Lattice construction of duality with non-Abelian gauge fields in 2+1D

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The lattice construction of Euclidean path integrals has been a successful approach of deriving 2+1D field theory dualities with a U(1) gauge field. In this work, we generalize this lattice construction to dualities with non-Abelian gauge fields. We construct the Euclidean space-time lattice path integral for a theory with strongly interacting SO(3) vector bosons and Majorana fermions coupled to an SO(3) gauge field and derive an exact duality between this theory and the theory of a free Majorana fermion on the space-time lattice. We argue that this lattice duality implies the desired infrared duality between the field theory with an SO(3) vector critical boson coupled to an $SO(3)_1$ Chern-Simons gauge theory, and a free massless Majorana fermion in 2+1D. We also generalize the lattice construction of dualities to models with O(3) gauge fields.

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I. INTRODUCTION

The classic particle-vortex duality [1,2] of bosonic systems revealed that different Lagrangians can secretly correspond to the same conformal field theory in the infrared limit. Recently the boson-fermion dualities [3–8], fermion-fermion dualities [9–12], and many descendant 2+1D field theory dualities [8, 13–29] have attracted a lot of attentions. The importance of these dualities are not just limited to their own theoretical interests. They also have close connections to gauge-gravity duality [5–7], mirror symmetry in supersymmetric field theory [30–32], fractional quantum Hall systems [9,33–37], as well as deconfined quantum critical points [13,38]. These dualities have tremendously advanced our understandings of 2+1D conformal field theories. In fact, a large class of them are weaved into a duality web through SL(2, $\mathbb{Z}$) transformations on U(1) currents and on the Chern-Simons terms of the U(1) gauge fields [8,39]. While the duality web itself provides interesting cross-check among different dualities with U(1) gauge fields, analytical derivation for certain members of the web are also obtained using wire constructions [12,40], loop model [41], deformation of exact supersymmetric dualities [23,31,32], and lattice constructions [2,42] of Euclidean path integrals.

Although some of the descendant dualities mentioned above have received rather positive numerical evidences [38,43], rigorously speaking most of the dualities are still conjectures. Apart from the consistency check on global symmetries, anomalies and matching global phase diagrams [14, 16–28], rigorous analytical results are much harder to obtain for dualities in the presence of non-Abelian gauge fields. Exact results, say, on partition functions, operator scaling dimensions, etc. in these dualities mostly rely on the large-N limit [5–7,44–46] or well-established level-rank dualities of Chern-Simons gauge theories without dynamical matter fields [47–49]. More analytical treatments to dualities with non-Abelian gauge group of finite rank and with dynamical matter fields will be certainly of great importance.

Deriving the dualities via the lattice construction of the Euclidean path integral [2,42] is an interesting and powerful method for its exactness and explicitness in connecting the degrees of freedom on the two sides of a duality without using any large-N limit or supersymmetry. On the other hand, this method often relies on the assumptions that the theories defined on the lattice indeed flow under the renormalization group to their expected continuum limit in the infrared (IR). Such assumptions can be hard to justify within the lattice models themselves especially in the presence of interactions. Nevertheless, the lattice construction of dualities, for example the Dasgupta-Halperin lattice construction for the particle-vortex dualtiy, are still often viewed as very suggestive evidence for the desired/conjectured dualities in the IR. Previously, the lattice construction has only been carried out for dualities with U(1) gauge fields [2,42]. In this paper, we generalize the lattice construction to dualities with orthogonal non-Abelian gauge groups. We study a lattice construction motivated by the IR duality between a critical SO(3) vector boson coupled to a $SO(3)_1$ Chern-Simons gauge theory and a free massless Majorana fermion in 2+1D, which was proposed in Refs. [16,19]. We first construct the Euclidean space-time lattice path integral of a theory with SO(3) vector bosons and Majorana fermions coupled to a SO(3) gauge field. As will be explained, a natural candidate of the IR limit of this lattice path integral is the continuum theory of a SO(3) vector boson coupled to a $SO(3)_1$ Chern-Simons gauge theory. We perform an *exact* mapping of this space-time lattice path integral to that of a free Majorana fermion whose continuum limit agrees with the expected dual of the critical bosons. Generalizing this construction, we obtain a slightly different lattice duality between an interacting model with

O(3) vector bosons and Majorana fermions coupled to O(3) gauge fields and the theory of free Majorana fermions.

II. LATTICE DUALITY WITH SO(3) GAUGE FIELDS

In Refs. [16,19], a duality between the SO(3) critical boson coupled to $\mathrm{SO}(3)_1$ Chern-Simons gauge theory and the free gapless Marjorana fermions in 2+1D was proposed. The continuum description of the boson side of the duality is given by the Lagrangian:

$$\mathcal{L}_b = |(\partial_\mu - ia_\mu)\phi|^2 + r|\phi|^2 + g|\phi|^4 + \frac{i}{2\cdot 4\pi}\mathrm{tr}_{\mathrm{SO}(3)}\left(a\wedge da - \frac{2i}{3}a\wedge a\wedge a\right), \quad (1)$$

where ϕ is a three-component real vector field coupled to an SO(3) gauge field a. The SO(3) gauge field a is subject to a Chern-Simons term at level 1 in which the trace "tr" is taken in the vector representation of SO(3). This theory has an dual description of a single two-component Majorana fermion ξ in $2+1\mathrm{D}$ given by the simple Lagrangian

$$\mathcal{L}_f = \bar{\xi}\gamma^\mu\partial_\mu\xi - m\bar{\xi}\xi. \quad (2)$$

This duality is directly shown to hold in the gapped phases with r and m carrying the same sign [16,19]. It is further conjectured to be valid even at the critical point where $r = m = 0$. In this section, we will start with formulating a lattice theory in close connection to the critical boson theory and constructing its path integral on the Euclidean space-time lattice. We will then map this path integral exactly to that of a free Majorana fermion. As we will see later this exact mapping is valid regardless of whether the resulting phases are gapped or at the critical point. We will also refer to this type of exact mapping between two lattice theories as a lattice duality. After establishing the lattice duality, we will discuss the correlation functions and the Z_2 global symmetry across the dualtiy.

A. Euclidean path integral on the lattice

We start with the basic ingredients for constructing the space-time lattice path integral for the boson side of the duality. We consider a discretized 3D Euclidean space-time lattice, which is taken to be a cubic lattice for simplicity. On each site n of the space-time lattice, there is a three-component unit vector $v_n = (v_{n,1}, v_{n,2}, v_{n,3})^\intercal$ that represents the SO(3) vector boson fields. The vector boson fields couple to their nearest neighbors and to the SO(3) gauge field residing on the links that connect them, leading to the following contribution to the Euclidean action:

$$S_{\mathrm{bg}}[v, O] = \sum_n \sum_{\mu=x,y,z} -J\, v_{n+\mu,i}O_{ij}^{n\mu}v_{n,j}, \quad (3)$$

where $\mu = x, y, z$ is summed over the unit lattice vector along the positive $x, y,$ and z direction. J is the coupling constant of the SO(3) vector bosons, which we assume to be always positive. $O_{ij}^{n\mu}$ is an SO(3) matrix that represents the SO(3) gauge connection along the links between sites n and $n+\mu$. The repeated SO(3) vector/matrix indices i, j are implicitly assumed to be summed automatically from 1 to 3.

According to the Lagrangian (1), we also need to introduce a Chern-Simons term for the gauge field $O^{n\mu}$. While it is difficult to directly write down the Chern-Simons term on the lattice, we can circumvent the difficulty by coupling the SO(3) gauge field to a massive Majorana fermion. This method is a direct generalization of the construction of the U(1) Chern-Simons term in a lattice duality studied in Ref. [42]. In this method, the Chern-Simons term can be viewed as the outcome of integrating out the massive Majorana fermions. Following Wilson's approach [50,51], we can write down the action for lattice Majorana fermions coupled to the SO(3) gauge field:

$$S_{\mathrm{fg}}[\chi, O] = \sum_n \sum_{\mu=x,y,z} \bar{\chi}_{n+\mu,i}(\sigma^\mu - R)O_{ij}^{n\mu}\chi_{n,j} + M\sum_n \bar{\chi}_{n,i}\chi_{n,i}, \quad (4)$$

Again, the repeated SO(3) indices i, j are automatically summed over. For each site n and each SO(3) color index i, the fermion field $\chi_{n,i} = (\chi_{n,i,1}, \chi_{n,i,2})^\intercal$ is a two-component spinor consists of two real Grassmann numbers. σ^μ stands for the Pauli matrices. $\bar{\chi}_{n+\mu,i}$ is defined as $\bar{\chi}_{n+\mu,i} = \chi^\intercal\sigma^y$. When we turn off the gauge field $O^{n\mu}$, the action Eq. (4) alone gives rise to $2^3 = 8$ Majorana fermions in the IR whose masses are controlled by the parameters R and M. The mass configuration of the massive Majorana fermions determines the Chern number C of the occupied bands of the Majorana fermions. To be more precise, by the band structure of Majorana fermions, we refer to the band structure obtained from quantizing the eight Majorana fermions. And by occupied bands, we mean all the negative energy states associated to the eight IR Majorana fermions after the quantization. The Chern number also serves as the level of the Chern-Simons term of the SO(3) gauge field when the fermion field χ_n is integrated out in the action $S_{\mathrm{fg}}[\chi, O]$. The relation between the Chern number C and parameters R and M that control the bare band structure of the Majorana fermions field χ_n is given by [52,53]

$$C = \begin{cases} 2\mathrm{sgn}(R) & 0 < |M| < |R|, \\ -\mathrm{sgn}(R) & |R| < |M| < 3|R|, \\ 0 & |M| > 3|R|. \end{cases} \quad (5)$$

Naively, by combining the action Eqs. (3) and (4), one would have already had all the essential ingredients for a lattice version of the field theory (1). However, for reasons that will become clear later, we would also like to include the interaction between the vector bosons and the Majorana fermions:

$$S_{\mathrm{int}}[\chi, v] = \frac{U_1}{4}\sum_{n,\mu}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}_{ij}^{n\mu}\mathcal{J}_{i'j'}^{n\mu}v_{n+\mu,i''}v_{n,j''} + \frac{U_2}{36}\sum_{n,\mu}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}_{ij}^{n\mu}\mathcal{J}_{i'j'}^{n\mu}\mathcal{J}_{i''j''}^{n\mu}, \quad (6)$$

where we have introduced the notation $\mathcal{J}_{ij}^{n\mu} \equiv \bar{\chi}_{n+\mu,i}(\sigma^\mu - R)\chi_{n,j}$ for the fermion current. $\varepsilon_{ii'i''}$ and $\varepsilon_{jj'j''}$ represent the totally antisymmetric tensor. U_1 and U_2 are coupling constants of these interactions in S_{int}. It is straightforward to verify that S_{int} is invariant under the SO(3) gauge transformations.

Now, we are ready to introduce our Euclidean space-time lattice path integral for one side of the duality:

$$Z_b = \int D[O^{n\mu}] \int D[\chi_{n,i}] \int D[v_n] e^{-S_{\rm bg}-S_{\rm fg}-S_{\rm int}}, \quad (7)$$

where the integration of the gauge field $\int D[O^{n\mu}]$ is implemented as the integration of the matrix $O^{n\mu}$ on each link over the Haar measure of SO(3). The boson field v_n on each site is integrated over the unit 2-sphere S^2, while the fermion fields $\chi_{n,i}$ are integrated as real Grassmann numbers. As we discussed, when the coupling constants U_1 and U_2 are set to zero, after the Majorana fermions are integrated out, this model naturally realizes the theory of SO(3) vector bosons coupled to a SO(3) gauge field with the Chern-Simons level given by Eq. (5). When U_1 and U_2 are finite, the relation between the Chern-Simons level generated by the Majorana fermions and parameters M and R is expected to be modified. We will address the effect of $S_{\rm int}$ and how it alters the interpretation of the lattice path integral Z_b as we proceed in obtaining its dual theory.

B. Exact mapping of lattice path integral

We notice that the SO(3) gauge field $O^{n\mu}$ in Eq. (7) can be integrated out analytically. In order to do so, we only need to consider the $S_{\rm bg}[v, O]$ and $S_{\rm fg}[\chi, O]$ parts of the action:

$$\begin{aligned}
&\int D[O^{n\mu}]\, e^{-S_{\rm bg}[v,O]} e^{-S_{\rm fg}[\chi,O]} \\
&= \exp\left(-\sum_n M\bar{\chi}_{n,i}\chi_{n,i}\right) \times \prod_{n,\mu}\left[\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{J^l}{l!}\frac{(-1)^m}{m!}\right. \\
&\quad\left.\times \int dO^{n\mu}\left(v_{n+\mu,i}O^{n\mu}_{ij}v_{n,j}\right)^l\left(\bar{\chi}_{n+\mu,i}(\sigma^\mu - R)O^{n\mu}_{ij}\chi_{n,j}\right)^m\right],
\end{aligned} \quad (8)$$

From now on, we will set $R = -1$ which renders the matrices $\sigma^\mu - R$ rank 1 and which combined with the fermionic statistics of the Grassmann numbers leads to vanishing contributions for all the terms with $m > 3$ (see Appendix A for a more detailed explanation). Notice that the integration over all gauge field configurations $\int D[O^{n\mu}]$ factorizes into the integration of SO(3) matrices $O^{n\mu}$ under the Haar measure on each link. After conducting a term by term integration, we obtain that

$$\int D[O^{n\mu}]\, e^{-S_{\rm bg}} e^{-S_{\rm fg}} = \left(\frac{\sinh J}{J}\right)^{3N_s} e^{-S_{\rm bfg}[v,\chi]} \quad (9)$$

where N_s is the number of sites in the Euclidean space-time lattice and the effective action $S_{\rm bfg}[v, \chi]$ takes the form:

$$\begin{aligned}
S_{\rm bfg}[v,\chi] = &\sum_n M\bar{\chi}_{n,i}\chi_{n,i} \\
&+ \sum_{n,\mu}\left\{K(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})\right. \\
&- \frac{K}{4}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''} \\
&+ \left.\frac{1-K^2}{36}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}\right\}.
\end{aligned} \quad (10)$$

with $K = \frac{J\cosh J - \sinh J}{J \sinh J}$ which is a positive number between 0 and 1 for all $J > 0$. The details of the derivation of Eq. (10) can be found in Appendix A. Notice that the effective action $S_{\rm bfg}[v, \chi]$ contains similar interactions to the ones we introduce in $S_{\rm int}$. We will choose

$$U_1 = K, \;\; U_2 = -1 + K^2, \quad (11)$$

so that the interactions in $S_{\rm bfg}[v, \chi]$ and in $S_{\rm int}[v, \chi]$ exactly cancel off each other.

For every fixed configuration of the vector field v_n, $S_{\rm bfg} + S_{\rm int}$ contains one and only one propagating Majorana fermion field

$$\xi_n \equiv \sqrt{K}\sum_i \chi_{n,i}v_{n,i}. \quad (12)$$

There are also two other Majorana fermion fields, denoted as ξ'_n and ξ''_n, orthogonal to ξ_n in the SO(3) "color space." For a fixed configuration of the vector boson v_n, we can perform a basis rotation from the χ_n field to ξ_n, ξ'_n and ξ''_n in the path integral. Notice that the fermion fields ξ'_n and ξ''_n only have local mass terms but not any kinetic terms [i.e., the $(\sigma^\mu - R)$ term]. They can be directly integrated out producing a factor that only depends on the mass parameter M. Therefore, after integrating out ξ'_n and ξ''_n, we can write

$$\begin{aligned}
&\int D[O^{n\mu}] \int D[\chi_{n,i}] e^{-S_{\rm bg}-S_{\rm fg}-S_{\rm int}} \\
&= \mathcal{N}\int D[\xi_n]\exp\left(-\sum_{n,\mu}\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n - \sum_n M'\bar{\xi}_n\xi_n\right),
\end{aligned} \quad (13)$$

where $\mathcal{N}$ is an overall normalization constant and the mass parameter M' for the Majorana fermion mode ξ_n is given by

$$M' = M/K. \quad (14)$$

Now, we have arrived at the dual theory which exactly describes free Majorana fermions on the Euclidean space-time lattice. Although Eq. (13) is derived on fixed configuration of the boson field v_n, its right-hand side does not depend on v_n. The independence on the vector boson configuration v_n is expected also from the fact that any fixed vector boson configuration v_n can be connected to any other by SO(3) gauge transformations before we perform any integration on the left-hand side. Therefore further integration over the boson field v_n in Eq. (13) will only introduce an extra overall multiplicative factor and will not change the nature of the duality. At this point, we have constructed an exact mapping between the theory in Eq. (7) and free Majorana fermions described by the action in Eq. (13). This exact mapping is valid regardless of whether or not the model Z_b (and its dual Majorana fermion theory) is at the critical point or not.

Having already set $R = -1$, if we further take $M' = 3$, the dual Majorana fermion ξ_n, based on the change of Chern number for $M' > 3$ and $M' < 3$ givein in Eq. (5), is exactly at a critical point described by a single free gapless Majorana fermion in the IR. It implies that the model (7) with the following choices of parameter is also critical:

$$M = 3K, \quad U_1 = K, \quad U_2 = -1 + K^2. \tag{15}$$

Now, we would like to return to the discussion of the continuum limit of the model (7) at the criticality. First of all, $0 < K < 1$ for any positive coupling J. Hence, M is always smaller than 3 at the critical point. If we choose J such that $1 < M < 3$ and neglect the effect of the interaction in S_{int} as we first integrate out the fermions χ_n in the model (7), we would naturally identify the resulting model as a SO(3) critical bosons coupled to an $SO(3)_1$ Chern-Simons gauge theory. The presence of finite $U_{1,2}$ complicates the integration over the fermions field χ_n in Eq. (7). The difficulty arises because when the bare mass of the Majorana fermion χ_n is of order 1, namely when the deviation of M from 3 (as well as 1) where the χ_n fermion becomes massless is of order 1, Eq. (15) will require $U_{1,2}$ to be also of order 1 at the same time. We notice that the $U_{1,2}$ interactions can be viewed as the hopping of three-particle bound states (of two χ_n fermions and a vector boson v_n or of three χ_n fermions) from one space-time lattice site to the neighboring sites. A possibility is that the hoppings of three-particle bound states are irrelevant in the continuum limit or their effects are small enough such that the level of the Chern-Simons term (which is generated by the integration over χ_n) is unchanged due to its quantized nature. If this possibility is indeed correct, the Euclidean space-time lattice path integral (7) will, even in the presence of $U_{1,2}$, still correspond to the field theory (1) in the IR with g flows to a fixed point value.

C. Correlation functions and global symmetry

Regardless of the interpretation of the theory (7) in the continuum limit, the mapping discussed above is explicit and exact, which allow us to identify not only the partition functions on both sides of the duality but also the correlation functions. In particular, any correlation functions of the Majorana fermion ξ_n can be exactly reproduced in the model (7) by the operators defined in Eq. (12).

The exact mapping also helps us keep track of the global symmetry on the two sides of the duality. The model (7) possesses a global Z_2 symmetry:

$$Z_2 : v_n \to -v_n, \quad \chi_n \to -\chi_n, \tag{16}$$

while it is evident from Eq. (12) that the free Majorana fermion ξ_n is neutral under this Z_2 symmetry. This apparent mismatch of the Z_2 global symmetry on the two lattice theories raises an potential subtlety in the IR duality of their speculated continuum counterparts Eqs. (1) and (2), where the Z_2 symmetry in the continuum field theories acts as $Z_2 : \phi \to -\phi, \xi \to \xi$. This subtlety is not induced by the lattice construction and was already noticed in Ref. [19] for the continuum field theories. The conjectured resolution of this symmetry mismatch is that the Z_2 symmetry is not spontaneously broken and the energy gap of the Z_2 charged excitations remains finite across the critical point on the boson side of the duality.

If we assume that the lattice theory (7) correctly captures in the continuum boson theory (1) in the IR, we can direct test the aforementioned conjecture. In the lattice theory E(7), there are two SO(3) gauge-invariant Z_2-charged (local) operators:

$$\epsilon_{ii'i''} v_{n,i} \chi_{n,i'} \chi_{n,i''} \quad \text{and} \quad \epsilon_{ii'i''} \chi_{n,i} \chi_{n,i'} \chi_{n,i''}, \tag{17}$$

where we have suppressed the spinor indices of the fermion field χ_n. Being an bosonic operator that is SO(3) gauge-invariant and odd under the Z_2 symmetry, $\epsilon_{ii'i''} v_{n,i} \chi_{n,i'} \chi_{n,i''}$ can serve as an order parameter for any potential spontaneous breaking of the global Z_2 symmetry in the theory (7). Interestingly, we notice that the operator $\epsilon_{ii'i''} v_{n,i} \chi_{n,i'} \chi_{n,i''}$ is always proportional to the product of the fermion fields ξ'_n and ξ''_n that are introduced after we integrate out the SO(3) gauge field in Eq. (10) and that are orthogonal to the low-energy fermion field ξ_n defined in Eq. (12). As stated previously, unlike the fermion field ξ_n, the fields ξ'_n and ξ''_n only have local mass terms but no kinetic term. Therefore any correlation of the fields ξ'_n and ξ''_n is short-range, which implies that the correlation of the operator $\epsilon_{ii'i''} v_{n,i} \chi_{n,i'} \chi_{n,i''}$ is also short-range and, consequently, that the Z_2 global symmetry is unbroken. This statement is in agreement with Ref. [19]. Furthermore, it is evident directly from the derivation of the exact mapping in the previous subsection, that the only low-energy field at the critical point is the fermion field ξ_n which is neutral under the global symmetry Z_2. That in turn implies that all the Z_2 charged excitations in the lattice theory (7) are gapped across the critical point. This observation offers another strong evidence for the conjecture on the Z_2 global symmetry proposed in Ref. [19].

D. An alternative dual model

In Sec. II B, we enforce the condition (11) to ensure that the dual Majorana fermion ξ_n is exactly free. In this section, we proceed with the lattice duality without this condition (11). In fact, for generic values of $U_{1,2}$ with $U_1 < K$, the path integral $e^{-S_{\text{bfg}} - S_{\text{int}}}$ (in a fixed background of vector boson configuration v_n) can be viewed as containing an SO(2) gauge field in disguise on the dual side (see Appendix B). We will discuss the physical meaning of this SO(2) gauge field later. To elucidate the SO(2) gauge structure, we first introduce an SO(2) gauge field represented by a 2×2 orthogonal matrix $U^{n\mu}_{ab}$ (with $a, b = 1, 2$) on each link (connecting site n and site $n + \mu$). Then, we construct the Majorana field $\eta_{n,a}$ that is charged under the SO(2) gauge field:

$$\eta_{n,1} = (K - U_1)^{\frac{1}{4}} \sum_i w'_{n,i} \chi_{n,},$$
$$\eta_{n,2} = (K - U_1)^{\frac{1}{4}} \sum_i w''_{n,i} \chi_{n,i}, \tag{18}$$

where w'_n and w''_n are two mutually orthogonal three-component unit vectors that are both orthogonal to the fixed vector bosons field background v_n. The subscript a in $\eta_{n,a}$ represents the SO(2) "color" index. In fact, the fermion fields $\eta_{n,a}$ are essential the same as the fermion fields ξ'_n and ξ''_n discussed in Sec. II B but with an extra $(K - U_1)^{\frac{1}{4}}$ prefactor that makes the fields $\eta_{n,a}$ well-defined only when $K > U_1$. As we will see, unlike the nonpropagating fermion fields ξ'_n and ξ''_n in Sec. II B, $\eta_{n,a}$ can be interpreted as propagating fermion fields in dual theory thanks to $K > U_1$. Having introduced these ingredients, we can rewrite $e^{-S_{\text{bfg}} - S_{\text{int}}}$ as (see Appendix B for detailed derivation)

$$e^{-S_{\rm bfg}-S_{\rm int}} = \exp\left[-\left(\sum_n M'\bar{\xi}_n\xi_n + \sum_{n,\mu}\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n\right)\right]\int D[U^{n\mu}]\exp\left[-\left(\sum_n M''\bar{\eta}_{n,a}\eta_{n,a} + \sum_{n,\mu}\bar{\eta}_{n+\mu,a}(\sigma^\mu - R)U^{n\mu}_{ab}\eta_{n,b}\right)\right]$$
$$\times \exp\left[-V\sum_{n,\mu}(\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n)(\bar{\eta}_{n+\mu,1}(\sigma^\mu - R)\eta_{n,1})(\bar{\eta}_{n+\mu,2}(\sigma^\mu - R)\eta_{n,2})\right], \quad (19)$$

where the repeated SO(2) indices a, b are automatically summed from 1 to 2. The integration of the SO(2) gauge configuration $\int D[U^{n\mu}]$ should be understood as the integration of the matrix $U^{n\mu}$ over the Haar measure of SO(2) on every link of the space-time lattice. The mass parameters M', M'' of the fermion ξ_n and η_n are given by

$$M' = M/K, \quad M'' = M(K - U_1)^{-\frac{1}{2}}, \quad (20)$$

and the coupling constant V by

$$V = \frac{1 + U_2 - K^2}{(K - U_1)K} \quad (21)$$

We can see that the right-hand side of Eq. (19) is a theory that describes a Majorana fermion ξ_n and Majorana fermions $\eta_{n,a}$ coupled to an SO(2) gauge field. The two types of fermions interact with each other via the SO(2) gauge-invariant interaction given in the third line of Eq. (19).

Physically, the SO(2) gauge field introduced in the dual theory in Eq. (19) essentially recovers the residue SO(2) subgroup of the SO(3) gauge group in the original model Eq. (7) in the presence of a fixed vector boson configuration v_n. The way to understand it is to notice that the fermions $\eta_{n,1}$ and $\eta_{n,2}$ that are charged under the SO(2) gauge field are essentially the two fermion fields orthogonal to ξ_n (as well as the fixed vector field value v_n) is the SO(3) color space. The SO(2) gauge group can be viewed as the residue gauge group of SO(3) that preserves the vector field configuration v_n. As we tune to the parameter regime where M' is close to 3, K is close to U_1 and V is small. The Majorana fermion χ_n will have a small bare mass, while the mass parameter M'' of the Majorana fermion η_n can be set to be $M'' \gg 3$ which ensures not only a big energy gap but also a trivial bare band stucture of η_n. Furthermore, if V is tuned to a small value by tuning U_2, the dual theory will only have an "almost free" (or weakly interacting) Majorana fermion ξ_n at low energy. Since the dual low-energy Majorana fermion ξ_n is now weakly interacting, the exact position of the critical point of the dual theory (19) [as well as the original model (7)] is expected to be shifted from $M' = 3$.

III. LATTICE DUALITY WITH O(3) GAUGE FIELDS

In this section, we generalize the model (7) studied in the previous section to the case with O(3) gauge fields. The action $S_{\rm bg}[v, O]$ given in Eq. (3) and $S_{\rm fg}[\chi, O]$ in Eq. (4) can be directly promoted to accommodate O(3) gauge field $O^{n\mu}$. We will rename them as $S'_{\rm bg}[v, O]$ and $S'_{\rm fg}[\chi, O]$ in order to distinguish them from their SO(3)-gauge-group counterpart. All the interactions in $S_{\rm int}$ are not invariant under the O(3) gauge transforms. Although one can always consider a "O(3)-gauged" version of $S_{\rm int}$, we will exclude them from the generalized model for simplicity. The Euclidean lattice path integral of the generalized model is given by

$$Z'_b = \int D[O^{n\mu}]\int D[\chi_{n,i}]\int D[v_n]e^{-S'_{\rm bg}-S'_{\rm fg}}, \quad (22)$$

where the integration of the boson field v_n and the fermion field χ_n follow the same rules as before. The integration of the gauge field $\int D[O^{n\mu}]$ is now carried over the Haar measure of O(3) instead of SO(3). However, since we can write O(3) = SO(3) × Z_2, the integration over O(3) can be viewed as the integration over the Haar measure of SO(3) together with an "integration" over the Z_2 subgroup.

Similar to the treatment of the previous section, we can first integrate out the O(3) gauge field

$$\int D[O^{n\mu}]\, e^{-S'_{\rm bg}}e^{-S'_{\rm fg}} = \left(\frac{2\sinh J}{J}\right)^{3N_s} e^{-S'_{\rm bfg}[v,\chi]}, \quad (23)$$

where the effective action $S'_{\rm bfg}[v, \chi]$ takes the form

$$S'_{\rm bfg}[v,\chi] = \sum_n M\bar{\chi}_{n,i}\chi_{n,i} + \sum_{n,\mu} K(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j}), \quad (24)$$

which is much simpler compared to Eq. (10) of the SO(3) case. Again, we introduce the fermion field ξ_n given in Eq. (12) and integrate out the orthogonal fermion fields ξ'_n and ξ''_n to obtain the following exact mapping

$$\int D[O^{n\mu}]\, e^{-S'_{\rm bg}}e^{-S'_{\rm fg}} = \mathcal{N}'\int D[\xi]\exp\left(-\sum_{n,\mu}\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n - \sum_n M'\bar{\xi}_n\xi_n\right), \quad (25)$$

where $\mathcal{N}'$ is an overall normalization constant and the mass parameter M' given by

$$M' = M/K. \quad (26)$$

Now, we have established an exact mapping from the model (22) to a model with the free Majorana freemion ξ_n on the dual side. Again, we emphasize that this lattice duality is exact and is valid for any choice of coupling constant J and mass parameter M. Based on the exact mapping, by setting $M' = M/K = 3$, we can tune both sides of the duality to the critical point where one side of the duality is captured by a massless free Majorana fermion in the IR.

While one side of the duality is naturally identified with the free massless Majorana fermion in the continuum limit, the IR nature of the model (22) remains unclear. One may follow the same strategy as the previous section to integrate over the fermion field χ first and interpret the continuum theory as a vector boson coupled to a Chern-Simons term. However, it is unclear how the Z_2 part of the O(3) gauge field behaves after we integrate out the fermions field χ_n. A possible reconciliation with the duality studied in the previous section and the IR duality between the continuum field theories (1) and (2) is that only the SO(3) subgroup of the O(3) is deconfined. Another puzzling feature of this lattice duality with O(3) gauge field is that while the dual Majorana fermion stays at the critical point, we can freely tune the parameters K such that the bare band structure of the fermions χ_n in the model (22) can have different Chern numbers (and the naively expected Chern-Simons levels): $C = 1$ when $1 < M < 3$ and $C = -2$ when $M < 1$. We will defer resolution of these puzzles to future studies.

Similar boson-fermion lattice duality with different gauge groups can also be constructed following the same method we discussed here. For example, as detailed in Appendix C, when the gauge group is reduced to Z_2, we can obtain a lattice duality between a free massless Majorana fermion and a Z_2 matter field coupled to a Z_2 gauge field.

The lattice duality constructed in this section has an O(3) gauge field on one side and a single Majorana fermion on the other. Interestingly, Ref. [28] propose an IR duality with similar but yet distinct features. The proposed IR duality is between a critical vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory (following the notations of Ref. [28]) and a single Majorana fermion plus a decoupled Z_2 gauge theory. In fact, as detailed in Appendix D, a modified version of the model (22), can be exactly mapped to a free Majorana fermion plus a decoupled Z_2 gauge theory on the lattice. The modified model is on its own is speculated to be connected to in the IR the critical vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory.

IV. SUMMARY AND DISCUSSION

In this work, we construct the Euclidean space-time lattice path integral of a theory with strongly interacting bosons and Majorana fermions coupled to SO(3) gauge field and exactly map this theory to the theory of free Majorana fermions. This exact mapping or lattice duality is argued to be potentially connected to the IR duality between critical bosons coupled to a $SO(3)_1$ Chern-Simons term and a single free Majorana fermion proposed in Ref. [16,19]. This lattice duality provides an exact mapping of the Euclidean path integrals and correlation functions between both sides of the duality, which allows us to obtain another suggestive evidence for the conjecture in Ref. [19] regarding the apparent mismatch of a global Z_2 symmetry on the two sides of the IR duality. This model is also generalized to lattice theories with O(3) gauge fields and with Z_2 gauge fields, respectively, which are both again exactly dualized to a free Majorana fermion. A different generalization of the lattice model with O(3) gauge fields is shown to be dual to a free Majorana fermion plus a decoupled Z_2 gauge field.

While the lattice dualities are exact, the fate of these strongly interacting lattice theories in the IR still needs more attention. In Sec. II, we argue that the model (7) with strongly interacting boson and fermions coupled to SO(3) gauge field should be related to the critical boson coupled to a Chern-Simons term in the IR. To solidify this argument, we need to understand the effect of $S_{\rm int}$, in particular, on the Chern-Simons level when we integrate out the fermion fields χ_n in Eq. (7). Knowing that the gauge invariant fermion in the lattice duality has a critical point, the bosonic side of the duality should certainly also have a critical point, but whether this critical point really corresponds to the one described by the field theory (1) is difficult to prove. This is certainly a question to be answered in the future. Similarly, as we discussed in Sec. III, the IR nature of the theory of strongly-interacting boson and fermions coupled to the O(3) gauge field given in Eq. (22) also remains unclear. In particular, the role of the Z_2 subgroup of the O(3) gauge group and the Chern number of the bare band structure of the fermion fields χ_n both require further investigation.

In a broader picture, it is an interesting direction to generalize the lattice construction of dualities to interacting theories with more general non-Abelian gauge groups and more flavors of interacting bosons or fermions. While being exact without the use of any large-N limit, holography or supersymmetry, the lattice construction is potentially a powerful tool in deriving previously proposal IR dualities with non-Abelian gauge fields as well as discovering new ones. Once the lattice construction of these duality is constructed, one should be able to read out the correspondence between the operators on both sides of the duality.

Note added. After the completion of this work, we learned of Ref. [54], which is a similar and independent attempt to generalize the lattice construction to tackle dualities with non-Abelian gauge fields.

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APPENDIX A: INTEGRATION OVER THE SO(3) GAUGE FIELD

To integrate out the SO(3) gauge field in the action (7), we only need to focus on the $S_{\rm bg}$ and $S_{\rm fg}$ parts of the action:

$$\int D[O^{n\mu}]\, e^{-S_{\rm bg}[v,O]} e^{-S_{\rm fg}[\chi,O]}$$
$$= \exp\left(-\sum_i \sum_n M\,\bar{\chi}_{n,i}\chi_{n,i}\right) \prod_n \prod_{\mu=x,y,z}$$

$$\times\left(\sum_{l=0}^{\infty}\sum_{m=0}^{3}\frac{J^l}{l!}\frac{(-1)^m}{m!}\int_{SO(3)}dO^{n\mu}\left(v_{n+\mu,i}O^{n\mu}_{ij}v_{n,j}\right)^l \times\left(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)O^{n\mu}_{ij}\chi_{n,j}\right)^m\right). \quad \text{(A1)}$$

Generically, since the fermion field χ_n on each site carries a twofold spinor index and a threefold SO(3) color index, the terms in the expansion are expected to vanish because of Fermi statistics only when $m > 6$. However, when we take $R = -1$ (as we did in the main text), all the terms with $m > 3$ will vanish. The simplification comes from the fact that $\sigma^\mu - R$ for any fixed μ is a rank-1 matrix acting on the spinor space of the fermion χ_n when $R = -1$. In another word, for a specific link labeled by n and μ, $(\sigma^\mu - R)$ only "hops" a single spinor mode of $\chi_{n,i}$ out of its two dimensional spinor space from site n to $n+\mu$. Therefore the terms with $m > 3$ all vanish because of Fermi statistics in Eq. (A1). Since the one spinor mode that is singled out by $\sigma^\mu - R$ for is different in different directions μ, the fermion kinetic term $\sigma^\mu - R$ overall *does not* leave any fermion mode nonpropagating. We will choose $R = -1$ throughout the discussion.

In the following, we will tackle the integration in Eq. (A1) for the terms with $m = 0,\ 1,\ 2,\ 3$ separately. The following identity on the integration over the SO(3) group will be helpful:

$$\int_{SO(3)}dO\,(y^\intercal Ox)^n(v^\intercal Ou)(s^\intercal Or) = \begin{cases}\frac{1}{2(n+2)}[x\cdot(u\times r)][y\cdot(v\times s)] & n \text{ odd}\\ \frac{1}{2(n+1)(n+3)}\{(n+2)(u^\intercal r)(v^\intercal s)-n[(x^\intercal u)(x^\intercal r)(v^\intercal s)+(y^\intercal v)(y^\intercal s)(u^\intercal r)]+3n(x^\intercal u)(x^\intercal r)(y^\intercal v)(y^\intercal s)\} & n \text{ even}\end{cases}. \quad \text{(A2)}$$

Here, the SO(3) matrix O is being integrated under the Haar measure of SO(3). $x, y \in \mathbb{R}^3$ are three-component vectors of unit length, while $r, s, v, u \in \mathbb{R}^3$ are any three-component vectors of arbitrary length.

For the terms with $m = 0$ in Eq. (A1), we have

$$\sum_{l=0}^{\infty}\frac{J^l}{l!}\int_{SO(3)}dO^{n\mu}\left(v^\intercal_{n+\mu}O^{n\mu}v_n\right)^l = \sum_{l\text{ even}}\frac{J^l}{l!}\frac{1}{l+1} = \frac{\sinh J}{J}. \quad \text{(A3)}$$

For $m = 1$, we have

$$\sum_{l=0}^{\infty}-\frac{J^l}{l!}\int_{SO(3)}dO^{n\mu}\left(v^\intercal_{n+\mu}O^{n\mu}v_n\right)^l\left(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)O^{n\mu}_{ij}\chi_{n,j}\right)$$
$$= \sum_{l\text{ odd}}-\frac{J^l}{l!}\frac{1}{l+2}((\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu-R)(\chi_{n,j}v_{n,j})) = -\frac{J\cosh J-\sinh J}{J^2}(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu-R)(\chi_{n,j}v_{n,j}). \quad \text{(A4)}$$

For $m = 2$, we have

$$\sum_{l=0}^{\infty}\frac{J^l}{l!}\frac{1}{2}\int_{SO(3)}dO^{n\mu}\left(v^\intercal_{n+\mu}O^{n\mu}v_n\right)^l\left(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)O^{n\mu}_{ij}\chi_{n,j}\right)^2$$
$$= \sum_{l\text{ odd}}\frac{J^l}{l!}\frac{1}{2}\frac{1}{2(l+2)}\varepsilon_{ii'i''}\varepsilon_{jj'j''}(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)\chi_{n,j})(\bar{\chi}_{n+\mu,i'}(\sigma^\mu-R)\chi_{n,j'})v_{n+\mu,i''}v_{n,j''}$$
$$= \frac{J\cosh J-\sinh J}{4J^2}\varepsilon_{ii'i''}\varepsilon_{jj'j''}(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)\chi_{n,j})(\bar{\chi}_{n+\mu,i'}(\sigma^\mu-R)\chi_{n,j'})v_{n+\mu,i''}v_{n,j''}.$$
$$= \frac{J\cosh J-\sinh J}{4J^2}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''}. \quad \text{(A5)}$$

Notice that the even l contributions in the equation above all vanish because of fermion statistics. Lastly, for $m = 3$, we have

$$\sum_{l=0}^{\infty}-\frac{J^l}{l!}\frac{1}{6}\int_{SO(3)}dO^{n\mu}\left(v^\intercal_{n+\mu}O^{n\mu}v_n\right)^l\left(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)O^{n\mu}_{ij}\chi_{n,j}\right)^3$$
$$= \sum_{l=0}^{\infty}-\frac{J^l}{l!}\frac{1}{6}\int_{SO(3)}dO^{n\mu}\left(v^\intercal_{n+\mu}O^{n\mu}v_n\right)^l(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)\chi_{n,i})^3$$
$$= -\frac{1}{6}\frac{\sinh J}{J}(\bar{\chi}_{n+\mu,i}(\sigma^\mu-R)\chi_{n,i})^3,$$
$$= -\frac{1}{36}\frac{\sinh J}{J}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}, \quad \text{(A6)}$$

where we have used the fact that $(\bar{\chi}_{n+\mu,i}(\sigma^\mu - R)O^{n\mu}_{ij}\chi_{n,j})^3 = (\det O^{n\mu})(\bar{\chi}_{n+\mu,i}(\sigma^\mu - R)\chi_{n,i})^3$ for $R = -1$ and $\det O^{n\mu} = 1$. Now, we can conclude that

$$\begin{aligned}\int D[O^{n\mu}]\, e^{-S_{\rm bg}[v,O]}e^{-S_{\rm fg}[\chi,O]} &= \exp\left(-\sum_n M\bar{\chi}_{n,i}\chi_{n,i}\right)\times\prod_{n,\mu}\left[\frac{\sinh J}{J} - \frac{J\cosh J - \sinh J}{J^2}(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})\right.\\ &\quad\left. + \frac{J\cosh J - \sinh J}{4J^2}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''} - \frac{1}{36}\frac{\sinh J}{J}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}\right]\\ &= e^{-S_{\rm bfg}[v,\chi]}\times\prod_{n,\mu}\frac{\sinh J}{J}\end{aligned}\tag{A7}$$

with the effective action $S_{\rm bfg}[v,\chi]$:

$$\begin{aligned}S_{\rm bfg}[v,\chi] = \sum_n &\Big\{M\bar{\chi}_{n,i}\chi_{n,i} + K(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})\\ &- \frac{K}{4}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''} + \frac{1-K^2}{36}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}\Big\},\end{aligned}\tag{A8}$$

where we have defined the variable $K = \frac{J\cosh J - \sinh J}{J\sinh J}$. In obtaining this effective action, we have applied the identity $(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})\times(\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''}) = \frac{1}{9}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}$ for $R = -1$.

APPENDIX B: DERIVATION OF THE ALTERNATIVE DUAL MODEL

Without the cancellation condition (11), we can generally write $e^{-S_{\rm bfg}[v,\chi]-S_{\rm int}[v,\chi]}$ as

$$\begin{aligned}e^{-S_{\rm bfg}[v,\chi]-S_{\rm int}[v,\chi]} &= \exp\left(-\sum_n M\bar{\chi}_{n,i}\chi_{n,i}\right)\times\prod_{n,\mu}\Big\{1 - K(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})\\ &\quad + \frac{K-U_1}{4}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}v_{n+\mu,i''}v_{n,j''} - \frac{1+U_2-U_1K}{36}\varepsilon_{ii'i''}\varepsilon_{jj'j''}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}\Big\},\end{aligned}\tag{B1}$$

We have introduced the fermion field ξ_n in Eq. (12) and fermion field $\eta_{n,a}$ (with $a = 1, 2$) in Eq. (18). These three fermion fields are orthogonal to each other in the SO(3) color space. Under a fixed configuration of the vector boson v_n, we can write

$$\begin{aligned}e^{-S_{\rm bfg}-S_{\rm int}} &= \exp\left[\sum_n(M'\bar{\xi}_{n,i}\xi_{n,i} + M''\bar{\eta}_{n,1}\eta_{n,1} + M''\bar{\eta}_{n,2}\eta_{n,2})\right]\\ &\quad\times\prod_{n,\mu}\Big\{1 - \bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n + (\bar{\eta}_{n+\mu,1}(\sigma^\mu - R)\eta_{n,1})(\bar{\eta}_{n+\mu,2}(\sigma^\mu - R)\eta_{n,2})\\ &\quad - \frac{1+U_2-U_1K}{(K-U_1)K}(\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n)(\bar{\eta}_{n+\mu,1}(\sigma^\mu - R)\eta_{n,1})(\bar{\eta}_{n+\mu,2}(\sigma^\mu - R)\eta_{n,2})\Big\},\end{aligned}\tag{B2}$$

where

$$M' = M/K,\quad M'' = M(K-U_1)^{-\frac{1}{2}}.\tag{B3}$$

Notice that we have been assuming that $K > U_1$ in this discussion. To simplify the expression further, a trick is to introduce an SO(2) gauge field $U^{n\mu}$, represented by a 2×2 orthogonal matrix $U^{n\mu}_{ab}$ with $a, b = 1, 2$, on every link:

$$\begin{aligned}e^{-S_{\rm bfg}-S_{\rm int}} &= \exp\left[-\sum_n(M'\bar{\xi}_n\xi_n + M''\bar{\eta}_{n,a}\eta_{n,a})\right]\times\prod_{n,\mu}\Big\{\int dU^{n\mu}\exp(-\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n)\times\exp\big(-\bar{\eta}_{n+\mu,a}(\sigma^\mu - R)U^{n\mu}_{ab}\eta_{n,b}\big)\\ &\quad\times\exp(-V(\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n)(\bar{\eta}_{n+\mu,1}(\sigma^\mu - R)\eta_{n,1})(\bar{\eta}_{n+\mu,2}(\sigma^\mu - R)\eta_{n,2}))\Big\}\end{aligned}\tag{B4}$$

with $V = \frac{1+U_2-K^2}{(K-U_1)K}$. Here, the repeated SO(2) indices a, b are automatically summed from 1 to 2. Although this rewriting is introduced as a trick, its physical meaning is discussed in Sec. II D. To show that Eq. (B4) is equivalent to Eq. (B2), we simply need to perform a Taylor expansion to the exponential term $\exp(-\bar{\eta}_{n+\mu,a}(\sigma^\mu - R)U^{n\mu}_{ab}\eta_{n,b})$ in Eq. (B4). Here, we have also chosen $R = -1$ which in this case leads to only three nonvanishing terms, i.e., the zeroth-, the first-, and the second-order terms, in the expansion. Other terms vanish because of Fermi statistics. Now, we can perform the integration $\int dU^{n\mu}$ term by term. The first order term further vanishes because it contains an odd power of $U^{n\mu}$. The integration for the zeroth- and the second-order terms are also simple because they are in fact independent of $U^{n\mu}$. Putting these together, we can verify that Eq. (B4) is consistent with Eq. (B2). Equation (B4) is essentially Eq. 19 after regrouping different terms on the right-hand side of the equation.

APPENDIX C: LATTICE DUALITY WITH Z_2 GAUGE FIELD

In this Appendix, we provide the lattice construction of a similar boson-fermion duality with the gauge group (on the boson side) reduced from O(3) to Z_2. Loosely speaking, we will show that a single-component real scalar boson coupled to an "O(1)$_1$ Chern-Simons" gauge field is dual to a free Majorana fermion in 2+1D. The precise meaning of the "O(1)$_1$ Chern-Simons" term is actually a Z_2 gauge field coupled to massive Majorana fermions in a Chern band with Chern number $C = 1$, which is also known as the Ising topological order.

We start with the path integral formulation on the 3D Euclidian space-time lattice. On the bosonic side of the duality, we introduce on each site n a single-component scalar boson σ_n (which can treated as a Z_2 variable $\sigma_n = \pm 1$) and two real Grassmanian variables $\chi_n = (\chi_{n1}, \chi_{n2})^\intercal$. They both couple to a Z_2 gauge field $B^{n\mu} = \pm 1$ on the link $n\mu$ that connects site n and $n + \mu$ (where $\mu = 0,\ 1,\ 2$ labels the link direction). The partition function of the Euclidean lattice path integral is given by

$$Z_b = \int D[\chi] \sum_{[\sigma,B]} e^{-S_{\text{bg}}[\sigma,B]-S_{\text{fg}}[\chi,B]}, \qquad \text{(C1)}$$

where the actions S_{bg} and S_{fg} are given by

$$S_{\text{bg}}[\sigma,B] = -J\sum_{n\mu} \sigma_{n+\mu}B^{n\mu}\sigma_n,$$
$$S_{\text{fg}}[\chi,B] = \sum_{n\mu} \bar{\chi}_{n+\mu}(\gamma^\mu - R)B^{n\mu}\chi_n + \sum_n M\bar{\chi}_n\chi_n. \qquad \text{(C2)}$$

Here the gamma matrices are defined as $(\gamma^0, \gamma^1, \gamma^2) = (\sigma^2, \sigma^3, \sigma^1)$ and $\bar{\chi}_n \equiv \chi_n^\intercal \gamma^0$. The Majorana Chern number of the lattice fermion χ is still given by Eq. (5). The theory describes the an Ising Higgs model twisted by auxiliary Majorana fermions. The interaction among auxiliary fermions χ can be circumvented when the gauge group is Z_2, which simplifies the derivation of the duality, similar to the case of O(3) gauge group in Eq. (22). The partition function Z_b in Eq. (C1) can be expanded on the lattice to the following form:

$$Z_b = \int D[\chi] \sum_{[\sigma,B]} \prod_{n\mu} T_{n\mu}(R)[\chi,B] W_{n\mu}(J)[\sigma,B] \times \prod_n V_n(M)[\chi], \ w$$

$$T_{n\mu}(R)[\chi,B] = e^{-\bar{\chi}_{n+\mu}(\gamma^\mu - R)B^{n\mu}\chi_n} = 1 - \bar{\chi}_{n+\mu}(\gamma^\mu - R)B^{n\mu}\chi_n,$$
$$W_{n\mu}(J)[\sigma,B] = e^{J\sigma_{n+\mu}B^{n\mu}\sigma_n} \propto 1 + (\tanh J)\sigma_{n+\mu}B^{n\mu}\sigma_n,$$
$$V_n(M)[\chi] = e^{-M\bar{\chi}_n\chi_n}. \qquad \text{(C3)}$$

In the expansion of $T_{n\mu}(R)[\chi, B]$, we have assumed $R = -1$ such that the expansion terminates at the quadratic order. On each link $n\mu$, we can first integrating out the Z_2 gauge field $B^{n\mu} = \pm 1$, and arrive at a new link term

$$T'_{n\mu}(R,J)[\chi,\sigma] = \sum_{B^{n\mu}} T_{n\mu}(R)[\chi,B] W_{n\mu}(J)[\sigma,B] = 1 - (\tanh J)\ \sigma_{n+\mu}\bar{\chi}_{n+\mu}(\gamma^\mu - R)\chi_n\sigma_n. \qquad \text{(C4)}$$

With this, the partition function in Eq. (C3) reduces to

$$Z_b = \int D[\chi] \sum_{[\sigma]} \prod_{n\mu} T'_{n\mu}(R,J)[\chi,\sigma] \prod_n V_n(M)[\chi]. \qquad \text{(C5)}$$

Integrating out the scalar (Ising) field σ_n simply imposes the current conservation of the fermion χ_n, which is already built-in in the fermion path integral formalism. If we redefine a new set of real Grassmannian variables

$$\xi_n = \sqrt{\tanh J}\ \chi_n\sigma_n, \qquad \text{(C6)}$$

the partition function in Eq. (C5) will simply become a theory of ξ_n fermion with renormalized mass,

$$Z_f = \int D[\xi] \prod_{n\mu} T_{n\mu}(R)[\xi] \prod_n V_n(M')[\xi],$$
$$T_{n\mu}(R)[\xi] = 1 - \bar{\xi}_{n+\mu}(\gamma^\mu - R)\xi_n = e^{-\bar{\xi}_{n+\mu}(\gamma^\mu - R)\xi_n},$$
$$V_n(M')[\xi] = e^{-M'\bar{\xi}_n\xi_n}, \qquad \text{(C7)}$$

where the renormalized mass M' is given by

$$M' = M/\tanh J. \qquad \text{(C8)}$$

Now we have arrived at a theory that exactly describes a free Majorana on the Euclidean space-time lattice, which can be equivalently written in the action form as follows:

$$Z_f = \int D[\xi] e^{-S_{\text{f}}[\xi]},$$
$$S_{\text{f}}[\xi] = \sum_{n\mu} \bar{\xi}_{n+\mu}(\gamma^\mu - R)\xi_n + \sum_n M'\bar{\xi}_n\xi_n. \qquad \text{(C9)}$$

Thus we have established an exact lattice duality between the twisted Ising Higgs model Z_b in Eq. (C1) and the free Majorana model Z_f in Eq. (C9). This exact mapping is valid regardless whether the model Z_b (or its dual model Z_f) is at the critical point or not. Knowing that the dual Majorana fermion ξ has a critical point at $R = -1$ and $M' = 3$, the original model Z_b should have the same critical point at $R = -1$, $M = M' \tanh J = 3 \tanh J$. For any positive coupling J, M is always smaller than 3. If we choose J such that $1 < M < 3$, as we integrate out the fermion χ, the model Z_b can be interpreted as a Z_2 matter field coupled to a Z_2 gauge field (with Ising topological order).

APPENDIX D: DUALITY TOWARDS A FREE MAJORANA FERMION PLUS A DECOUPLED Z_2 GAUGE FIELD

In Ref. [28], an IR duality between the critical vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory (following the notation of Ref. [28]) and the theory with a free Majorana fermion plus a decoupled Z_2 gauge theory is proposed. Before we discuss a related lattice duality, a brief explanation of the $O(3)^0_{1,1}$ Chern-Simons term is in order. In the convention of Ref. [28], $O(3)^0_{1,1}$ Chern-Simons term is an O(3) Chern-Simons term at level 1 plus another topological term $f[w_1]$ at level 1. Here, w_1 is the first Stiefel-Whitney class of the O(3) gauge bundle, which in this case can be identified with the Z_2 gauge fields obtained from restricting the $O(3) = SO(3) \times Z_2$ gauge group to its Z_2 subgroup. When we viewed w_1 as a Z_2 gauge field, $f[w_1]$ is the topological term generated by coupling the Z_2 gauge field w_1 to a single copy of Z_2-charged massive Majorana fermion in a band structure with Chern number 1 (or equivalently to a single copy of $p + ip$ superconductor with the fermions carrying a Z_2 charge). Since we can decompose the gauge group according to $O(3) = SO(3) \times Z_2$, we can rewrite following Ref. [28] that $O(3)^0_{1,1}$ Chern-Simons term = $SO(3)_1$ Chern-Simons term + $\pi w_2[SO(3)] \cup w_1 + 3f[w_1]$, where w_1 is identified with the Z_2 subgroup of the O(3) gauge group and $w_2[SO(3)]$ is the second Stiefel-Whitney class of the SO(3) part of the gauge field. From this relation, one can see that the $O(3)^0_{1,1}$ Chern-Simons term can be generated by coupling three copies of massive Majorana fermions (forming an vector representation under the O(3) gauge group) each in a band structure with Chern number 1 to an O(3) gauge field. As we will see, this understanding of the $O(3)^0_{1,1}$ Chern-Simons term will also be helpful in understanding the IR nature of the lattice model we will discuss in the following.

This proposed IR duality in Ref. [28] is similar to the lattice duality studied in Sec. III on the corresponding "boson sides" as they both describe a vector boson coupled to an O(3) Chern-Simons gauge theory. However, on the "fermion side," while both studies contain the theory of a free Majorana fermion, the proposed duality of Ref. [28] also includes an extra decoupled Z_2 gauge theory. Inspired by the proposed IR duality, we will introduce a lattice model that is slightly different from Eq. (22) and construct an exact mapping to a dual theory containing a free Majorana fermion and a decoupled Z_2 gauge theory. We will agree that this lattice duality is connected to the IR duality between the critical vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory and a free Majorana fermion plus a decoupled Z_2 gauge theory.

Now, we introduce the ingredients of the lattice model. We consider a model with the same degrees of freedom as the model (22): a vector boson field v_n and a Majorana fermion field χ_n on each site both coupled to the O(3) gauge field $O^{n\mu}$ on the links. Since $O(3) = SO(3) \times Z_2$, we can always decompose the O(3) gauge field as $O^{n\mu} = B^{n\mu}\tilde{O}^{n\mu}$, where $B^{n\mu} = \pm 1$ describes a Z_2 gauge field and $\tilde{O}^{n\mu}$ is an SO(3) matrix that describes a SO(3) gauge field on the link $n\mu$. The integration over the O(3) gauge field $O^{n\mu}$ is equivalent to the integration over the SO(3) gauge field $\tilde{O}^{n\mu}$ under the Haar measure of SO(3) followed by the summation over $B^{n\mu} = \pm 1$, i.e., $\int D[O^{n\mu}] = \int D[\tilde{O}^{n\mu}] \sum_{[B^{n\mu}]}$, where $\sum_{[B^{n\mu}]}$ represents the summation over all configuration of $B^{n\mu} = \pm 1$. We will include the actions S'_{bg} and S'_{fg} introduced in Sec. III in the current lattice model. Using these new gauge field variables, we can rewrite them as

$$S'_{bg}[v, O] = S'_{bg}[v, \tilde{O}, B] = \sum_n \sum_{\mu=x,y,z} -J\, v_{n+\mu,i} B^{n\mu} \tilde{O}^{n\mu}_{ij} v_{n,j} \tag{D1}$$

and

$$S'_{fg}[\chi, O] = S'_{fg}[\chi, \tilde{O}, B] = \sum_n \sum_{\mu=x,y,z} \bar{\chi}_{n+\mu,i}(\sigma^\mu - R) \times B^{n\mu}\tilde{O}^{n\mu}_{ij}\chi_{n,j} + M\sum_n \bar{\chi}_{n,i}\chi_{n,i}. \tag{D2}$$

While the model studied in Sec. III only contains the terms S'_{bg} and S'_{fg}, we will further introduce an interaction term S'_{int} and a gauge field term S'_{Z_2} for the current model of interest. The interaction terms S'_{int} is given by

$$S'_{int}[\chi, v, B] = \frac{U_1}{4}\sum_{n,\mu} \varepsilon_{ii'i''}\varepsilon_{jj'j''} B^{n\mu} \mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'} v_{n+\mu,i''} v_{n,j''} + \frac{U_2}{36}\sum_{n,\mu} \varepsilon_{ii'i''}\varepsilon_{jj'j''} B^{n\mu}\mathcal{J}^{n\mu}_{ij}\mathcal{J}^{n\mu}_{i'j'}\mathcal{J}^{n\mu}_{i''j''}. \tag{D3}$$

Notice that even though $S'_{int}[\chi, v, B]$ only depends on the Z_2 gauge field $B^{n\mu}$, it is fully O(3) gauge-invariant. In fact, one can view S'_{int} as the "O(3)-gauged" version of Eq. (6). The Z_2 gauge field $B^{n\mu}$ can have its only dynamics described by the standard Z_2 lattice gauge theory action:

$$S'_{Z_2}[B] = \sum_{\text{plaquette } p} t \prod_{\text{link } l\in p} B^l, \tag{D4}$$

where p labels the two-dimensional square plaquettes of the 3D space-time cubic lattice, the product $\prod_{\text{link } l\in p}$ represents the product over all links l that belong to the plaquette p, and t denotes the coupling constant of the Z_2 gauge field $B^{n\mu}$. Notice that such a pure gauge dynamical term was not introduced in any previous lattice models studied in the main text where the only gauge field dynamics were thought of as generated by the coupling to the Majorana fermions. In the discussion here, we not only couple the gauge field to the fermions χ_n, but also include the term $S'_{Z_2}[B]$ into consideration. As we will see, such a gauge dynamical term will help keeping the Z_2 gauge field deconfined across the duality. Having introduced all the ingredients, we can write down the model of interest

$$Z''_b = \sum_{[B^{n\mu}]} \int D[\tilde{O}^{n\mu}] \int D[\chi_{n,i}] \int D[v_n] e^{-S'_{bg}-S'_{fg}-S'_{int}-S'_{Z_2}}. \tag{D5}$$

To construct the lattice duality, we first integrate out the SO(3) gauge field $\tilde{O}_{n\mu}$ in Eq. (D5). The technical details of this integration very much follow those in Sec. II and Appendix A. Similar to Sec. II, by choosing the parameters

Eq. 11, we have

$$\int D[\tilde{O}^{n\mu}]e^{-S'_{\rm bg}-S'_{\rm fg}-S'_{\rm int}} = \exp\left(-\sum_n M\bar{\chi}_{n,i}\chi_{n,i}\right) \times \prod_{n,\mu}[1-K(\bar{\chi}_{n+\mu,i}v_{n+\mu,i})(\sigma^\mu - R)(\chi_{n,j}v_{n,j})]. \quad \text{(D6)}$$

Interestingly, even though all of $S'_{\rm bg}$, $S'_{\rm fg}$ and $S'_{\rm int}$ depend on the Z_2 gauge field $B^{n\mu}$, the right-hand side of this equation is independent of $B^{n\mu}$ as the result of both the integration over the SO(3) gauge field $\tilde{O}^{n\mu}$ and the parameter choice Eq. (11). Again, we introduce the Majorana fermion field ξ_n following Eq. (12) and integrate out the Majorana fermion fields ξ'_n and ξ''_n that are orthogonal to ξ'_n in the O(3) color space. We obtain that

$$\begin{aligned}\sum_{[B^{n\mu}]}\int D[\tilde{O}^{n\mu}]\int D[\chi_{n,i}]e^{-S'_{\rm bg}-S'_{\rm fg}-S'_{\rm int}-S'_{Z_2}} \\ = \mathcal{N}''\sum_{[B^{n\mu}]}\int D[\xi_n]\exp\bigg(-\sum_{n,\mu}\bar{\xi}_{n+\mu}(\sigma^\mu - R)\xi_n \\ -\sum_n M'\bar{\xi}_n\xi_n\bigg)\times\exp\left(-S'_{Z_2}\right),\end{aligned} \quad \text{(D7)}$$

where $M' = M/K$ and $\mathcal{N}''$ is an overall normalization constant. We notice that the right-hand side of Eq. (D7) describes a dual theory with a free Majorana fermion plus a decoupled Z_2 gauge theory. In fact, this lattice duality can be viewed as the one studied in Sec. II B with the Z_2 global symmetry (introduced in Sec. II C) promoted to a dynamical Z_2 gauge theory.

When we tune $M' = 3$ and keep the coupling constant t in Eq. (D4) of the Z_2 gauge field sufficiently large (and positive), the dual theory is at a critical point that contains a massless Majorana fermion and a decoupled deconfined Z_2 gauge theory in the IR. Now, we turn to the discussion of the IR nature of the model (D5) at this critical point. When $M' = 3$, the mass parameter M is always $M < 3$. When we choose the coupling constant J such that $1 < M < 3$, the Chern number C of the bare band structure of the Majorana fermion fields χ_n in the model (D5) becomes $C = 1$. In this regime, if we integrate out the Majorana fermions χ_n in Eq. (D5), while neglecting the effect of $S'_{\rm int}$ in this integration, an $O(3)^0_{1,1}$ Chern-Simons term will be generated and the resulting theory will be naturally identified as a vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory. However, it is unclear how exactly the interaction terms $S'_{\rm int}$, which is inevitable by the condition (11), affects this statement. An observation is that, in a naively continuum limit, all the terms in $S'_{\rm int}$ contain high powers of space-time derivatives and may be considered irrelevant in a continuum field theory. Also, we notice that the $O(3)^0_{1,1}$ Chern-Simons term does not have any continuous tuning parameters. These two observations make it plausible that the $O(3)^0_{1,1}$ Chern-Simons term is not affected by the interactions $S'_{\rm int}$ when we integrate out the fermion χ_n with a bare mass $3 - M$ of order 1 (and with $M - 1$ order 1 as well) in the model (D5). Therefore we speculate that the theory (D5) still corresponds to the theory of a vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory in the IR despite of the interactions $S'_{\rm int}$. If this speculation is correct, the lattice duality discussed in this Appendix can be viewed as an UV regulated version of the IR duality between a critical vector boson coupled to an $O(3)^0_{1,1}$ Chern-Simons gauge theory and a free Majorana fermion plus a decoupled Z_2 gauge field proposed in Ref. [28].

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