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Quantum Brownian motion in a quasiperiodic potential

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We consider a quantum particle subject to Ohmic dissipation, moving in a bichromatic quasiperiodic potential. In a periodic potential the particle undergoes a zero-temperature localization-delocalization transition as dissipation strength is decreased. We show that the delocalized phase is absent in the quasiperiodic case, even when the deviation from periodicity is infinitesimal. Using the renormalization group, we determine how the effective localization length depends on the dissipation. We show that a similar problem can emerge in the strong-coupling limit of a mobile impurity moving in a periodic lattice and immersed in a one-dimensional quantum gas.

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Introduction. Localization has been a subject of interest for over half a century, following Anderson's seminal work on electron propagation in disordered media [1]. Recently, the recognition that the many-body localized (MBL) insulator is a stable state of matter with a robust nonequilibrium phase structure has sparked renewed interest in the topic [2–6]. Although much of this effort has focused on isolated systems with uncorrelated disorder, two departures from these prevalent paradigms have emerged as significant. First, studying localization in open quantum systems coupled to an external "bath" is both intrinsically interesting [7–9] and relevant to many experiments [10–12]. Second, quasiperiodic systems can also display localization, but unlike their disordered cousins, may be less susceptible to rare region effects that disrupt MBL in $d > 1$ [13–19]. Quasiperiodic potentials can be engineered robustly and controllably in cold-atom experiments, either by superposing two mutually incommensurate optical lattices, or by "cut-and-project" techniques. Experiments have now begun to probe the interplay of localization, interactions, and coupling to a bath in quasiperiodic systems [10,11,19–24].

Here, we show that the properties of a quasiperiodic system can be altered by coupling to a bath with nontrivial dynamics, even without interactions. As MBL focuses on excited eigenstates and hence high temperature T, baths in that context are approximated as Markovian, i.e., memoryless on long timescales [8]. In contrast, for $T \to 0$, the bath autocorrelation time can diverge, so that memory effects become significant. Such non-Markovian baths can arise naturally from quantum dissipation, induced, e.g., by coupling to a continuum of gapless excitations [25,26]. The simplest examples involve dissipative dynamics of a *single* quantum degree of freedom [25–31]. This can be the position of a particle, but similar models arise more generally in "quantum impurity problems," describing, e.g., the phase of a resistively and capacitively shunted Josephson junction, a Kondo spin in a metal, or the scattering phase shift at a quantum point contact or across a mobile impurity in a quantum fluid [32–34].

Despite their simplicity, these models can nevertheless exhibit phase transitions, e.g., as a function of dissipation strength [27,30,35,36]. For instance, a particle in a periodic potential can undergo a $T = 0$ phase transition as the strength of Ohmic dissipation α is tuned: For $\alpha > \alpha_c$ the particle is localized in one of the potential minima, while for $\alpha < \alpha_c$ it is delocalized and undergoes quantum Brownian motion over long distances, where α_c is a critical value of dissipation set by the periodicity of the potential [27]. We examine the fate of this $T = 0$ transition for *quasiperiodic* potentials. We show that the delocalized phase present at weak dissipation $\alpha < \alpha_c$ for a single periodic potential [27] is destabilized by an additional periodic perturbation, even when the latter has a *higher* critical dissipation strength in isolation. The resulting phase diagram depends on the ratio between the periods of the potentials. In the commensurate case, the delocalized phase survives, but with a lower critical dissipation strength than for either potential in isolation; for the incommensurate (quasiperiodic) case, it is destroyed. Notably, with dissipation the delocalized phase is absent even for infinitesimally weak quasiperiodic perturbations, in striking contrast to the dissipationless case [13] where it survives up to a critical value of the quasiperiodicity. Although the problem formally maps to a "double-frequency" boundary sine-Gordon model with no exact solution, we can compute an approximate localization length using renormalization-group (RG) techniques. We showcase this approach for examples of commensurate and incommensurate perturbations.

We also find a surprising application of our analysis to the currently more experimentally realizable setting of a mobile impurity moving in a periodic lattice in one dimension, immersed in a quantum fluid that it scatters strongly via contact interactions. Here, our model describes the dissipative dynamics of the scattering phase across the impurity, the relevant commensurability is between the gas density and the lattice, and the transition corresponds to a change in the impurity dispersion [energy-momentum relation $E(P)$], from flat to periodic.

Model. We begin by considering a single quantum particle interacting with a bath of harmonic oscillators [25,26]. The joint Hamiltonian is

$$H = H_0(q) + \frac{1}{2}\sum_a \frac{p_a^2}{m_a} + m_a\omega_a^2\left(x_a + \frac{f_a[q]}{m_a\omega_a^2}\right)^2, \quad (1)$$

where a indexes the oscillators, q is the spatial coordinate of the particle, and $H_0 = p^2/2m + V(q)$, with $V(q)$ a local potential. We assume linear particle-bath coupling $f[q] = \lambda_a q$, and characterize the bath via its spectral function $J(\omega) = \frac{\pi}{2}\sum_a \frac{\lambda_a^2}{m_a\omega_a}\delta(\omega - \omega_a)$. We restrict to Ohmic dissipation, $J(\omega) = \eta|\omega|$, which in the classical/high-temperature limit yields Brownian motion described by a Langevin equation [25,26]. Integrating out the bath in the partition function yields an (imaginary-time) effective action for the particle [37], which for Ohmic dissipation and $V = 0$ is

$$S_0 = \int_0^{\beta\hbar} d\tau\left[\frac{m}{2}\dot{q}^2(\tau) + \frac{\eta}{2\pi}\int_{-\infty}^{\infty} d\tau' \frac{q(\tau)q(\tau')}{(\tau - \tau')^2}\right]. \quad (2)$$

We scale out a microscopic length q_0 (this will be set by the potential) and take $\theta(\tau) = 2\pi q(\tau)/q_0$. We identify the characteristic energy scale $E_0 = (2\pi\hbar)^2/mq_0^2$ required to confine the particle to q_0, so that $\Lambda = E_0/\hbar$ sets the scale of the bare kinetic energy. Since this is irrelevant under the RG by power counting (compared to the nonlocal bath contribution) we replace it by a cutoff Λ on the bath term [27–29,31]

$$S_0[\theta(\omega)] = \frac{\alpha}{4\pi}\int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi}\, |\omega||\theta(\omega)|^2. \quad (3)$$

Appropriate choices of $V(q)$ realize a number of interesting scenarios. We will exclusively consider potentials of the form $V(q) = -\sum_\mu V_\mu \cos(\lambda_\mu q)$, with one or two V_μ initially nonzero. In this case, we choose $q_0 = 2\pi/\min[\lambda_\mu]$, and rescale parameters to obtain $V[\theta] = \sum_\mu V_\mu \cos(\lambda_\mu\theta)$, where now $\lambda_\mu \geqslant 1$ and $V_1 \neq 0$. We will analyze the phase diagram of $S_0 + S_V$, where $S_V = \int d\tau\, V[\theta(\tau)]$, for different choices of λ_μ.

Single frequency. We first consider a single harmonic, i.e., $V_\mu = 0$ for $\mu \neq 1$, corresponding to a particle in a periodic potential [27–31,35], with

$$S_V[\theta(\tau)] = -V_1\int d\tau \cos[\theta(\tau)], \quad (4)$$

meaning $S_0 + S_v$ is a boundary sine-Gordon model. Therefore, the perturbative effect of the potential to the "free fixed point" (3) can be straightforwardly diagnosed using momentum-shell RG [27,38], as follows. First, we split the fields into "slow" and "fast" modes $\theta(\omega)=\theta_s(\omega)\Theta[\Lambda/b-\omega]+\theta_f(\omega)\Theta[\omega - \Lambda/b]$, where Θ is the unit step function, and $b = e^\ell$. We then integrate out the fast modes, possibly generating new terms, using a cumulant expansion about the Gaussian S_0, and rescale frequencies via $\omega \mapsto b\omega$ to keep S_0 fixed. Finally, we define rescaled fields via $\theta(\tilde{\omega}) = b^{-1}\theta_s(\omega)$. Iterating this transformation, we obtain the RG flow equation for V_1,

$$\frac{dV_1}{d\ell} = \left(1 - \frac{1}{\alpha}\right)V_1 + O(V_1^3). \quad (5)$$

This shows that the model has a phase transition at $\alpha_c = 1$: For $\alpha < \alpha_c$, V_1 flows to zero under the RG (corresponding to the free phase), whereas for $\alpha > \alpha_c$, V_1 is relevant and the flow is to strong coupling. In this limit, a variational estimate suggests that the localization length ξ^* diverges as $(\alpha - \alpha_c)^{-1/2}$ [27]. The constancy of α under RG follows from two facts. First, note that V_1 is local in time, and coarse graining preserves locality; in contrast, S_0 is nonlocal in time for $T \to 0$, and so cannot emerge in the perturbative RG. Second, the coefficient of θ is fixed by translational symmetry, $\theta \to \theta + 2\pi\mathbb{Z}$. Thus, α does not flow [27]. Additionally, while V_1 itself does not receive corrections at second order in V_1, a V_2 term *is* generated at $O(V_1^2)$. However, it is less relevant than V_1, which is always the most relevant term generated by the flow to *all* orders. (This will no longer be true if a second harmonic V_γ with $\gamma \notin \mathbb{Z}$ is included.)

Generalized RG flows. We now study the double-frequency (bichromatic) boundary sine-Gordon model,

$$S_V[\theta(\tau)] = -\int d\tau\{V_1\cos[\theta(\tau)] + V_\gamma\cos[\gamma\theta(\tau)]\}, \quad (6)$$

where, without loss of generality, we take $\gamma > 1$. Observe that with this choice, for $\alpha < 1$, both V_1 and V_γ are irrelevant if considered in isolation. For $\gamma \in \mathbb{Z}$, any term generated by the RG has a higher scaling dimension than V_1, and is therefore also irrelevant. For $\gamma \notin \mathbb{Z}$, we must consider the terms generated at second order in the RG equations. Intuitively, this is because "beating" between two cosines can yield a cosine with a shorter wavelength, potentially relevant even when V_1, V_γ are not. This picture already signals that rational and irrational γ are physically distinct: In the former case, there are finitely many such beats; in the latter there are infinitely many. This is a consequence of the fact that a quasiperiodic potential has no shortest reciprocal lattice vector [39].

To study these effects quantitatively, we determine the RG flow equations. We consider all wave vectors generated by the RG, corresponding to the set $\mathcal{L} = \{\lambda : \lambda = |m + \gamma n|, m, n \in \mathbb{Z}\}$ [40]. While an explicit derivation of RG equations requires a tedious (albeit standard) cumulant expansion [38], their structure is fixed by the operator product expansion of boundary sine-Gordon theory,

$$\frac{dV_\lambda}{d\ell} = \left(1 - \frac{\lambda^2}{\alpha}\right)V_\lambda + \sum_{\lambda',\lambda''} C_\lambda^{\lambda'\lambda''} V_{\lambda'}V_{\lambda''} + \cdots, \quad (7)$$

where $C_\lambda^{\lambda'\lambda''} = \frac{\lambda'\lambda''}{2\alpha}(\delta_{\lambda,\lambda'+\lambda''} - \delta_{\lambda,\lambda'-\lambda''})$, and "$\cdots$" denotes higher-order terms that we neglect in this perturbative analysis. Evidently, this coupled set of equations (7) captures the beat phenomenon described above, since at $O(V^2)$ the RG generates new terms that are absent at the bare level. These in turn generate other terms as the flow proceeds. The absence of $\theta \mapsto \theta + 2\pi$ symmetry may allow additional terms that in principle could affect the RG flows; however, the set (7) remains valid at a perturbative level, and we proceed assuming their validity. To understand their solution, we consider the scenario where $V_1 = u_0$, $V_\gamma = \epsilon u_0$ at the bare level; for a given α, the question then is to determine (i) the new critical dissipation strength $\alpha_c' < \alpha_c$, (ii) the RG time $\ell^*(\alpha)$ at which, for $\alpha_c' < \alpha < \alpha_c$, a relevant potential generated by these bare values flows to $O(1)$, and (iii) the corresponding localization length associated with this relevant potential. For $\ell \gtrsim \ell^*$ we enter the strong-coupling regime where our perturbative RG is no longer reliable. Unlike in the conventional single-frequency boundary sine-Gordon problem, there is no exact solution or duality to leverage here. Though we have assumed a flow to strong coupling, we cannot rule out the possibility of an intermediate fixed point stabilized by higher-order terms neglected in (7); this is a question for future analysis.

Taking $\gamma = m/n \geqslant 1$ to be an irreducible *rational* number, the minimum nonzero wave vector is given by $\lambda_c = 1/n$, and all V_λ for $\lambda \in \mathcal{L}$ are irrelevant if $\alpha < \alpha_c' \equiv \lambda_*^2$, i.e., the delocalized phase survives, but shrinks in extent. However, for $\alpha_c' < \alpha < 1$, the localization is driven by high-order "beats": Bare V_1, V_γ are irrelevant, but generate other V_λ's as they flow to zero; eventually, a relevant term emerges and grows to $O(1)$. The corresponding scale ℓ^* controls the crossover to localization: Intuitively, it is the scale at which the particle "sees" the potential. To understand this, we consider (7) for a minimal set of V_λ needed to generate a relevant term. We ignore second-order terms for each unless they help generate the relevant term, which is justified by numerical iteration of (7). We then integrate the flows of $V_1(\ell)$ and $V_\gamma(\ell)$ directly [38]. For $\gamma = 3/2$ and $1/4 < \alpha < 1$, since a relevant term ($V_{1/2}$) is generated by these two directly, we find it grows to $O(1)$ in an RG "time,"

$$\ell^* = \frac{\alpha}{\alpha_c' - \alpha} \ln\left[\epsilon u_0^2\right] + \cdots, \tag{8}$$

where the omitted terms "$\cdots$" do not involve u_0 or ϵ. We can extract from this scale a localization length $\xi^* \propto \sqrt{\langle\theta^2(\tau)\rangle}$, where in evaluating the average we only consider the modes between the current RG scale $\Lambda e^{-\ell^*}$ and the original cutoff Λ. We find $\xi^* = \frac{q_0}{2\pi}\sqrt{\frac{2\ell^*}{\alpha}} \propto (\alpha - \alpha_c)^{-1/2}$ [38], which mirrors a variational calculation for the single-harmonic problem [27]. A similar relation for ℓ^* may be obtained for generic commensurate γ, but with the difference that higher powers of ϵ and u_0 appear in the logarithm, corresponding to the fact that the relevant operator emerges at a higher order.

Quasiperiodic case. We now turn to the quasiperiodic (incommensurate) problem. For irrational $\gamma \notin \mathbb{Q}$, we see immediately that the minimum nonzero wave vector λ_c in $\mathcal{L}$ is ill defined. Therefore, the critical dissipation strength for localization is zero, so that arbitrarily weak dissipation leads to localization. Intuitively, for rational $\gamma = m/n$, the combined potential $V(\theta) = V_1 \cos\theta + V_\gamma \cos\gamma\theta$ always contains a periodic set of equally spaced minima (e.g., at spacing $2\pi n$); if the dissipation is sufficiently weak that coherent tunneling between these minima remains possible, the delocalized phase survives. Conversely, for irrational γ, $V(\theta)$ hosts no such periodic set of minima—indeed, there is no real-space periodicity. Therefore, the coherent tunneling is disrupted on long length scales, so that no matter how small the dissipation, the particle will eventually come to rest in some potential minimum.

For concreteness, we consider the *Fibonacci potential*, given by $\gamma = \varphi = \frac{1}{2}(1 + \sqrt{5})$, the golden mean. Within $\mathcal{L}$, we note that the decreasing sequence $\lambda_n \equiv (-1)^n(F_{n+1} - \varphi F_n) = \varphi^{-n}$, where F_n is the nth element of the Fibonacci sequence, goes to zero rapidly as $n \to \infty$. We will refer to these as Fibonacci wave numbers: Taking $\lambda_0 = 1$, $\lambda_1 = \varphi - 1$ is the first new term generated by the RG with a smaller wave number than those present at the bare level, and subsequent λ_n are quickly generated by successive RG iterations, $\lambda_n = \lambda_{n-2} - \lambda_{n-1}$. Although for a given α there exist many arbitrary $\mu_{m,n} = m - \varphi n$ such that $\mu_{m,n}^2 < \alpha$, a smaller Fibonacci wave number will always have been generated earlier in the RG, and thus will have had more time to grow in strength and spawn further λ_n. Thus, determining the most relevant wave number is simplified relative to a generic irrational γ (though by analogy to the Fibonacci case, we conjecture they will be generated by successive "best rational approximants" of γ).

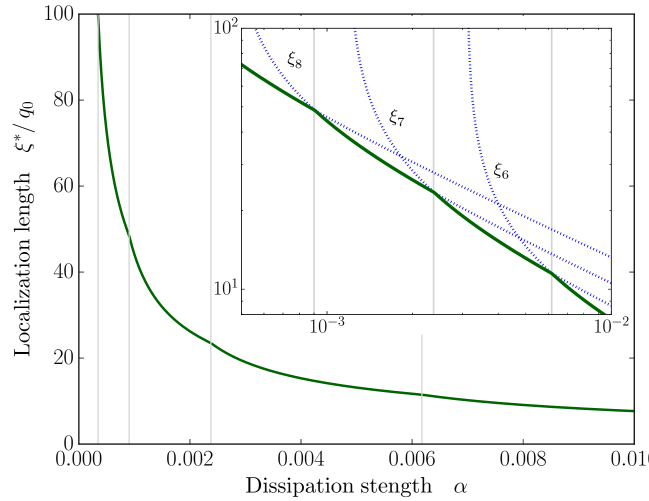

FIG. 1. Localization length ξ^* as a function of dissipation α for quasiperiodic potential with $\gamma = \varphi$ (solid green). Inset: Same plot on a log-log scale. As α is decreased, ξ^* is a piecewise function that changes nonanalytically for $\alpha \sim \alpha_n = \varphi^{-2n}$ between successive $\xi_n = \frac{q_0}{2\pi}\sqrt{\frac{2\ell_n}{\alpha}}$ [see Eq. (9)].

The crossover to localization is controlled by a critical scale ℓ^*, the RG time for *some* relevant term to become $O(1)$. We denote λ_{n^*} as the first relevant term become $O(1)$ when all λ_n are allowed to be nonzero. Each λ_n requires RG time ℓ_n to grow to $O(1)$, and ℓ^* corresponds to the smallest among the ℓ_n for a given α, where ℓ_n are determined by analogy to (8),

$$\ell_n = \frac{\alpha}{\varphi^{-2n} - \alpha} \ln\left[V_\gamma^{F_n} V_1^{F_{n+1}}\right], \tag{9}$$

as may be verified by direct integration of (7) [38]. Omitted from (9) are subleading corrections that vanish in the limit $\alpha \ll 1$ [38]. As α is decreased, ℓ^* is set by successive ℓ_{n^*} with larger Fibonacci indices: Taking $V_1 = -\ln V_\gamma = 1$, we see that $\xi^* = \frac{q_0}{2\pi}\sqrt{\frac{2\ell^*}{\alpha}}$ is determined by successive ℓ_n in a piecewise manner, with ℓ^* changing from ℓ_n to ℓ_{n+1} at $\alpha \sim \varphi^{-2n} = \lambda_n^2$. This leads to nonanalyticity in ξ^* (Fig. 1). Although there is always a relevant, localizing potential with wave number λ_{n^*}, it requires increasingly long for this term to be generated, corresponding to $\ell^* \to \infty$. Dynamically, it will take increasingly longer for the particle to "feel" the localization.

Realization via mobile impurity. So far, we have assumed that our model directly describes a particle in a quasiperiodic landscape. This can be challenging to engineer and observe in cold-atom simulations. We now discuss an alternative route to the same physics in a mobile impurity problem [32–34,41]. Consider a single mobile impurity, with coordinate X and momentum P, in a periodic optical lattice (spacing $a = 1$ and length L), and immersed in a quantum fluid. Describing the latter as a Luttinger liquid with interaction parameter K and velocity v,

$$H_g = \frac{v}{2\pi}\int_{-L/2}^{L/2} dx \left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right], \tag{10}$$

with $[\phi(x), \partial_y\theta(y)] = i\pi\delta(x-y)$ captures its dynamics. We assume that the optical lattice is sufficiently strong that the impurity has tight-binding dispersion given by $H_i = -t_i\cos(P)$, and that the particle and the gas interact via contact interactions $H_{\rm int} = u\rho(X)$, where $\rho(X)$ is the density of the gas, and t_i and u are coupling strengths. The full Hamiltonian is $H = H_i + H_g + H_{\rm int}$. It is convenient to make a unitary transformation $\mathcal{U}_X = e^{iP_gX}$ to the frame comoving with the impurity, so that $H \mapsto \mathcal{U}_X H \mathcal{U}_X^{-1} = H_g + u\rho(0) - t_i\cos(P - P_g)$. Since X is now absent from H, P is conserved and corresponds to the total momentum. We now take the $u \to \infty$ limit, corresponding to a strongly scattering impurity, where the leading term at $O(1/u)$ involves the tunneling of gas particles across the impurity. This yields the Josephson-like term $H_r \approx -t_g\cos(\Theta)$, where $\Theta = \theta(0^+) - \theta(0^-)$ describes the phase shift across the impurity. We may relate P_g to Θ by using the usual Luttinger liquid relations for the density $\rho = \pi^{-1}\partial_x\phi$ and momentum $\pi_\phi = \partial_x\theta$,

$$P_g = \int_{|x|>\epsilon} dx\rho\pi_\phi = \frac{1}{\pi}\int_{|x|>\epsilon} dx\,\partial_x\phi\partial_x\theta = -\nu\Theta, \tag{11}$$

where the integral excludes the origin as there is a break in the fluid at the impurity. We have used the mode expansion $\phi(x) = \phi_0 + \pi\frac{N}{L}x + \tilde{\phi}(x)$, $\theta(x) = \theta_0 + \pi\frac{J}{L}x + \tilde{\theta}(x)$, where N, J are the total particle number and current, respectively, and $\nu = N/L$ is the average density or filling. Finally, we integrate out the gapless sound modes of H_g subject to the boundary condition $\theta(0^+, t) - \theta(0^-, t) = \Theta(t)$; this generates dissipative dynamics for Θ. Working in imaginary time we arrive at the impurity effective action,

$$S_i = \int d\tau[t_i\cos(P + \gamma\Theta) + t_g\cos\Theta] + \frac{\alpha}{4\pi}\int d\omega|\omega||\Theta_\omega|^2, \tag{12}$$

with $\alpha = 1/K$ [42], $\gamma = \nu$; $P \neq 0$ does not affect the RG flows, and hence we see that the impurity is described by the double-frequency sine-Gordon action, with the wave vector of one of the cosines tuned by the gas density. Reinstating the lattice spacing a, we see that $\gamma = \nu a$ corresponds to the number of gas atoms in each unit cell of the potential seen by the impurity; evidently, there is no particular restriction to commensurate γ. In this language, the regime where the cosines are irrelevant corresponds to an impurity that is nondispersive, i.e., whose energy is independent of P, while the one where the cosines are relevant corresponds to a dispersive impurity. When the gas density is commensurate with the impurity potential, the impurity is able to move recoillessly between minima while simultaneously allowing an integer number of gas particles to tunnel across it; for sufficiently weak dissipation this "dressed" process continues to show quantum Brownian motion. This effect is absent in the quasiperiodic case, but depending on the scale at which the system is probed, the dispersion will show different periodicity set by the potential that controls ξ^*. We defer further investigation of the impurity realization of the quasiperiodic problem to future work.

Discussion. In conclusion, we have shown that a quantum particle moving in a quasiperiodic potential is always localized by a dissipative bath as $T \to 0$. This is in sharp contrast with the well-known quantum phase transition in the periodic case. We also argued that this physics could be realized in the strong-coupling regime of a mobile impurity in a one-dimensional Fermi gas moving in a periodic lattice. On the formal side, we note that while the infrared behavior of the single-frequency boundary sine-Gordon field theory can be studied using instanton expansions and integrability, much less is known about multifrequency variants. It would be very interesting—and of direct relevance to an experimentally accessible regime of mobile impurity problems—to develop analytic tools to analyze the flow to strong coupling in this theory, and investigate the possibility of a different class of intermediate-coupling fixed points.

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[38] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.100.060301 for details of the RG procedure and plots of flow of couplings under the RG.
[39] Though they are not periodic in space, quasicrystals *do* have a regular structure of Bragg peaks.
[40] In mathematical terms, $\mathcal{L}$ is a $\mathbb{Z}$ module on $\{1, \gamma\}$.
[41] A. Lamacraft, Phys. Rev. B **79**, 241105(R) (2009).
[42] We take K to be the effective value obtained after eliminating derivative terms of the form $\partial_x \Theta$.