{[\Re z(c)]^2 + [\Im z(c)]^2}\right] = I_1 + I_2 + I_3, \tag{E21}$$

where

$$I_1 \equiv \int_0^{y(\gamma)/\log\gamma} dc\left[\frac{\Im z(c)}{[\Re z(c)]^2 + [\Im z(c)]^2}\right]$$
$$= \int_0^{y(\gamma)/\log\gamma} dc\left[\frac{y(\gamma)}{[c\log(\gamma)/2]^2 + [y(\gamma)]^2}\right], \tag{E22}$$

$$\text{define } u \equiv c\frac{\log\gamma}{y(\gamma)},$$

$$I_1 = \frac{1}{\log\gamma}\int_0^1 du\left[\frac{1}{(u/2)^2 + 1}\right] \propto \frac{1}{\log\gamma}, \tag{E23}$$

$$I_2 \equiv \int_{y(\gamma)/\log y(\gamma)}^{y(\gamma)} dc\left[\frac{\Im z(c)}{[\Re z(c)]^2 + [\Im z(c)]^2}\right]$$
$$= \int_{y(\gamma)/\log\gamma}^{y(\gamma)} dc\left[\frac{\pi\gamma/c}{y(\gamma)^2 + [\pi\gamma/c]^2}\right], \tag{E24}$$

define $u = c\cdot y(\gamma)/\gamma \approx c\dot{\log}(\gamma)/y(\gamma)$,

$$\begin{aligned} I_2 &= \frac{\gamma}{y(\gamma)^2}\int_1^{\log y(\gamma)} du\left[\frac{\pi/u}{1+[\pi/u]^2}\right] \\ &\approx \frac{\gamma}{y(\gamma)^2}\int_1^{\log y(\gamma)} du\frac{\pi}{u} \propto \frac{1}{\log\gamma}\log\log y(\gamma) \\ &\propto \frac{1}{\log\gamma}. \end{aligned} \tag{E25}$$

Here, we are treating $\log\log\gamma$ as an $\mathcal{O}(1)$ constant, which has been the order of our precision throughout:

$$I_3 \equiv \int_{y(\gamma)}^{\infty} dc\left[\frac{\Im z(c)}{[\Re z(c)]^2+[\Im z(c)]^2}\right] \tag{E26}$$

$$\begin{aligned} &= \int_{y(\gamma)}^{\infty} dc\left[\frac{\gamma\pi/c}{c^2+[\gamma\pi/c]^2}\right] \\ &\approx \int_{y(\gamma)}^{\infty} dc\frac{\gamma\pi}{c^3} = 2\pi\frac{\gamma}{y(\gamma)^2} = \frac{2\pi}{\log y(\gamma)} \\ &\approx \frac{4\pi}{\log\gamma}. \end{aligned} \tag{E27}$$

Thus, we see that all terms contribute $\propto 1/\log\gamma$, and therefore we have that

$$\nu(\omega=0) \propto \frac{\nu_0}{\log(\mathrm{Gi}/8\pi r)}. \tag{E28}$$

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