Electric field assisted amplification of magnetic fields in tilted Dirac cone systems

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We show that the continuum limit of the tilted Dirac cone in materials such as 8-*Pmmn* borophene and layered organic conductor α-(BEDT-TTF)$_2$I$_3$ corresponds to deformation of the Minkowski space-time of Dirac materials. From its Killing vectors we construct an emergent tilted-Lorentz (t-Lorentz) symmetry group for such systems. As an example of the t-Lorentz transformations we obtain the exact solution of the Landau bands for a crossed configuration of electric and magnetic fields. For any given tilt parameter $0 \leqslant \zeta < 1$, if the ratio $\chi = v_F B_z/E_y$ of the crossed magnetic and electric fields satisfies $\chi \geqslant 1+\zeta$, one can always find appropriate t-boosts in both valleys labeled by $\tau = \pm 1$ in such a way that the electric field can be t-boosted away, whereby the resulting pure effective magnetic field B_z^τ governs the Landau level spectrum around each valley τ. The effective magnetic field in one of the valleys is always larger than the applied perpendicular magnetic field. This amplification comes at the expense of of diminishing the effective field in the opposite valley and can be detected in various quantum oscillation phenomena in tilted Dirac cone systems. Tuning the ratio of electric and magnetic fields to $\chi_{\rm min} = 1+\zeta$ leads to valley selective collapse of Landau levels. Our geometric description of the tilt in Dirac systems reveals an important connection between the tilt and an incipient "rotating source" when the tilt parameter can be made to depend on space-time in a certain way.

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I. INTRODUCTION

Lorentz transformations that translate observations in two reference frames in such a way that $ds^2 = -c^2dt^2 + dx^2$ remains invariant [1] are at the heart of relativistic quantum field theories. The Lorentz symmetry (group) is part of the Poincaré group which is a fundamental symmetry of a world in which the elementary particles live. In this world, the speed c of light is the universal upper limit of speeds, and is furthermore isotropic, giving rise to upright light cones. Now, imagine an alternative world in which the light cones are tilted. This world does not exist in the standard model of particle physics, but certain lattices in solid-state systems afford to mimic such space-times. In such a new world, the emergent symmetry at long wavelengths is not the Lorentz symmetry anymore. However, such a space-time is invariant under some deformation of the Lorentz group which in this paper will be called the t-Lorentz. This emergent symmetry group is parametrized by t-boosts where "t" emphasizes the importance of tilt in this strange world.

Which condensed-matter systems can realize such a continuous deformation of the Minkowski space-time? Owing to rich lattice structures of the solid-state systems, the Dirac or Weyl equations [2] can emerge as effective description of low-energy electronic degrees of freedom in one [3], two [4], and three spatial dimensions [2,5,6]. The structure of the space-time felt by electrons in a generic Dirac/Weyl material is precisely the Minkowski space-time, albeit the difference being that the speed of light will be replaced by Fermi velocity v_F of the material at hand which is usually two to three orders of magnitude smaller than c [7–9]. The small ratio of v_F/c does not harm the Lorentz symmetry.[1] The life becomes more interesting when it comes to lattices with nonsymmorphic symmetry elements. After all, the condensed-matter systems are mounted on a background crystal, not on the vacuum. Therefore, the Lorentz symmetry is not necessarily the symmetry group of linear band touching in solid-state systems. In fact, the rich point-group symmetry of crystals can provide classes of fermions which have no counterparts in the standard model of particle physics [10]. For example, unconventional fermions living on crystals with nonsymmorphic symmetry elements can boldly violate the spin-statistic theorem [11] which rests on the Lorentz symmetry [12]. Therefore, relaxing the Lorentz symmetry seems to produce opportunities not available in physics of elementary particles.

In this paper we would like to show that the effect of nonsymmorphic lattice structures is not limited to generation of strange forms of fermions in condensed matter. When the Dirac/Weyl equations are brought to their mundane sub-eV solid-state framework, the nonsymmorphic symmetry elements which arise from the underlying lattice can generate a finite amount of tilt in the *intrinsic* Dirac cone spectrum of electronic degrees of freedom [13]. Such a finite tilt should be contrasted to very small tilt that can be *extrinsically* induced in graphene by appropriate strains [14,15]. This solid-state-based world of tilted Dirac cones is the subject of this paper. Although the tilt deformation of Dirac equation destroys the Lorentz symmetry, still a deformed version of Lorentz group survives in the form of t-Lorentz group. The purpose of this

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[1]It has more interesting effect of enhancing the fine structure constant for Dirac fermions of condensed matter [7].

work is (1) to study the isometries of the space-time with tilted Dirac cones and construct the t-Lorentz group and identify its algebraic structure and (2) to show that the tilt can help to *amplify* the effective magnetic field in one of the valleys, at the cost of reducing the magnetic field in the other valley. As an example of the use of the emergent t-Lorentz transformations we exactly solve the Landau band problem in crossed magnetic and electric field background [16]. The t-Lorentz transformations serve to provide a neat solution for the Landau bands and also reveal a hidden transfer of magnetic field between the two valleys which can only be mediated by the tilt that has no analog in tiltless Dirac systems.

The candidate materials related to the space-time discussed in our work are quasi-two-dimensional (molecular orbital based) systems such as organic α-(BEDT-TTF)$_2$I$_3$ [17,18], or an atom thick sheet of 8-*Pmmn* borophene [13]. The advantage of the latter system, aside from being in two space dimensions which offers functionalization and manipulation opportunities, is that (i) its intrinsic tilt parameter is quite large, (ii) the tilt can be further controlled with perpendicular electric field from the undertilted regime of the pristine borophene to the overtilted regime, and (iii) the particular nonsymmorphic structure of the space group protects the Dirac node [19] as long as the intrinsic spin-orbit coupling is small. Owing to very small atomic number of boron, the intrinsic spin-orbit interaction is $\sim$0.02 [20] meV. So, for all practical purposes, the intrinsic tilted Dirac cone in pristine borophene can be assumed to be massless. An essential feature of tilted Dirac cones is that being mounted on a lattice, they always come in pairs with opposite tilts $\pm\zeta$. This sign difference is behind the amplification mechanism that we will discuss in this paper.

II. t-LORENTZ TRANSFORMATIONS

Let us start by minimal form of tilted Dirac equations for one of the valleys [20–24]

$$H = \hbar v_F \begin{pmatrix} \zeta k_x & k_x - ik_y \\ k_x + ik_y & \zeta k_x \end{pmatrix} = \hbar v_F(\zeta k_x \sigma_0 + \boldsymbol{k}.\boldsymbol{\sigma}), \quad (1)$$

where the Pauli matrices σ_μ with $\mu = 0, 1, 2$ act on the orbital space and σ_0 is the unit 2×2 matrix in this space. This theory is characterized by two velocity scales: v_F determines the conelike dispersion, while $v_t = \zeta v_F$ determines the tilt of the energy axis with respect to the $k_x k_y$ plane. We have used our freedom to choose coordinate system such that the k_x axis is along the tilt direction. The pristine borophene with intrinsic $\zeta \sim 0.4$ lies in the undertilted regime (i.e., $0 < \zeta < 1$). It can be tuned by a perpendicular electric field to the overtilted regime with $1 < \zeta$ [13]. From the effective theory of 8-*Pmmn* borophene it follows that the other valley is obtained by $\zeta \to -\zeta$ and $\sigma_x \to -\sigma_x$. As far as the following geometric construction is concerned, the anisotropy of the Fermi velocity v_F in any realistic material [9,25] can be absorbed into a rescaling of momenta (or coordinates) which will give rise to a constant Jacobian and does not alter the physics.[2] The anisotropy can be restored at the end by simply setting $v_F \to \sqrt{v_{F_x} v_{F_y}}$. The eigenvalues and eigenstates of the tilted Dirac cone Hamiltonian are given by

$$E_s(\boldsymbol{k}) = k(s + \zeta \cos\theta_{\boldsymbol{k}}), \qquad |\boldsymbol{k}, \pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm e^{i\theta_k} \end{pmatrix}, \quad (2)$$

where $s = \pm 1$ refers to positive- (E_+) and negative- (E_-) energy branches, and $\theta_{\boldsymbol{k}}$ is the polar angle of the wave vector $\boldsymbol{k}$, with respect to the x direction.

Following Volovik [27,28], the dispersion of gapless tilted Dirac cone can be viewed as a null surface in a Painelevé-Gullstrand space-time

$$ds^2 = -v_F^2 dt^2 + (d\boldsymbol{r} - \boldsymbol{v}_t dt)^2. \quad (3)$$

The space part of this metric has acquired a Galilean boost by velocity $\boldsymbol{v}_t$. The dispersion relation of massless particles in this space-time is given by $g^{\mu\nu}k_\mu k_\nu = 0$ which upon identifying $k_\mu = (E/v_F, \boldsymbol{k})$ gives $(E - \boldsymbol{k}\cdot\boldsymbol{v}_t)^2 - v_F^2 k^2 = 0$. This is nothing but the dispersion relation (2) of the tilted Dirac fermions. *Therefore, for the tilted Dirac fermions, the space-time is given by metric (3).* If the tilt velocity $v_t = \zeta v_F$ can be made to depend on the radial coordinate [13], the condition $v_t > v_F$ for the overtilted Dirac cone corresponds to the black hole in $3+1$ dimensions, and Bañados-Teitelboim-Zanelli (BTZ) black holes [29] in $(2+1)$-dimensional space-time. The explicit form of this metric in $1+1$ dimensions is given by

$$g_{\mu\nu} = \begin{bmatrix} -\lambda^2 & -\zeta \\ -\zeta & 1 \end{bmatrix} \leftrightarrow g^{\mu\nu} = \begin{bmatrix} -1 & \zeta \\ \zeta & \lambda^2 \end{bmatrix}, \quad (4)$$

where we have introduced $\lambda^2 = 1 - \zeta^2$. In this work we confine ourselves to a much simpler form of this metric where the tilt velocity $\boldsymbol{v}_t$ is constant all over the space-time, and as such is a flat space-time. So, in this work we will study the fundamental symmetry and physics arising from a constant tilt parameter ζ. Further evidence in favor of the above geometry comes from the fact that the brute force calculation of the polarization tensor $\Pi^{\mu\nu} \sim \langle j^\mu j^\nu \rangle$ shows that it acquires the covariant form $(g^{\mu\nu}q^2 - q^\mu q^\nu)\pi(q)$, *only* when the above tilted geometry (4) is used [30]. From now on we will assume that the Fermi velocity $v_F = 1$ and will restore it when required.

For clarity, let us first derive the t-Lorentz transformation in $1+1$ dimensions. Since we will need to ensure that $\zeta = 0$ reduces to the standard Minkowski space-time, let us start by a quick reminder of the Lorentz transformation in this space: In this case, a small Lorentz transformation parametrized by κ is $\Lambda_0 = \mathbb{1} + i\kappa K_0$, where K_0 is the generator of transformation. Invariance of $ds^2 = -dt^2 + dx^2$ [equivalent to $g_0 = \mathrm{diag}(-1, 1)$ metric] means $g_0 K_0 + K_0^T g_0 = 0$. This fixes $iK_0 = \sigma_x$, where σ_x is the first Pauli matrix [11,31]. From this the Lorentz transformation for finite κ in $1+1$ dimension becomes

$$\Lambda_0(\kappa) = \begin{bmatrix} \cosh(\kappa) & \sinh(\kappa) \\ \sinh(\kappa) & \cosh(\kappa) \end{bmatrix}, \quad (5)$$

where the boost parameter κ is related to the velocity β by $\cosh(\kappa) = (1 - \beta^2)^{-1/2} \equiv \gamma$ and $\sinh(\kappa) = -\beta\gamma$. The same logic allows us to derive a generalized Lorentz transformation

[2]When the Fermi velocity scale is random, interesting "gravitational-lensing"-like phenomena appear in tilted $(2+1)$-dimensional Dirac systems [26].

in $1+1$ dimensions in presence of a nonzero tilt parameter ζ. We expand $\Lambda = \mathbb{1} + i\kappa K$, and require it to leave the metric (4) of tilted Dirac fermions invariant. Again, this fixes the generator of the t-Lorentz transformation

$$iK = \begin{bmatrix} -\zeta & 1 \\ \lambda^2 & \zeta \end{bmatrix}. \tag{6}$$

From this one can immediately find the large t-Lorentz transformation. For a t-boost along the x (tilt) direction we obtain

$$\Lambda^x_{1+1} = \gamma \begin{bmatrix} 1+\zeta\beta & -\beta \\ -\lambda^2\beta & 1-\zeta\beta \end{bmatrix}. \tag{7}$$

Needless to say, for $\zeta = 0$ this equation reduces to Eq. (5).

Now, we are ready to construct the t-Lorentz transformations in $2+1$ dimensions. The coordinates x^μ are defined by $(x^0 = v_F t,\ x^1 = x,\ x^2 = y)$. We choose the x direction along the tilt direction such that the tilt is given by the two-vector $\boldsymbol{\zeta} = (\zeta_1 = \zeta,\ \zeta_2 = 0)$. For this choice the metric in $2+1$ dimensions will be a generalization of Eq. (4) and is given by [30]

$$g_{\mu\nu} = \begin{bmatrix} -\lambda^2 & -\zeta & 0 \\ -\zeta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

For a t-boost along the x direction where the tilt lies, the corresponding $(2+1)$-dimensional generator is obtained from the $(1+1)$-dimensional generator by padding with 0's, while the other generators are obtained from $gK + K^T g = 0$ as

$$iK_x = \begin{bmatrix} -\zeta & 1 & 0 \\ \lambda^2 & \zeta & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad iK_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \lambda^2 & \zeta & 0 \end{bmatrix},$$

$$iJ_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \zeta & -1 & 0 \end{bmatrix}.$$

The names are chosen to emphasize that in the limit $\zeta = 0$ they reduce to corresponding generators of boosts along x, y and rotation around z axes, respectively. The algebra of these generators turns out to be the following deformation of Lorentz algebra [32]:

$$[K_x, K_y] = i\zeta K_y - i(1-\zeta^2)J_z, \tag{9}$$

$$[J_z, K_x] = iK_y + i\zeta J_z, \tag{10}$$

$$[J_z, K_y] = -iK_x, \tag{11}$$

and in the limit $\zeta = 0$ reduces to what one expects for the Lorentz group [32,33].

With the above generators we can construct the t-Lorentz transformations along x (tilt direction) and y (transverse to tilt direction) by simply exponentiating $iK_x\kappa$ and $iK_y\kappa$. Similarly, the t-rotation around the z axis will be obtained by exponentiation of $iJ_z\theta$. The result is

$$\Lambda^x_{2+1} = \begin{bmatrix} \cosh\kappa - \zeta\sinh\kappa & \sinh\kappa & 0 \\ \lambda^2\sinh\kappa & \cosh\kappa + \zeta\sinh\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

$$\Lambda^y_{2+1} = \begin{bmatrix} \cosh\lambda\kappa & \frac{\zeta}{\lambda^2}[\cosh\lambda\kappa - 1] & \frac{1}{\lambda}\sinh\lambda\kappa \\ 0 & 1 & 0 \\ \lambda\sinh\lambda\kappa & \frac{\zeta}{\lambda}\sinh\lambda\kappa & \cosh\lambda\kappa \end{bmatrix}, \tag{13}$$

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ \zeta(1-\cos\theta) & \cos\theta & \sin\theta \\ \zeta\sin\theta & -\sin\theta & \cos\theta \end{bmatrix}. \tag{14}$$

The first equation, namely Eq. (12), is a straightforward generalization of Eq. (7). The second equation, (13), also reduces to the standard Lorentz transformation of the Minkowski space-time in the limit of $\zeta = 0$ ($\lambda = 1$). The third equation, (14), is also a generalization of rotation around the z axis which again reduces to the rotation in the Minkowski space in the limit of $\zeta = 0$.

In Appendix A we give a more systematic derivation of the above transformations using the method of Killing vectors to ensure that the above derivation does not miss any isometry of the space-time of tilted Dirac fermions.

III. AMPLIFICATION OF MAGNETIC FIELDS

One of the fascinating results of the special theory of relativity is that the electric and magnetic fields are actually components of the same tensor which covariantly transforms under the Lorentz transformation. Lukose and co-workers [34] have used this fact to obtain a beautiful exact solution of the Landau bands in crossed E_y and B_z fields. In a tilted Minkowski space with metric (8) if the electric and magnetic fields are due to sources living in the same space, the field strength tensor $F^{\mu\nu}$ will also transform appropriately under t-Lorentz transformations. The nice property of two-dimensional Dirac materials is that the Fermi surface can be tuned to regimes where hydrodynamic regime with emergent electromagnetic fields can be achieved. The t-Lorentz transformations enables us to study the Landau bands in crossed electric and magnetic fields in $(2+1)$-dimensional tilted Dirac cone systems such as borophene.

To study the transformation of emergent electric and magnetic fields under t-Lorentz transformation, let us restore the Fermi velocity v_F to manifestly see its interplay with the speed of light c. Assuming a B field along the z direction and an E field along the y direction in borophene, then a t-boost along the x direction by the velocity βv_F changes (E_y, B_z) according to [7,34],

$$E'_y = \gamma[(1+\zeta\beta)E_y - \beta v_F B_z], \tag{15}$$

$$v_F B'_z = \gamma[(1-\zeta\beta)v_F B_z - (1-\zeta^2)\beta E_y]. \tag{16}$$

Again, the $\zeta = 0$ limit agrees with the corresponding result in graphene [34]. When the electromagnetic fields do not arise from electric charges in space-time (8) one has to set $\zeta = 0$ in the transformation of E and B fields. In the limit of

$\zeta = 0$, a boost along the x direction does not change E_x [33]. The same holds for a t-Lorentz transformation along the x (tilt) direction. For a t-Lorentz transformation $\Lambda = \Lambda^x_{2+1}$ with arbitrary ζ we have

$$E'_x = F'^{01} = \Lambda^0_\rho \Lambda^1_\sigma F^{\rho\sigma} = \Lambda^0_0 \Lambda^1_1 F^{01} + \Lambda^0_1 \Lambda^1_0 F^{10}$$
$$= \gamma^2(1+\beta\zeta)(1-\beta\zeta)E_x + (-\beta)(-\lambda^2\beta)(-E_x) = E_x,$$

which is similar to the standard Lorentz transformation.

Following Lukose and co-workers [34], we choose the t-boosted frame such in the t-boosted frame the electric field can be eliminated. In valley $\tau = \pm$, this can be achieved for the t-boost parameter,

$$\beta_{*\tau} = \frac{E_y}{v_F B_z - \tau\zeta E_y} = \frac{1}{\chi - \tau\zeta}, \quad \chi = \frac{v_F B_z}{E_y}, \tag{17}$$

where we have used the fact that the signs of the tilt parameter ζ for the two valleys are opposite. For a given material the ζ is fixed (which can be assumed to be positive) and for type-I tilted Dirac systems is further less than 1. In this case, the condition $\chi > 1 + \tau\zeta$ guarantees that $|\beta_{*,\tau}| < 1$ and hence a (separate) t-boost (for each valley with parameters $\beta_{*,\pm}$) can be found that eliminates the electric field. In such a t-boosted frame a purely effective magnetic field B^τ_z of the following form will be felt:

$$\frac{B^\tau_z}{B_z} = \frac{\sqrt{1-\beta^2_{*,\tau}}}{1+\tau\zeta\beta_{*,\pm}} = \frac{\sqrt{(\chi-\tau\zeta)^2-1}}{\chi} \equiv \rho_\tau, \tag{18}$$

where factor ρ_τ denotes the ratio B^τ_z/B_z. In $\zeta = 0$ limit, B^τ_z reduces to the effective field $B_z(1-\beta^2_*)^{1/2}$, in agreement with the work of Lukose and co-workers [34]. The condition $\chi > 1 + \zeta$ guarantees that both t-boost parameters $\beta_{*,\pm}$ have magnitudes less than one and pure magnetic fields $B^\pm_z$ in both valleys can be realized. This corresponds to magnetic field dominated regime, where always the electric field can be eliminated [16].[3] In particular at $\chi_{\min} = 1 + \zeta$ the B^+_z vanishes, while B^-_z becomes $2B_z\sqrt{\zeta/(1+\zeta)}$. The vanishing of B^+_z at this particular value of $\chi_{\min}$ means that the Landau orbits around the $\tau = +$ valley collapse [34] while the Landau orbits around the other valley survive. The magnetic field around the $\tau = -$ valley will be larger than B_z when $\zeta > \frac{1}{3}$. This is in contrast to the situation in graphene with $\zeta = 0$, where the behavior of Landau levels in both valleys is the same. By tuning the ratio χ of crossed B_z and E_y fields beyond $\chi_{\min}$, the B^+_z starts to increase from zero, but always remains less than B^-_z. By reversing the direction of either E_y or B_z which amounts to flipping the sign of χ, the collapsed Landau levels will be centered around the other valley.

So far, the Landau levels are calculated in the t-boosted frame where E_y is zero. In this frame we have a pure Landau level problem for the effective field B^τ_z. This pure Landau level (i.e., with B field only) has been already studied [21,22] and the Landau levels are given by $\varepsilon^\tau_n = \mathrm{sgn}(n)\sqrt{2|n|}\frac{\hbar v_F}{\ell_B}\sqrt{\rho_\tau}(1-\zeta^2)^{1/4} = \varepsilon_n\sqrt{\rho_\tau\lambda}$, where ℓ_B is the magnetic length associated with the applied field B_z, namely, $\ell_B = \sqrt{\hbar/eB_z}$ and ε_n are Landau levels of upright Dirac cone in the absence of electric field, namely, $\rho_\tau \to 1$. In Appendix B we give a simple derivation of this result.

To obtain the Landau bands that are observed in the laboratory frame, one must t-boost back the Landau energy-momentum 3-vector $(\varepsilon^\tau_n, k_x, k_y)$ to the laboratory frame $(\bar\varepsilon_n, \bar k_x, \bar k_y)$. The t-boost along the x direction Eq. (12) when inverted gives $\bar k_y = k_y$ and linear t-Lorentz transformations $(\varepsilon_n, k_x) \to (\bar\varepsilon_n, \bar k_x)$. Elimination of k_x gives the dispersion relation

$$\bar\varepsilon^\tau_n = \frac{\sqrt{1-\beta^2_{*,\tau}}}{1+\tau\zeta\beta_{*,\tau}}\varepsilon^\tau_n + \frac{\beta_{*,\tau}}{1+\tau\zeta\beta_{*,\tau}}\bar k_x. \tag{19}$$

Using Eq. (17), the above dispersion relation is simplified to

$$\bar\varepsilon^\tau_n(\bar k_x) = \rho^{3/2}_\tau(1-\zeta^2)^{1/2}\varepsilon_n + \frac{E_y}{B_z}\bar k_x, \tag{20}$$

where to emphasize the dispersive nature of the Landau spectrum, we have explicitly indicated the $\bar k_x$ dependence of $\bar\varepsilon_n$. This is in agreement with the work of Goerbig and co-workers [16] and in the limit of upright Dirac cone with $\zeta \to 0$, reduces to the result of Lukose and co-workers [34].

The combination $\chi - \tau\zeta$ appearing in the t-boost parameter [Eq. (17)] and effective magnetic fields B^τ_z felt in the t-boosted frames at valleys τ can be interpreted in either of the following two ways: (i) a problem with intrinsic tilt ζ is equivalent to a crossed (E, B) problem with effective $\tilde\chi = \chi - \tau\zeta$ [35], or (ii) one can forget about χ and absorb it into an effective tilt parameter $\tilde\zeta = \zeta - \tau\chi$. The latter interpretation has been adopted by Goerbig and co-workers to imply that the in-plane electric field can generate valley-asymmetric effective tilts [16]. The former interpretation, however, is more interesting and extends beyond a simple Landau level problem. Indeed, constructing a Maxwell theory for the space-time given by metric (8) indicates that effective role of a *constant* $\boldsymbol\zeta$ to leading order is to replace [35]

$$v_F\boldsymbol{B} \to v_F\boldsymbol{B} - \tau\boldsymbol\zeta \times \boldsymbol{E}. \tag{21}$$

From this perspective, it is not surprising that the tilt $\boldsymbol\zeta$ takes advantage of $\boldsymbol{E}$ to reduce the magnetic field in one valley, and to add it back in the other valley.

The situation becomes even more interesting if the tilt parameter $\boldsymbol\zeta$ is not a constant. Indeed, the metric (8) in the small-ζ limit where ζ^2 effects can be ignored coincides with the metric of a rotating gravitational source if the ζ can be made to depend on space coordinate according to $\zeta_x \propto y$ and $\zeta_y \propto -x$ [33]. The effect of rotation of the source is such that it gives rise to nonzero g_{0i} components in the metric. These components are responsible for Lens-Thirring effect which is the precession of spins arising from the vector field $\boldsymbol\zeta$ [33]. In this limit, for all practical purposes the role of $\boldsymbol\zeta$ will be formally equivalent to a "magnetic" field given by $\nabla \times \boldsymbol\zeta$ [33]. The opposite $\boldsymbol\zeta$ for the two valleys creates opposite "gravitomagnetic" effects. For $\boldsymbol\zeta \propto (y, -x)$ it would not be surprising that the tilt parameter ζ helps to diminish the effective magnetic field in one valley and amplify it in the other valley. In this case, the exchange of magnetic field strength between the two valleys can be attributed to a hidden

[3]Otherwise, when the electric field is dominant, t-Lorentz transformation can eliminate B field and leave valley-dependent pure E fields around each valley. This might have consequences for valleytronics.

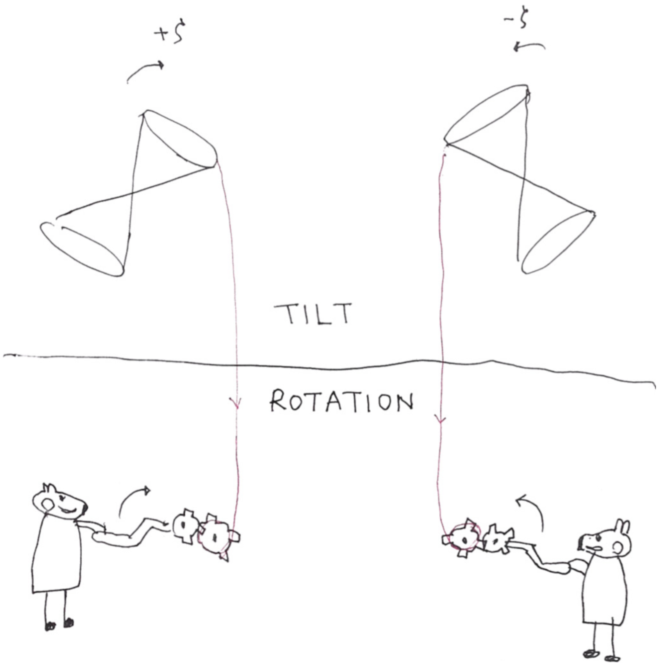

FIG. 1. Connection between the space-dependent tilt of the Dirac cone and rotating source in the metric. Sketch by Hasti Jafari.

"rotating source" as in Fig. 1. Even in the present case where $\boldsymbol{\zeta}$ does not depend on space-time, the above amplification of the effective magnetic fields rests on Eq. (21) and remains a genuine property of t-Lorentz covariance. Note that in the case of graphene ($\zeta = 0$) the effective magnetic field is always less than the applied B_z, while in the present case the magnetic field felt in the t-boosted frame for one of the valleys can be enhanced. This is because the tilt agent $\boldsymbol{\zeta}$ steals a "magnetic field" $\boldsymbol{\zeta} \times \boldsymbol{E}$ from one valley and traffics it to the other valley.

IV. SUMMARY AND DISCUSSION

Isometries of the metric compatible with tilted Dirac cone are tilted-Lorentz transformations. Transformation of the emergent electromagnetic field strength tensor under t-Lorentz transformations allows us to exactly solve, as an example, the problem of tilted Dirac fermions in crossed electric and magnetic fields. The solution consists of two interpenetrating Landau bands. The Landau levels ε_n^τ in the two valleys $\tau = \pm$ are asymmetric [16]. The effective magnetic field felt in one valley is always smaller than the background B_z while in the other valley stronger magnetic field is felt. This amplification has no analog in upright Dirac cone systems. The agreement with Ref. [16] demonstrates that thinking of tilted Dirac cone materials as deformations of Minkowski space-time is more than an alternative way of solving the same problem. It suggests that the non-Minkowski space-time in these materials is a reality and can be employed to explore further gravitational analogies.

The way the tilt parameter appears in the metric is similar to the metric of a rotating gravitational source (see Fig. 1) if the tilt parameter depends on space coordinate as $\boldsymbol{\zeta} \propto (y, -x)$. This can be achieved in 8-*Pmmn* borophene [13]. In this way, the spatiotemporal variation of the tilt vector $\boldsymbol{\zeta}$ is expected to couple to the spin of the electrons via spin-rotation coupling [36]. This is expected to give rise to electric field control of spin current, where the spin and the electric field couple through the metric of the space-time [37]. In the light of recent reports of torsional anomaly in Weyl semimetals with tilted cone [38] and gravitational lensinglike effect in tilted Dirac cone [26], and our recent calculation of the covariant structure of the polarization tensor in tilted cone systems [30], it appears that the tilted Dirac/Weyl cone systems are a fertile search grounds for gravitational analogies in condensed matter.

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APPENDIX A: KILLING VECTORS

Killing vectors satisfy the Killing equation [33]

$$g_{\mu\nu,\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda_{,\nu} + g_{\lambda\nu}\xi^\lambda_{,\mu} = 0. \tag{A1}$$

For simplicity, let us first work out the Killing vectors for the $(1+1)$-dimensional space with a constant tilt parameter ζ. The metric is given by Eq. (4). Using this metric in the Killing equation (A1), for $\mu\nu = 00$ we obtain

$$g_{0\lambda}\xi^\lambda_{,0} = 0 \Rightarrow (\zeta^2 - 1)\xi^0_{,0} = \zeta\xi^1_{,0}$$

which implies

$$\xi^1 = \frac{\zeta^2 - 1}{\zeta}\xi^0 + b(x) \tag{A2}$$

with $b(x)$ some yet unknown function of x only. Similarly, the $\mu\nu = 11$ case gives $g_{1\lambda}\xi^\lambda_{,1} = 0$ which implies $\zeta\xi^0_{,1} = \xi^1_{,1}$ from which it follows that

$$\xi^1 = \zeta\xi^0 + c(t) \tag{A3}$$

with $c(t)$ a yet unknown function of t only. Combining Eqs. (A2) and (A3) gives

$$\xi^0 = \zeta b(x) - \zeta c(t), \quad \xi^1 = \zeta^2 b(x) + (1 - \zeta^2)c(t). \tag{A4}$$

Finally, for the $\mu\nu = 01$ case, assuming that $g_{01} = -\zeta$ is independent of coordinates and hence $g_{01,\lambda} = 0$ we obtain $g_{0\lambda}\xi^\lambda_{,1} + g_{\lambda 1}\xi^\lambda_{,0} = 0$, which when expanded gives

$$-(1 - \zeta^2)\xi^0_{,1} + (-\zeta)\xi^1_{,1} + (-\zeta)\xi^0_{,0} + (1)\xi^1_{,0} = 0. \tag{A5}$$

Substituting from Eq. (A4) in this Eq. (A5) will imply $\partial_t c(t) = \zeta\partial_x b(x)$. The $\mu\nu = 10$ case also will give the same result. The solution of this equation is given by

$$b(x) = Ax + B, \quad c(t) = \zeta At + C. \tag{A6}$$

These solutions then give the final form for the components (ξ^0, ξ^1) of the Killing vector $\boldsymbol{\xi}$ as follows:

$$\xi^0 = A\zeta(x - \zeta t) + B\zeta - C\zeta, \tag{A7}$$

$$\xi^1 = A[\zeta^2 x + \zeta(1 - \zeta^2)t] + B\zeta^2 + C(1 - \zeta^2), \tag{A8}$$

which depend on three parameters and therefore there are three Killing vectors. The following choices give the Killing vectors:

$$(A, B, C) = (0, 1, 0) \rightarrow \xi_{\hat{0}} \propto (1, \zeta), \tag{A9}$$

$$(A, B, C) = (0, 0, 1) \rightarrow \xi_{\hat{1}} \propto (-\zeta, \lambda^2), \tag{A10}$$

$$(A, B, C) = (1, 0, 0) \rightarrow \xi_{\hat{2}} \propto (x - \zeta t, \lambda^2 t + \zeta x). \tag{A11}$$

Therefore, the generators of symmetry operations are

$$\xi_{\hat{0}} \rightarrow \partial_t + \zeta \partial_x, \tag{A12}$$

$$\xi_{\hat{1}} \rightarrow -\zeta \partial_t + \lambda^2 \partial_x, \tag{A13}$$

$$\xi_{\hat{2}} \rightarrow (x - \zeta t)\partial_t + (\lambda^2 t + \zeta x)\partial_x. \tag{A14}$$

When the tilt parameter ζ is zero, the above generators will correspond to time translation, space translation, and Lorentz transformations, respectively. Therefore, the above generators are generators of the symmetries of the space-time in which the dispersion of massless particles is given by the tilted Dirac cone. Furthermore, the conserved quantities in such space are not energy H (corresponding to generator ∂_t) and momentum P (corresponding to generators ∂_x). In such space, the conserved quantities will be $H + \zeta P$ and $\lambda^2 P - \zeta H$ which of course reduce to H and P in the limit of $\zeta = 0$.

To construct the explicit form of the Λ^x_{1+1}, we use the generator arising from the Killing vector $\xi_{\hat{2}}$ and denote it by $i\hat{K}$, namely,

$$i\hat{K} = \left((x - \zeta t)\frac{\partial}{\partial t} + (\lambda^2 t + \zeta x)\frac{\partial}{\partial x} \right) \tag{A15}$$

which gives

$$i\hat{K}x = \lambda^2 t + \zeta x,$$
$$i^2\hat{K}^2 x = i\hat{K}(\lambda^2 t + \zeta x) = x.$$

Repeating the above relation gives the operator equations $i^{2n}\hat{K}^{2n}x = x$ and $i^{2n+1}\hat{K}^{2n+1}x = \lambda^2 t + \zeta x$. Therefore,

$$\begin{aligned} e^{iK\kappa}x &= \sum_n \frac{\kappa^{2n}}{(2n)!}x + \sum_n \frac{\kappa^{2n+1}}{(2n+1)!}(\lambda^2 t + \zeta x) \\ &= \cosh\kappa x + \sinh\kappa(\lambda^2 t + \zeta x) \\ &= [\cos\kappa + \zeta\sinh\kappa]x + [\lambda^2\sinh\kappa]t. \end{aligned} \tag{A16}$$

Upon identifying $\cosh\kappa = \gamma$ and $\sinh\kappa = -\beta\gamma$, this transformation of x coordinate corresponds precisely to the second row of Eq. (7). Similarly, by operating with $i\hat{K}$ on the time coordinate t, one obtains $i\hat{K}t = x - \zeta t$ from which it follows $i^2\hat{K}^2 t = i\hat{K}(x - \zeta t)$ which is $\lambda^2 t + \zeta x - \zeta(x - \zeta t)$. Using $\lambda^2 + \zeta^2 = 1$, it finally gives $i^2\hat{K}^2 t = t$. Therefore, $i^{2n}\hat{K}^{2n}t = t$ and $i^{2n+1}\hat{K}^{2n+1} = (x - \zeta t)$. These relations allow us to simplify

$$e^{iK\kappa}t = [\cosh\kappa - \zeta\sinh\kappa] + [\sinh\kappa]x. \tag{A17}$$

This will precisely correspond to the first row of Eq. (7). Therefore, the Killing vector $\xi_{\hat{2}}$ in Eq. (A11) or equivalently Eq. (A14) determines the generator of t-Lorentz transformation.

The method of Killing to construct isometries of a given geometry (metric) is quite general and can even be applied to space- and/or time-dependent tilt vector $\boldsymbol{\zeta}$.

APPENDIX B: SEMICLASSICAL LANDAU QUANTIZATION FOR TILTED DIRAC CONE FERMIONS

Let us assume that we are in the positive-energy branch of the dispersion relation (2) branch. The Fermi surface corresponding to a constant energy $\varepsilon > 0$ the momentum is

$$k = \frac{\varepsilon}{1 + \zeta\cos\theta_{\boldsymbol{k}}}, \tag{B1}$$

which is equation of an ellipse in the polar coordinate with origin of $\boldsymbol{k}$ located on the ellipse focal point. Major and minor axes of ellipse are given by $2a = \varepsilon(1+\zeta)^{-1} + \varepsilon(1-\zeta)^{-1}$ and $b = \varepsilon$. After restoring the constants $\hbar$ and v_F, the k-space area at energy ε will give the following semiclassical quantization:

$$S(\varepsilon) = \pi\frac{\varepsilon^2}{\lambda^2(\hbar v_F)^2} = \frac{2\pi eB}{\hbar}n, \tag{B2}$$

where, in the right-hand side, the Berry phase of tilted Dirac fermions has been incorporated. Therefore, the quantized energies of Landau orbitals of tilted Dirac cone system with tilt parameter ζ are given by the new cyclotron energies $\varepsilon_c \equiv \hbar\omega_c$ which can be obtained from corresponding energy scale of the upright Dirac fermions as [21,22]

$$\begin{aligned} \varepsilon_{n,\zeta} &= \varepsilon_{c,\zeta}\sqrt{2n}, \quad \varepsilon_{c,\zeta} = \lambda\hbar v_F\sqrt{\frac{eB}{\hbar}} = (1-\zeta^2)^{1/2}\varepsilon_c \\ &\equiv (1-\zeta^2)^{1/2}\frac{\hbar v_F}{\ell_B}. \end{aligned} \tag{B3}$$

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