Topological nonlinear σ-model, higher gauge theory, and a systematic construction of 3+1D topological orders for boson systems

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A discrete nonlinear σ-model is obtained by triangulate both the space-time M^{d+1} and the target space K. If the path integral is given by the sum of all the simplicial homomorphisms $\phi : M^{d+1} \to K$ (i.e., maps without any topological defects), with an partition function that is independent of space-time triangulation, then the corresponding nonlinear σ-model will be called topological nonlinear σ-model which is exactly soluble. These exactly soluble models suggest that phase transitions induced by fluctuations with no topological defects usually produce a topologically ordered state and are topological phase transitions. In contrast, phase transitions induced by fluctuations with all topological defects give rise to trivial product states and are not topological phase transitions. Under the classification conjecture of Lan-Kong-Wen [Phys. Rev. X **8**, 021074 (2018)], it is shown that, if K is a space with only nontrivial first homotopy group G, which is finite, then these topological nonlinear σ-models can already realize all 3+1D bosonic topological orders without emergent fermions, which are described by Dijkgraaf-Witten theory with gauge group $\pi_1(K) = G$. Under the similar conjecture, we show that the 3+1D bosonic topological orders with emergent fermions can be realized by topological nonlinear σ-models with $\pi_1(K) =$ finite groups, $\pi_2(K) = Z_2$, and $\pi_{n>2}(K) = 0$. A subset of these topological nonlinear σ-models corresponds to 2-gauge theories, which realize and may classify bosonic topological orders with emergent fermions that have no emergent Majorana zero modes at triple string intersections.

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I. INTRODUCTION

A. Background

The study of topological phase of matter has become a very active field of research in condensed matter physics, quantum computation, as well as in part of quantum field theory and mathematics. However, "topological" may have very different meanings, even in the same context of topological phase of matter.

In *topological* insulator/superconductor [1–6], "topological" means the twist in the band structure of orbitals (see Fig. 1), which is described by the curvature, Chern number, finite dimensional fiber bundle, etc. [7–10]. Such "topological" properties can be defined even without any particles.

However, in *topological* order [11–13], "topological" means the pattern of quantum entanglement [14–16] in many-body wave functions of $N \sim 10^{20}$ variables:

$$\Psi(m_1, m_2, \dots, m_N). \tag{1}$$

It is hard to visualize the patterns of many-body entanglement in such complicated many-body systems. We may use Celtic knots to help us to get some spirit of topological order or pattern of many-body entanglement (see Fig. 2).

So the "topology" in topological order is very different from the classical topology that distinguishes a sphere from a torus. We will refer this new kind of "topology" as quantum topology. It turns out that the mathematical foundation for quantum topology is related to topological quantum field theory, braided fusion category, cohomology, etc. [17–25].

To develop a quantitative theory for topological order and the related pattern of many-body entanglement, we need to identify physical probes that can measure topological order [11–13], i.e., identify topological invariants that can characterize topological order. We know that, for crystal order, x-ray scattering is a universal probe that can measure all crystal orders (see Fig. 3). So we like to ask: do we have a single universal probe that can measure all topological orders?

One potential universal probe (topological invariant) for topological orders is the partition function Z. Let us consider bosonic systems described by the path integral of nonlinear σ-models:

$$Z(M^{d+1}; K, \mathcal{L}) = \sum_{\phi(x)} e^{-\int_{M^{d+1}} d^{d+1}x\, \mathcal{L}(\phi(x), \partial\phi(x), \dots)}. \tag{2}$$

Here, M^{d+1} is a d+1D space-time manifold and K a target manifold. $\sum_{\phi(x)}$ sum over all the maps $\phi : M^{d+1} \to K$, $x \in M^{d+1}$ and $\phi(x) \in K$. $d^{d+1}x\, \mathcal{L}(\phi(x), \partial\phi(x), \dots)$ is a $(d+1)$-form at x that depends on $\phi(x)$, $\partial\phi(x)$ etc. $\mathcal{L}(\phi(x), \partial\phi(x), \dots)$ is also called the Lagrangian density in physics.

The pair $(K, \mathcal{L})$ labels the bosonic systems, and the partition function Z is a map from space-time manifolds to complex numbers

$$Z(-; K, \mathcal{L}) : \{M^{d+1}\} \to \mathbb{C}. \tag{3}$$

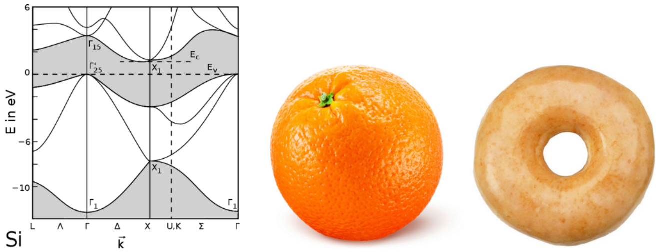

FIG. 1. "Topology" in topological insulator/superconductor (2005) corresponds to the twist in the band structure of orbitals, which is similar to the topological structure that distinguishes a sphere from a torus. This kind of topology is *classical topology*.

So the partition function Z is a physical probe that measure the bosonic system. However, $Z(-;K,\mathcal{L})$ does not measure topological order, since two systems $(K,\mathcal{L})$ and $(K',\mathcal{L}')$ that are in the same topologically ordered phase can have different partition functions: $Z(-;K,\mathcal{L}) \neq Z(-;K',\mathcal{L}')$. In other words, the partition function $Z(-;K,\mathcal{L})$ is not a topological invariant.

We know that the leading term in the partition function comes from the energy density $\varepsilon(x)$:

$$Z(M^{d+1};K,\mathcal{L})) = e^{-\int_{M^{d+1}} d^{d+1}x\,\varepsilon(x)} Z^{\text{top}}(M^{d+1};K,\mathcal{L})), \quad (4)$$

where the subleading term $Z^{\text{top}}(M^{d+1};K,\mathcal{L})$ is of order 1 in large space-time volume limit. The leading term $e^{-\int_{M^{d+1}} d^{d+1}x\,\varepsilon(x)}$ is not topological, since even when two systems $(K,\mathcal{L})$ and $(K',\mathcal{L}')$ are in the same topologically ordered phase, their energy densities $\varepsilon(x)$ and $\varepsilon'(x)$ can be different.

However, the idea of using partition function to characterize topological order is not totally wrong. In particular, the subleading term is believed to be topological [26]. So $Z^{\text{top}}(M^{d+1};K,\mathcal{L})$ are topological invariants that can be used to measure/define topological order. Reference [27] describes ways to extract topological invariant $Z^{\text{top}}(M^{d+1};K,\mathcal{L})$ from nontopological partition function $Z(M^{d+1};K,\mathcal{L})$ via surgery operations.

After identifying the topological invariants that characterize and define topological orders, the next issue is to systematically construct bosonic systems $(K,\mathcal{L})$ that realize all kinds of topological orders, which is the topic of this paper. Only 10 years ago, systemic and classifying understanding of strongly correlated systems appeared to an impossible task. At that time, the only systemic understanding is Landau symmetry breaking theory. Since then, we have obtained systemic and classifying understanding of strongly correlated 1+1D gapped phases at zero temperature [28,29]. This paper presents a systemic and a classifying understanding of strongly correlated 3+1D gapped liquid phases [30,31] at zero temperature (under the conjectures proposed in Refs. [32,33]).

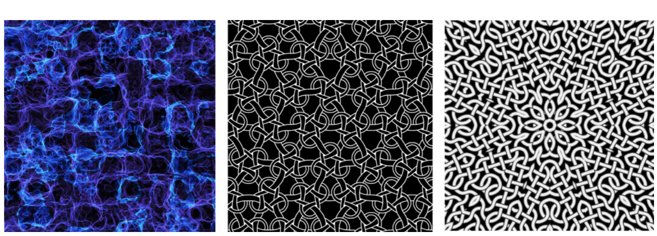

FIG. 2. "Topology" in topological order (1989) corresponds to the pattern of many-body entanglement in many-body wave function $\Psi(m_1, m_2, \ldots, m_N)$, which is robust against any local perturbations that can break any symmetry. Such robustness is the meaning of "topological" in topological order. This kind of topology is *quantum topology*.

(1) It is known that some topological orders are described by gauge theory with finite gauge group. Knowing the theoretical existence of higher gauge theory, one may wonder, do we have condensed matter systems on lattice that can produce topological orders described by higher gauge theory. In this paper, we will describe in details a general way to construct exactly soluble bosonic models on lattice: topological nonlinear σ-models, and their special cases—higher gauge theories. We believe that, under the conjectures proposed in Refs. [32,33], topological nonlinear σ-models can realize all 3+1D bosonic topological orders with gappable boundary. In particular, higher gauge theories realize and classify all bosonic topological orders with the following property: the topological orders have a gapped boundary that all pointlike, stringlike and other higher dimensional excitations on the boundary have a unit quantum dimension.

(2) We find that many higher gauge theories and topological nonlinear σ-models are equivalent and describe the same topological order. We identified a small subset of 2-gauge theories and topological nonlinear σ-models, and argue that the subset can realize all topological order in 3+1D bosonic systems. In particular, 3+1D higher gauge theory with higher gauge group $\mathcal{B}(\Pi_1, \Pi_2, \ldots)$ (where Π_i are finite) is equivalent [i.e., produces the same topological invariants $Z^{\text{top}}(M^{d+1})$] to a 3+1D higher gauge theory with higher gauge group $\mathcal{B}(\Pi'_1, Z_2)$. (For notation, see Sec. ID.) This result allows us to classify, under the conjectures in Refs. [32,33], all topological order in 3+1D via concrete models. Using these models, we can study universal experimental properties of all 3+1D topological orders. More specifically, We use exactly soluble 2-gauge theories to systematically realize and classify EF1 topological orders—3+1D bosonic topological orders with emergent bosons and fermions where triple string intersections carry no Majorana zero modes. The rest of 3+1D bosonic topological orders with emergent bosons and fermions are EF2 topological orders where some triple string intersections must carry Majorana zero modes [33]. We find that EF2 topological orders can be realized by topological nonlinear σ-models which are beyond 2-gauge theories.

There are many works [34–43] on higher gauge theories and their connection to topological phases of matter. In this paper, we present a detailed description of "lattice higher gauge theories," in a way to make their connection to nonlinear σ-model explicit. In our presentation, we do not require higher gauge symmetry and higher gauge holonomy. We even do not mod out higher gauge transformations. Our "lattice higher gauge theories" are just lattice nonlinear σ-models with only lattice scalar fields (i.e., lattice qubits). However, lattice nonlinear σ-models (without higher gauge symmetry) can realize topological orders whose low-energy effective theories are higher gauge theories with emergent higher gauge symmetry. In other words, we describe how higher gauge theories can emerge from lattice qubit models (i.e., quantum

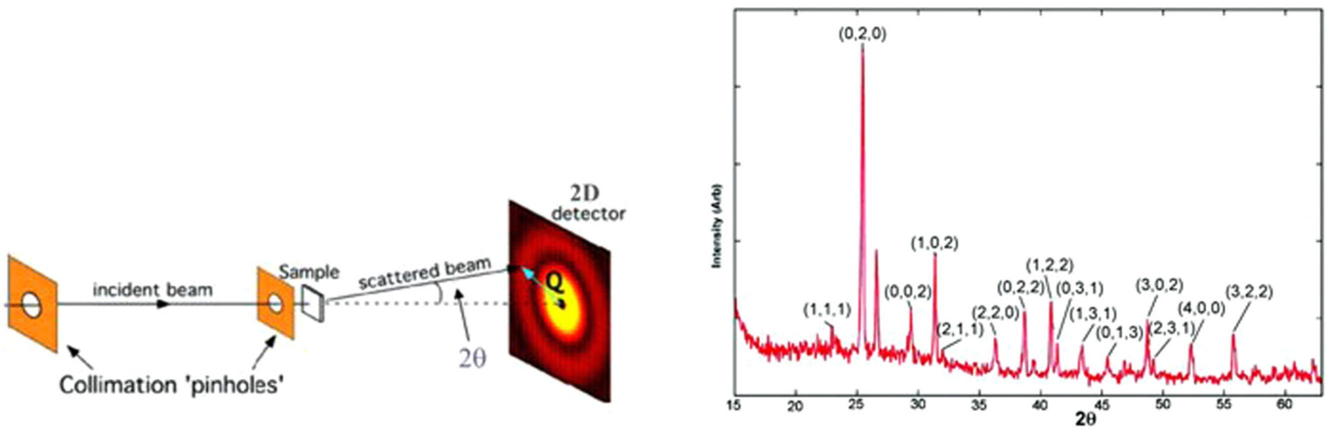

FIG. 3. X-ray scattering is a universal probe for all crystal orders.

spin models in condensed matter). In this paper, we also apply 2-gauge theories to classify a subclass of 3+1D bosonic topological orders with emergent fermions. We point out that the rest of 3+1D bosonic topological orders with emergent fermions are beyond 2-gauge theories and can be realized by more general topological nonlinear σ-models.

B. Realize topological orders via disordered symmetry breaking states without topological defects

In this paper, we show that all the higher gauge theories can be viewed as nonlinear σ-models with some complicated target space and carefully designed action. Such a duality relation between nonlinear σ-models and higher gauge theories suggests that we may be able to use disordered symmetry breaking states (which are described by nonlinear σ-models) to realize a large class of topological orders. In other words, starting with a symmetry breaking state and letting the order parameter have a strong quantum fluctuation, we may get a symmetric disordered ground state with topological order.

However, this picture seems to contradict with many previous results that a symmetric disordered ground state is usually just a trivial product state rather than a topological state. The study in this paper suggests that the reason why we get a trivial disordered state is because the strongly fluctuating order parameter in the disordered state contains a lot of topological defects, such as vortex lines, monopoles, etc.

The importance of the topological defects [44] in producing short-range correlated disordered states have been emphasized by Kosterlitz and Thouless in Ref. [45], which shared the 2016 Nobel prize "for theoretical discoveries of topological phase transitions and topological phases of matter."

In this paper, we show that the phase transitions driven by fluctuations with all possible topological defects produce disordered states that have no topological order, and correspond to nontopological phase transitions. While transitions driven by fluctuations without any topological defects usually produce disordered states that have nontrivial topological orders, and correspond to topological phase transitions. This phenomenon has been demonstrated in a 2+1D $\mathbb{R}P^3 = SO_3$ nonlinear σ-model [46,47]. Thus it may be confusing to refer the transition driven by topological defects as a topological phase transitions, since the appearance of topological defects decrease the chance to produce topological phases of matter.

More precisely, if the fluctuating order parameter in a disordered state has no topological defects, then the corresponding disordered state will usually have a nontrivial topological order. The type of the topological order depends on the topology of the degenerate manifold K of the order parameter (i.e., the target space of the nonlinear σ-model). For example, if $\pi_1(K)$ is a finite group and $\pi_{n>1}(K) = 0$, then the disordered phase may have a topological order described by a gauge theory of gauge group $G = \pi_1(K)$. If $\pi_1(K), \pi_2(K)$ are finite groups and $\pi_{n>2}(K) = 0$, then the disordered phase may have a topological order described by a 2-gauge theory of 2-gauge group $\mathcal{B}(\pi_1(K), \pi_2(K))$.

It is the absence of topological defects that enable the symmetric disordered state to have a nontrivial topological order. When there are a lot of topological defects, they will destroy the topology of the degenerate manifold of the order parameter (i.e., the degenerate manifold effectively becomes a discrete set with trivial topology). In this case, the symmetric disordered state becomes a product state with no topological order. Certainly, if the fluctuating order parameter contains only a subclass of topological defects, then only part of the topological structure of the degenerate manifold is destroyed by the defects. The corresponding symmetric disordered state may still have a topological order, as discussed in Ref. [48].

C. Realizations of all 3+1D bosonic topological orders

It was shown [32,33] that all 3+1D bosonic topological orders belong to two classes: AB topological orders where all pointlike excitations are bosonic and EF topological orders where some pointlike excitations are fermionic. Reference [33] shows that all EF topological orders have a unique gapped boundary with the following properties.

(1) All stringlike boundary excitations have a unit quantum dimension. These boundary strings form a finite group $\hat{G}_b$ under string fusion. The group $\hat{G}_b$ is an extension of a finite group G_b by Z_2^m: $\hat{G}_b = Z_2^m \leftthreetimes G_b$. (See Sec. I D for the definition of $Z_2^m \leftthreetimes G_b$.)

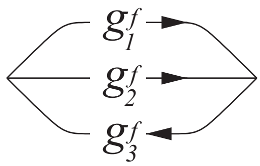

FIG. 4. A string configuration in the bulk described by a triple $(\chi_{g_1^f}, \chi_{g_2^f}, \chi_{g_3^f})$, where χ_{g^f} is a conjugacy class in G_f containing $g^f \in G_f$ and the triple satisfy $g_1^f g_2^f = g_3^f$.

(2) There is one nontrivial type of pointlike boundary excitations which is fermionic and has a unit quantum dimension.

(3) There are on-string pointlike excitations—Majorana zero modes of quantum dimension $\sqrt{2}$. The Majorana zero mode always lives at the pointlike domain wall where a string labeled by g joins a string labeled by gm. Here, $g \in \hat{G}_b$ and m is the nontrivial element in Z_2^m.

We note that the boundary fermions can form a topological p-wave superconducting (pSC) chain [49]. The boundary strings labeled by $\hat{G}_b$ can be viewed as the boundary strings labeled by G_b plus the pSC chain. In particular, a string labeled by g and a string labeled by gm differ by a pSC chain.

If $\hat{G}_b$ is the trivial extension of G_b by Z_2^m: $\hat{G}_b = Z_2^m \times G_b$, the corresponding bulk topological order is called a EF1 topological order. If $\hat{G}_b$ is a nontrivial extension of G_b by Z_2^m: $\hat{G}_b = Z_2^m \leftthreetimes_{\rho_2} G_b$ where $\rho_2 \in H^2(\mathcal{B}G_b; \mathbb{Z}_2^m)$, the corresponding bulk topological order is called a EF2 topological order. Here, we have used a conjecture—a holographic principle [25,26,50]—that the boundary topological order completely determines the bulk topological order.

When $\hat{G}_b$ is the trivial extension: $\hat{G}_b = Z_2^m \times G_b$, we can drop boundary strings that come from the pSC chain (by regarding the pSC chain as a kind of trivial strings). Thus the EF1 topological order has a simpler gapped boundary: in addition to the boundary strings of unit quantum dimension labeled by a finite group G_b, there is one and only one nontrivial type of pointlike boundary excitations which is fermionic and has a unit quantum dimension [33].

In the above, we have defined EF1 and EF2 topological orders via their boundary properties. To distinguish EF1 and EF2 topological order through their bulk properties, we consider a stringlike excitation in the bulk that has triple string intersections (see Fig. 4). Note that a triple string intersection is described by the conjugacy classes $\chi_{g_1^f}, \chi_{g_2^f}, \chi_{g_3^f} \subset G_f$ that satisfy $g_1^f g_2^f = g_3^f$. By measuring the appearance of Majorana zero mode at triple string intersections for different triples $\chi_{g_1^f}, \chi_{g_2^f}, \chi_{g_3^f}$, we can determine the cohomology class of ρ_2 [33]. If the measured ρ_2 is a coboundary, the bulk topological order is an EF1 or an AB topological order. Otherwise, the bulk topological order is an EF2 topological order.

It has been shown that all 3+1D AB topological orders are classified and realized by 1-gauge theories (i.e., Dijkgraaf-Witten gauge theories) [32]. In this paper, we show that all 3+1D EF1 topological orders are classified and realized by 2-gauge theories with 2-gauge-group $\mathcal{B}(G_b, Z_2^f)$. The pointlike topological excitations (including emergent fermions) are described by symmetric fusion category $s\mathcal{R}\text{ep}(Z_2^f \leftthreetimes G_b)$, where $Z_2^f \leftthreetimes G_b$ is an extension of G_b by Z_2^f.

We will also discuss how to systematically realize 3+1D EF2 topological orders through topological nonlinear σ-models whose target space K satisfies $\pi_1(K) = G_b$ and $\pi_2(K) = \mathbb{Z}_2$. These topological nonlinear σ-models are beyond 2-gauge theories. The resulting EF2 topological orders have pointlike topological excitations described by $s\mathcal{R}\text{ep}(Z_2^f \leftthreetimes G_b)$.

Our results suggest the following more general picture:

Statement I.1. Exactly soluble n-gauge theories can realize all bosonic topological orders in $n+1$ spatial dimensions that have a gaped boundary where all boundary excitations (including on d-brane excitations) have a unit quantum dimension.

This is because higher groups can be viewed as higher monoidal categories where all objects and higher morphisms are invertible. For more general bosonic topological orders whose gapped boundary excitations have nonunit quantum dimensions, we need to use more general exactly soluble models, such as topological nonlinear σ-model or even more general tensor network models, to realize them [26].

Combining the above realization results and the boundary results in Ref. [33], we obtain the following classification of EF topological orders:

Statement I.2. 3+1D EF topological orders are classified by unitary fusion 2-categories [51] that have the following properties.

(1) The simple objects are labeled by $\hat{G}_b = Z_2^m \leftthreetimes_{\rho_2} G_b$, and their fusion is described by the group $\hat{G}_b$.

(2) For each simple object g there is one nontrivial invertible 1-morphism corresponding to a fermion f_g.

(3) In addition, there are quantum-dimension-$\sqrt{2}$ 1-morphisms $\sigma_{g,gm}$ that connect two objects g and gm, where $g \in \hat{G}_b$ and m is the generator of Z_2^m.

(4) The fusion of 1-morphisms is given by $f_g f_g = \mathbb{1}$ and $\sigma_{g,gm}\sigma_{gm,g} = \mathbb{1} \oplus f_g$.

D. Notations and conventions

Let us first explain some notations used in this paper. We will use extensively the mathematical formalism of cochains, coboundaries, and cocycles, as well as their higher cup product $\underset{k}{\smile}$, Steenrod square Sq^k, and the Bockstein homomorphism $\mathcal{B}_n$. A brief introduction can be found in Appendix A. We will abbreviate the cup product $a \smile b$ as ab by dropping $\smile$. We will use a symbol with bar, such as $\bar{a}$ to denote a cochain on the target complex $\mathcal{K}$. We will use a to denote the corresponding pullback cochain on space-time $\mathcal{M}^{d+1}$: $a = \phi^* \bar{a}$, where ϕ is a homomorphism of complexes $\phi: \mathcal{M}^{d+1} \to \mathcal{K}$.

We will use $\overset{n}{=}$ to mean equal up to a multiple of n, and use $\overset{d}{=}$ to mean equal up to df (i.e., up to a coboundary). We will use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x, and $\langle l, m \rangle$ for the greatest common divisor of l and m ($\langle 0, m \rangle \equiv m$).

Also, we will use $Z_n = \{1, e^{i\frac{2\pi}{n}}, e^{i2\frac{2\pi}{n}}, \ldots, e^{i(n-1)\frac{2\pi}{n}}\}$ to denote an Abelian group, where the group multiplication is "$*$." We use $\mathbb{Z}_n = \{\lfloor -\frac{n}{2}+1 \rfloor, \lfloor -\frac{n}{2}+1 \rfloor + 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ to denote an integer lifting of Z_n, where "$+$" is done without mod-n. In this sense, $\mathbb{Z}_n$ is not a group under "$+$." However, under a modified equality $\stackrel{n}{=}$, $\mathbb{Z}_n$ is the Z_n group under "+." Similarly, we will use $\mathbb{R}/\mathbb{Z} = (-\frac{1}{2}, \frac{1}{2}]$ to denote an $\mathbb{R}$-lifting of U_1 group. Under a modified equality $\stackrel{1}{=}$, $\mathbb{R}/\mathbb{Z}$ is the U_1 group under "+." In this paper, there are many expressions containing the addition "+" of $\mathbb{Z}_n$-valued or $\mathbb{R}/\mathbb{Z}$-valued, such as $a_1^{\mathbb{Z}_n} + a_2^{\mathbb{Z}_n}$, where $a_1^{\mathbb{Z}_n}$ and $a_2^{\mathbb{Z}_n}$ are $\mathbb{Z}_n$-valued. These additions "+" are done without mod n or mod 1. In this paper, we also have expressions like $\frac{1}{n}a_1^{\mathbb{Z}_n}$. Such an expression convert a $\mathbb{Z}_n$-valued $a_1^{\mathbb{Z}_n}$ to a $\mathbb{R}/\mathbb{Z}$-valued $\frac{1}{n}a_1^{\mathbb{Z}_n}$, by viewing the $\mathbb{Z}_n$ value as a $\mathbb{Z}$ value. (In fact, $\mathbb{Z}_n$ is a $\mathbb{Z}$ lifting of Z_n.)

We introduced a symbol $\leftthreetimes$ to construct fiber bundle X from the fiber F and the base space B:

$$pt \to F \to X = F \leftthreetimes B \to B \to pt. \tag{5}$$

We will also use $\leftthreetimes$ to construct group extension of H by N [52]:

$$1 \to N \to N \leftthreetimes_{e_2,\alpha} H \to H \to 1. \tag{6}$$

Here, $e_2 \in H^2[H; Z(N)]$ and $Z(N)$ is the center of N. Also H may have a nontrivial action on $Z(N)$ via $\alpha : H \to \text{Aut}(N)$. e_2 and α characterize different group extensions.

We will use $K(\Pi_1, \Pi_2, \dots, \Pi_n)$ to denote a connected topological space with homotopy group $\pi_i(K(\Pi_1, \Pi_2, \dots, \Pi_n)) = \Pi_i$ for $1 \leqslant i \leqslant n$, and $\pi_i(K(\Pi_1, \Pi_2, \dots, \Pi_n)) = 0$ for $i > n$. In this paper, *we assume that all Π_n's are finite.* We note that π_i is Abelian for $i > 1$. If only one of the homotopy groups, say Π_d, is nontrivial, then $K(\Pi_1, \Pi_2, \dots, \Pi_n)$ is the Eilenberg-MacLane space, which is denoted as $K(\Pi_d, d)$. If only two of the homotopy groups, say Π_d, $\Pi_{d'}$, is nontrivial, then we denote the space as $K(\Pi_d, d; \Pi_{d'}, d')$, etc. We will use $\mathcal{K}(\Pi_1; \Pi_2; \dots; \Pi_n)$, $\mathcal{K}(\Pi_d, d)$, and $\mathcal{K}(\Pi_d, d; \Pi_{d'}, d')$ to denote the simplicial complexes that describe a triangulation of $K(\Pi_1, \Pi_2, \dots, \Pi_n)$, $K(\Pi_d, d)$, and $K(\Pi_d, d; \Pi_{d'}, d')$ respectively. We will use $\mathcal{B}(\Pi_1; \Pi_2; \dots; \Pi_n)$, $\mathcal{B}(\Pi_d, d)$, and $\mathcal{B}(\Pi_d, d; \Pi_{d'}, d')$ to denote the simplicial sets with only one vertex satisfying Kan conditions that describe a special triangulation of $K(\Pi_1, \Pi_2, \dots, \Pi_n)$, $K(\Pi_d, d)$, and $K(\Pi_d, d; \Pi_{d'}, d')$ respectively. Since simplicial sets satisfying Kan conditions are viewed as higher groupoids in higher category theory, the simplicial sets $\mathcal{B}(\Pi_1; \Pi_2; \dots; \Pi_n)$, $\mathcal{B}(\Pi_d, d)$, and $\mathcal{B}(\Pi_d, d; \Pi_{d'}, d')$, with only one vertex (unit), can be viewed as higher groups. In this paper, higher groups are treated therefore as this sort of special simplicial sets.

II. TOPOLOGICAL NONLINEAR σ-MODELS AND TOPOLOGICAL TENSOR NETWORK MODELS

A. Discrete defectless nonlinear σ-models

The nonlinear σ-model (2) is widely used in field theory to describe a bosonic system. If we require the map $\phi(x)$ to be continuous, then the nonlinear σ-model will be defectless, i.e., the fluctuations contain no defects. But the corresponding path integral (2) is not well defined since the summation $\sum_{\phi(x)}$ over ∞^∞ number of the continuous maps is not well defined. To obtain a well defined theory, we discretize both the space-time M^{d+1} and the target space K. We replace them by simplicial complexes $\mathcal{M}^{d+1}$ and $\mathcal{K}$.

1. A detailed description of simplicial complex

Let us first describe the simplicial complexes systematically. We introduce $M_0, M_1, M_2, \dots$ as the sets of vertices, links, triangles, etc. that form the space-time complex $\mathcal{M}^{d+1}$. The complex $\mathcal{M}^{d+1}$ is formally described by

$$M_0 \overset{d_0,d_1}{\Longleftarrow} M_1 \overset{d_0,d_1,d_2}{\Lleftarrow} M_2 \overset{d_0,\dots,d_3}{\underset{\cdot}{\longleftarrow}} M_3 \overset{d_0,\dots,d_4}{\underset{\cdot}{\longleftarrow}} M_4 \cdots, \tag{7}$$

where d_i are the face maps, describing how the $(n-1)$-simplices are attached to a n-simplex. Similarly, the complex $\mathcal{K}$ is formally described by

$$K_0 \overset{d_0,d_1}{\Longleftarrow} K_1 \overset{d_0,d_1,d_2}{\Lleftarrow} K_2 \overset{d_0,\dots,d_3}{\underset{\cdot}{\longleftarrow}} K_3 \overset{d_0,\dots,d_4}{\underset{\cdot}{\longleftarrow}} K_4 \cdots, \tag{8}$$

where $K_0, K_1, K_2, \dots$ are the sets of vertices, links, triangles, etc. that form the target complex $\mathcal{K}$.

In this paper, we will use $v_1, v_2, \dots \in K_0$ to label different vertices in the complex $\mathcal{K}$. We will use $l_1, l_2, \dots \in K_1$ to label different links in the complex $\mathcal{K}$, and $t_1, t_2, \dots \in K_2$ different triangles, etc. We choose a fine triangulation on $\mathcal{M}^{d+1}$ such that the links, triangles, etc. can be be labeled by their vertices. In other words, we will use i to label vertices in M_0. We will use (ij) to label links in M_1, and (ijk) to label triangles in M_2, etc.

The continuous maps between manifolds $\phi(x) : M^{d+1} \to K$ is replaced by homomorphisms between complexes $\phi : \mathcal{M}^{d+1} \to \mathcal{K}$. The homomorphism ϕ is a set of maps $\phi^{(0)} : M_0 \to K_0, \phi^{(1)} : M_1 \to K_1, \phi^{(2)} : M_2 \to K_2$, etc. that preserve the attachment structure of simplices described by the face maps d_i. For example, if (ij) is attached to (ijk) by the face map d_3 in space-time complex $\mathcal{M}^{d+1}$, then $\phi^{(1)}((ij))$ is attached to $\phi^{(2)}((ijk))$ by the face map d_3 in target space complex $\mathcal{K}$. The homomorphism is the discrete version of continuous map. Physically, the continuous map or the homomorphism describes fluctuations without any topological defects and any kind of "tears."

2. A simple definition of discrete nonlinear σ-model

Now, a *discrete nonlinear σ-model* is defined via the following path integral:

$$Z(\mathcal{M}^{d+1}; \mathcal{K}, \bar{\omega}_{d+1}) = \sum_{\phi} e^{2\pi i \int_{\mathcal{M}^{d+1}} \phi^* \bar{\omega}_{d+1}}, \tag{9}$$

where $\sum_\phi$ sums over all the homomorphisms $\phi : \mathcal{M}^{d+1} \to \mathcal{K}$. It is clear that the map ϕ assign a label v_i to each vertex $i \in M_0$, a label l_{ij} to each link $(ij) \in M_1$, a label t_{ijk} to each triangle $(ijk) \in M_2$, etc. Thus we can view the map ϕ as a collection of fields on the space-time complex $\mathcal{M}$: a field v_i on the vertices M_0, a field e_{ij} on the links M_1, a field t_{ijk} on the triangles M_2, etc. We can rewrite the path integral as a integration of these fields:

$$Z(\mathcal{M}^{d+1}; \mathcal{K}, \bar{\omega}_{d+1}) = \sum_{v_i, l_{ij}, t_{ijk}, \dots} e^{2\pi i \int_{\mathcal{M}^{d+1}} \omega_{d+1}(v,l,t,\dots)}. \tag{10}$$

Although these fields $v_i, l_{ij}, t_{ijk}, \dots$ satisfy certain local constraints described by the face maps d_i, we can impose these local constraints by energy penalty: the field configurations that do not satisfy attachment conditions will cost a large energy. Thus we can view these fields as independent fields.

The term $e^{2\pi\, i \int_{\mathcal{M}^{d+1}} \phi^* \bar{\omega}_{d+1}}$ in the path integral is the action amplitude. Here, $\phi^*\bar{\omega}_{d+1} \equiv \omega_{d+1}$ is a real-valued $(d+1)$-cochain on $\mathcal{M}^{d+1}$ which is a pull back of a real-valued $(d+1)$-cochain $\bar{\omega}_{d+1}$ on $\mathcal{K}$. The resulting path integral defines a discrete nonlinear σ-model whose fluctuations have no defects.

However, the above definition of discrete nonlinear σ-model has an inconvenience: different choices of space-time triangulation may lead to different phases of the bosonic systems. To avoid this problem, we like to choose some special triangulation $\mathcal{K}$ of the target space K, and some special $\bar{\omega}_{d+1}$'s on $\mathcal{K}$ such that, for a given pair $(\mathcal{K}, \bar{\omega}_{d+1})$, the corresponding discrete defectless nonlinear σ-model will realize the same phase for any space-time triangulations, as long as they are very fine triangulations (i.e., in the thermodynamic limit). Such kind of choice of $(\mathcal{K}, \bar{\omega}_{d+1})$ turns out to give rise exactly soluble models. To describe how we choose $(\mathcal{K}, \bar{\omega}_{d+1})$, we will first discuss a more general class of discrete bosonic discrete nonlinear σ-models—tensor network models.

In the above definition of discrete nonlinear σ-models, we assign each $d+1$-simplex Δ^{d+1} a field-dependent complex number $e^{i\,2\pi \int_{\Delta^{d+1}} \omega_{d+1}}$, and multiply all these numbers together to get an action amplitude. In the more general tensor network models, we also assign each n-simplex Δ^n, $n < d+1$, a field-dependent real positive number, and multiply all these numbers together to get additional contributions to the action amplitude. In the following, we will describe tensor network models in details.

B. Exactly soluble tensor network models

Let us describe a tensor network model in 2+1D space-time complex $\mathcal{M}^3$ as an example. The tensor network model is constructed from a tensor set $\mathbb{T}$ of two real and one complex tensors: $\mathbb{T} = (w_{v_0}, w_{l_{01}}^{v_0 v_1}, C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}})$. We will call $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$ the top tensor and w_{v_0}, $w_{l_{01}}^{v_0 v_1}$ the weight tensors. The complex tensor $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$ can be associated with a tetrahedron (0123), which has a branching structure (see Fig. 5). A branching structure is a choice of orientation of each link in the complex so that there

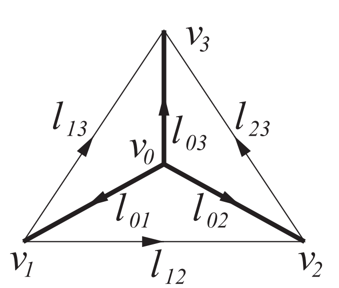

FIG. 5. The tensor $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$ is associated with a tetrahedron, which has a branching structure. If the vertex-0 is above the triangle-123, the tetrahedron has an orientation $s_{0123} = *$. If the vertex-0 is below the triangle-123, the tetrahedron has an orientation $s_{0123} = 1$. The branching structure gives the vertices a local order: the ith vertex has i incoming links.

is no oriented loop on any triangle (see Fig. 5). Here the v_0 index is associated with the vertex-0, the l_{01} index is associated with the link-01, and the t_{012} index is associated with the triangle-012. They represents the degrees of freedom on the vertices, links, and the triangles. Similarly, the real tensor $w_{l_{01}}^{v_0 v_1}$ is associated with a link (01), and w_{v_0} with a vertex 0.

Using the tensors, we can define a path integral on any 3-complex that has no boundary [26]:

$$Z(\mathcal{M}^3; \mathbb{T}) = \sum_{v_i,\ldots;l_{ij},\ldots;t_{ijk},\ldots} \prod_i w_{v_i} \prod_{(ij)} w_{l_{ij}}^{v_i v_j} \times \prod_{(ijkm)} \left[C^{l_{ij}l_{ik}l_{im}l_{jk}l_{jm}l_{km};t_{ijk}}_{v_iv_jv_kv_m;t_{ikm}t_{ijm}t_{jkm}}\right]^{s_{ijkm}}, \tag{11}$$

where $\sum_{v_i;l_{ij};t_{ijk}}$ sums over all the vertex indices, the link indices, and the triangle indices, $s_{ijkm} = 1$ or $*$ depending on the orientation of tetrahedron $(ijkm)$ (see Fig. 5).

On the complex $\mathcal{M}^3$ with boundary: $\mathcal{B}^2 = \partial\mathcal{M}^3$, the partition function is defined differently:

$$Z = \sum_{\{v_i;l_{ij};t_{ijk}\}} \prod_{i\notin\mathcal{B}^2} w_{v_i} \prod_{(ij)\notin\mathcal{B}^2} w_{l_{ij}}^{v_i v_j} \prod_{(ijkm)} \left[C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}\right]^{s_{ijkm}}, \tag{12}$$

where $\sum_{v_i;l_{ij};t_{ijk}}$ only sums over the vertex indices, the link indices, and the triangle indices that are not on the boundary. The resulting Z is actually a complex function of v_i's, l_{ij}'s, and t_{ijk}'s on the boundary $\mathcal{B}^2$: $Z = Z(\{v_i; l_{ij}; t_{ijk}\})$. Such a function is a vector in a Hilbert space $\mathcal{H}_{\mathcal{B}^2}$. We will denote such a vector by $|\Psi(\mathcal{M}^3)\rangle$.

Consider two complexes $\mathcal{M}_1^3$ and $\mathcal{M}_2^3$ with the same boundary $\mathcal{B} = \partial\mathcal{M}_1^3 = -\partial\mathcal{M}_2^3$, the inner product between $|\Psi(\mathcal{M}_1^3)\rangle$ and $|\Psi(\mathcal{M}_2^3)\rangle$ can be obtained by gluing $\mathcal{M}_1^3$ and $\mathcal{M}_2^3$ together $\mathcal{M}^3 = \mathcal{M}_1^3 \cup \mathcal{M}_2^3$ and perform the path integral on $\mathcal{M}^3$

$$\langle\Psi(\mathcal{M}_2^3)|\Psi(\mathcal{M}_1^3)\rangle = Z(\mathcal{M}^3; \mathbb{T}). \tag{13}$$

This is because the inner product of two wave functions $|\Psi(\mathcal{M}_2^3)\rangle$ and $|\Psi(\mathcal{M}_1^3)\rangle$ performs the summation of the boundary indices $\{v_i; l_{ij}; t_{ijk}\}$. We note that, in the definition of $|\Psi(\mathcal{M}_1^3)\rangle$ and $|\Psi(\mathcal{M}_2^3)\rangle$, the tensors w_{v_i} and $w_{l_{ij}}^{v_i v_j}$ are absent for the vertices and the links on the boundary. When we glue two boundaries together, these tensors w_{v_i} and $w_{l_{ij}}^{v_i v_j}$ need to be added back. So the tensors w_{v_i} and $w_{l_{ij}}^{v_i v_j}$ defines the inner product in the boundary Hilbert space $\mathcal{H}_{\mathcal{B}^2}$. Therefore we require w_{v_i} and $w_{l_{ij}}^{v_i v_j}$ to satisfy the following unitary condition (or the reflection positivity condition):

$$w_{v_i} > 0, \quad w_{l_{ij}}^{v_i v_j} > 0. \tag{14}$$

The tensor network model (11) are also inconvenient since for a fixed tensor set $\mathbb{T}$, different choices of the triangulations of the space-time $\mathcal{M}^3$ may lead to different phases. To solve this problem, we want to choose the tensors $(w_{v_0}, w_{l_{01}}^{v_0 v_1}, C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}})$ such that the path integral is retriangulation invariant. The corresponding models will be called a topological tensor network model, which can realize the same phase for any triangulations of the space-time $\mathcal{M}^3$. In general such a phase has a nontrivial topological order that has gappable boundary.

The invariance of Z under the re-triangulation in Fig. 6 requires that

$$\sum_{\phi_{123}} C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}} C^{l_{12}l_{13}l_{14}l_{23}l_{24}l_{34};t_{123}}_{v_1v_2v_3v_4;t_{134}t_{124}t_{234}} = \sum_{l_{04}} w^{v_0v_4}_{l_{04}} \sum_{t_{014}t_{024}t_{034}} C^{l_{01}l_{02}l_{04}l_{12}l_{14}l_{24};t_{012}}_{v_0v_1v_2v_4;t_{024}t_{014}t_{124}} C^{*\,l_{01}l_{03}l_{04}l_{13}l_{14}l_{34};t_{013}}_{v_0v_1v_3v_4;t_{034}t_{014}t_{134}} C^{l_{02}l_{03}l_{04}l_{23}l_{24}l_{34};t_{023}}_{v_0v_2v_3v_4;t_{034}t_{024}t_{234}}. \tag{15}$$

The invariance of Z under the retriangulation in Fig. 7 requires that

$$C^{l_{02}l_{03}l_{04}l_{23}l_{24}l_{34};t_{023}}_{v_0v_2v_3v_4;t_{034}t_{024}t_{234}} = \sum_{l_{01}l_{12}l_{13}l_{14},v_1} w_{v_1} w^{v_0v_1}_{l_{01}} w^{v_1v_2}_{l_{12}} w^{v_1v_3}_{l_{13}} w^{v_1v_4}_{l_{14}} \sum_{t_{012}t_{013}t_{014}t_{123}t_{124}t_{134}} \times C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}} C^{*\,l_{01}l_{02}l_{04}l_{12}l_{14}l_{24};t_{012}}_{v_0v_1v_2v_4;t_{024}t_{014}t_{124}} C^{l_{01}l_{03}l_{04}l_{13}l_{14}l_{34};t_{013}}_{v_0v_1v_3v_4;t_{034}t_{014}t_{134}} C^{l_{12}l_{13}l_{14}l_{23}l_{24}l_{34};t_{123}}_{v_1v_2v_3v_4;t_{134}t_{124}t_{234}}. \tag{16}$$

There are other similar conditions for different choices of the branching structures. To obtain these conditions, we start with a 4-simplex (01234), and divide the five 3-simplices on the boundary of the 4-simplex (01234) into two groups. Then the partition function (12) on one group of the 3-simplices must equal to the partition function on the other group of the 3-simplices, after a complex conjugation.

The above two types of the conditions are sufficient to determine the tensor set $\mathbb{T}$ that produces a topologically invariant partition function Z for any triangulated space-time $\mathcal{M}^3$. For such a tensor set, its partition function $Z = Z^{\text{top}}$ [i.e., the energy density in Eq. (4) $\varepsilon(x) = 0$]. Such topological partition function $Z^{\text{top}}(\mathcal{M}^3)$ is nothing but the topological invariant for three manifolds introduced by Turaev and Viro [53].

C. Topological nonlinear σ-models

A subclass of topological tensor network models happen to have a form of discrete defectless nonlinear σ-models. Such topological tensor network models (i.e., exactly soluble discrete nonlinear σ-models) are called topological nonlinear σ-models.

In the following, we will explain why a subclass of topological tensor network models can be viewed as discrete defectless nonlinear σ-models. Again we will use a 2+1D nonlinear σ-model as example. The target complex $\mathcal{K}$ has a set of vertices labeled by v, a set of links labeled by l, a set of triangles labeled by t, etc. We assume that each tetrahedron in $\mathcal{K}$ is uniquely determined by its vertices v_0, v_1, v_2, v_3, its links $l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}$, and its triangles $t_{012}, t_{023}, t_{013}$.

We first assign a complex number to each tetrahedron in $\mathcal{K}$, which can be written as $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$. When the indices $v_0, v_1, v_2, v_3, l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}, t_{012}, t_{023}, t_{013}$ are not vertices, links, and triangles of a tetrahedron in $\mathcal{K}$, then the corresponding $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}} = 0$. Similarly, we also choose a real tensor $w^{v_0v_1}_{l_{01}}$ whose value is positive when v_0, v_1, l_{01}, are the vertices and the link of a triangle in $\mathcal{K}$. Otherwise $w^{v_0v_1}_{l_{01}} = 0$. We also assign a real positive value w_v to each vertex v in $\mathcal{K}$. For such a choice of tensor set $\mathbb{T}$, the partition function (11) actually describes a discrete defectless nonlinear σ-model.

To see this we note that a homomorphism $\phi : \mathcal{M}^3 \to \mathcal{K}$ assigns a value v_i (a vertex in $\mathcal{K}$) to each vertex i in $\mathcal{M}^3$. ϕ also assigns a value l_{ij} to each link (ij) and assigns a value t_{ijk} to each triangle (ijk) in $\mathcal{M}^3$. The terms in the summation in Eq. (11) are nonzero only when the fields v_i, l_{ij}, t_{ijk} correspond to a homomorphism $\phi : \mathcal{M}^3 \to \mathcal{K}$. Thus the summation $\sum_{\{v_i;l_{ij};t_{ijk}\}}$ in Eq. (11) corresponds to a summation $\sum_\phi$ over all the homomorphisms $\phi : \mathcal{M}^3 \to \mathcal{K}$. In this case, Eq. (11) can be viewed as a discrete defectless nonlinear σ-model. If the tensors w_{v_0} $w^{v_0v_1}_{l_{01}}$, $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$ also satisfy the conditions Eqs. (15) and (16), then the corresponding discrete defectless nonlinear σ-model will be a topological nonlinear σ-model.

D. Labeling simplices in a complex

In the above example, most components of the tensor $C^{l_{01}l_{02}l_{03}l_{12}l_{13}l_{23};t_{012}}_{v_0v_1v_2v_3;t_{023}t_{013}t_{123}}$ are zero. This is because most combinations of $v_0, v_1, v_2, v_3, l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}, t_{012}, t_{023}, t_{013}$ are not vertices, links, and triangles of a tetrahedron in $\mathcal{K}$. In the following, we will describe a more economical way to label simplices in a complex, such that each label will have a smaller range and a larger fraction of the tensor elements will be nonzero.

We still use v to label different vertices in the complex $\mathcal{K}$. Thus $K_0 = \{v\}$. To label links in $\mathcal{K}$, we will first try to use two vertices v_0, v_1 on the two ends of the link to label it.

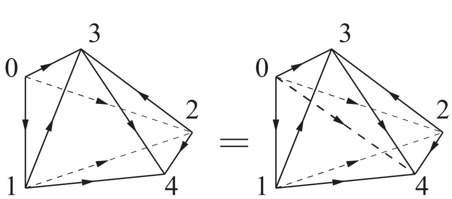

FIG. 6. A retriangulation of a 3D complex, obtained by dividing the five 3-simplices on the boundary of the 4-simplex (01234) into [(0123), (1234)] and [(0124), (0134), (0234)].

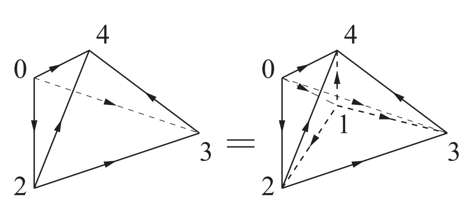

FIG. 7. A retriangulation of another 3D complex, obtained dividing the five 3-simplices on the boundary of the 4-simplex (01234) into [(0234)] and [(0123), (0124), (0134), (1234)].

If there are many links with the same end points v_0, v_1, we will introduce additional label a_{01} to label these links with the same set of end points. Thus different links in $\mathcal{K}$ are labeled by (v_0, v_1, a_{01}), and $K_1 = \{(v_0, v_1, a_{01})\}$. We see that the new link label a_{01} has a smaller range than the original link label $l_{01} \sim (v_0, v_1, a_{01})$.

In general, the set of the extra labels, $\{a_{01}\}$, depends on the end points v_0, v_1. In this paper, we will only consider a special type of complex $\mathcal{K}$ such that the set of the extra labels, $\{a_{01}\}$, does not depend on the end points v_0, v_1. In this case, a_{01} can be treated as a new label that is independent of vertex label v_i.

Similarly, different triangles t_{012} in $\mathcal{K}$ are labeled by $t_{012} \sim (v_0, v_1, v_2, a_{01}, a_{12}, a_{02}, b_{012})$, and $K_2 = \{(v_0, v_1, v_2, a_{01}, a_{12}, a_{02}, b_{012})\}$. Again the complex $\mathcal{K}$ has a property that b_{012} is a new label independent of vertex and link labels v_i, a_{jk}. We like to stress that not all combinations $\{(v_0, v_1, v_2, a_{01}, a_{12}, a_{02}, b_{012})\}$ correspond to valid triangles in $\mathcal{K}$. Only when $v_0, v_1, v_2, a_{01}, a_{12}, a_{02}, b_{012}$'s satisfy certain conditions, can they label the triangles in $\mathcal{K}$. Using the new set of labels, the tensors that define topological nonlinear σ-model can be rewritten as w_{v_0}, $w_{a_{01}}^{v_0v_1}$, and $C^{a_{01}a_{02}a_{03}a_{12}a_{13}a_{23};b_{012}}_{v_0v_1v_2v_3;b_{023}b_{013}b_{123}}$, where the indices have a smaller range. The new notation is more economical in the sense that the space of each new additional label is smaller than that of old label. For example, the set $\{a_{ij}\}$ is usually smaller than the set $\{l_{ij}\}$ because $\{l_{ij}\}$ can be very roughly understood as the product $\{v_i\}\times\{v_i\}\times\{a_{ij}\}$. See also (36) for an explicit example in the case when the target is a 2-group.

III. DIJKGRAAF-WITTEN GAUGE THEORIES FROM TOPOLOGICAL NONLINEAR σ-MODELS

In this section, we will introduce 1-gauge theories (i.e., Dijkgraaf-Witten gauge theories), as topological nonlinear σ-models. We will show that 1-gauge theories are nothing but a special kind of topological nonlinear σ-models whose target space K is modeled by a special one-vertex complex $\mathcal{K}$ and satisfy $\pi_1(K) = G$, $\pi_{k>1}(K) = 0$. Such a one-vertex complex $\mathcal{K}$ is a simplicial set and is denoted by $\mathcal{B}G$. Similarly n-gauge theories are nothing but a special kind of topological nonlinear σ-models whose target spaces K is modeled by a simplicial set $\mathcal{B}(\pi_1(K), \pi_2(K), \ldots)$ and satisfy $\pi_{k>n}(K) = 0$.

A. Lattice gauge theories from topological nonlinear σ-models

The simplest class of topological nonlinear σ-models has a simple target space $K(G)$, the Eilenberg-MacLane spaces with only nontrivial $\pi_1(K(G)) = G$. For a finite G, $K(G)$ is the classifying space BG. To construct a discrete nonlinear σ-model from the classifying space $BG = K(G)$, we need to choose a triangulation of $BG = K(G)$ which is a simplicial complex. Here we will choose a triangulation that contains only one vertex. The corresponding triangulation is a simplicial set denoted by $\mathcal{B}G$ or $\mathcal{B}(G)$. We will show that for such a one-vertex triangulation, the topological nonlinear σ-model becomes a (Dijkfraaf-Witten) lattice gauge theory, which is also called 1-gauge theory.

The triangulation $\mathcal{B}G = \mathcal{B}(G)$ is obtained in the following way.

(1) There is only one vertex $(\mathcal{B}G)_0 = \{pt\}$ (called the base point) in BG.

(2) The links are the loops starting and ending at the base point. We pick one loop in each homotopic class of loops: $(\mathcal{B}G)_1 = \pi_1(BG)$. Thus the links are labeled by the group elements $a_{ij} \in G$: $(\mathcal{B}G)_1 = G$.

(3) For arbitrary three links a_{01}, a_{12}, a_{02} they may not form the links around a triangle. Only when they satisfy $a_{01}a_{12} = a_{02}$, the composition of the three links is a contractible loop. In this case, there is a triangle t_{012} bounded by the links a_{01}, a_{12}, a_{02}. Note that, for a finite G, $\pi_n(BG) = 0$ for $n \geqslant 2$. Thus all different choices of triangles are homotopy equivalent. Here we just pick a particular one. This gives rise to the set of 2-simplices labeled by the three links a_{01}, a_{12}, a_{02} that satisfy $a_{01}a_{12} = a_{02}$. Thus the set of 2-simplices is $(\mathcal{B}G)_2 = G^{\times 2}$, labeled by a_{01}, a_{12}.

(4) The set of 3-simplices $(\mathcal{B}G)_3$ is obtained by filling all four triangles in a tetrahedron that share their sides in the expected way. Using a similar consideration, we find the set of 3-simplices to be $(\mathcal{B}G)_3 = G^{\times 3}$, labeled by a_{01}, a_{12}, a_{23}.

The sets of higher simplices $(\mathcal{B}G)_n = G^{\times n}$ are obtained in the same way. To summarize, the complex $\mathcal{B}G$ has the following nerve:

$$pt \overset{d_0,d_1}{\Longleftarrow} G \overset{d_0,d_1,d_2}{\Lleftarrow} G^{\times 2} \overset{d_0,\ldots,d_3}{\longleftarrow} G^{\times 3} \overset{d_0,\ldots,d_4}{\longleftarrow} G^{\times 4} \ldots \tag{17}$$

Next, let us determine the set of tensors that satisfy the retriangulation invariance conditions like (15) and (16). We assume the space-time dimension to be $d+1$. For each $d+1$-simplex labeled by $(a_{01}, a_{12}, \ldots, a_{d,d+1})$ in $\mathcal{B}G$, we assign a complex number

$$T_{d+1}(a_{ij}) = w_{d+1}\, e^{i2\pi\bar{\omega}_{d+1}(a_{01},a_{12},\ldots,a_{d,d+1})}, \tag{18}$$

where $\bar{\omega}_{d+1}(a_{01}, a_{12}, \ldots, a_{d,d+1})$ is a $\mathbb{R}/\mathbb{Z}$-valued cocycle on $\mathcal{B}G$: $\bar{\omega}_{d+1} \in H^{d+1}(\mathcal{B}G; \mathbb{R}/\mathbb{Z})$. T is the top tensor in the tensor set $\mathbb{T}$, like the tensor $C^{a_{01}a_{02}a_{03}a_{12}a_{13}a_{23};b_{012}}_{v_0v_1v_2v_3;b_{023}b_{013}b_{123}}$ in Sec. III A. For each n-simplex, $n \leqslant d$, we assign a positive number w_n. w_n's correspond to the weight tensors w_{v_0} and $w_{a_{01}}^{v_0v_1}$ in Sec. III A. The partition function of the corresponding topological nonlinear σ-model is then given by

$$Z = \sum_{\phi}\left[\prod_{n=0}^{d+1}(w_n)^{N_n}\right] e^{i2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar{\omega}_{d+1}}, \tag{19}$$

where N_n is the number of n-simplices in $\mathcal{M}^{d+1}$ and $\sum_\phi$ sums over all the homomorphisms $\phi : \mathcal{M}^{d+1} \to \mathcal{B}G$. Because $\bar{\omega}_{d+1}$ is a cocycle on $\mathcal{B}G$, the term $e^{i2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar{\omega}_{d+1}}$ is independent on how we triangulate the space-time $\mathcal{M}^{d+1}$. However, the term $\sum_\phi\prod_{n=0}^{d+1}(w_n)^{N_n}$ does dependent on the triangulation of $\mathcal{M}^{d+1}$. The idea is to choose the weight tensors w_n to cancel such triangulation dependence.

Let us define two homomorphisms ϕ and ϕ' to be homotopic if there exist a homomorphism $\Phi : I\times\mathcal{M}^{d+1} \to \mathcal{B}G$ such that, when restricted to the two boundaries of $I\times\mathcal{M}^{d+1}$, Φ becomes ϕ and ϕ'. For such two homomorphisms, we have

$$e^{2\pi i\int_{\mathcal{M}^{d+1}}\phi^*\bar{\omega}_{d+1}} = e^{2\pi i\int_{\mathcal{M}^{d+1}}\phi'^*\bar{\omega}_{d+1}} \tag{20}$$

if the space-time $\mathcal{M}^{d+1}$ has no boundary. Such a property is called gauge invariance. Since the phase factor

$e^{2\pi i\int_{\mathcal{M}^{d+1}}\phi^*\bar{\omega}_{d+1}}$ only depends on the homotopic classes $[\phi]$, we can rewrite it as $e^{2\pi i\int_{\mathcal{M}^{d+1}}[\phi]^*\bar{\omega}_{d+1}}$. For two homotopic homomorphisms ϕ and ϕ', their corresponding field configurations a and a' are said to be gauge equivalent.

Let us describe the homotopic classes $[\phi]$ in more detail. First, there is a surjective map

$$\phi \twoheadrightarrow \text{Hom}(\pi_1(\mathcal{M}^{d+1}), G) \tag{21}$$

where $\text{Hom}(\pi_1(\mathcal{M}^{d+1}), G)$ is the set of group homomorphisms. There is another surjective map

$$\text{Hom}(\pi_1(\mathcal{M}^{d+1}), G) \twoheadrightarrow \{[\phi]\}, \tag{22}$$

where $\{[\phi]\}$ is the set of homotopic classes of the simplicial homomorphisms $\mathcal{M}^{d+1} \xrightarrow{\phi} \mathcal{B}G$. Two group homomorphisms $\gamma, \gamma' \in \text{Hom}(\pi_1(\mathcal{M}^{d+1}), G)$ are said to be equivalent if their are related by

$$\gamma = g\gamma' g^{-1}, \quad g \in G. \tag{23}$$

Let $[\gamma]$ be an equivalent class of the group homomorphisms $\text{Hom}(\pi_1(\mathcal{M}^{d+1}), G)$. It turns out that

$$\{[\gamma]\} = \{[\phi]\}, \tag{24}$$

where $\{[\gamma]\}$ is the set of equivalent classes of the group homomorphisms.

Now, $\sum_\phi$ is reduced to a summation over the homotopic classes of the homomorphisms ϕ, $\sum_{[\phi]}$, which is a sum with only a few terms:

$$Z = \sum_{[\phi]}\left[\prod_{n=0}^{d+1}(w_n)^{N_n}\right]N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G)\, e^{2\pi i\int_{\mathcal{M}^{d+1}}[\phi]^*\bar{\omega}_{d+1}}, \tag{25}$$

where $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G)$ is the number of the homomorphisms $\phi : \mathcal{M}^{d+1} \to \mathcal{B}G$ in the homotopic class $[\phi]$. Due to the one-to-one correspondence between $[\phi]$ and $[\gamma]$, we can also write $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G)$ as $N([\gamma], \mathcal{M}^{d+1}, \mathcal{B}G)$. The total number of the homomorphisms ϕ is given by

$$N(\mathcal{M}^{d+1}, \mathcal{B}G) = \sum_{[\phi]} N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G). \tag{26}$$

To count $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G)$, we note that, in the above discrete nonlinear σ-models, the map ϕ sends all vertices in $\mathcal{M}^{d+1}$ (labeled by $i = 0, \dots, N_v - 1$) to the base point pt in $\mathcal{B}G$. The map ϕ sends an link $(ij) \in \mathcal{M}^{d+1}$ to an link $a_{ij} \in \mathcal{B}G$. Thus on each link (ij) of space-time complex $\mathcal{M}^{d+1}$, we have a degree of freedom a_{ij}. Note that if three links in space-time complex, (01), (12), and (02), form the boundary of a triangle (012), then the map ϕ will sends such a triangle to the triangle $t_{012} \in \mathcal{B}G$ bounded by a_{01}, a_{12}, a_{02}. This implies that there is no extra degrees of freedom on the triangles except those that come from the links a_{01}, a_{12}, a_{02}. It also implies that a_{ij} on the three links (ij) satisfy a flat condition:

$$a_{ij}a_{jk} = a_{ik}. \tag{27}$$

This is an example of the conditions discussed above. Using similar considerations, we see that there are no extra degrees of freedom on the 3-simplices and higher simplices. Thus the summation $\sum_\phi$ can be rewritten as $\sum_{a_{ij}}$ where $\sum_{a_{ij}}$ sum over all $a_{ij} \in G$ on link $(ij) \in \mathcal{M}^{d+1}$, so that a_{ij} satisfy the flat condition (27).

Since the set of a_{ij} describes a flat G-gauge connection, we see that $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G)$ is the number gauge equivalent flat G-gauge connections on $\mathcal{M}^{d+1}$. We find that

$$\begin{aligned}N([\phi], \mathcal{M}^{d+1}, \mathcal{B}G) &= N([\gamma], \mathcal{M}^{d+1}, \mathcal{B}G)\\ &= |G|^{N_0}W_{\text{top}}([\gamma], M^{d+1}, \mathcal{B}G),\end{aligned} \tag{28}$$

$$W_{\text{top}}([\gamma], M^{d+1}, \mathcal{B}G) = W_{\text{top}}([\phi], M^{d+1}, \mathcal{B}G) = ||[\gamma]||/|G|,$$

where $|G|$ is the number of the elements in the group G and $||[\gamma]||$ is the number of the elements in the equivalent class $[\gamma]$. Here the factor $|G|^{N_0}$ comes from the numbers of gauge transformations

$$a_{ij} \to g^i a_{ij} g_j^{-1} \tag{29}$$

generated by $g_i \in G$ on each vertex i in $\mathcal{M}^{d+1}$. Also $1/W_{\text{top}}([\phi], M^{d+1}, \mathcal{B}G)$ is the number of gauge transformations that leave a gauge field a (or ϕ) invariant. So $1/W_{\text{top}}([\gamma], M^{d+1}, \mathcal{B}G)$ is given by the number of the elements in the subgroup of G thats leave γ invariant, which is $|G|/|[\gamma]|$. Thus $W_{\text{top}}([\phi], M^{d+1}, \mathcal{B}G)$ is independent of the triangulation on $\mathcal{M}^{d+1}$.

N_0 in $|G|^{N_0}W_{\text{top}}([\phi], M^{d+1}, \mathcal{B}G)$ depends on the triangulation of M^{d+1}. We want to choose w_n to cancel the N_0 dependence, which turns out to be

$$w_0 = |G|^{-1}, \quad \text{other } w_n = 1. \tag{30}$$

In this case, the partition function (19) becomes

$$\begin{aligned}Z &= \sum_{a_{ij}}\left(\prod_i |G|^{-1}\right)e^{i2\pi\int_{\mathcal{M}^{d+1}}\omega_{d+1}(a_{01},a_{12},\dots,a_{d,d+1})}\\ &= \sum_{[\phi]} W_{\text{top}}([\phi], M^{d+1}, \mathcal{B}G)\, e^{i2\pi\int_{\mathcal{M}^{d+1}}[\phi]^*\bar{\omega}_{d+1}},\end{aligned} \tag{31}$$

which is invariant under the retriangulation of space-time M^{d+1}. Such choice of tensors give us a topological nonlinear σ-model.

We see that the topological nonlinear σ-models with $\mathcal{B}G$ as the target complex are classified by the $(d+1)$-cohomology classes $H^{d+1}(\mathcal{B}G; \mathbb{R}/\mathbb{Z})$. When $\omega_{d+1} = 0$, the partition function is given by the equal weight summation of all flat connections a_{ij} on the links of space-time complex, which give rise to a G-gauge theory in the deconfined phase. If we choose a nontrivial cocycle $\omega_{d+1} \in H^{d+1}(\mathcal{B}G; \mathbb{R}/\mathbb{Z})$, then the path integral (31) will gives rise to a Dijkgraaf-Witten lattice gauge theory.

B. Classification of exactly soluble 1-gauge theories

We have seen that by choosing a classifying space $BG = K(G)$ as the target space and choosing a particular triangulation of $K(G)$, $\mathcal{B}(G)$, as the target complex, we obtain the Dijkgraaf-Witten gauge theories for a finite gauge group G. For each finite gauge group G, we only have one corresponding $K(G)$. The different $(d+1)$-cohomology classes $\omega_{d+1} \in H^{d+1}(K(G), \mathbb{R}/\mathbb{Z})$ give rise to different Dijkgraaf-Witten gauge theories. Thus Dijkgraaf-Witten gauge theories (or 1-gauge theories) are classified by pairs (G, ω_{d+1}).

We have seen that Dijkgraaf-Witten gauge theories are topological nonlinear σ-models. It is natural to ask if topological nonlinear σ-models with target complex $\mathcal{B}G$ are Dijkgraaf-Witten gauge theories. In other words, we have shown that the tensor set

$$T_{d+1}(a_{ij}) = e^{i2\pi\bar{\omega}_{d+1}(a_{01},a_{12},\dots,a_{d,d+1})}, \quad w_0 = |G|^{-1} \tag{32}$$

satisfy the retriangulation invariance conditions, such as Eqs. (15) and (16). The question is that if all the solutions of the retriangulation invariance conditions [such as Eqs. (15) and (16)] have the form Eq. (32) as described by a cocycle $\bar{\omega}_{d+1}$. There is another related question: given a triangulation $\mathcal{K}$ of the classifying space BG ($\mathcal{K}$ may not be a simplicial set), are all the topological nonlinear σ-models with target complex $\mathcal{K}$ equivalent to Dijkgraaf-Witten gauge theories (i.e., produce the same topological invariant Z^{top} or produce the same topological order)? We left the questions for future work (see Ref. [54] and references therein for some discussions).

IV. 2-GAUGE THEORIES FROM TOPOLOGICAL NONLINEAR σ-MODELS

In this section, we are going to discuss exactly soluble 2-gauge theories and their classification, from a point of view of topological nonlinear σ-model. We have seen that if the target space K has only nontrivial $\pi_1(K)$, we can get a 1-gauge theory from the topological nonlinear σ-model. If the target space K has only nontrivial $\pi_1(K)$ and $\pi_2(K)$, then we can get a 2-gauge theory.

A. 2-groups

1. Classification of 2-groups

To obtain a 2-gauge theory via a topological nonlinear σ-model, we choose a special triangulation of $K(G,\Pi_2)$, the simplicial set $\mathcal{B}(G,\Pi_2)$, as the target complex. The simplicial set $\mathcal{B}(G,\Pi_2)$ is called a 2-group. The corresponding topological nonlinear σ-model can be a 2-gauge theory. In this section, we concentrate on 2-groups $\mathcal{B}(G,\Pi_2)$, where G is a finite group and Π_2 a finite Abelian group.

The simplicial set $\mathcal{B}(G;\Pi_2)$ (the 2-group) can be viewed as a fiber bundle with $\mathcal{B}(0;\Pi_2) = \mathcal{B}(\Pi_2, 2)$ as the fiber and $\mathcal{B}(G)$ as the base space:

$$\mathcal{B}(\Pi_2, 2) \to \mathcal{B}(G;\Pi_2) \to \mathcal{B}(G). \tag{33}$$

Thus a classification of $\mathcal{B}(G;\Pi_2)$ can be obtain using the following general result.

Lemma IV.1. The simplicial set $\mathcal{B}(\pi_1;\dots;\pi_n)$ has the following fibration:

$$\mathcal{B}(\pi_n, n) \to \mathcal{B}(\pi_1;\dots;\pi_n) \to \mathcal{B}(\pi_1;\dots;\pi_{n-1}),$$

Thus $\mathcal{B}(\pi_1;\dots;\pi_n)$ for fixed π_i's are classified by $H^{n+1}[\mathcal{B}(\pi_1;\dots;\pi_{n-1});\pi_n]$ with local coefficient π_n.

The $n=2$ case was discussed in Ref. [55], Theorem 43.

Using the above result, we find that, for a fixed pair (G,Π_2), the 2-groups $\mathcal{B}(G;\Pi_2)$ are classified by $H^3(\mathcal{B}(G),\Pi_2) = H^3(\mathcal{B}G,\Pi_2^{\alpha_2})$. The local coefficient Π_2 in topological cohomology classes $H^3(\mathcal{B}G,\Pi_2^{\alpha_2})$ means that G may have a nontrivial action on Π_2, which is described by $\alpha_2 : G \to \text{Aut}(\Pi_2)$. Such an action is indicated by the superscript α_2 in $\Pi_2^{\alpha_2}$.

To summarize, 2-groups $\mathcal{B}(G;\Pi_2)$ are classified by the following data:

$$G;\ \Pi_2, \alpha_2, \bar{n}_3 \tag{34}$$

where G is a finite group, Π_2 a finite Abelian group, α_2 a group action $\alpha_2 : G \to \text{Aut}(\Pi_2)$, and $\bar{n}_3(a_{01},a_{12},a_{23})$ is a *group-cocycle* in $\mathcal{H}^3(G,\Pi_2^{\alpha_2})$. The group-cocycle condition that determines $\bar{n}_3(a_{01},a_{12},a_{23})$ is given by

$$\begin{aligned}
0 &= \alpha_2(a_{01})\cdot\bar{n}_3(a_{12},a_{23},a_{34}) - \bar{n}_3(a_{02},a_{23},a_{34})\\
&\quad + \bar{n}_3(a_{01},a_{13},a_{34}) - \bar{n}_3(a_{01},a_{12},a_{24}) + \bar{n}_3(a_{01},a_{12},a_{23})\\
&= \alpha_2(a_{01})\cdot\bar{n}_3(a_{12},a_{23},a_{34}) - \bar{n}_3(a_{01}a_{12},a_{23},a_{34})\\
&\quad + \bar{n}_3(a_{01},a_{12}a_{23},a_{34}) - \bar{n}_3(a_{01},a_{12},a_{23}a_{34})\\
&\quad + \bar{n}_3(a_{01},a_{12},a_{23})
\end{aligned} \tag{35}$$

for all a_{01}, a_{12}, a_{23}, and a_{34}.

2. A description of one-vertex triangulation $\mathcal{B}(G;\Pi_2)$

After knowing how to label all the 2-groups $\mathcal{B}(G;\Pi_2)$ using the data (34), the next important question is to obtain a detailed description of the simplicial set $\mathcal{B}(G;\Pi_2)$ from the classifying data (34). The simplicial set $\mathcal{B}(G;\Pi_2)$ has the following sets of simplices:

$$pt \overset{d_0,d_1}{\Longleftarrow} G \overset{d_0,\dots,d_2}{\Lleftarrow} G^{\times 2}\times\Pi_2 \overset{d_0,\dots,d_3}{\underset{\leftarrow}{\overset{\leftarrow}{\cdot}}} G^{\times 3}\times\Pi_2^{\times 3} \overset{d_0,\dots,d_4}{\underset{\leftarrow}{\overset{\leftarrow}{\cdot}}} G^{\times 4}\times\Pi_2^{\times 6}\cdots \overset{d_0,\dots,d_5}{\underset{\leftarrow}{\overset{\leftarrow}{\cdot}}} G^{\times 5}\times\Pi_2^{\times 10}\dots \tag{36}$$

Let us describe the sets of simplices and the face map d_m in more details. First, there is only one vertex pt in $\mathcal{B}(G;\Pi_2)$. The links in $\mathcal{B}(G;\Pi_2)$ are labeled by elements a_{ij} in G. All the links connect pt to pt, and correspond to noncontractable loops in $\pi(\mathcal{B}(G;\Pi_2)) = G$. Thus the face maps are give by

$$d_0(a_{01}) = pt, \quad d_1(a_{01}) = pt. \tag{37}$$

The boundary map is given by $\partial = d_0 - d_1$. We see that $\partial(a_{01}) = 0$ and the link (a_{01}) is a 1-cycle, for all $a_{01}\in G$.

The composition of two links a_{01} and a_{12} can be deformed into the link a_{02} if and only if

$$a_{01}a_{12} = a_{02}. \tag{38}$$

Thus a_{01}, a_{12}, and a_{02} are boundary of a triangle if and only if $a_{01}a_{12} = a_{02}$.

The G-valued a_{ij} on each link of $\mathcal{B}(G,\Pi_2)$ define a G-valued 1-cochain $\bar{a}$, which is called a canonical 1-cochain.

Using $\bar{a}$, the above condition can be written as

$$\delta\bar{a} \equiv a_{01}a_{12}a_{02}^{-1} = \mathbb{1}. \tag{39}$$

This implies that the canonical 1-cochain $\bar{a}$ is a G-valued 1-cocycle.

When $a_{01}a_{12} = a_{02}$, there may be many triangles with the same boundaries a_{01}, a_{12}, and a_{02}. These triangles are labeled by elements in Π_2. Thus all the triangles are labeled by $(a_{01}, a_{12}, a_{02}; b_{012})$ where a_{ij} satisfy Eq. (38). If we use independent a_{ij}, we find all the triangles are labeled by $[a_{01}, a_{12}; b_{012}]$, which leads to the set of triangles $G^{\times 2}\times\Pi_2$. The face maps are given by

$$\begin{aligned} d_0(a_{01}, a_{12}, a_{02}; b_{012}) &= (a_{12}),\\ d_1(a_{01}, a_{12}, a_{02}; b_{012}) &= (a_{02}),\\ d_2(a_{01}, a_{12}, a_{02}; b_{012}) &= (a_{01}), \end{aligned} \tag{40}$$

which map the triangle to one of its links. From the face maps d_m, we obtain the boundary map ∂:

$$\partial = d_0 - d_1 + d_2. \tag{41}$$

Thus the boundary of triangle $(a_{01}, a_{12}, a_{02}; b_{012})$ is given by

$$\partial(a_{01}, a_{12}, a_{02}; b_{012}) = (a_{12}) - (a_{02}) + (a_{01}). \tag{42}$$

Using the above boundary map, we find that four triangles $-(a_{01}, a_{12}, a_{02}; b_{012})$, $(a_{01}, a_{13}, a_{03}; b_{013})$, $-(a_{02}, a_{23}, a_{03}; b_{023})$, $(a_{12}, a_{23}, a_{13}; b_{123})$, form a 2-cycle since their boundaries cancel each other. Note that a_{ij} in each triangle must satisfy Eq. (38). Otherwise, they will not form triangles. However, the 2-cycle formed by the four triangles may not be the boundary of a tetrahedron in $\mathcal{B}(G; \Pi_2)$. In order to have a tetrahedron in $\mathcal{B}(G; \Pi_2)$ that fill the 2-cycle, b_{ijk}'s must satisfy a condition. In other words, a_{ij}'s and b_{ijk}'s that label the links and triangles in a tetrahedron in $\mathcal{B}(G; \Pi_2)$ must satisfy a condition. Such a condition can be described using the cochain language (see Appendix A) if we introduce a Π_2-valued canonical 2-cochain $\bar{b}$, as defined by the values b_{ijk} on all the triangles of $\mathcal{B}(G; \Pi_2)$. Using $\bar{b}$, the condition on b_{ijk} can be written as

$$d\bar{b} = \bar{n}_3(\bar{a}), \tag{43}$$

So, the canonical 2-cochain $\bar{b}$ may not be a cocycle. Its derivative is given by a function of canonical 1-cocycle $\bar{a}$. When α_2 is trivial, the above have the following explicit expression: a_{ij}'s and b_{ijk}'s that label the links and triangles in a tetrahedron satisfy

$$b_{123} - b_{023} + b_{013} - b_{012} = \bar{n}_3(a_{01}, a_{12}, a_{23}). \tag{44}$$

When α_2 is nontrivial, $d\bar{b} = \bar{n}_3(\bar{a})$ becomes

$$\alpha_2(a_{01}) \cdot b_{123} - b_{023} + b_{013} - b_{012} = \bar{n}_3(a_{01}, a_{12}, a_{23}). \tag{45}$$

We see that the tetrahedrons in $\mathcal{B}(G; \Pi_2)$ are labeled by $(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})$ that satisfy Eqs. (38) and (45). In other words, the tetrahedrons in $\mathcal{B}(G; \Pi_2)$ are labeled by independent indices $[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}]$. These tetrahedrons form the set $G^{\times 3}\times\Pi_2^{\times 3}$ in Eq. (36).

The face maps d_m's on tetrahedrons are given by

$$\begin{aligned} &d_0(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})\\ &\quad= (a_{12}, a_{23}, a_{13}; b_{123}),\\ &d_1(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})\\ &\quad= (a_{02}, a_{23}, a_{03}; b_{023}),\\ &d_2(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})\\ &\quad= (a_{01}, a_{13}, a_{03}; b_{013}),\\ &d_3(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})\\ &\quad= (a_{01}, a_{12}, a_{02}; b_{012}). \end{aligned} \tag{46}$$

Let us introduce $s[01]$ to describe the link (a_{01}), $s[012]$ the triangle $(a_{01}, a_{12}, a_{02}; b_{012})$, $s[0123]$ the tetrahedron $(a_{01}, a_{12}, a_{23}, a_{02}, a_{13}, a_{03}; b_{012}, b_{023}, b_{013}, b_{123})$, etc. Then, the above expression can be put in a more compact form

$$\begin{aligned} d_0 s[0123] &= s[123], \quad d_1 s[0123] = s[023],\\ d_2 s[0123] &= s[013], \quad d_3 s[0123] = s[012]. \end{aligned} \tag{47}$$

Using independent labels, Eq. (46) can be rewritten as

$$\begin{aligned} &d_0[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}] = [a_{12}, a_{23}; b_{123}] = \left[a_{12}, a_{23}; \alpha_2^{-1}(a_{01}) \cdot (b_{023} - b_{013} + b_{012} + \bar{n}_3(a_{01}, a_{12}, a_{23}))\right],\\ &d_1[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}] = [a_{02}, a_{23}; b_{023}], \quad d_2[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}] = [a_{01}, a_{13}; b_{013}],\\ &d_3[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}] = [a_{01}, a_{12}; b_{012}]. \end{aligned} \tag{48}$$

The boundary map ∂ for tetrahedron is given by

$$\partial = d_0 - d_1 + d_2 - d_3. \tag{49}$$

Thus

$$\begin{aligned} &\partial[a_{01}, a_{12}, a_{23}; b_{012}, b_{023}, b_{013}]\\ &\quad= [a_{12}, a_{23}; \alpha_2^{-1}(a_{01}) \cdot (b_{023} - b_{013} + b_{012}\\ &\qquad + \bar{n}_3(a_{01}, a_{12}, a_{23}))] - [a_{02}, a_{23}; b_{023}]\\ &\qquad + [a_{01}, a_{13}; b_{013}] - [a_{01}, a_{12}; b_{012}]. \end{aligned} \tag{50}$$

In general, the n-simplices in $G^{\times n}\times\Pi_2^{\binom{n}{2}}$ are labeled by (a_{ij}, b_{klm}), $i < j$, $k < l < m$ $i, j, k, l, m = 0, 1, \dots, n$, that satisfy the conditions (38) (after replacing 012 by $i<j<k$) and Eq. (45) (after replacing 0123 by $i < j < k < l$). We see that all the a_{ij}'s are determined by the independent $a_{01}, a_{12}, \dots, a_{n-1,n}$. Similarly, all the b_{ijk}'s are given by an independent subset of b_{ijk}'s. Such independent subset is obtained by picking $i = 0$, and $j < k$.

Using the labeling scheme (a_{ij}, b_{ijk}), $i, j, k = 0, 1, \dots, n$, where a_{ij}, b_{ijk} satisfy Eqs. (38) and (45), we can obtain a simple description of the face map d_m in Eq. (36) that sends an n-simplex to an $(n-1)$-simplex. To describe the action of d_m, we start with an n-simplex (a_{ij}, b_{ijk}). The resulting $(n-1)$-simplex is obtained by dropping all in a_{ij}, b_{ijk} in the set (a_{ij}, b_{ijk}) that contain the vertex m. This changes (a_{ij}, b_{ijk}) to its subset which is written as

$$d_m(a_{ij}, b_{ijk}|0 \leqslant i, j, k \leqslant n) = (a_{ij}, b_{ijk}|i, j, k \neq m). \quad (51)$$

a_{ij}, b_{ijk} in the subset also satisfy Eq. (38) and (45). The subset $d_m(a_{ij}, b_{ijk})$ describes the resulting $(n-1)$-simplex after the d_m map. We see that the explicit expression for $d_m(a_{ij}, b_{ijk}|0 \leqslant i, j, k \leqslant n)$ is simple to construct using non-independent a_{ij}, b_{ijk}'s.

3. A trivialization

We have mentioned that $\bar{n}_3$ in $d\bar{b} = \bar{n}_3(\bar{a})$ is a group-cocycle in $\mathcal{H}^3(G;\Pi_2)$. Such a group cocycle correspond to a topological cocycle $\tilde{n}_3$ on space $\mathcal{B}G$: $\tilde{n}_3 \in H^3(\mathcal{B}G;\Pi_2)$. We may also view $\bar{n}_3(\bar{a})$ as a function of $\bar{a}$ and as a topological cocycle on $\mathcal{B}(G,\Pi_2)$.

We note that there is unique complex homomorphism $\varphi : \mathcal{B}(G,\Pi_2) \to \mathcal{B}G$, which sends the triangle $(a_{01}, a_{12}, a_{02}; b_{012})$ in $\mathcal{B}(G,\Pi_2)$ to the triangle (a_{01}, a_{12}, a_{02}) in $\mathcal{B}G$. Then, we may view $\bar{n}_3(\bar{a})$ on $\mathcal{B}(G,\Pi_2)$ as a pullback of $\tilde{n}_3$ on $\mathcal{B}G$ by the homomorphism φ:

$$\bar{n}_3 = \varphi^*\tilde{n}_3. \quad (52)$$

We note that although $\tilde{n}_3$ is a nontrivial cocycle on $\mathcal{B}G$, its pullback $\bar{n}_3 = \varphi^*\tilde{n}_3$ is always a coboundary on $\mathcal{B}(G,\Pi_2)$: $d\bar{b} = \bar{n}_3(\bar{a})$. In other words, given a Π_2-valued 3-cocycle $\tilde{n}_3$ on $\mathcal{B}G$, let $\mathcal{B}(G,\Pi_2) \xrightarrow{\varphi} \mathcal{B}G$ be the fibration corresponding to $\tilde{n}_3$, which always exists as stated in Lemma IV.1. Then $\varphi^*\tilde{n}_3$ is a coboundary on $\mathcal{B}(G,\Pi_2)$.

The above result can be generalized: given a Π_{m+1}-valued $(m+2)$-cocycle $\tilde{n}_{m+2}$ on $\mathcal{B}(G,\dots,\Pi_m)$, let $\mathcal{B}(G,\dots,\Pi_m,\Pi_{m+1}) \xrightarrow{\varphi} \mathcal{B}(G,\dots,\Pi_m)$ be the fibration corresponding to $\tilde{n}_{m+2}$ as stated in Lemma IV.1. Then $\varphi^*\tilde{n}_{m+2}$ is a coboundary on $\mathcal{B}(G,\dots,\Pi_m,\Pi_{m+1})$: $d\bar{b}_{m+1} = \varphi^*\tilde{n}_{m+2}$.

B. 2-gauge theories

To define a d+1D topological nonlinear σ-model (we will assume $d \geqslant 2$ since there is no 2-gauge theory in 1+1D), we need to specify the tensor set $\mathbb{T}$. To do so, for each $d+1$-simplex labeled by (a_{ij}, b_{ijk}) in $\mathcal{B}(G,\Pi_2)$ we assign a complex number

$$T_{d+1}(a_{ij}, b_{ijk}) = w_{d+1} e^{i2\pi\bar{\omega}_{d+1}(a_{ij},b_{ijk})}, \quad (53)$$

where $\bar{\omega}_{d+1}(a_{ij}, b_{ijk})$ is a $\mathbb{R}/\mathbb{Z}$-valued cocycle on $\mathcal{B}(G,\Pi_2)$: $\bar{\omega}_{d+1} \in H^{d+1}(\mathcal{B}(G,\Pi_2);\mathbb{R}/\mathbb{Z})$. T is the top tensor in the tensor set $\mathbb{T}$. For each n-simplex in $\mathcal{B}(G,\Pi_2)$, $n \leqslant d$, we assign a positive number w_n, which correspond to the weight tensors in the tensor set. The partition function of the corresponding topological nonlinear σ-model is then given by

$$Z = \sum_{\phi}\left[\prod_{n=0}^{d+1}(w_n)^{N_n}\right] e^{i2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar{\omega}_{d+1}} \quad (54)$$

where N_n is the number of n-simplices in $\mathcal{M}^{d+1}$ and $\sum_\phi$ sums over all the homomorphisms $\phi : \mathcal{M}^{d+1} \to \mathcal{B}(G,\Pi_2)$.

The pullbacks of the canonical cochains $\bar{a}$ and $\bar{b}$ on $\mathcal{B}(G,\Pi_2)$ by the homomorphisms ϕ give rise to cochains a and b on $\mathcal{M}^{d+1}$:

$$a = \phi^*\bar{a}, \quad b = \phi^*\bar{b}. \quad (55)$$

a and b are referred as gauge field and rank-2 gauge field in physics, which satisfy

$$\delta a = \mathbb{1}, \quad db = n_3(a). \quad (56)$$

In fact there is a one-to-one correspondence between the allowed field configurations a and b and the homomorphisms. Thus we can replace $\sum_\phi$ and $\sum_{a,b}$:

$$Z = \sum_{a,b}\left[\prod_{n=0}^{d+1}(w_n)^{N_n}\right] e^{i2\pi\int_{\mathcal{M}^{d+1}}\omega_{d+1}(a,b)}. \quad (57)$$

As shown in Eq. (20), homotopic homomorphisms ϕ's give rise to the same action amplitude $e^{2\pi\, i\int_{\mathcal{M}^4}\phi^*\bar{\omega}_{d+1}}$. Thus the partition function can be written as

$$Z = \sum_{[\phi]}\left[\prod_{n=0}^{d+1}(w_n)^{N_n}\right] N([\phi], \mathcal{M}^{d+1}, \mathcal{B}(G,\Pi_2)) \times e^{i2\pi\int_{\mathcal{M}^{d+1}}[\phi]^*\bar{\omega}_{d+1}}, \quad (58)$$

where $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}(G,\Pi_2))$ is the number of homomorphisms $\phi : \mathcal{M}^{d+1} \to \mathcal{B}(G,\Pi_2)$ in the homotopic class $[\phi]$.

Let two field configurations $a_{ij}, b_{ijk}, \dots$ and $a'_{ij}, b'_{ijk}, \dots$ on $\mathcal{M}^{d+1}$ come from two homotopic homomorphisms ϕ and ϕ'. Thus the two field configurations have the same the action amplitude $e^{2\pi\, i\int_{\mathcal{M}^4}\phi^*\bar{\omega}_{d+1}}$. We say that the two configurations differ by a gauge transformation.

The gauge equivalent field configurations are generated by two kinds of gauge transformations. The first one is generated by g_i on each vertex

$$a_{ij} \to a'_{ij} = g_i a_{ij} g_j^{-1},$$
$$b_{ijk} \to b'_{ijk} = b_{ijk} + \zeta_2(a_{ij}, a_{jk}, g_i, g_j, g_k), \quad (59)$$

where $\zeta_2(a_{ij}, a_{jk}, g_i, g_j, g_k)$ is a Π_2-valued function that satisfy

$$\begin{aligned}(d\zeta_2)&(a_{ij}, a_{jk}, a_{kl}, g_i, g_j, g_k, g_l)\\ &= -\zeta_2(a_{ij}, a_{jk}, g_i, g_j, g_k) + \zeta_2(a_{ik}, a_{kl}, g_i, g_k, g_l)\\ &\quad - \zeta_2(a_{ij}, a_{jl}, g_i, g_j, g_l) + \alpha_2(g_{ij})\cdot\zeta_2(a_{jk}, a_{kl}, g_j, g_k, g_l)\\ &= n_3\big(g_i a_{ij} g_j^{-1}, g_j a_{jk} g_k^{-1}, g_k a_{kl} g_l^{-1}\big) - n_3(a_{ij}, a_{jk}, a_{ik}).\end{aligned} \quad (60)$$

Since n_3 is a cocycle, the above equation always has a solution. The second one is generated by Π_2-valued λ_{ij} on each link

$$a_{ij} \to a'_{ij} = a_{ij},$$
$$b_{ijk} \to b'_{ijk} = b_{ijk} + \lambda_{ij} - \lambda_{ik} + \alpha_2(g_{ij})\cdot\lambda_{jk}. \quad (61)$$

Equations (59) and (61) generate the 2-gauge transformations. The action amplitude $e^{2\pi\, i\int_{\mathcal{M}^4}\omega_{d+1}(a,b)}$ is invariant under the 2-gauge transformations.

Since $N([\phi], \mathcal{M}^{d+1}, \mathcal{B}(G,\Pi_2))$ counts 2-gauge equivalent field configurations, from the above form of 2-gauge transformations, we see that

$$N([\phi],\mathcal{M}^{d+1},\mathcal{B}(G,\Pi_2)) = |G|^{N_0}|\Pi_2|^{N_1}W_{\text{top}}([\phi],M^{d+1},\mathcal{B}(G,\Pi_2)). \quad (62)$$

To cancel the triangulation dependence N_0 and N_1, we choose the weight tensors to be

$$w_0 = |G|^{-1}, \quad w_1 = |\Pi_2|^{-1}, \quad \text{other } w_n = 1. \quad (63)$$

Such choice of top and weight tensors, (53) and (63), give rise to a topological nonlinear σ-model which is a 2-gauge theory.

We like to remark that Eqs. (53) and (63) represent one class of the solutions to the retriangulation invariance conditions [like Eqs. (15) and (16)]. It is not clear if Eqs. (53) and (63) represent all the solutions to the retriangulation invariance conditions. In other words, it is not clear if topological nonlinear σ-models with target complex $\mathcal{B}(G,\Pi_2)$ are always 2-gauge theories described by [see Eq. (57)]

$$Z = \sum_{a,b}\left(\prod_i |G|^{-1}\prod_{(ij)}|\Pi_2|^{-1}\right)e^{i2\pi\int_{\mathcal{M}^{d+1}}\omega_{d+1}(a,b)} = \sum_{[\phi]} W_{\text{top}}([\phi],M^{d+1},\mathcal{B}(G,\Pi_2))e^{i2\pi\int_{\mathcal{M}^{d+1}}[\phi]^*\bar\omega_{d+1}}. \quad (64)$$

Since the data $(G;\ \Pi_2,\alpha_2,\bar n_3)$ classify the 2-groups, the d+1D 2-gauge theories are then classified by the following data:

$$G;\ \Pi_2,\alpha_2,\bar n_3;\ \bar\omega_{d+1}, \quad (65)$$

where $\bar\omega_{d+1}\in H^{d+1}(\mathcal{B}(G,\Pi_2),\mathbb{R}/\mathbb{Z})$. Using the above data, we can construct a 2-gauge theory Eq. (64).

C. 2-group cocycles

$\bar\omega_{d+1}$ in Eq. (64) is called a 2-group cocycle. In the following, we give an explicit description of 2-group cocycles, based on the discussion in Sec. IV A. First, a d+1D 2-group cochain $\bar\omega_{d+1}$ with value $\mathbb{M}$ is a function $\bar\omega_{d+1}: G^{\times d}\times\Pi_2^{\binom{d}{2}}\to\mathbb{M}$. Then we can define the differential operator d acting on the 2-group cochains as the following [see Eq. (47) or (48)]:

$$(d\bar\omega_{d+1})(s[0\dots d+1]) = \sum_{m=0}^{d+1}(-)^m\bar\omega_{d+1}(s[1\dots\hat m\dots d{+}1]). \quad (66)$$

In each dimension, we obtain

$$(d\bar\omega_0)(a_{01}) = 0, \quad (67)$$

$$(d\bar\omega_1)(a_{01},a_{12},b_{012}) = \bar\omega_1(a_{01}) - \bar\omega_1(a_{02}) + \bar\omega_1(a_{12}), \quad (68)$$

$$(d\bar\omega_2)(a_{01},a_{12},a_{23},b_{012},b_{013},b_{023}) = -\bar\omega_2(a_{01},a_{12},b_{012}) + \bar\omega_2(a_{01},a_{13},b_{013}) - \bar\omega_2(a_{02},a_{23},b_{023}) + \bar\omega_2(a_{12},a_{23},b_{123}), \quad (69)$$

$$\begin{aligned}(d\bar\omega_3)&(a_{01},a_{12},a_{23},a_{34},b_{012},b_{013},\ b_{014},b_{023},b_{024},b_{034})\\ &= +\bar\omega_3(a_{01},a_{12},a_{23},b_{012},b_{013},b_{023}) - \bar\omega_3(a_{01},a_{12},a_{24},b_{012},b_{014},b_{024}) + \bar\omega_3(a_{01},a_{13},a_{34},b_{013},b_{014},b_{034})\\ &\quad - \bar\omega_3(a_{02},a_{23},a_{34},b_{023},b_{024},b_{034}) + \bar\omega_3(a_{12},a_{23},a_{34},b_{123},b_{124},b_{134}),\end{aligned} \quad (70)$$

$$\begin{aligned}(d\bar\omega_4)&(a_{01},a_{12},a_{23},a_{34},a_{45},b_{012},b_{013},b_{014},b_{015},b_{023},b_{024},b_{025},b_{034},b_{035},b_{045})\\ &= -\bar\omega_4(a_{01},a_{12},a_{23},a_{34},b_{012},b_{013},b_{014},b_{023},b_{024},b_{034}) + \bar\omega_4(a_{01},a_{12},a_{23},a_{35},b_{012},b_{013},b_{015},b_{023},b_{025},b_{035})\\ &\quad -\bar\omega_4(a_{01},a_{12},a_{24},a_{45},b_{012},b_{014},b_{015},b_{024},b_{025},b_{045}) + \bar\omega_4(a_{01},a_{13},a_{34},a_{45},b_{013},b_{014},b_{015},b_{034},b_{035},b_{045})\\ &\quad -\bar\omega_4(a_{02},a_{23},a_{34},a_{45},b_{023},b_{024},b_{025},b_{034},b_{035},b_{045}) + \bar\omega_4(a_{12},a_{23},a_{34},a_{45},b_{123},b_{124},b_{125},b_{134},b_{135},b_{145}),\end{aligned} \quad (71)$$

$$\begin{aligned}(d\bar\omega_5)&(a_{01},a_{12},a_{23},a_{34},a_{45},a_{56},b_{012},b_{013},b_{014},b_{015},b_{016},b_{023},b_{024},b_{025},b_{026},b_{034},b_{035},b_{036},b_{045},b_{046},b_{056})\\ &= +\bar\omega_5(a_{01},a_{12},a_{23},a_{34},a_{45},b_{012},b_{013},b_{014},b_{015},b_{023},b_{024},b_{025},b_{034},b_{035},b_{045})\\ &\quad -\bar\omega_5(a_{01},a_{12},a_{23},a_{34},a_{46},b_{012},b_{013},b_{014},b_{016},b_{023},b_{024},b_{026},b_{034},b_{036},b_{046})\\ &\quad +\bar\omega_5(a_{01},a_{12},a_{23},a_{35},a_{56},b_{012},b_{013},b_{015},b_{016},b_{023},b_{025},b_{026},b_{035},b_{036},b_{056})\\ &\quad -\bar\omega_5(a_{01},a_{12},a_{24},a_{45},a_{56},b_{012},b_{014},b_{015},b_{016},b_{024},b_{025},b_{026},b_{045},b_{046},b_{056})\\ &\quad +\bar\omega_5(a_{01},a_{13},a_{34},a_{45},a_{56},b_{013},b_{014},b_{015},b_{016},b_{034},b_{035},b_{036},b_{045},b_{046},b_{056})\\ &\quad -\bar\omega_5(a_{02},a_{23},a_{34},a_{45},a_{56},b_{023},b_{024},b_{025},b_{026},b_{034},b_{035},b_{036},b_{045},b_{046},b_{056})\\ &\quad +\bar\omega_5(a_{12},a_{23},a_{34},a_{45},a_{56},b_{123},b_{124},b_{125},b_{126},b_{134},b_{135},b_{136},b_{145},b_{146},b_{156}).\end{aligned} \quad (72)$$

In the above, the variables a_{ij} with $j-i>1$ and b_{ijk} with $i\neq 0$ do not appear on the left-hand side of the equation but appear on the right-hand side of the equation. In fact, these a_{ij} and b_{ijk} are given by $a_{i,i+1}$'s and b_{0mn}'s that do appear on the left-hand side of the equation:

$$\begin{aligned}a_{ij} &= a_{i,i+1}\dots a_{j-1,j}, \quad \text{if } j-i\geqslant 2,\\ b_{ijk} &= \alpha_2^{-1}(a_{01})\cdot[b_{0jk} - b_{0ik} + b_{0ij} + \bar n_3(a_{0i},a_{ij},a_{jk})].\end{aligned} \quad (73)$$

So the above are conditions on the functions of $a_{i,i+1}$'s and b_{0mn}'s.

With the above definition of d operator, we can define the 2-group cocycles as the 2-group cochains that satisfy $d\bar{\omega}_{d+1} = 0$. This generalizes the notion of group cocycle to 2-group cocycle. Two different 2-group cocycles $\bar{\omega}_{d+1}$ and $\bar{\omega}'_{d+1}$ are equivalent if they differ by a 2-group coboundary $d\bar{\nu}_d$. The set of equivalent classes of d+1D 2-group cocycles is denoted as $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{M})$.

D. Cohomology of 2-group

One way to understand the structure of $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{M})$ is to use the fibration $\mathcal{B}(\Pi_2,2)\to\mathcal{B}(G_b;\Pi_2)\to\mathcal{B}G_b$ [see Eq. (33), and use spectral sequence to reduce the cohomology of $\mathcal{B}(G_b;\Pi_2)$ to cohomology groups of G_b and $\mathcal{B}(\Pi_2,2)$]. In particular, from Appendix B, we see that every element in $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{R}/\mathbb{Z})$ can be labeled by $(k_0,k_1,\dots,k_d)$ where $k_l\in H^l[\mathcal{B}G_b, H^{d+1-l}(\mathcal{B}(\Pi_2,2);\mathbb{R}/\mathbb{Z})_G]$, although some $(k_0,k_1,\dots,k_d)$'s may not correspond to any elements in $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{R}/\mathbb{Z})$, and some different $(k_0,k_1,\dots,k_d)$'s may correspond to the same element in $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{R}/\mathbb{Z})$. [When $\mathcal{B}(G_b;\Pi_2)=\mathcal{B}(\Pi_2,2)\times\mathcal{B}G_b$, $(k_0,k_1,\dots,k_d)$ will be the one-to-one label of all the elements in $H^{d+1}(\mathcal{B}(G_b;\Pi_2),\mathbb{R}/\mathbb{Z})$.]

Next, let us concentrate on a special case of $\Pi_2=\mathbb{Z}_2$, and try to compute $H^{d+1}(\mathcal{B}(G_b;\mathbb{Z}_2),\mathbb{R}/\mathbb{Z})$. Since $\mathbb{Z}_2$ group has no nontrivial automorphism, α_2 is always trivial. However, $\bar{n}_3\in H^3(\mathcal{B}G_b;\mathbb{Z}_2)$ is in general nontrivial. Thus a 2-group $\mathcal{B}(G_b;\mathbb{Z}_2)$ is characterized by a pair $G,\bar{n}_3$. The cohomology $H^*(\mathcal{B}(\mathbb{Z}_2,2),\mathbb{Z})$ is given by [56]

$$\begin{array}{ccccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7, \\ H^d(\mathcal{B}(\mathbb{Z}_2,2),\mathbb{Z}): & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2. \end{array} \quad (74)$$

Using the universal coefficient theorem

$$H^n(X,\mathbb{M})\simeq H^n(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{M}\oplus\mathrm{Tor}(H^{n+1}(X;\mathbb{Z}),\mathbb{M}) \quad (75)$$

and $\mathbb{Z}_n\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}=0$, $\mathrm{Tor}(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_n$, we find that $H^n(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})=H^{n+1}(\mathcal{B}(\mathbb{Z}_2,2),\mathbb{Z})$:

$$\begin{array}{cccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6, \\ H^d(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z}): & \mathbb{R}/\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2. \end{array} \quad (76)$$

Using the above result, we find that $H^4(\mathcal{B}(G_b;\mathbb{Z}_2),\mathbb{R}/\mathbb{Z})$ can be labeled by

$$\begin{aligned} H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z}) &= \mathbb{Z}_4=\{\bar{k}_0\},\\ H^1[\mathcal{B}G_b;H^3(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})] &= \{0\},\\ H^2[\mathcal{B}G_b;H^2(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})] &= H^2(\mathcal{B}G_b;\mathbb{Z}_2)=\{k_2\},\\ H^3[\mathcal{B}G_b;H^1(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})] &= \{0\},\\ H^4[\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})] &= \{k_4\}. \end{aligned} \quad (77)$$

Since 2-gauge theories in 3+1D are classified pairs $(\bar{n}_3,\bar{\omega}_4)$, $\omega_4\in H^4(\mathcal{B}(G_b;\mathbb{Z}_2),\mathbb{R}/\mathbb{Z})\}$, we find that each 3+1D 2-gauge theory corresponds to one or more elements in a subset of

$$H^3[\mathcal{B}G_b;\mathbb{Z}_2)]\times H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z}) \times H^2[\mathcal{B}G_b;\mathbb{Z}_2)]\times H^4[\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})]. \quad (78)$$

The first H comes from $\bar{n}_3$ and the rest H's from $\bar{\omega}_4$.

If the index $\bar{k}_0\in H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})=\mathbb{Z}_4$ is $\bar{k}_0=2$, the 2-gauge theory has emergent fermions. The index k_2 in $H^2(\mathcal{B}G_b;\mathbb{Z}_2)$ describes the extension of G_b by $\mathbb{Z}_2$ to obtain G_f. $s\mathcal{R}ep(G_f)$ describes the particle-like excitations in the 2-gauge theory. For details, see Sec. VI.

V. PURE 2-GAUGE THEORY OF 2-GAUGE-GROUP $\mathcal{B}(\Pi_2,2)$

In the last section, we discuss some general properties of 2-gauge theory. In this section, we are going to discuss a special 2-gauge theory, *pure 2-gauge theory*.

A. Pure 2-group and pure 2-gauge theory

If we choose the target complex of the topological nonlinear σ-model to be $\mathcal{B}(0;\Pi_2)=\mathcal{B}(\Pi_2,2)$, we will get a pure 2-gauge theory of 2-gauge-group $\mathcal{B}(\Pi_2,2)$, where Π_2 is a finite Abelian group. There is only one complex of $\mathcal{B}(\Pi_2,2)$-type. The complex $\mathcal{B}(\Pi_2,2)$ has a structure

$$pt \xleftarrow{d_0,d_1} pt \xleftarrow{d_0,d_1,d_2} \Pi_2 \xleftarrow{d_0,\dots,d_3} \Pi_2^{\times 3} \xleftarrow{d_0,\dots,d_4} \Pi_2^{\times 6} \xleftarrow{d_0,\dots,d_5} \Pi_2^{\times 10}\dots \quad (79)$$

In this case, $\bar{n}_3=0$, α_2 is trivial, and b_{ijk} satisfy

$$b_{123}-b_{023}+b_{013}-b_{012}=0. \quad (80)$$

We see that canonical 2-cochain $\bar{b}$ is a Π_2-valued 2-cocycle on target complex $\mathcal{B}(\Pi_2,2)$. The action of d on the cochains in $\mathcal{B}(\Pi_2,2)$ are given by

$$(d\omega_0)()=0, \quad (81)$$

$$(d\omega_1)(b_{012})=\omega_1(), \quad (82)$$

$$(d\omega_2)(b_{012},b_{013},b_{023})=-\omega_2(b_{012})+\omega_2(b_{013})-\omega_2(b_{023})+\omega_2(b_{123}), \quad (83)$$

$$\begin{aligned}&(d\omega_3)(b_{012},b_{013},b_{014},b_{023},b_{024},b_{034})\\ &=\omega_3(b_{012},b_{013},b_{023})-\omega_3(b_{012},b_{014},b_{024})+\omega_3(b_{013},b_{014},b_{034})-\omega_3(b_{023},b_{024},b_{034})+\omega_3(b_{123},b_{124},b_{134}),\end{aligned} \quad (84)$$

$$(d\omega_4)(b_{012},b_{013},b_{014},b_{015},b_{023},b_{024},b_{025},b_{034},b_{035},b_{045})$$
$$= -\omega_4(b_{013},b_{014},b_{023},b_{024},b_{034}) + \omega_4(b_{012},b_{013},b_{015},b_{023},b_{025},b_{035}) - \omega_4(b_{012},b_{014},b_{015},b_{024},b_{025},b_{045})$$
$$+ \omega_4(b_{013},b_{014},b_{015},b_{034},b_{035},b_{045}) - \omega_4(b_{023},b_{024},b_{025},b_{034},b_{035},b_{045}) + \omega_4(b_{123},b_{124},b_{125},b_{134},b_{135},b_{145}). \quad (85)$$

In the above, the variables b_{ijk} for $i \neq 0$ do not appear on the left-hand side of the equation, but appear on the right-hand side of the equation. In fact, these b_{ijk} are given by b_{0mn}'s that do appear on the left-hand side of the equation:

$$b_{ijk} = b_{0jk} - b_{0ik} + b_{0ij}. \quad (86)$$

So the above are the conditions on functions of b_{0mn}'s.

Clearly,

$$H^0(\mathcal{B}(\Pi_2,2);\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}, \quad H^1(\mathcal{B}(\Pi_2,2);\mathbb{R}/\mathbb{Z}) = 0. \quad (87)$$

From Eq. (83), we see that, for $\Pi_2 = \mathbb{Z}_n$, a 2-group 2-cocycle has a form

$$(\omega_2)_{ijk} = \frac{m}{n} b_{ijk} + c, \quad m = 0,\dots,n-1, \quad (88)$$

The constant term c is a coboundary. Thus $H^2(\mathcal{B}(\mathbb{Z}_n,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$. This allows us to show that for a finite Π_2

$$H^2(\mathcal{B}(\Pi_2,2);\mathbb{R}/\mathbb{Z}) = \Pi_2, \quad (89)$$

which agrees with $H^2(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ [see Eq. (76)].

To compute $H^4(\mathcal{B}(\Pi_2,2);\mathbb{R}/\mathbb{Z})$, let us first assume $\Pi_2 = \mathbb{Z}_2$. From Eq. (76), we see that $H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_4$. One of the four-dimensional 2-group cocycle is given by

$$\omega_4(b) = \tfrac{1}{2} b^2. \quad (90)$$

We note that $2\omega_4 \overset{1}{=} 0$. Thus ω_4 only generate $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4 = H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})$.

To obtain the generator of $H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})$, we note that, if we view b as $\mathbb{Z}$-valued 2-cochain, we have $db = 2c$ where c is a $\mathbb{Z}$-valued 3-cochain. Then, from Eqs. (A19) and (A20), we see that

$$d\mathrm{Sq}^2 b = \mathrm{Sq}^2 db + 2\mathrm{Sq}^3 b = 4(c \underset{1}{\smile} c + bc). \quad (91)$$

Thus

$$\omega_4(b) = \tfrac{1}{4}\mathrm{Sq}^2 b \quad (92)$$

is a $\mathbb{R}/\mathbb{Z}$-valued 4-cocycle: $d\omega_4(b) \overset{1}{=} 0$. Such a ω_4 generates the full group $\mathbb{Z}_4 = H^4(\mathcal{B}(\mathbb{Z}_2,2);\mathbb{R}/\mathbb{Z})$.

In general, if b is a $\mathbb{Z}_n$-valued 2-cocycle, we have $db = nc$ where c is a $\mathbb{Z}$-valued 3-cochain. From Eq. (A17), we see that

$$d\mathrm{Sq}^2 b = \mathrm{Sq}^2 db + 2\mathrm{Sq}^3 b = n^2 c \underset{1}{\smile} c + 2nbc. \quad (93)$$

This result tells us that when $n =$ odd,

$$\omega_4(b) = \frac{1}{n}\mathrm{Sq}^2 b \quad (94)$$

is a $\mathbb{R}/\mathbb{Z}$-valued 4-cocycle, while when $n =$ even

$$\omega_4(b) = \frac{1}{2n}\mathrm{Sq}^2 b \quad (95)$$

is a $\mathbb{R}/\mathbb{Z}$-valued 4-cocycle. $\omega_4(b)$ generates a $\mathbb{Z}_n$ group when $n =$ odd, and a $\mathbb{Z}_{2n}$ group when $n =$ even. This suggests that $H^4(\mathcal{B}(\mathbb{Z}_n,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ when $n =$ odd, and $H^4(\mathcal{B}(\mathbb{Z}_n,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{2n}$ when $n =$ even.

B. Pure 2-gauge theory in 3+1D

1. n = odd case

We see that, when $n =$ odd, we have n different 3+1D $\mathcal{B}(\mathbb{Z}_n,2)$ 2-gauge theories, described by partition function

$$Z(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),k) = \sum_{db^{\mathbb{Z}_n} \overset{n}{=} 0} e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n}(b^{\mathbb{Z}_n})^2} \quad (96)$$

where $k = 0,1,\dots,n-1$ and $b^{\mathbb{Z}_n}$ is a $\mathbb{Z}_n$-value 2-cocycle. Clearly, the action amplitude $e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{n}(b^{\mathbb{Z}_n})^2}$ is invariant under the 2-gauge transformation $b^{\mathbb{Z}_n} \to b^{\mathbb{Z}_n} + d\lambda$. The above 2-gauge theory was studied in Ref. [57]. It was found that the theory realizes a 3+1D $Z_{\langle 2k,n\rangle}$-gauge theory. It is an untwist $Z_{\langle 2k,n\rangle}$-gauge theory since $2kn/\langle 2k,n\rangle^2$ is always even.

2. n = even case

When $n =$ even, we have $2n$ different 3+1D $\mathcal{B}(Z_n,2)$ 2-gauge theories, described by partition function

$$Z(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),m) = \sum_{db^{\mathbb{Z}_n} \overset{n}{=} 0} e^{2\pi i \int_{\mathcal{M}^4} \frac{m}{2n}\mathrm{Sq}^2 b^{\mathbb{Z}_n}} \quad (97)$$

where $m = 0,1,\dots,2n-1$. Noticing that the $\mathbb{Z}_n$-valued 2-cocycle $b^{\mathbb{Z}_n}$ satisfies $db^{\mathbb{Z}_n} = nc$. Under the 2-gauge transformation $b^{\mathbb{Z}_n} \to b^{\mathbb{Z}_n} + d\lambda$ generated by $\mathbb{Z}_n$-valued 1-cochain λ, we see that, from Eq. (A24) and using $db^{\mathbb{Z}_n} = nc$

$$\mathrm{Sq}^2(b^{\mathbb{Z}_n} + d\lambda) - \mathrm{Sq}^2 b^{\mathbb{Z}_n} \overset{2n,d}{=} 0. \quad (98)$$

This implies the 2-gauge invariance of the action amplitude $e^{2\pi i \int_{\mathcal{M}^4} \frac{k}{2n}\mathrm{Sq}^2 b}$ for the $n =$ even case.

3. Properties and duality relations

The pure 2-gauge theories (96) and (97) were studied for $n =$ odd cases and for $n =$ even and $m = 2k$ cases in Ref. [57]. In these cases, it was found that the theory realizes a 3+1D $Z_{\langle 2k,n\rangle}$-gauge theory. The $Z_{\langle 2k,n\rangle}$-gauge theory has emergent fermions if $2kn/\langle 2k,n\rangle^2 =$ odd, and it is a untwist $Z_{\langle 2k,n\rangle}$-gauge theory if $2kn/\langle 2k,n\rangle^2 =$ even. To understand the properties of the model (97) for $n =$ even and $m =$ odd cases, we compute the partition function (97) in Appendix C. The result is summarized in Table I. We see that, for $n =$ even, the 3+1D pure 2-gauge theory is equivalent to $Z_{\langle m,n\rangle}$-gauge theory. The theory has emergent fermion iff $mn/\langle m,n\rangle^2 =$ odd.

The higher gauge theories are labeled by a pair (K,ω_{d+1}): a target space K and a cocycle ω_{d+1} on it. Some times two different higher gauge theories may realize the same topologically ordered phase. In this case, we say that the two theories are equivalent or dual to each other. The results

TABLE I. Volume independent partition function $Z^{\text{top}}(\mathcal{M}^4;\mathcal{B},\omega_4)$ for the constructed local bosonic models, on closed four-dimensional space-time manifolds. The space-time $\mathcal{M}^4$ considered have vanishing Euler number and Pontryagin number $\chi(\mathcal{M}^4)=P_1(\mathcal{M}^4)=0$, which makes $Z^{\text{top}}(M^4)$ to be a topological invariant [26]. Here, $L^3(p)$ is the three-dimensional lens space and $F^4=(S^1\times S^3)\#(S^1\times S^3)\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. F^4 is not spin.

Models\\M^4:	T^4	$T^2\times S^2$	$S^1\times L^3(p)$	F^4	Low-energy effective theory
$Z^{\text{top}}(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),\frac{m}{2n}\text{Sq}^2b^{\mathbb{Z}_n})$ (97) $n=$ even, $m=0,\dots,2n-1$	$\langle m,n\rangle^3$	$\langle m,n\rangle$	$\langle m,n,p\rangle$	$\langle m,n\rangle$ if $\frac{mn}{\langle m,n\rangle^2}=$ even; 0 if $\frac{mn}{\langle m,n\rangle^2}=$ odd	$Z_{\langle m,n\rangle}$ gauge theory (with fermions iff $\frac{mn}{\langle m,n\rangle^2}=$ odd)
$Z^{\text{top}}(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),\frac{k}{n}\text{Sq}^2b^{\mathbb{Z}_n})$ (96) $n=$ odd, $k=0,\dots,n-1$	$\langle 2k,n\rangle^3$	$\langle 2k,n\rangle$	$\langle 2k,n,p\rangle$	$\langle 2k,n\rangle$	untwisted $Z_{\langle 2k,n\rangle}$ gauge theory (no emergent fermions)
$Z^{\text{top}}(\mathcal{M}^4;\mathcal{B}\mathbb{Z}_n,0)$	n^3	n	$\langle n,p\rangle$	n	untwisted Z_n gauge theory (no emergent fermions)

in Table I suggest the following duality relations, where we use $[\mathcal{B}(\Pi_1.\Pi_2,\dots),\bar\omega_{d+1}]$ to label different higher gauge theories.

(1) For $n=$ even and $\frac{mn}{\langle m,n\rangle^2}=$ even,

$$\left[\mathcal{B}(\mathbb{Z}_n,2),\frac{m}{2n}\text{Sq}^2b^{\mathbb{Z}_n}\right]\sim[\mathcal{B}(\mathbb{Z}_{\langle m,n\rangle}),0]. \tag{99}$$

(2) For $n=$ odd,

$$\left[\mathcal{B}(\mathbb{Z}_n,2),\frac{k}{n}\text{Sq}^2b^{\mathbb{Z}_n}\right]\sim[\mathcal{B}(\mathbb{Z}_{\langle 2k,n\rangle}),0]. \tag{100}$$

We note that $[\mathcal{B}(\mathbb{Z}_n),0]$ is an untwisted Z_n-gauge theory.

VI. 3+1D 2-GAUGE THEORY OF 2-GAUGE-GROUP $\mathcal{B}(G_b,Z_2^f)$

In this section, we are going to consider more general 3+1D 2-gauge theories which have 2-gauge-group $\mathcal{B}(G_b,Z_2^f)$.

A. The Lagrangian and space-time path integral

Since $\mathbb{Z}_2^f$ has no nontrivial automorphism, so α_2 is trivial. As a result, such 2-gauge theories are classified by

$$G_b;\bar n_3;\bar\omega_4, \tag{101}$$

where $\bar n_3\in H^3(\mathcal{B}G_b;\mathbb{Z}_2^f)$ and $\bar\omega_4\in H^4(\mathcal{B}(G_b;\mathbb{Z}_2^f);\mathbb{R}/\mathbb{Z})$.

To write down the Lagrangian and space-time path integral for the 2-gauge theories, the key is to find $\bar\omega_4$. To do so, we note that the links in $\mathcal{B}(G_b;\mathbb{Z}_2^f)$ are labeled by (a), $a\in G_b$. The triangles in $\mathcal{B}(G_b;\mathbb{Z}_2^f)$ are labeled by $(a_{ij},a_{jk},a_{ik},b_{ijk})$ that satisfy Eqs. (38) and (45). We see that on each link of $\mathcal{B}(G_b;\mathbb{Z}_2^f)$, we have a label a_{ij}, and on each triangle we have a label b_{ijk}. We may view a_{ij} as the canonical G_b-valued 1-cocycle $\bar a$ [due to Eq. (38)], and b_{ijk} as the canonical $\mathbb{Z}_2^f$-valued 2-cochain $\bar b$ on $\mathcal{B}(G_b;\mathbb{Z}_2^f)$. The canonical 1-cocycle and the 2-cochain are related

$$d\bar b=\bar n_3(\bar a). \tag{102}$$

We may use the 1-cocycle $\bar a$ and the 2-cochain $\bar b$ to write down $\bar\omega_4$.

We note that each $\bar\omega_4\in H^4(\mathcal{B}(G_b;\mathbb{Z}_2^f);\mathbb{R}/\mathbb{Z})$ corresponds [see Eq. (77)] to one or more elements in a subset of

$$H^4(\mathcal{B}(\mathbb{Z}_2^f,2);\mathbb{R}/\mathbb{Z})\times H^2(\mathcal{B}G_b;\mathbb{Z}_2^f)\times H^4(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z}). \tag{103}$$

To construct a $\bar\omega_4$, we may guess $\bar\omega_4=\frac{\bar k_0}{4}\text{Sq}^2\bar b$. Using Eq. (A20), we find that

$$d\text{Sq}^2\bar b=\text{Sq}^2\bar n_3(\bar a)+2\text{Sq}^3\bar b=\text{Sq}^2\bar n_3(\bar a)+2\bar b\bar n_3(\bar a). \tag{104}$$

So $\bar\omega_4=\frac{\bar k_0}{4}\text{Sq}^2\bar b$ is not a cocycle. However, the error is only a function of 1-cocycle $\bar a$ if $\bar k_0=2$. In this case, we can fix the error by adding a function of $\bar a$, $\bar\nu_4(\bar a)$. Similarly, we can try $\bar\omega_4=\frac12\bar b\bar e_2(\bar a)$, where $\bar e_2(\bar a)\in Z^2(\mathcal{B}G_b;\mathbb{Z}_2^f)$. But $d[\bar b\bar e_2(\bar a)]=\bar n_3(\bar a)\bar e_2(\bar a)$. Again $\bar\omega_4=\frac12\bar b\bar e_2(\bar a)$ is not a cocycle. Again we can fix it by adding a function $\bar\nu_4(\bar a)$. Thus we come up with the following general expression of $\bar\omega_4$:

$$\bar\omega_4(\bar a,\bar b)=\frac{k_0}{2}\text{Sq}^2\bar b+\frac12\bar b\bar e_2(\bar a)+\bar\nu_4(\bar a), \tag{105}$$

where $\bar\nu_4(\bar a)$ is a $\mathbb{R}/\mathbb{Z}$-valued cochain in $C^4(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$ that satisfy

$$-d\bar\nu_4(\bar a)=\frac{k_0}{2}\text{Sq}^2\bar n_3(\bar a)+\frac12\bar n_3(\bar a)\bar e_2(\bar a). \tag{106}$$

In this case, $\bar\omega_4(\bar a,\bar b)$ will be a cocycle $d\bar\omega_4\overset{1}{=}0$. The three terms in Eq. (105) correspond to the three cohomology classes in Eq. (103). Thus our construction of $\bar\omega_4$ is complete (for $\bar n_3\neq 0$).

Using the expression (105) for $\bar\omega_4$, we can construct a topological nonlinear σ-model (i.e., a 2-gauge theory):

$$\begin{aligned}Z\big(\mathcal{M}^4;\mathcal{B}\big(G_b;\mathbb{Z}_2^f\big),\bar\omega_4\big)&=\sum_\phi\left(\prod_i|G_b|^{-1}\prod_{(ij)}2^{-1}\right)e^{2\pi i\int_{\mathcal{M}^4}\phi^*\bar\omega_4}\\&=|G_b|^{-N_0}2^{-N_1}\sum_{\delta a=\mathbb{1},\,db=n_3}e^{2\pi i\int_{\mathcal{M}^4}\nu_4(a)+\frac{k_0}{2}\text{Sq}^2b+\frac12be_2(a)},\end{aligned} \tag{107}$$

where $\sum_{\delta a=\mathbb{1},\,db=n_3}$ sum over the G_b-valued 1-cochains a_{ij} and the $\mathbb{Z}_2^f$-valued 2-cochains b_{ijk} on the space-time complex $\mathcal{M}^4$, that satisfy

$$(\delta a)_{ijk}\equiv a_{ij}a_{jk}a_{ik}^{-1}=\mathbb{1},\qquad db=n_3(a). \tag{108}$$

In the above $k_0=0,1$ labels the elements of the $\mathbb{Z}_2$ subgroup of $H^4(\mathcal{B}(\mathbb{Z}_2^f,2);\mathbb{R}/\mathbb{Z})=\mathbb{Z}_4$, $\bar e_2(a)$ labels the elements in $H^2(\mathcal{B}G_b;\mathbb{Z}_2^f)$, and different $\bar\nu_4(a)$ differ by the elements in $H^4(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$. Plus $\bar n_3 \in H^3(\mathcal{B}G_b;\mathbb{Z}_2^f)$, the four pieces of data, $(k_0, \bar e_2, \bar n_3, \bar\nu_4)$, classify 2-gauge theories of 2-gauge-group $\mathcal{B}(G_b;\mathbb{Z}_2^f)$.

B. The equivalence between $[k_0, \bar e_2(\bar a), \bar n_3(\bar a), \bar\nu_4(\bar a)]$'s

The Lagrangian of the 2-gauge theory (107) is labeled by the data $[k_0, \bar e_2(\bar a), \bar n_3(\bar a), \bar\nu_4(\bar a)]$:

$$\begin{aligned} \bar e_2(a_{01}, a_{12}) &\in Z^2(\mathcal{B}G_b;\mathbb{Z}_2),\\ \bar n_3(a_{01}, a_{12}, a_{23}) &\in Z^3(\mathcal{B}G_b;\mathbb{Z}_2),\\ \bar\nu_4(a_{01}, a_{12}, a_{23}, a_{34}) &\in C^{d+1}(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z}),\end{aligned} \tag{109}$$

that satisfy

$$d\bar\nu_4(\bar a) \overset{1}{=} \tfrac{1}{2}[\mathrm{Sq}^2\bar n_3(\bar a) + \bar n_3(\bar a)\bar e_2(\bar a)]. \tag{110}$$

As local bosonic systems, the different 2-gauge theories labeled by different data may realize the same bosonic topological phase. We say that these 2-gauge theories or these data are equivalent.

Note that the Lagrangian is a 2-group cocycle, and two Lagrangians differing by a 2-group coboundary should be equivalent. This kind of equivalent relation is generated by the following three kinds of transformations.

(1) A transformation generated by a 1-cochain $\bar l_1 \in C^1(\mathcal{B}G_b;\mathbb{Z}_2)$

$$\begin{aligned} \bar e_2 &\to \bar e_2 + d\bar l_1,\\ \bar n_3 &\to \bar n_3,\\ \bar\nu_4 &\to \bar\nu_4 + \tfrac{1}{2}\bar n_3\bar l_1.\end{aligned} \tag{111}$$

(2) A transformation generated by a 2-cochain $\bar u_2 \in C^2(\mathcal{B}G_b;\mathbb{Z}_2)$

$$\begin{aligned} \bar e_2 &\to \bar e_2,\\ \bar n_3 &\to \bar n_3 + d\bar u_2,\\ \bar\nu_4 &\to \bar\nu_4 + \frac{k_0}{2}(d\bar u_2 \underset{2}{\smile} \bar n_3 + \mathrm{Sq}^2\bar u_2 + \bar u_2\bar e_2).\end{aligned} \tag{112}$$

(3) A transformation generated by a 3-cochain $\bar\eta_3 \in C^3(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$:

$$\begin{aligned} \bar e_2 &\to \bar e_2,\\ \bar n_3 &\to \bar n_3,\\ \bar\nu_4 &\to \bar\nu_4 + d\bar\eta_3.\end{aligned} \tag{113}$$

Under these transformations, the Lagrangian $\nu_4(a) + \frac{k_0}{2}\mathrm{Sq}^2 b + \frac{1}{2}be_2(a)$ only changes by a coboundary. These transformations do not change the topological partition function and do not change the topological order in the ground state.

We like to point out that the different transformations of the second type do not commute. These transformations may generate changes $(\bar e_2, \bar n_3, \bar\nu_4) \to (\bar e_2, \bar n_3, \bar\nu_4 + \Delta\bar\omega_4)$, where $\Delta\bar\omega_4$ is a cocycle in $Z^4(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$.

We also want to mention that the above transformations can not generate all possible equivalent relations. In particular, an isomorphism of the target space $\mathcal{B}(G_b, Z_2^f) \to \mathcal{B}(G_b, Z_2^f)$ (2-group isomorphism) may relate two Lagrangians whose difference is not a 2-group coboundary. We are not sure if there are more general "duality" equivalent relations between 2-gauge theories. This will be left for future work.

C. 2-gauge transformations in the cocycle σ-model

As a local bosonic model, the discrete nonlinear σ-model (107) do not have to have any symmetry. However, in Eq. (107), we choose a very special Lagrangian, the pullback of a cocycle on the target space. For such a special Lagrangian, the model is exactly soluble. Such a special Lagrangian has a large set of accidental symmetries: invariant under 2-gauge transformations. These accidental symmetries are called 2-gauge symmetries.

The first type of 2-gauge transformation is given by 1-cochain $\lambda_1 \in C^1(M^{d+1};\mathbb{Z}_2)$:

$$b \to b + d\lambda_1, \quad a \to a; \tag{114}$$

We find that, using Eqs. (A23) and (A21)

$$\begin{aligned} &k_0\mathrm{Sq}^2(b + d\lambda_1) + (b + d\lambda_1)e_2(a) - k_0\mathrm{Sq}^2 b - be_2(a)\\ &\overset{2,d}{=} k_0\mathrm{Sq}^2 d\lambda_1 \overset{2,d}{=} 0.\end{aligned} \tag{115}$$

Therefore the Lagrangian changes by only a total derivative term under the first type of 2-gauge transformation.

The second type of 2-gauge transformation is given by 0-cochain $g_i \in C^0(M^{d+1};G_b)$:

$$b \to b + \zeta_2(a, g), \quad a_{ij} \to a^g = g_i a_{ij} g_j^{-1}. \tag{116}$$

Under the above transformation,

$$\begin{aligned} n_3(a) &\to n_3(a^g) \overset{2}{=} n_3(a) + d\zeta_2(a, g),\\ e_2(a) &\to e_2(a^g) \overset{2}{=} e_2(a) + d\xi_1(a, g),\end{aligned} \tag{117}$$

which defines $\zeta_2(a, g)$. Thus the condition $db \overset{2}{=} n_3(a)$ is maintained under the 2-gauge transformation. We find that, using Eqs. (A22) and (A21),

$$\begin{aligned} &k_0\mathrm{Sq}^2(b + \zeta_2) + (b + \zeta_2)(e_2 + d\xi_1) - k_0\mathrm{Sq}^2 b - be_2\\ &\overset{2,d}{=} k_0\, db \underset{2}{\smile} d\zeta_2 + bd\xi_1 + \zeta_2 e_2 + \zeta_2 d\xi_1\\ &\overset{2,d}{=} k_0 n_3 \underset{2}{\smile} d\zeta_2 + n_3\xi_1 + \zeta_2 e_2 + \zeta_2 d\xi_1.\end{aligned} \tag{118}$$

We note that the above only depends on a and g. Thus, if $\nu(a)$ satisfies

$$\nu(a^g) - \nu(a) \overset{2,d}{=} k_0 n_3 \underset{2}{\smile} d\zeta_2 + n_3\xi_1 + \zeta_2 e_2 + \zeta_2 d\xi_1, \tag{119}$$

the Lagrangian changes by only a total derivative term under the second type of 2-gauge transformation.

D. The pointlike excitations in the 2-gauge theory

There are two types of pointlike excitations in the 2-gauge theory. Let S^1 be the world line of a pointlike excitation of the first type. The presence of the pointlike excitation modifies the path integral via a Wilson loop:

$$Z(\mathcal{M}^4;\mathcal{B}(G_b;\mathbb{Z}_2^f)) = |G_b|^{-N_0}2^{-N_1} \times \sum_{\delta a=\mathbb{1},\, db=n_3} \left[\mathrm{Tr}\prod_{S^1} R_{G_b}(a_{ij})\right] e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 b+\frac{1}{2}be_2(a)}, \quad (120)$$

where $R_{G_b}(a)$, $a \in G_b$, is a representation of G_b and $\prod_{S^1} R_{G_b}(a_{ij})$ is a product $R_{G_b}(a_{ij})$ along the loop S^1.

To describe the second type of pointlike excitations, let f_3 be the Poincaré dual of the worldline C^1 of the pointlike excitations. Then the second type of pointlike excitations are created by modifying the condition $db = n_3(a)$ to

$$db = n_3(a) + f_3. \quad (121)$$

Now the path integral with the second type of pointlike excitations becomes

$$Z(\mathcal{M}^4;\mathcal{B}(G_b;\mathbb{Z}_2^f)) = |G_b|^{-N_0}2^{-N_1} \sum_{\delta a=\mathbb{1},\, db=n_3+f_3} e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 b+\frac{1}{2}be_2(a)}. \quad (122)$$

To understand the property of the second type of excitations, let us assume the worldline S^1 to be the boundary of a disk D^2. Let a $\mathbb{Z}_2$-valued 2-cochain s_2 to be the Poincaré dual of D^2. Then we have $f_3 = ds_2$. The above path integral can be rewritten as

$$\begin{aligned}
Z(\mathcal{M}^4;\mathcal{B}(G_b;\mathbb{Z}_2^f))|G_b|^{N_0}2^{N_1}
&= \sum_{\delta a=\mathbb{1},\, db=n_3+ds_2} e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 b+\frac{1}{2}be_2(a)}\\
&= \sum_{\delta a=\mathbb{1},\, db=n_3} e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 (b+s_2)+\frac{1}{2}(b+s_2)e_2(a)}\\
&= e^{k_0\pi\, i\int_{\mathcal{M}^4}\mathrm{Sq}^2 s_2}\\
&\quad\times \sum_{\delta a=\mathbb{1},\, db=n_3} e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 b+\frac{1}{2}be_2}\, e^{\pi\, i\int_{\mathcal{M}^4} k_0 f_3 \smile_2 n_3+s_2e_2},
\end{aligned} \quad (123)$$

where we have used Eq. (A22). We note that the term $e^{\pi\, i\int_{\mathcal{M}^4} s_2e_2(a)}$ is the only one on the disk D^2 that depends on the 1-cocycle field a. This term can be rewritten as

$$e^{\pi\, i\int_{\mathcal{M}^4} s_2e_2(a)} = e^{\pi\, i\int_{D^2} e_2(a)}. \quad (124)$$

However, on the surface, $e^{\pi\, i\int_{D^2} e_2(a)}$ may not be a function of the world line $C^1 = \partial D^2$. It may depend on how we extend the C^1 to D^2 (i.e., changing s_2 by a cocycle). In other words, $e^{\pi\, i\int_{\mathcal{M}^4} s_2e_2(a)}$ may change if change s_2 by a cocycle. On the other hand, the path integral (123) depends on s_2 via $f_3 = ds_2$. So it should not change is we change s_2 by a cocycle. In other words,

$$e^{\pi\, i\int_{\mathcal{M}^4} k_0\mathrm{Sq}^2 s_2+s_2e_2(a)} \quad (125)$$

should not change if we change s_2 by a cocycle.

If we s_2 by a cocycle β_2, the term $e^{\pi\, i\int_{\mathcal{M}^4} k_0\mathrm{Sq}^2 s_2+s_2e_2(a)}$ changes by a factor

$$e^{\pi\, i\int_{\mathcal{M}^4} k_0\mathrm{Sq}^2\beta_2+\beta_2e_2(a)} = e^{\pi\, i\int_{\mathcal{M}^4} k_0(w_2+w_1^2)\beta_2+\beta_2e_2(a)} \quad (126)$$

where we have used Eq. (A22), and the fact $\mathrm{Sq}^2\beta_2 \overset{2,dd}{=} (w_2 + w_1^2)\beta_2$. We also assume that $\mathcal{M}^4$ is closed. Next we will show that $k_0(w_2 + w_1^2) + e_2 \overset{2,d}{=} 0$, and the above factor is always 1.

To show $k_0(w_2 + w_1^2) + e_2 \overset{2,d}{=} 0$, let us fix a and do the path integral of b. We have seen that if we change b by a coboundary, the action amplitude $e^{2\pi\, i\int_{\mathcal{M}^4} \nu_4(a)+\frac{k_0}{2}\mathrm{Sq}^2 b+\frac{1}{2}be_2(a)}$ does not change. However, if we change b by a cocycle b_0, the action amplitude will change. Using Eqs. (A23) and (A21), we find that

$$\begin{aligned}
&k_0\mathrm{Sq}^2(b+b_0) + (b+b_0)e_2 - k_0\mathrm{Sq}^2 b - be_2\\
&\overset{2,d}{=} k_0\mathrm{Sq}^2 b_0 + b_0e_2 \overset{2,d}{=} [k_0(w_2+w_1^2)+e_2]b_0.
\end{aligned} \quad (127)$$

Thus the action amplitude depends on b_0 via $e^{\pi\, i\int_{\mathcal{M}^4}[k_0(w_2+w_1^2)+e_2]b_0}$. Since $\mathcal{M}^4$ is orientable and compact, its intersection form for $\mathbb{Z}_2$-valued 2-cocycle classes is nondegenerate. Therefore, when we integral over b (i.e., b_0) in the path integral, such a term will cause the partition function to vanish if

$$k_0(w_2 + w_1^2) + e_2 \neq \mathbb{Z}_2\text{-valued coboundary}. \quad (128)$$

As a result, $k_0(w_2 + w_1^2) + e_2 \overset{2,d}{=} 0$ in order for the partition function to be nonzero. This completes our proof.

For simplicity let us assume $k_0 = 0$ for the time being. We consider a particle described by a world line C^1 that is a combination of the first type and the second type. In this case, C^1 dependent factor in the path integral is given by

$$\left[\mathrm{Tr}\prod_{S^1} R_{G_b}(a_{ij})\right] e^{\pi\, i\int_{D^2} e_2(a)}, \quad C^1 = \partial D^2. \quad (129)$$

Since $e_2(a) \overset{2,d}{=} 0$, the term $e^{\pi\, i\int_{D^2} e_2(a)}$ only depend on C^1, and does not depend on how we extend C^1 to D^2. The term $e^{\pi\, i\int_{D^2} e_2(a)}$ introduces ± 1 phase to $R_{G_b}(a_{ij})$ and promotes it into a representations of $G_f = Z_2 \leftthreetimes_{e_2} G_b$. $e_2(a)$ is the two cocycle that describes the Z_2 extension of G_b, since on the space-time $\mathcal{M}^4$, $e_2(a)$ is trivialized. We see that the pointlike excitations are described by G_f representations.

We know that when $n_3 = e_2 = k_0 = 0$, the a and b fields in the 2-gauge theory (107) decouple. In this case, a describes a G_b gauge theory (with a cocycle twist), and b describes a Z_2 gauge theory (in the dual form) [57]. Thus Eq. (107) describes a $Z_2\times G_b$ gauge theory, whose charges are described by $Z_2\times G_b$ representations. The above result suggests that when $e_2 \neq 0$, Eq. (107) describes a $G_f = Z_2 \leftthreetimes_{e_2} G_b$ gauge theory, whose charges are described by G_f representations.

When $n_3 = e_2 = 0$ but $k_0 = 1$, the a and b fields in the 2-gauge theory (107) still decouple. In this case, b describes a twisted Z_2 gauge theory (in the dual form) where the Z_2 charge is a fermion (see Sec. V B) [57]. Thus Eq. (107) describes a $Z_2\times G_b$ gauge theory, whose charges are described by representations $\mathrm{s}\mathcal{R}\mathrm{ep}(Z_2\times G_b)$, where the nontrivial Z_2 representations are fermions. When $e_2 \neq 0$, we expect Eq. (107) to describe a $G_f = Z_2 \leftthreetimes_{e_2} G_b$ gauge theory, whose charges are described by representations $\mathrm{s}\mathcal{R}\mathrm{ep}(G_f)$. In fact, Eq. (107), with $k_0 = 1$, is an example of high dimensional bosonization for fermions that carry a G_f quantum number [57–59].

To summarize, the pointlike excitations in the 2-gauge theory (107) are described by $\mathcal{R}\text{ep}(G_f)$ when $k_0 = 0$ and by s$\mathcal{R}\text{ep}(G_f)$ when $k_0 = 1$. Here, $\mathcal{R}\text{ep}(G_f)$ is the symmetric fusion category formed by the representations of G_f where all the representations are bosons. s$\mathcal{R}\text{ep}(G_f)$ is the symmetric fusion category formed by the representations of G_f where all the representations that represent the extended Z_2 trivially are bosons and the others are fermions. The representations that represent the extended Z_2 trivially correspond to the first type of pointlike excitations, which are always bosons regardless the value of k_0. The representations that represent the extended Z_2 nontrivially correspond to the second type of pointlike excitations. The second type of pointlike excitations are fermions when $k_0 = 1$, and bosons when $k_0 = 0$.

We see that when $k_0 = 0$, there is no fermionic particle-like excitations, and the 2-gauge theory Eq. (107) is dual to Dijkgraaf-Witten model with gauge group G_f. This is consistent with the result in Ref. [32].

VII. CLASSIFY AND REALIZE 3+1D EF1 TOPOLOGICAL ORDERS BY 2-GAUGE THEORIES OF 2-GAUGE-GROUP $\mathcal{B}(G_b, Z_2^f)$

It was argued that 3+1D AB and EF topological orders with emergent bosons and/or fermions have a unique canonical boundary [32,33]. On the canonical boundary, the boundary stringlike excitations are labeled by the elements in a finite group. All these boundary string excitations have a unit quantum dimension. For EF1 topological orders with emergent fermions, the canonical boundary also has an emergent fermionic pointlike excitation with quantum dimension 1 [33]. These boundary excitations are described by a pointed unitary fusion 2-category. Such a pointed unitary fusion 2-category is classified by a 2-group $\mathcal{B}(G_b, Z_2^f)$ and a $\mathbb{R}/\mathbb{Z}$-valued 4-cocycle ω_4 on the 2-group. Here G_b is the group that labels the types of boundary string excitations. Therefore *all EF1* 3+1D *topological orders are classified by a pair* $\mathcal{B}(G_b, Z_2^f), \bar{\omega}_4$ *– a 2-group and a* $\mathbb{R}/\mathbb{Z}$*-valued 4-cocycle on the 2-group*.

To see why pointed fusion 2-categories are classified by the pairs $(\mathcal{B}(G_b, Z_2^f), \bar{\omega}_4)$, we note that the pointed fusion 2-category has objects labeled by elements in G_b, 1-morphisms labeled by elements in Z_2 and 2-morphisms corresponding to physical operators. The 2-morphisms are not all invertible, but for the structural morphisms we only need to consider the invertible 2-morphisms, thus no generality is lost by restricting 2-morphisms to $U(1) \simeq \mathbb{R}/\mathbb{Z}$. This way we obtain a 3-group $\mathcal{B}(G_b, Z_2, \mathbb{R}/\mathbb{Z})$, which has the same classification data as the pointed fusion 2-category. We explain now in more detail.

On one hand, by lLemma IV.1, we have

$$\mathcal{B}(\mathbb{R}/\mathbb{Z}, 3) \to \mathcal{B}(G_b, Z_2, \mathbb{R}/\mathbb{Z}) \to \mathcal{B}(G_b, Z_2), \qquad (130)$$

and $\mathcal{B}(G_b, Z_2, \mathbb{R}/\mathbb{Z})$ is classified by the base 2-group $\mathcal{B}(G_b, Z_2)$ and an element $\bar{\omega}_4$ in $H^4(\mathcal{B}(G_b, Z_2), \mathbb{R}/\mathbb{Z})$. Then the 2-group $\mathcal{B}(G_b, Z_2)$ is in turn characterised by $G_b, Z_2, \bar{n}_3 \in H^3(\mathcal{B}G_b; \mathbb{Z}_2)$. Thus 3-group $\mathcal{B}(G_b, Z_2, \mathbb{R}/\mathbb{Z})$ is characterised by $(G_b, Z_2, \bar{n}_3 \in H^3(G_b, Z_2), \bar{\omega}_4 \in H^4(\mathcal{B}(G_b, Z_2), \mathbb{R}/\mathbb{Z}))$.

On the other hand, recall the classification data of the pointed fusion 2-category that is listed in Ref. [33].

(1) Objects $g \in G_b$, 1-morphisms $p_g \in Z_2 \subset \text{Hom}(g, g)$.

(2) Interchange law: 2-isomorphisms [$U(1)$ phase factors] $\tilde{b}(p'_g, q'_h, p_g, q_h)$ that determines the particle statistics.

(3) Associator: 1-morphism $n_3(g, h, j) : (gh)j \to g(hj)$ in $H^3(\mathcal{B}G_b; \mathbb{Z}_2)$ and 2-isomorphisms $\tilde{n}_3(p_g, q_h, r_j)$.

(4) Pentagonator: 2-isomorphisms $\nu_4(g, h, j, k) \in C^4(\mathcal{B}G_b, \mathbb{R}/\mathbb{Z})$.

We thus find an exact correspondence between the above and the classification data on the higher group side $(G_b, Z_2, \bar{n}_3 \in H^3(\mathcal{B}G_b; \mathbb{Z}_2), \bar{\omega}_4 \in H^4(\mathcal{B}(G_b, Z_2), \mathbb{R}/\mathbb{Z}))$ as below: G_b, Z_2, n_3 are exactly the same. The 2-group 4-cocycle $\bar{\omega}_4$ has three components $k_0, \bar{e}_2, \bar{\nu}_4$.

(1) k_0 corresponds to $\tilde{b}(p'_g, q'_h, p_g, q_h)$ on the 2-category side. It has four different choices, corresponding to boson, fermion, semion, and antisemion statistics, respectively. For EF1 topological orders we stick to the choice of fermion statistics, which is indicated in our notation by using Z_2^f instead of Z_2.

(2) $\bar{e}_2$ determines the Z_2^f extension from G_b to G_f. Together with k_0 it determines the associator 2-morphisms $\tilde{n}_3(p_g, q_h, r_j)$ on the 2-category side.

(3) The last component $\bar{\nu}_4$ is just the pentagonator $\nu_4(g, h, j, k)$ on the 2-category side.

(4) Moreover, on both sides they satisfy the same consistent condition (110).

Since all 3+1D EF1 topological orders are classified by $\mathcal{B}(G_b, Z_2^f), \bar{\omega}_4$, and since for each pair $\mathcal{B}(G_b, Z_2^f), \bar{\omega}_4$ we can construct a 2-gauge theory to realize a EF1 topological order, we conclude that exactly soluble 2-gauge theories of 2-gauge-group $\mathcal{B}(G_b, Z_2^f)$ realize and classify all 3+1D EF1 topological orders.

VIII. REALIZE 3+1D EF2 TOPOLOGICAL ORDERS BY TOPOLOGICAL NONLINEAR σ-MODELS

A. Construction of topological nonlinear σ-models

In Ref. [26], it was conjectured that all topological orders with gappable boundary can be realized by exactly soluble tensor network model defined on space-time complex [27,53,60,61]. In Ref. [33], it was shown that all EF topological orders have a unique canonical boundary described by a unitary fusion 2-category in Statement I.2. Motivated by the results in Refs. [32,60], here we like to show that all the EF 3+1D bosonic topological orders can be realized by topological nonlinear σ-models, a particular type of tensor network models defined on space-time complex [26,27,61]. The topological nonlinear σ-models are constructed using the data of unitary fusion 2-categories described in Statement I.2.

Let us remind the readers that the canonical boundary of a EF topological order is described by a unitary fusion 2-category $\mathcal{A}_b^3$. The boundary stringlike excitations (the simple objects in $\mathcal{A}_b^3$) are labeled by the elements in $\hat{G}_b = G_b \leftthreetimes Z_2^m$ [33]. All the strings have a unit quantum dimension and their fusion is described by the group $\hat{G}_b$:

$$g_1 g_2 = g_3, \quad g_1, g_2, g_3 \in \hat{G}_b. \qquad (131)$$

Also two strings (two objects) labeled by g and gm (where $g \in \hat{G}_b$ and m is the generator of Z_2^m) are connected by an 1-morphism $\sigma_{g,gm}$ of quantum dimension $\sqrt{2}$. This 1-morphism correspond to an on-string pointlike excitation. There is another 1-morphism f_g of quantum dimension 1 that connect every string g to itself. The second 1-morphism correspond to a fermionic pointlike excitation. The fusion of 1-morphisms is given by

$$f_g \otimes f_g = \mathbb{1}, \quad f_g \otimes \sigma_{g,gm} = \sigma_{g,gm},$$
$$\sigma_{g,gm} \otimes \sigma_{gm,g} = \mathbb{1} \oplus f_g. \tag{132}$$

We note that the fusion 2-category $\mathcal{A}_b^3$ has three layers. The first layer is formed by objects in a fusion category. For our case, the simple objects in fusion ring form a finite group $\hat{G}_b$ [see Eq. (131)]. The second layer is formed by 1-morphisms generated by $\mathbb{1}$, f_g, $\sigma_{g,gm}$. The objects and the 1-morphisms are described by a fusion category [see Eq. (132)]. The third layer is formed by 2-morphisms, which are complex vector spaces for our case. The objects plus the 1-morphisms and 2-morphisms are described by the fusion 2-category. In the first part of this section, we are going to show that the simple objects and simple morphisms in the fusion category Eqs. (131) and (132) (i.e., the object and 1-morphism layers) are described by a simplicial set $\hat{\mathcal{K}}(\hat{G}_b, Z_2^f)$. And from this simplicial set, we can recover the entire fusion category (including semisimple objects). In the second part of this section, we will show that the 2-morphism layer is described by a set of tensors. So the fusion 2-category is described by a topological nonlinear σ-model with a target complex $\hat{\mathcal{K}}(\hat{G}_b, Z_2^f)$.

To obtain the bulk topological nonlinear σ-model that realize the fusion 2-category $\mathcal{A}_b^3$, let us first ignore the quantum-dimension-$\sqrt{2}$ 1-morphisms $\sigma_{g,gm}$. In this case, the canonical boundary will be described by a pointed unitary fusion 2-category, i.e., by a 2-group $\mathcal{B}(\hat{G}_b, Z_2^f)$ and a $\mathbb{R}/\mathbb{Z}$-valued 4-cocycle $\bar{\omega}_4(\hat{a}, \bar{b})$ on the 2-group, where $\hat{a}$ and $\bar{b}$ are canonical 1-cochain and 2-cochain of $\mathcal{B}(\hat{G}_b, Z_2^f)$. The tensor network model that realize this reduced boundary will be a 2-gauge theory of 2-gauge-group $\mathcal{B}(\hat{G}_b, Z_2^f)$. In other words, the links in the tensor network model have an index $\hat{a}_{ij} \in \hat{G}_b$ which defines $\hat{a}$, and the triangles in the tensor network model have an index $b_{ijk} \in Z_2^m$ which defines $\bar{b}$. $\hat{a}$ and $\bar{b}$ satisfy

$$\delta\hat{a} = \mathbb{1}, \quad d\bar{b} = \hat{n}_3(\hat{a}), \tag{133}$$

where $\hat{n}_3 \in \mathcal{H}^3(\hat{G}_b; \mathbb{Z}_2^f)$. The corresponding path integral is given by

$$Z(\mathcal{M}^4) = |\hat{G}_b|^{-N_0} 2^{-N_1} \sum_{\delta\hat{a}=\mathbb{1},\, db=\hat{n}_3(\hat{a})} e^{2\pi\, i \int_{\mathcal{M}^4} \omega_4(\hat{a},b)}. \tag{134}$$

Now, let us include the 1-morphisms $\sigma_{g,gm}$ that connect two strings g and gm. But at the moment, we will assume such 1-morphisms to have a unit quantum dimension and a fusion $\sigma_{g,gm} \otimes \sigma_{gm,g} = \mathbb{1}$. Since the extra 1-morphism can connect two strings differ by m, the flat condition on $\hat{a}$ is modified and becomes a quasiflat condition $\delta\hat{a} \in Z_2^m$. In $\mathcal{B}(\hat{G}_b, Z_2^f)$, three links $\hat{a}_{ij}$, $\hat{a}_{jk}$, $\hat{a}_{ki} = (\hat{a}_{ik})^{-1}$ bound a triangle only when $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = \mathbb{1}$. Now we add some triangles to the complex $\mathcal{B}(\hat{G}_b, Z_2^f)$ so that three links $\hat{a}_{ij}$, $\hat{a}_{jk}$, $\hat{a}_{ki}$ bound a triangle even when $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = m \in Z_2^m$. Including these extra triangles change the first homotopy group of the target complex to $\pi_1 = \hat{G}_b/Z_2^m = G_b$. The new target complex is denoted as $\hat{\mathcal{B}}(G_b, Z_2^f)$, which is a triangulation of $K(G_b, Z_2^f)$.

Let us compare two triangulations, $\hat{\mathcal{B}}(G_b, Z_2^f)$ and $\mathcal{B}(G_b, Z_2^f)$, of the same space $K(G_b, Z_2^f)$. In $\mathcal{B}(G_b, Z_2^f)$, the links are labeled by $a_{ij} \in G_b$, while in $\hat{\mathcal{K}}(G_b, Z_2^f)$ we double the number of links, which now are labeled by $\hat{a}_{ij} \in \hat{G}_b = Z_2^m \rtimes G_b$. The triangles in $\mathcal{B}(G_b, Z_2^f)$ are labeled by $[a_{01}, a_{12}, a_{02}; b_{012}]$ where a_{01}, a_{12}, a_{02} satisfy $a_{01}a_{12}(a_{02})^{-1} = \mathbb{1}$. On the other hand, the triangles in $\hat{\mathcal{B}}(G_b, Z_2^f)$ are labeled by $[\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{02}; b_{012}]$ where $\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{02}$ satisfy $\hat{a}_{01}\hat{a}_{12}(\hat{a}_{02})^{-1} \in Z_2^m$. The full structure of $\hat{\mathcal{K}}(G_b, Z_2^f)$ is determined by its canonical 1-cochain $\hat{a}$ and 2-cochain $\bar{b}$ that satisfy

$$\delta\hat{a} \in Z_2^m, \quad d\bar{b} = \hat{n}_3(\hat{a}). \tag{135}$$

where $\hat{n}_3(\hat{a})$ is a 3-cocycle in $\hat{\mathcal{K}}(\hat{G}_b, Z_2^f)$ satisfying

$$\hat{n}_3(\hat{a}) = n_3(\pi^m(\hat{a})), \quad \pi^m : \hat{G}_b \to G_b,$$
$$n_3(\bar{a}) \in H^3(\mathcal{B}G_b, Z_2^f). \tag{136}$$

To have a more rigorous construction of $\hat{\mathcal{B}}(G_b, Z_2^f)$, we note that given a morphism of groups $A_2 \xrightarrow{p_2} \hat{G}$, $\ker p_2 \xrightarrow{0} G := \hat{G}/\mathrm{Im}p_2$ together with G action α on $\ker p_2$ and $n_3 \in H^3(G, \ker p_2^\alpha)$ decide a 2-group $\mathcal{B}(G, \ker p_2)$, which as a simplicial set has the following form: $K_n = G^{\times n} \times (\ker p_2)^{\times\binom{n}{2}}$, where

$$K_1 = \{(a_{01})|a_{01} \in \hat{G}\},$$
$$K_2 = \left\{(a_{01}, a_{12}, a_{02}; b_{012})|a_{01}a_{12}a_{02}^{-1} = \mathbb{1}, b_{012} \in \ker p_2\right\},$$
$$K_3 = \{(a_{01}, a_{12}, a_{23}; b_{012}, b_{013}, b_{023}, b_{123})|\alpha(a_{01})b_{123} - b_{023}$$
$$+ b_{013} - b_{012} = n_3(a_{01}, a_{12}, a_{23}) \in \ker p_2\}, \tag{137}$$

and K_n in general is made up of these n-simplices whose 2-faces are elements of K_2 and such that each set of four 2-faces gluing together as a 3-simplex is an element of K_3. This is the so-called coskeleton construction.

Then we pullback this 2-group structure via the projection map $\hat{G} \xrightarrow{\pi^m} G$, we obtain another 2-group. The pullback simplicial set $\hat{K}_\bullet$ of $K_\bullet$ through $\hat{K}_1 \to K_1$ (both $\hat{K}_0 = K_0 = pt$) is inductively defined as $\hat{K}_n = K_n \times_{\mathrm{Hom}(\partial\Delta[n],K)} \mathrm{Hom}(\partial\Delta[n], K)$. Here $\partial\Delta[n]$ is the boundary simplicial set of the standard simplicial simplex $\Delta[n]$. Pullback of a 2-group still satisfies the same Kan conditions, thus still a 2-group. Then after calculation, we see that the pullback 2-group as a simplicial set has the following form: $\hat{K}_n = \hat{G}^{\times n} \times A_2^{\times\binom{n}{2}}$, where

$$\hat{K}_1 = \{(\hat{a}_{01})|\hat{a}_{01} \in \hat{G}\},$$
$$\hat{K}_2 = \left\{(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{02}; b_{012})|\hat{a}_{01}\hat{a}_{12}\hat{a}_{02}^{-1} \in \mathrm{Im}p_2, b_{012} \in \ker p_2\right\},$$
$$\hat{K}_3 = \{(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{23}; b_{012}, b_{013}, b_{023}, b_{123})|\alpha(\pi^m(\hat{a}_{01}))b_{123}$$
$$- b_{023} + b_{013} - b_{012} = \hat{n}_3(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{23}) \in \ker p_2\},$$
$$\tag{138}$$

and $\hat{K}_n$ is similarly defined by coskeleton construction. Here, $\hat{n}_3 = (\pi^m)^* n_3$ is the pullback 3-cocycle. We denote this 2-group by $\hat{\mathcal{B}}(G, \ker p_2)$. Since the pullback construction introduces equivalent 2-groups, $\hat{\mathcal{B}}(G, \ker p_2)$ and $\mathcal{B}(G, \ker p_2)$ are equivalent 2-groups. To apply in the above situation, we take

$G = G_b$, $A_2 = Z_2^f \times Z_2^m$ and $p_2 = 0\times i$ where $i : Z_2^m \to \hat{G}_b$ is the embedding, thus $\ker p_2 = Z_2^f$ and $\mathrm{Im} p_2 = Z_2^m$.

Through the above examples, we see that pointed unitary fusion 2-categories have a "geometric" picture in terms of 2-groups. The fusion rules in the 2-categories are described by the complex of the 2-groups. The complicated coherent relations in the 2-categories are described by the cocycle conditions on the 2-groups side. In the following, we will develop a "geometric" picture, i.e., a complex $\hat{\mathcal{K}}(G_b; Z_2^f)$, for the unitary fusion 2-category $\mathcal{A}_b^3$ that contains noninvertible 1-morphisms.

The complex $\hat{\mathcal{K}}(G_b; Z_2^f)$ has one vertex. The links in $\hat{\mathcal{K}}(G_b; Z_2^f)$ are labeled by elements $\hat{a}_{ij}$ in group $\hat{G}_b = Z_2^m \leftthreetimes_{\rho_2} G_b$, with $\rho_2 \in H^2(\mathcal{B}G_b; \mathbb{Z}_2)$. The complex $\hat{\mathcal{K}}(G_b; Z_2^f)$ has the same set of links as $\mathcal{B}(\hat{G}_b, Z_2^f)$, but has a different set of triangles to describe a different set of 1-morphisms. In $\hat{\mathcal{K}}(G_b, Z_2^f)$, three links $\hat{a}_{ij}$, $\hat{a}_{jk}$, $\hat{a}_{ki} = (\hat{a}_{ik})^{-1}$ bound a triangle when $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} \in Z_2^m$. When $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = \mathbb{1}$, the three links bound two triangles labeled by $b_{ijk} = 0, 1$. When $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = m$, where m generates Z_2^m, the three links bound only one triangle which has a fixed $b_{ijk} = 1$.

The tetrahedrons in $\hat{\mathcal{K}}(G_b; Z_2^f)$ describe the fusion channels of 1-morphisms Eq. (132). Consider a 2-sphere in $\hat{\mathcal{K}}(G_b; Z_2^f)$ formed by four triangles who share their edges. If all four triangles carry no m-flux, i.e., satisfy $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = \mathbb{1}$, then the 2-sphere is filled by a tetrahedron if the label b_{ijk} on the four triangles satisfy $\sum b_{ijk} \overset{2}{=} \bar{n}_3(a_{ij})$. Here $\bar{n}_3(\hat{a}_{ij})$ is a function that depends on labels $\hat{a}_{ij}$ of the six links on the 2-sphere. Note that $\bar{n}_3(\hat{a}_{ij})$ is defined only when $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = \mathbb{1}$ for all four triangles. If two of four triangles carry m-flux, i.e., satisfy $\hat{a}_{ij}\hat{a}_{jk}\hat{a}_{ki} = m$, then the 2-sphere is filled by a tetrahedron regardless the values of the labels b_{ijk} on the four triangles.

If all four triangles carry m-flux, then the 2-sphere is filled by two different tetrahedrons, labeled by $c_{0123} = 0, 1$. This is because each triangle with m-flux corresponds to the 1-morphism σ. The fusion of three σ is given by $\sigma \otimes \sigma \otimes \sigma = (\mathbb{1} \oplus f) \otimes \sigma = 2\sigma$. The factor 2 means there are two fusion channels, and thus two different tetrahedrons to fill the 2-sphere.

At higher dimensions, every 3-sphere formed by five tetrahedrons glued along their 2-faces is filled by a 4-simplex, every 4-spheres formed by six 4-simplexes glued along their 3-faces is filled by a 5-simplex, etc. In this way, we obtain the simplicial set $\hat{\mathcal{K}}(G_b; Z_2^f)$ (which is thus 3-coskeleton):

$$K_0 \overset{d_0,d_1}{\leftleftarrows} K_1 \overset{d_0,d_1,d_2}{\Lleftarrow} K_2 \overset{d_0,\dots,d_3}{\underset{\cdot}{\leftleftarrows}} K_3 \overset{d_0,\dots,d_4}{\underset{\cdot}{\leftleftarrows}} K_4 \cdots, \tag{139}$$

where the simplexes at each dimensions are given by

$$\begin{aligned}
\hat{K}_0 &= \{pt.\},\\
\hat{K}_1 &= \{(\hat{a}_{01}) | \hat{a}_{01} \in \hat{G}\},\\
\hat{K}_2 &= \{(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{02}; b_{012}) | \hat{a}_{01}\hat{a}_{12} = \hat{a}_{02},\ b_{012} = 0, 1\\
&\quad \text{or} \quad \hat{a}_{01}\hat{a}_{12} = m\hat{a}_{02},\ b_{012} = 0.\},\\
\hat{K}_3 &= \{(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{23}, \hat{a}_{02}, \hat{a}_{13}, \hat{a}_{03}; b_{012}, b_{013},\\
&\quad b_{023}, b_{123}; c_{0123})\\
&\quad |\text{if all } \delta\hat{a} = \mathbb{1}:\ b_{123} - b_{023} + b_{013} - b_{012} = \hat{n}_3(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{23}),\\
&\quad c_{0123} = 0; \quad \text{if two } \delta\hat{a} = m:\ c_{0123} = 0.\},
\end{aligned} \tag{140}$$

where $\hat{n}_3 \in H^3(\mathcal{B}\hat{G}_b; \mathbb{Z}_2)$. The complex $\hat{\mathcal{K}}(G_b; Z_2^f)$ describes a fusion category formed by the objects and 1-morphisms in the unitary fusion 2-category $\mathcal{A}_b^3$. (The 2-morphisms in $\mathcal{A}_b^3$ will be discussed in the later part of this section.)

Since it is a coskeleton construction of a 3-step tower, $\hat{\mathcal{K}}(G_b; Z_2^f)$ is certainly a simplicial set. In general, the geometric realization $|Y|$ of the simplicial set Y is a topological space. By construction, $|Y|$ is given by $|Y| := \sqcup Y_i \times \Delta^i / \sim$, where $\sim$ is provided by gluing along lower dimensional faces provided by the information given by s the degeneracy maps. However, $|Y|$ may not be a manifold. Also, $\hat{\mathcal{K}}(G_b; Z_2^f)$ is not a 2-group any more. First of all, strict $Kan(3, j)!$ are not satisfied, and even nonstrict $Kan(4, j)$ are not satisfied. Nevertheless, $\pi_{\geqslant 3}(\hat{\mathcal{K}}(G_b; Z_2^f)) = 0$. Moreover, we still have $\pi_2(\hat{\mathcal{K}}(G_b; Z_2^f)) = Z_2^f$ and $\pi_1(\hat{\mathcal{K}}(G_b; Z_2^f)) = G_b$.

Although $\hat{\mathcal{K}}(G_b; Z_2^f)$ does not correspond to a 2-group, in the following, we will show that from the data of $\hat{\mathcal{K}}(G_b; Z_2^f)$, one can recover the fusion category, which is the original fusion 2-category $\mathcal{A}_b^3$ without the 2-morphism layer. We first let the set of simple objects to be the links in $\hat{\mathcal{K}}(G_b; Z_2^f)$, $C_0 := \hat{\mathcal{K}}(G_b; Z_2^f)_1 = \hat{G}_b$. And let the set of simple 1-morphisms to be the triangles with one side degenerate in $\hat{\mathcal{K}}(G_b; Z_2^f)$. One can picture them as bigons (see Fig. 8),

$$\begin{aligned}
C_1 :=& \{(1, \hat{a}_{12}, \hat{a}_{02}; b_{012}) \in \hat{K}_2\}\\
=& \{(g, g'; b) | g = g', b = 0, 1; g' = gm, b = 0\}
\end{aligned}$$

Then the composition $\cdot_v$ of 1-morphisms can be read from the information of $\hat{K}_3$, which tells which tetrahedrons are allowed, indicated by Fig. 9. For example, we have a unique tetrahedron $(1, 1, g, 1, g, g; 0, b, b + b', b')$ in $\hat{K}_3$ to fill its (3,1)-horn. Then this implies that $(g, g; b) \cdot_v (g, g; b') = (g, g; b + b')$, here $+$ is the addition in Z_2. Then the only nonunique case is for $(g, gm; 0) \cdot_v (gm, g; 0)$: there are both $(1, 1, g, 1, gm, g; 0, 0, 0, 0)$ or $(1, 1, g, 1, gm, g; 0, 0, 1, 0)$ to fill the (3, 1)-horn. This makes $(g, gm; 0) \cdot_v (gm, g; 0) = (g, g; 0 \oplus 1)$ a nonsimple element. We thus can extend $\cdot_v$ to an associative product to all semi-simple objects and 1-morphisms. We call the result category $\mathcal{A}_b^3$.

Now we will read from $\hat{K}_3$ the fusion product for $\mathcal{A}_b^3$, which makes $\mathcal{A}_b^3$ further into a fusion category. We only need to take care of fusion of simple objects and simple 1-morphisms, then we can extend the fusion by distribution law to semisimple objects and 1-morphisms. The fusion of simple objects is simply the group multiplication of $\hat{G}_b$; the fusion of simple

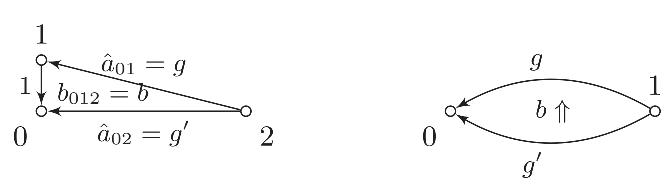

FIG. 8. Links are simple objects and triangles with degenerate (0,1)-sides are simple 1-morphisms.

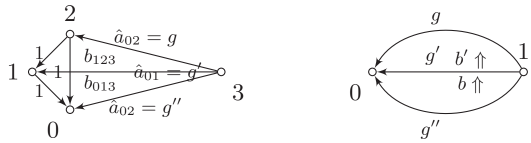

FIG. 9. The composition $\cdot_v$ of 1-morphisms.

1-morphisms is again read from tetrahedrons in $\hat{K}_3$. If we want to fuse $(g_1, g_1'; b_1)$ and $(g_2, g_2'; b_2)$, the first step is to transfer the (0,1)-side degenerate triangle $(g_1, g_1'; b_1) = (1, g_1', g_1; b_1)$ to an (2,3)-side degenerate triangle, by filling the (3,0)-horn of the tetrahedron (0,1,2,3) with a unique element

$$(1, g_1', g_1', g_1, g_1, 1; b_1, 0, b_1, 0) \in \hat{K}_3.$$

The second step is to fill the (2,1)-horn of the triangle (0, 1, 4) without flux with $(g_1', g_2', g_1'g_2'; 0)$. The third step is to finally fill the (3,1)-horn of the tetrahedron (0, 2, 3, 4) and obtain a triangle (0, 3, 4) with three sides $(g_1, g_2, g_1'g_2')$. The fourth step is to transfer this triangle to a triangle with sides $(1, g_1g_2, g_1'g_2')$ by filling the (3, 2)-horn of a tetrahedron. The filling can be nonunique only in the third step. This procedure is illustrated with Fig. 10.

Following this strategy, the calculation shows that the only nonunique case happens when we fuse $(g_1, g_1m; 0)$ and $(g_2, g_2m, 0)$, and $(g_1, g_1m; 0) \otimes (g_2, g_2m, 0) = (g_1g_2, g_1g_2; 0 \oplus 1)$. The associator for the fusion product is still given by n_3. Thus we have recovered a fusion 2-category from the simplicial set $\hat{\mathcal{K}}(G_b, Z_2^f)$.

To obtain the coherence relations (i.e., the 2-morphism layer) in the unitary fusion 2-category $\mathcal{A}_b^3$, we try to construct topological nonlinear σ-models with target complex $\hat{\mathcal{K}}(G_b, Z_2^f)$. To do so, we assign a complex number to each 4-simplex in $\hat{\mathcal{K}}(G_b, Z_2^f)$. A 4-simplex is labeled by $(\hat{a}_{ij}; b_{ijk}, c_{ijkl} | i, j, k, l = 0, 1, 2, 3, 4)$, that satisfy

$$\begin{aligned}
&\hat{a}_{ij} \in \hat{G}_b, \quad b_{ijk} \in \mathbb{Z}_2, \quad c_{ijkl} \in \mathbb{Z}_2;\\
&b_{123} - b_{023} + b_{013} - b_{012} = \hat{n}_3(\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{23})\\
&\qquad\qquad\qquad \text{when all four } \delta\hat{a} = \mathbb{1},\\
&b_{ijk} = 0 \quad \text{when } (\delta\bar{a})_{ijk} = m.\\
&c_{ijkl} = 0 \quad \text{when one of } \delta\bar{a} = \mathbb{1}.
\end{aligned} \tag{141}$$

We see that c_{ijkl} can take two values 0,1 only when all four $\delta\bar{a} = m$. So we can write such a complex number as

$$\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}} \tag{142}$$

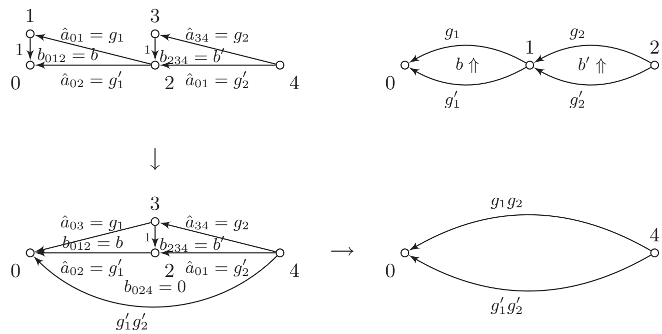

FIG. 10. Fusion of 1-morphisms.

which corresponds to the top tensor of the tensor set. The above number is nonzero only when $\hat{a}_{ij}, b_{ijk}, c_{ijkl}$ satisfy Eq. (141). We also assign a positive number w_0 to the vertex in $\hat{\mathcal{K}}(G_b, Z_2^f)$. To the links labeled by $[\hat{a}_{01}]$ we assign the same positive number w_1. To the triangle labeled by $[\hat{a}_{01}, \hat{a}_{12}, \hat{a}_{02}; b_{012}]$ we assign a positive number $w_2(\mathbb{1})$ or $w_2(m)$ depending on $\hat{a}_{01}\hat{a}_{12}(\hat{a}_{02})^{-1} = \mathbb{1}$ or m. The path integral that describes the topological nonlinear σ-model on space-time with boundary is given by

$$\begin{aligned}
Z(\mathcal{M}^4) = &\sum_{\delta\hat{a} \in Z_2^m,\, db = \hat{n}_3(\hat{a}), c} \prod_i{}' w_0 \prod_{(ij)}{}' w_1 \prod_{(ijk)}{}' w_2[\delta\hat{a})_{ijk}]\\
&\times \prod_{(ijklp)} \left(\hat{\Omega}_4{}^{\hat{a}_{ij}\hat{a}_{ik}\hat{a}_{il}\hat{a}_{ip}\hat{a}_{jk}\hat{a}_{jl}\hat{a}_{jp}\hat{a}_{kl}\hat{a}_{kp}\hat{a}_{lp};c_{jklp}c_{iklp}c_{ijlp}}_{b_{ijk}b_{ijl}b_{ijp}b_{ikl}b_{ikp}b_{ilp}b_{jkl}b_{jkp}b_{jlp}b_{klp};c_{ijkp}c_{ijkl}}\right)^{s_{ijklp}},
\end{aligned} \tag{143}$$

where $\prod_{(ijklp)}$ is a product over all the 4-simplices and s_{ijklp} is the orientation of the 4-simplices (see Fig. 12). Also, $\prod'_{(ijk)}$ is a product over all the interior triangles, $\prod'_{(ij)}$ is a product over all the interior links, and $\prod_i$ is a product over all the interior vertices.

The rank-25 tensor $\hat{\Omega}_4$, as well as the weight tensors w_0, w_1, and w_2, must satisfy certain conditions in order for the above path integral to be re-triangulation invariant. The conditions can be obtained in the following way: We start with a 5-simplex (012345). Then, divide the six 4-simplices on the boundary of the 5-simplex (012345) into two groups. Then the partition function on one group of the 4-simplices must equal to the partition function on the other group of the 4-simplices, after a complex conjugation.

For example, the two groups of the 4-simplices can be [(12345), (02345), (01345)] and [(01245), (01235), (01234)]. This partition leads to a condition

$$\begin{aligned}
&\sum_{b_{345}} \sum_{c_{0345}c_{1345}c_{2345}} w_2[(\delta\hat{a})_{345}]\, \hat{\Omega}_4{}^{\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{1345}c_{1245}}_{b_{123}b_{124}b_{125}b_{134}b_{135}b_{145}b_{234}b_{235}b_{245}b_{345};c_{1235}c_{1234}} \left(\hat{\Omega}_4{}^{\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{0345}c_{0245}}_{b_{023}b_{024}b_{025}b_{034}b_{035}b_{045}b_{234}b_{235}b_{245}b_{345};c_{0235}c_{0234}}\right)^*\\
&\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{1345}c_{0345}c_{0145}}_{b_{013}b_{014}b_{015}b_{034}b_{035}b_{045}b_{134}b_{135}b_{145}b_{345};c_{0135}c_{0134}}\\
&= \sum_{b_{012}} \sum_{c_{0123}c_{0124}c_{0125}} w_2[(\delta\hat{a})_{012}]\, \hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{04}\hat{a}_{05}\hat{a}_{12}\hat{a}_{14}\hat{a}_{15}\hat{a}_{24}\hat{a}_{25}\hat{a}_{45};c_{1245}c_{0245}c_{0145}}_{b_{012}b_{014}b_{015}b_{024}b_{025}b_{045}b_{124}b_{125}b_{145}b_{245};c_{0125}c_{0124}} \left(\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{05}\hat{a}_{12}\hat{a}_{13}\hat{a}_{15}\hat{a}_{23}\hat{a}_{25}\hat{a}_{35};c_{1235}c_{0235}c_{0135}}_{b_{012}b_{013}b_{015}b_{023}b_{025}b_{035}b_{123}b_{125}b_{135}b_{235};c_{0125}c_{0123}}\right)^*\\
&\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}.
\end{aligned} \tag{144}$$

For the partition [(12345), (02345)] and [(01345), (01245), (01235), (01234)], we obtain a condition

$$\begin{aligned}
&\sum_{c_{2345}} \hat{\Omega}_4{}^{\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{1345}c_{1245}}_{b_{123}b_{124}b_{125}b_{134}b_{135}b_{145}b_{234}b_{235}b_{245}b_{345};c_{1235}c_{1234}} \left(\hat{\Omega}_4{}^{\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{0345}c_{0245}}_{b_{023}b_{024}b_{025}b_{034}b_{035}b_{045}b_{234}b_{235}b_{245}b_{345};c_{0235}c_{0234}}\right)^* \\
&= w_1 \sum_{\hat{a}_{01};b_{012},b_{013},b_{014},b_{015};c_{0123}c_{0124}c_{0125}c_{0134}c_{0135}c_{0145}} w_2[(\delta\hat{a})_{012}]w_2[(\delta\hat{a})_{013}]w_2[(\delta\hat{a})_{014}] \\
&\quad\times w_2[(\delta\hat{a})_{015}]\left(\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{1345}c_{0345}c_{0145}}_{b_{013}b_{014}b_{015}b_{034}b_{035}b_{045}b_{134}b_{135}b_{145}b_{345};c_{0135}c_{0134}}\right)^* \hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{04}\hat{a}_{05}\hat{a}_{12}\hat{a}_{14}\hat{a}_{15}\hat{a}_{24}\hat{a}_{25}\hat{a}_{45};c_{1245}c_{0245}c_{0145}}_{b_{012}b_{014}b_{015}b_{024}b_{025}b_{045}b_{124}b_{125}b_{145}b_{245};c_{0125}c_{0124}} \\
&\quad\times \left(\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{05}\hat{a}_{12}\hat{a}_{13}\hat{a}_{15}\hat{a}_{23}\hat{a}_{25}\hat{a}_{35};c_{1235}c_{0235}c_{0135}}_{b_{012}b_{013}b_{015}b_{023}b_{025}b_{035}b_{123}b_{125}b_{135}b_{235};c_{0125}c_{0123}}\right)^* \hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}.
\end{aligned} \tag{145}$$

For the partition [(12345)] and [(02345), (01345), (01245), (01235), (01234)], we obtain a condition

$$\begin{aligned}
&\hat{\Omega}_4{}^{\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{1345}c_{1245}}_{b_{123}b_{124}b_{125}b_{134}b_{135}b_{145}b_{234}b_{235}b_{245}b_{345};c_{1235}c_{1234}} = w_0 w_1^5 \sum_{\hat{a}_{01},\hat{a}_{02},\hat{a}_{03},\hat{a}_{04},\hat{a}_{05}} \sum_{b_{012},b_{013}} w_2[(\delta\hat{a})_{012}]w_2[(\delta\hat{a})_{013}] \\
&\sum_{b_{014},b_{015},b_{045},b_{023};c_{0123}c_{0124}c_{0125}c_{0134}c_{0135}c_{0145}c_{0234}c_{0235}c_{0245}c_{0345}} w_2[(\delta\hat{a})_{014}]w_2[(\delta\hat{a})_{015}]w_2[(\delta\hat{a})_{045}]w_2[(\delta\hat{a})_{023}] \\
&\sum_{b_{024},b_{025},b_{034},b_{035};} w_2[(\delta\hat{a})_{024}]w_2[(\delta\hat{a})_{025}]w_2[(\delta\hat{a})_{034}]w_2[(\delta\hat{a})_{035}]\ \hat{\Omega}_4{}^{\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{23}\hat{a}_{24}\hat{a}_{25}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{2345}c_{0345}c_{0245}}_{b_{023}b_{024}b_{025}b_{034}b_{035}b_{045}b_{234}b_{235}b_{245}b_{345};c_{0235}c_{0234}} \\
&\left(\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{03}\hat{a}_{04}\hat{a}_{05}\hat{a}_{13}\hat{a}_{14}\hat{a}_{15}\hat{a}_{34}\hat{a}_{35}\hat{a}_{45};c_{1345}c_{0345}c_{0145}}_{b_{013}b_{014}b_{015}b_{034}b_{035}b_{045}b_{134}b_{135}b_{145}b_{345};c_{0135}c_{0134}}\right)^* \hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{04}\hat{a}_{05}\hat{a}_{12}\hat{a}_{14}\hat{a}_{15}\hat{a}_{24}\hat{a}_{25}\hat{a}_{45};c_{1245}c_{0245}c_{0145}}_{b_{012}b_{014}b_{015}b_{024}b_{025}b_{045}b_{124}b_{125}b_{145}b_{245};c_{0125}c_{0124}} \left(\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{05}\hat{a}_{12}\hat{a}_{13}\hat{a}_{15}\hat{a}_{23}\hat{a}_{25}\hat{a}_{35};c_{1235}c_{0235}c_{0135}}_{b_{012}b_{013}b_{015}b_{023}b_{025}b_{035}b_{123}b_{125}b_{135}b_{235};c_{0125}c_{0123}}\right)^* \\
&\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}.
\end{aligned} \tag{146}$$

There are many other similar conditions from different partitions.

Each solution of these conditions give us a topological nonlinear σ-model. Some of these models have emergent fermions and describe EF topological orders. We believe that all EF topological orders can be realized this way.

In general, it is very hard to find solutions of these conditions, since that corresponds to solve billions of nonlinear equations with millions of unknown variables, even for the simplest cases. One way to make progress is to note that when restricted to the indices $\hat{a}$ that satisfy $\delta\hat{a} = \mathbb{1}$, the tensor $\hat{\Omega}_4$ becomes a U(1)-valued 4-cocycle on the 2-group $\mathcal{B}(\hat{G}_b, Z_2^f)$. This is because some conditions for $\hat{\Omega}_4$, such as Eq. (144), act within these components of $\hat{\Omega}_4$ whose indices satisfy $\delta\hat{a} = \mathbb{1}$. When $\delta\hat{a} = \mathbb{1}$, $w_2(m)$ will not appear in these conditions. In this case, if we choose $\hat{\Omega}_4$ to be a U(1)-valued 4-cocycle on the 2-group, the terms in the summation in Eq. (144) will all have the same value. Thus we can replace the summation in Eq. (144) by factors that count the number of the terms in the summation. From Eq. (144), we see that these factors cancel out. In this case, the condition (144) reduces to the condition for the 4-cocycles on the 2-group. Thus the restricted $\hat{\Omega}_4$ must be U(1)-valued 4-cocycle on the 2-group $\mathcal{B}(\hat{G}_b, Z_2^f)$, which has a form

$$\begin{aligned}
&\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}\Big|_{\delta\hat{a}=\mathbb{1},c\text{'s}=0} \\
&= e^{2\pi\, i\int_{(01234)} \nu_4(\hat{a}) + \frac{k_0}{2}\mathrm{Sq}^2 b + \frac{1}{2} b e_2(\hat{a})}.
\end{aligned} \tag{147}$$

When $k_0 = 1$, the tensor $\hat{\Omega}_4$ and the associated topological nonlinear σ-model will describe an EF topological order. Starting from the parcial solution (147) we can use the equations Eqs. (144)–(146) to find other components of $\hat{\Omega}_4$ whose indices do not satisfy $\delta\hat{a} = \mathbb{1}$.

As we have seen that the topological nonlinear σ-model on the complex $\hat{\mathcal{K}}(G_b, Z_2^f)$ is closely related to the unitary fusion 2-category $\mathcal{A}_b^3$ that describes the canonical boundary of a EF topological order [33]. The links in $\hat{\mathcal{K}}(G_b, Z_2^f)$ correspond to the objects in the fusion 2-category. The 1-morphisms f_g that connect an object to itself corresponds to triangles with no flux, which are labeled by $\pi_2[\hat{\mathcal{K}}(G_b, Z_2^f)] = \mathbb{Z}_2$. The noninvertible 1-morphisms $\sigma_{g,gm}$ correspond to triangles with m-flux. If we treat the objects connected by 1-morphisms as equivalent, then the equivalent classes of the objects correspond to $\pi_1[\hat{\mathcal{K}}(G_b, Z_2^f)] = G_b$. The fusion of the objects in different orders may differ by an 1-morphism which lives in $\pi_2[\hat{\mathcal{K}}(G_b, Z_2^f)]$, It is called an associator. In both Ref. [33] and this paper, we use the same symbol $\hat{n}_3$ to describe the associator. The part of the $\hat{\Omega}_4$ tensor, $\hat{\nu}_4$, also correspond to $\hat{\nu}_4$ in Ref. [33] that is another piece of data to describe the unitary fusion 2-category $\mathcal{A}_b^3$. It is this correspondence between topological nonlinear σ-models on $\hat{\mathcal{K}}(G_b, Z_2^f)$ and the fusion 2-categories described in Ref. [33] that allows us to conclude that all EF topological orders are realized by topological nonlinear σ-models on $\hat{\mathcal{K}}(G_b, Z_2^f)$.

From a consideration of 2-gauge transformations [see Eqs. (59) and (61)], we expect w_0 and w_1 to contain factors $|\hat{G}_b|^{-1}$ and $\frac{1}{2}$ to cancel the volume of the 2-gauge transformations. If $w_2(\mathbb{1}) = w_2(m)$ with m being the generator of Z_2^m, the solutions should describe AB or EF1 topological orders. If $w_2(\mathbb{1}) \neq w_2(m)$, some of these solutions should describe EF2 topological orders. In particular, we expect $w_2(m)$ to be related to the quantum dimension of the noninvertible 1-morphism—the Majorana zero mode.

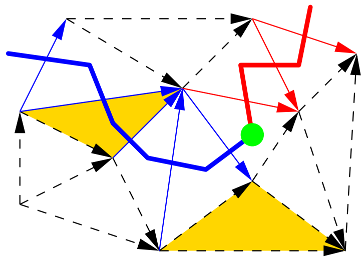

FIG. 11. A boundary configuration. The thin dash-lines corresponds to $\hat{a}_{ij} = \mathbb{1}$. The thin colored-lines corresponds to $\hat{a}_{ij} \neq \mathbb{1}$. The white triangles corresponds to $b_{ijk} = 0$. The yellow triangles corresponds to $b_{ijk} = 1$, which are boundary fermions. The nonzero $\hat{a}_{ij}$'s describe boundary strings on the dual lattice, represented by the thick lines. The strings with different colors are described by g and gm. The domain wall between two strings has a Majorana zero mode marked by a green dot.

B. The canonical boundary of topological nonlinear σ-models

In the last section, we constructed topological nonlinear σ-models using the data of unitary fusion 2-categories in Statement I.2. In this section, we like to show that the topological nonlinear σ-models have a canonical boundary described by corresponding unitary fusion 2-category $\mathcal{A}_b^3$.

The canonical boundaries of the topological nonlinear σ-models are very simple which are given by choosing $\hat{a}_{ij} = \mathbb{1}$ and $b_{ijk} = 0$ on the boundary. The states with $\hat{a}_{ij} \neq \mathbb{1}$ and $b_{ijk} \neq 0$ corresponds excited states with boundary stringlike and pointlike excitations (see Fig. 11).

We see that the boundary string are labeled by $\hat{a}_{ij}$ which is an element in $\hat{G}_b$. They correspond to objects in a unitary fusion 2-category. b_{ijk} on triangles correspond to 1-morphisms of unit quantum dimension. $b_{ijk} = 1$ implies the presence of a fermion on the triangle (ijk). The condition $db = \hat{n}_3(\hat{a})$ describes how a fermion worldline can starts or ends at certain configurations of $\hat{a}$, where $\hat{n}_3(\hat{a}) \neq 0$.

The Fermi statistics of the particle described by $b_{ijk} \neq 0$ is determined by the form of the top tensor $\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}$ in Eq. (147). $k_0 = 1$ will make the particle to be a fermion.

The triangles with $\delta\hat{a} = m$ will carry a Majorana zero mode, provided that the weight tensor $w_2(\delta\hat{a})$ satisfies $w_2(\mathbb{1}) \neq w_2(m)$. If $w_2(\mathbb{1}) = w_2(m)$, the triangles with $\delta\hat{a} = m$ will not correspond to a Majorana zero mode. These results suggest that the canonical boundaries of the topological nonlinear σ-models are described by unitary fusion 2-categories in Statement I.2.

To summarize, the topological nonlinear σ-models are described by the following data:

$$\hat{G}_b = Z_2^m \leftthreetimes_{\rho_2} G_b,\ \hat{n}_3(\hat{a}),\ w_0,\ w_1,\ w_2(\mathbb{1}),\ w_2(m),$$
$$\hat{\Omega}_4{}^{\hat{a}_{01}\hat{a}_{02}\hat{a}_{03}\hat{a}_{04}\hat{a}_{12}\hat{a}_{13}\hat{a}_{14}\hat{a}_{23}\hat{a}_{24}\hat{a}_{34};c_{1234}c_{0234}c_{0134}}_{b_{012}b_{013}b_{014}b_{023}b_{024}b_{034}b_{123}b_{124}b_{134}b_{234};c_{0124}c_{0123}}, \tag{148}$$

where $\hat{n}_3(\hat{a})$ is defined only when $\delta\hat{a} = \mathbb{1}$. In that case, it is a $\mathbb{Z}_2$-valued group 3-cocycle for $\hat{G}_b$: $\hat{n}_3|_{\delta\hat{a}=\mathbb{1}} \in \mathcal{H}^3(\hat{G}_b;\mathbb{Z}_2)$. Also, w_0, w_1, $w_2(Z_2^m)$, $\hat{\Omega}_4$ satisfy a set of nonlinear equations, such as Eqs. (144)–(146). If the tensor $\hat{\Omega}_4$ has a form (147) with $k_0 = 1$, then the data describe an EF topological order. Such data also classify the EF topological orders after quotient out certain equivalence relation. When $\hat{G}_b = Z_2^m \leftthreetimes_{\rho_2} G_b$ is a nontrivial extension of G_b by Z_2^m and when $w_2(\mathbb{1}) \neq w_2(m)$, the data classify the EF2 topological orders.

Although we have collected many evidences to support the above proposal, many details still need to be worked out to confirm it.

IX. TURAEV-VIRO CONSTRUCTION AND HIGHER CATEGORY

The above topological nonlinear σ-models are actually a special case of Turaev-Viro type state sum construction. So in this section, we will discuss such a construction in most general setting.

The most general Turaev-Viro type state sum construction of $n+1$D TQFT that one can imagine is to triangulate the space-time, color all the k-simplices for $k < n+1$, give each $n+1$-simplex a factor which depends the its colorings, then multiply the factors together to get the action amplitude, and then sum the action amplitudes over all possible colorings (i.e., do the path integral). The final answer is the partition function of the state sum model. In order for the partition function to be topological, the colorings and the factors must satisfy a series of self-consistent conditions, such that the action amplitudes are retriangulation invariant [see conditions Eqs. (144)–(146)].

More precisely, (1) the coloring of a 0-simplex (i.e., vertex) is in a labeling set L_0. (2) The coloring of a 1-simplex (i.e., link) between vertices $a, b \in L_0$ is in a labeling set $L_1(a, b)$. (3) The coloring of a 2-simplex (i.e., triangle) is in a labeling set which is determined by the colorings of the three vertices and three links. (4) ... (5) The factor Ω_i of an $n+1$-simplex i is a function of all the above colorings. (6) The weighting factor $w_j^{(k)}$ of an k-simplex $(k < n+1)$ j is a function of all the colorings of the k-simplex. (7) The partition function is given by

$$Z = \sum_{\text{colorings}} \left(\prod_{k=0}^{n}\prod_{j} w_j^{(k)}\right)\left(\prod_i \Omega_i\right). \tag{149}$$

However, it is not necessary to use so general a construction. Many of the above models turn out to describe the same topological phase. To produce all the possible phases it is sufficient to use only some simplified versions of state sum model. This can be seen by the following n-category picture.

Although not rigorously proved, we believe the above data of the most general state sum model exactly corresponds to a n-category which describes the topological defects and excitations on a gapped nD boundary of the $n+1$D TQFT. ($n+1$D Turaev-Viro type TQFT $=$ $n+1$D TQFT with gapped boundary.) The labeling sets L_k of k-simplices are just the sets of isomorphisms classes of simple k-morphisms in the n-category. An $n+1$-simplex can be read as a closed graph in the n-category, whose evaluation gives rise to a complex number which is the factor associated to the $n+1$ simplex.

The physical picture of such $n+1$-category is that k-morphisms correspond to codimension k topological defects. More precisely,

(1) 0-morphism: nD "defect," in fact not a defect but just a label of a uniform boundary region, like a boundary "phase."

(2) 1-morphism: $n-1$D defect between different boundary phases.

(3) ...

(4) $(n-2)$-morphism: 2D=1+1D defect, namely, line defect. If between trivial membrane defects, they are actually stringlike excitations.

(5) $(n-1)$-morphism: 1D=0+1D defect, point defect. If between trivial line defects, they are pointlike excitations, or particles.

(6) n-morphism: 0D defect, "instanton," a change in time, represented by physical operators.

As a first simplification, nD boundary phases that can have $n-1$D defects between them should be "Morita-equivalent"; they share the same bulk phase. This indicates that the vertex labels in the state sum model (corresponding to 0-morphisms in the n-category) should not produce more phases and can always be dropped. For the n-category this means fixing the 0-morphism, which turns the n-category into a *fusion* $(n-1)$-category.

For $n=2$, the above is the best one can do, and it is why $2+1$D Turaev-Viro TQFTs are built upon fusion categories. There are two levels of colorings:

(1) L_1 labels the links, also the objects in the fusion category, corresponding to the point-like excitations on the boundary.

(2) $L_2(a,b,c)$ labels the triangles, also the morphisms in the fusion category, corresponding to basis operators in the fusion space $\mathrm{Hom}(a\otimes b, c)$. When all such fusion spaces are 1-dimensional, this level of coloring can be dropped and these models are called *multiplicity-free*.

For $n=3$, the above means that $3+1$D Turaev-Viro TQFTs can be built upon fusion 2-categories. There are thus there levels of colorings:

(1) L_1 labels the links, also the objects in the fusion 2-category, corresponding to the string-like excitations on the boundary.

(2) $L_2(a,b,c)$ labels the triangles, also the 1-morphisms in the fusion 2-category, corresponding to the point-like defects on the junction of a, b, c strings.

(3) $L_3(\dots)$ labels the tetrahedrons, also the basis physical operators in the corresponding fusion space.

One possible further simplification is to consider boundaries without string-like excitations. Note that in this case particles (the only nontrivial topological defects) on the boundary form a pre-modular category. The corresponding model is the so-called Crane-Yetter TQFT [60] or Walker-Wang model [62]. But they are not sufficient to construct all 3+1D phases. In Refs. [63,64], it was generalised to use G-crossed extension of the premodular category as input data.

Another possibility is to consider boundaries whose point-like excitations are as simple as possible, by the results in Refs. [32,33]. This leads to more general and powerful simplification.

First, every 3+1D topological phase have a canonical gapped boundary (this in particular means that all 3+1D topological phase are of Turaev-Viro type) whose stringlike excitations fuse under a group multiplication law. Thus L_1 is a group. Note that the particles (point defects on the trivial string) form a premodular category. G-crossed premodular category is just a special case of of such fusion 2-category. In other words, constructions, based on fusion 2-category whose objects form a group, include the construction in Refs. [63,64].

Second, Refs. [32,33] further show that particles on the boundary can be reduced to Vec or sVec; in other words, either there is no nontrivial particle, or the only nontrivial particle is the fermion. This means that there are at most two labels in L_2. (As a result most L_3 also become trivial.) This is exactly the model discussed in this paper, which should be the most simplified version of state sum model in 3+1D, but still general enough to produce all 3+1D phases.

X. SUMMARY

In this paper, we show that higher gauge theories are nothing but familiar nonlinear σ-models in the topological-defect-free disordered phase. As a result, nonlinear σ-models whose target spaces K satisfy $\pi_1(K)=$ finite group and $\pi_{k>1}(K)=0$ can realize gauge theories, and nonlinear σ-models whose target spaces K satisfy $\pi_1(K), \pi_2(K)=$ finite group and $\pi_{k>2}(K)=0$ can realize 2-gauge theories, etc.

We discuss in detail how to characterize and classify higher gauge theories, such as 2-gauge theories. As an application, we use 2-gauge theories to realize and classify all 3+1D EF1 topological orders—3+1D topological orders for bosonic systems with emergent fermions, but no Majorana zero modes for triple string intersections. We also design topological nonlinear σ-models to realize and classify all 3+1D EF2 topological orders—3+1D topological orders for bosonic systems with emergent fermions that have Majorana zero modes for some triple string intersections. Since EF topological orders can be viewed as gauged fermionic SPT state in 3+1D, our result also give rise to a classification of 3+1D fermionic SPT orders.

To obtain the above results, we developed a "geometric" way to view the unitary fusion 2-category $\mathcal{A}_b^3$ for the canonical boundary of the EF topological orders. We used a special triangulation of a space $K(\hat{G}_b, \mathbb{Z}_2^f)$ to described the fusion category formed by the objects and 1-morphisms in $\mathcal{A}_b^3$. We used a tensor set defined for the triangulation to described the 2-morphism layer of 2-category $\mathcal{A}_b^3$.

ACKNOWLEDGMENTS

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APPENDIX A: SPACE-TIME COMPLEX, COCHAINS, AND COCYCLES

In this paper, we consider models defined on a space-time lattice. A space-time lattice is a triangulation of the $d+1$D space-time, which is denoted as $\mathcal{M}^{d+1}$. We will also call the triangulation $\mathcal{M}^{d+1}$ as a space-time complex, which is

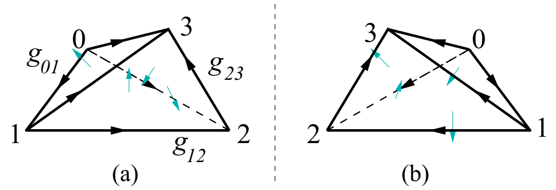

FIG. 12. Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

formed by simplices—the vertices, links, triangles, etc. We will use $i, j, \ldots$ to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled by $(i, j), (j, k), \ldots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i, j, k), (j, k, l), \ldots$.

In order to define a generic lattice theory on the space-time complex $\mathcal{M}^{d+1}$ using local tensors $T_{ij\ldots k}$ and $\omega_{d+1}(a^{G_f}_{ij}, a^{G_f}_{ik}, \ldots)$, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure [21,65,66]. A branching structure is a choice of orientation of each link in the d+1D complex so that there is no oriented loop on any triangle (see Fig. 12).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, etc. So the simplex in Fig. 12(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subsimplices) a canonical orientation. Figure 12 illustrates two 3-simplices with opposite canonical orientations compared with the three-dimension space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

Given an Abelian group $(\mathbb{M}, +)$, an n-cochain f_n is an assignment of values in $\mathbb{M}$ to each n-simplex, for example a value $f_{n;i,j,\ldots,k} \in \mathbb{M}$ is assigned to n-simplex $(i, j, \ldots, k)$. So *a cochain f_n can be viewed as a bosonic field on the space-time lattice*.

We like to remark that a simplex $(i, j, \ldots, k)$ can have two different orientations $s_{ij\ldots k} = \pm$. We can use $(i, j, \ldots, k)$ and $(j, i, \ldots, k) = -(i, j, \ldots, k)$ to denote the same simplex with opposite orientations. The value $f_{n;i,j,\ldots,k}$ assigned to the simplex with opposite orientations should differ by a sign: $f_{n;i,j,\ldots,k} = -f_{n;j,i,\ldots,k}$. So to be more precise f_n is a linear map f_n : n-simplex $\to \mathbb{M}$. We can denote the linear map as $\langle f_n, n\text{-simplex}\rangle$, or

$$\langle f_n, (i, j, \ldots, k)\rangle = f_{n;i,j,\ldots,k} \in \mathbb{M}. \tag{A1}$$

More generally, a *cochain* f_n is a linear map of n-chains:

$$f_n : n\text{-chains} \to \mathbb{M}, \tag{A2}$$

or (see Fig. 13)

$$\langle f_n, n\text{-chain}\rangle \in \mathbb{M}, \tag{A3}$$

where a *chain* is a composition of simplices. For example, a 2-chain can be a 2-simplex: (i, j, k), a sum of two 2-simplices: $(i, j, k) + (j, k, l)$, a more general composition of 2-simplices: $(i, j, k) - 2(j, k, l)$, etc. The map f_n is linear respect to such a composition. For example, if a chain is m copies of a simplex, then its assigned value will be m times that of the simplex. $m = -1$ correspond to an opposite orientation.

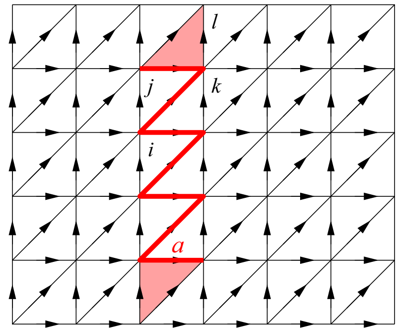

FIG. 13. A 1-cochain a has a value 1 on the red links: $a_{ik} = a_{jk} = 1$ and a value 0 on other links: $a_{ij} = a_{kl} = 0$. da is nonzero on the shaded triangles: $(da)_{jkl} = a_{jk} + a_{kl} - a_{jl}$. For such 1-cochain, we also have $a \smile a = 0$. So when viewed as a $\mathbb{Z}_2$-valued cochain, $\mathcal{B}_2 a \neq a \smile a$ mod 2.

We will use $C^n(\mathcal{M}^{d+1}; \mathbb{M})$ to denote the set of all n-cochains on $\mathcal{M}^{d+1}$. $C^n(\mathcal{M}^{d+1}; \mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-value fields (or paths) on $\mathcal{M}^{d+1}$. Note that $C^n(\mathcal{M}^{d+1}; \mathbb{M})$ is an Abelian group under the + operation.

The total space-time lattice $\mathcal{M}^{d+1}$ correspond to a $(d+1)$-chain. We will use the same $\mathcal{M}^{d+1}$ to denote it. Viewing f_{d+1} as a linear map of $(d+1)$-chains, we can define an "integral" over $\mathcal{M}^{d+1}$:

$$\int_{\mathcal{M}^{d+1}} f_{d+1} \equiv \langle f_{d+1}, \mathcal{M}^{d+1}\rangle. \tag{A4}$$

We can define a derivative operator d acting on an n-cochain f_n, which give us an $n+1$-cochain (see Fig. 13):

$$\langle df_n, (i_0 i_1 i_2 \ldots i_{n+1})\rangle = \sum_{m=0}^{n+1} (-)^m \langle f_n, (i_0 i_1 i_2 \ldots \hat{i}_m \ldots i_{n+1})\rangle, \tag{A5}$$

where $i_0 i_1 i_2 \ldots \hat{i}_m \ldots i_{n+1}$ is the sequence $i_0 i_1 i_2 \ldots i_{n+1}$ with i_m removed, and $i_0, i_1, i_2 \ldots i_{n+1}$ are the ordered vertices of the $(n+1)$-simplex $(i_0 i_1 i_2 \ldots i_{n+1})$.

A cochain $f_n \in C^n(\mathcal{M}^{d+1}; \mathbb{M})$ is called a *cocycle* if $df_n = 0$. The set of cocycles is denoted as $Z^n(\mathcal{M}^{d+1}; \mathbb{M})$. A cochain f_n is called a *coboundary* if there exist a cochain f_{n-1} such that $df_{n-1} = f_n$. The set of coboundaries is denoted as $B^n(\mathcal{M}^{d+1}; \mathbb{M})$. Both $Z^n(\mathcal{M}^{d+1}; \mathbb{M})$ and $B^n(\mathcal{M}^{d+1}; \mathbb{M})$ are Abelian groups as well. Since $d^2 = 0$, a coboundary is always a cocycle: $B^n(\mathcal{M}^{d+1}; \mathbb{M}) \subset Z^n(\mathcal{M}^{d+1}; \mathbb{M})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles, $[f_n]$, form the so called cohomology group denoted as

$$H^n(\mathcal{M}^{d+1}; \mathbb{M}) = Z^n(\mathcal{M}^{d+1}; \mathbb{M})/B^n(\mathcal{M}^{d+1}; \mathbb{M}), \tag{A6}$$

$H^n(\mathcal{M}^{d+1}; \mathbb{M})$, as a group quotient of $Z^n(\mathcal{M}^{d+1}; \mathbb{M})$ by $B^n(\mathcal{M}^{d+1}; \mathbb{M})$, is also an Abelian group.

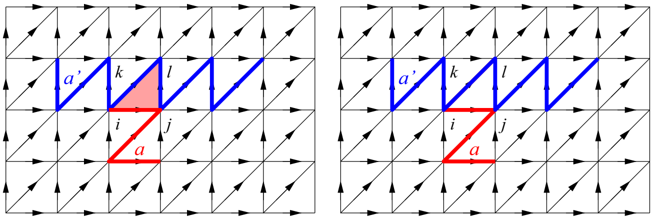

FIG. 14. A 1-cochain a has a value 1 on the red links, Another 1-cochain a' has a value 1 on the blue links. On the left, $a \smile a'$ is nonzero on the shade triangles: $(a \smile a')_{ijl} = a_{ij}a'_{jl} = 1$. On the right, $a' \smile a$ is zero on every triangle. Thus $a \smile a' + a' \smile a$ is not a coboundary.

For the $\mathbb{Z}_N$-valued cocycle x_n, $dx_n \overset{N}{=} 0$. Thus

$$\mathcal{B}_N x_n \equiv \frac{1}{N} dx_n \tag{A7}$$

is a $\mathbb{Z}$-valued cocycle. Here, $\mathcal{B}_N$ is Bockstein homomorphism.

From two cochains f_m and h_n, we can construct a third cochain p_{m+n} via the cup product (see Fig. 14):

$$p_{m+n} = f_m \smile h_n,$$
$$\langle p_{m+n}, (0 \to m+n)\rangle = \langle f_m, (0 \to m)\rangle \langle h_n, (m \to m+n)\rangle, \tag{A8}$$

where $i \to j$ is a consecutive sequence from i to j:

$$i \to j \equiv i, i+1, \dots, j-1, j. \tag{A9}$$

The cup product has the following property

$$d(h_n \smile f_m) = (dh_n) \smile f_m + (-)^n h_n \smile (df_m) \tag{A10}$$

We see that $h_n \smile f_m$ is a cocycle if both f_m and h_n are cocycles. If both f_m and h_n are cocycles, then $f_m \smile h_n$ is a coboundary if one of f_m and h_n is a coboundary. So the cup product is also an operation on cohomology groups $\smile$: $H^m(M^d;\mathbb{M}) \times H^n(M^d;\mathbb{M}) \to H^{m+n}(M^d;\mathbb{M})$. The cup product of two *cocycles* has the following property (see Fig. 14):

$$f_m \smile h_n = (-)^{mn} h_n \smile f_m + \text{coboundary}. \tag{A11}$$

We can also define higher cup product $f_m \underset{k}{\smile} h_n$, which gives rise to a $(m+n-k)$-cochain [67]:

$$\begin{aligned}
&\langle f_m \underset{k}{\smile} h_n, (0, 1, \dots, m+n-k)\rangle \\
&= \sum_{0 \leqslant i_0 < \dots < i_k \leqslant n+m-k} (-)^p \langle f_m, (0 \to i_0, i_1 \to i_2, \dots)\rangle \\
&\quad \times \langle h_n, (i_0 \to i_1, i_2 \to i_3, \dots)\rangle,
\end{aligned} \tag{A12}$$

and $f_m \underset{k}{\smile} h_n = 0$ for $k > m$ or n or $k < 0$. Here $i \to j$ is the sequence $i, i+1, \dots, j-1, j$, and p is the number of permutations to bring the sequence

$$0 \to i_0, i_1 \to i_2, \dots; i_0+1 \to i_1 - 1, i_2 + 1 \to i_3 - 1, \dots \tag{A13}$$

to the sequence

$$0 \to m+n-k. \tag{A14}$$

For example,

$$\begin{aligned}
\langle f_m \underset{1}{\smile} h_n, (0, 1, \dots, m+n-1)\rangle &= \sum_{i=0}^{m-1} (-)^{(m-i)(n+1)} \\
&\times \langle f_m, (0 \to i, i+n \to m+n-1)\rangle \langle h_n, (i \to i+n)\rangle.
\end{aligned} \tag{A15}$$

We can see that $\underset{0}{\smile} = \smile$. Unlike cup product at $k = 0$, the higher cup product of two cocycles may not be a cocycle. For cochains f_m, h_n, we have

$$\begin{aligned}
d(f_m \underset{k}{\smile} h_n) &= df_m \underset{k}{\smile} h_n + (-)^m f_m \underset{k}{\smile} dh_n \\
&\quad + (-)^{m+n-k} f_m \underset{k-1}{\smile} h_n + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m.
\end{aligned} \tag{A16}$$

Let f_m and h_n be cocycles and c_l be a chain, from Eq. (A16), we can obtain

$$\begin{aligned}
d(f_m \underset{k}{\smile} h_n) &= (-)^{m+n-k} f_m \underset{k-1}{\smile} h_n \\
&\quad + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m, \\
d(f_m \underset{k}{\smile} f_m) &= [(-)^k + (-)^m] f_m \underset{k-1}{\smile} f_m, \\
d(c_l \underset{k-1}{\smile} c_l + c_l \underset{k}{\smile} dc_l) &= dc_l \underset{k}{\smile} dc_l - [(-)^k - (-)^l] \\
&\quad \times (c_l \underset{k-2}{\smile} c_l + c_l \underset{k-1}{\smile} dc_l).
\end{aligned} \tag{A17}$$

From Eq. (A17), we see that, for $\mathbb{Z}_2$-valued cocycles z_n,

$$\text{Sq}^{n-k}(z_n) \equiv z_n \underset{k}{\smile} z_n \tag{A18}$$

is always a cocycle. Here, Sq is called the Steenrod square. More generally $h_n \underset{k}{\smile} h_n$ is a cocycle if $n + k =$ odd and h_n is a cocycle. Usually, the Steenrod square is defined only for $\mathbb{Z}_2$ valued cocycles or cohomology classes. Here, we like to define Steenrod square for $\mathbb{M}$-valued cochains c_n:

$$\text{Sq}^{n-k} c_n \equiv c_n \underset{k}{\smile} c_n + c_n \underset{k+1}{\smile} dc_n. \tag{A19}$$

From Eq. (A17), we see that

$$d\text{Sq}^k c_n = d(c_n \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} dc_n) = \text{Sq}^k dc_n + (-)^n \begin{cases} 0, & k = \text{odd} \\ 2\text{Sq}^{k+1} c_n & k = \text{even} \end{cases}. \tag{A20}$$

In particular, when c_n is a $\mathbb{Z}_2$-valued cochain, we have

$$d\text{Sq}^k c_n \overset{2}{=} \text{Sq}^k dc_n. \tag{A21}$$

Next, let us consider the action of Sq^k on the sum of two $\mathbb{M}$-valued cochains c_n and c_n':

$$\begin{aligned}
\mathrm{Sq}^k(c_n+c_n') &= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + c_n \underset{n-k}{\smile} c_n' + c_n' \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} dc_n' + c_n' \underset{n-k+1}{\smile} dc_n\\
&= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + [1+(-)^k] c_n \underset{n-k}{\smile} c_n' - (-)^{n-k}[-(-)^{n-k} c_n' \underset{n-k}{\smile} c_n + (-)^n c_n \underset{n-k}{\smile} c_n']\\
&\quad + c_n \underset{n-k+1}{\smile} dc_n' + c_n' \underset{n-k+1}{\smile} dc_n\\
&= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + [1+(-)^k] c_n \underset{n-k}{\smile} c_n' + (-)^{n-k}[dc_n' \underset{n-k+1}{\smile} c_n + (-)^n c_n' \underset{n-k+1}{\smile} dc_n]\\
&\quad - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n) + c_n \underset{n-k+1}{\smile} dc_n' + c_n' \underset{n-k+1}{\smile} dc_n\\
&= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + [1+(-)^k] c_n \underset{n-k}{\smile} c_n' + [1+(-)^k] c_n' \underset{n-k+1}{\smile} dc_n - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n)\\
&\quad - [(-)^{n-k+1} dc_n' \underset{n-k+1}{\smile} c_n - c_n \underset{n-k+1}{\smile} dc_n']\\
&= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + [1+(-)^k] c_n \underset{n-k}{\smile} c_n' + [1+(-)^k] c_n' \underset{n-k+1}{\smile} dc_n - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n)\\
&\quad - d(dc_n' \underset{n-k+2}{\smile} c_n) + dc_n' \underset{n-k+2}{\smile} dc_n\\
&= \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n' + dc_n' \underset{n-k+2}{\smile} dc_n + [1+(-)^k][c_n \underset{n-k}{\smile} c_n' + c_n' \underset{n-k+1}{\smile} dc_n]\\
&\quad - (-)^{n-k} d(c_n' \underset{n-k+1}{\smile} c_n) - d(dc_n' \underset{n-k+2}{\smile} c_n).
\end{aligned} \tag{A22}$$

We see that, if one of the c_n and c_n' is a cocycle,

$$\mathrm{Sq}^k(c_n+c_n') \overset{2,d}{=} \mathrm{Sq}^k c_n + \mathrm{Sq}^k c_n'. \tag{A23}$$

We also see that

$$\begin{aligned}
\mathrm{Sq}^k(c_n+df_{n-1}) &= \mathrm{Sq}^k c_n + \mathrm{Sq}^k df_{n-1} + [1+(-)^k] df_{n-1} \underset{n-k}{\smile} c_n - (-)^{n-k} d(c_n \underset{n-k+1}{\smile} df_{n-1}) - d(dc_n \underset{n-k+2}{\smile} df_{n-1})\\
&= \mathrm{Sq}^k c_n + [1+(-)^k][df_{n-1} \underset{n-k}{\smile} c_n + (-)^n \mathrm{Sq}^{k+1} f_{n-1}] + d[\mathrm{Sq}^k f_{n-1} - (-)^{n-k} c_n \underset{n-k+1}{\smile} df_{n-1} - dc_n \underset{n-k+2}{\smile} df_{n-1}].
\end{aligned} \tag{A24}$$

Using Eq. (A25), we can also obtain the following result if $dc_n =$ even

$$\mathrm{Sq}^k(c_n+2c_n') \overset{4}{=} \mathrm{Sq}^k c_n + 2d(c_n \underset{n-k+1}{\smile} c_n') + 2dc_n \underset{n-k+1}{\smile} c_n' \overset{4}{=} \mathrm{Sq}^k c_n + 2d(c_n \underset{n-k+1}{\smile} c_n'). \tag{A25}$$

As another application, we note that, for a $\mathbb{Z}_2$ cochain m_d and using Eq. (A16),

$$\begin{aligned}
\mathrm{Sq}^1(m_d) &= m_d \underset{d-1}{\smile} m_d + m_d \underset{d}{\smile} dm_d = \tfrac{1}{2}(-)^d[d(m_d \underset{d}{\smile} m_d) - dm_d \underset{d}{\smile} m_d] + \tfrac{1}{2} m_d \underset{d}{\smile} dm_d\\
&= (-)^d \mathcal{B}_2(m_d \underset{d}{\smile} m_d) - (-)^d \mathcal{B}_2 m_d \underset{d}{\smile} m_d + m_d \underset{d}{\smile} \mathcal{B}_2 m_d\\
&= (-)^d \mathcal{B}_2 m_d - 2(-)^d \mathcal{B}_2 m_d \underset{d+1}{\smile} \mathcal{B}_2 m_d = (-)^d \mathcal{B}_2 m_d - 2(-)^d \mathrm{Sq}^1 \mathcal{B}_2 m_d,
\end{aligned} \tag{A26}$$

where we have used $m_d \underset{d}{\smile} m_d = m_d$. This way, we obtain a relation between the Steenrod square and Bockstein homomorphism, when m_d is a $\mathbb{Z}_2$ valued cocycle

$$\mathrm{Sq}^1(m_d) \overset{2}{=} \mathcal{B}_2 m_d. \tag{A27}$$

APPENDIX B: LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE

The Lyndon-Hochschild-Serre spectral sequence (see Ref. [68], pages 280 and 291 and Ref. [69]) allows us to understand the structure of of the cohomology of a fiber bundle $F \to X \to B$, $H^*(X;\mathbb{R}/\mathbb{Z})$, from $H^*(F;\mathbb{R}/\mathbb{Z})$ and $H^*(B;\mathbb{R}/\mathbb{Z})$. In general, $H^d(X;\mathbb{M})$, when viewed as an Abelian group, contains a chain of subgroups

$$\{0\} = H_{d+1} \subset H_d \subset \ldots \subset H_0 = H^d(X;\mathbb{M}) \tag{B1}$$

such that H_l/H_{l+1} is a subgroup of a factor group of $H^l[B, H^{d-l}(F;\mathbb{M})_B]$, i.e., $H^l[B, H^{d-l}(F;\mathbb{M})_B]$ contains a subgroup Γ^k, such that

$$H_l/H_{l+1} \subset H^l[B, H^{d-l}(F;\mathbb{M})_B]/\Gamma^l, \quad l = 0, \ldots, d. \tag{B2}$$

Note that $\pi_1(B)$ may have a nontrivial action on $\mathbb{M}$ and $\pi_1(B)$ may have a nontrivial action on $H^{d-l}(F;\mathbb{M})$ as determined by the structure $F \to X \to B$. We add the subscript B to $H^{d-l}(F;\mathbb{M})$ to indicate this action. We also have

$$H_0/H_1 \subset H^0[B, H^d(F;\mathbb{M})_B],$$
$$H_d/H_{d+1} = H_d = H^d(B;\mathbb{M})/\Gamma^d. \tag{B3}$$

In other words, all the elements in $H^d(X;\mathbb{M})$ can be one-to-one labeled by $(x_0, x_1, \dots, x_d)$ with

$$x_l \in H_l/H_{l+1} \subset H^l[B, H^{d-l}(F;\mathbb{M})_B]/\Gamma^l. \tag{B4}$$

Note that here $\mathbb{M}$ can be $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$ etc. Let $x_{l,\alpha}$, $\alpha = 1, 2, \dots$, be the generators of H^l/H^{l+1}. Then we say $x_{i,\alpha}$ for all l, α are the generators of $H^d(X;\mathbb{M})$. We also call H_l/H_{l+1}, $l = 0, \dots, d$, the generating subfactor groups of $H^d(X;\mathbb{M})$.

The above result implies that we can use $(k_0, k_1, \dots, k_d)$ with $k_l \in H^l[B, H^{d-l}(F;\mathbb{R}/\mathbb{Z})_B]$ to label all the elements in $H^d(X;\mathbb{R}/\mathbb{Z})$. However, such a labeling scheme may not be one-to-one, and it may happen that only some of $(k_0, k_1, \dots, k_d)$ correspond to the elements in $H^d(X;\mathbb{R}/\mathbb{Z})$. However, on the other hand, for every element in $H^d(X;\mathbb{R}/\mathbb{Z})$, we can find a $(k_0, k_1, \dots, k_d)$ that corresponds to it.

For the special case $X = B\times F$, $(k_0, k_1, \dots, k_d)$ will give us a one-to-one labeling of the elements in $H^d(B\times F;\mathbb{R}/\mathbb{Z})$. In fact,

$$H^d(B \times F;\mathbb{R}/\mathbb{Z}) = \bigoplus_{l=0}^{d} H^l[B, H^{d-l}(F;\mathbb{R}/\mathbb{Z})]. \tag{B5}$$

APPENDIX C: PARTITION FUNCTIONS FOR 3+1D PURE 2-GAUGE THEORY

In this section, we compute the partition function for the pure 2-gauge theory (97) with n = even and m = odd. Let $C^d(\mathcal{M};\mathbb{M})$ be the set of $\mathbb{M}$-valued $(d+1)$-cochains on the complex $\mathcal{M}$, $Z^d(\mathcal{M};\mathbb{M})$ the set of $(d+1)$-cocycles, and $B^d(\mathcal{M};\mathbb{M})$ the set of $(d+1)$-coboundaries. When $m = 0$, the partition function is given by the number of $\mathbb{Z}_n$-valued 2-cocycles $|Z^2(\mathcal{M}^4;\mathbb{Z}_n)|$, which is $|H^2(\mathcal{M}^4;\mathbb{Z}_n)|$ times the number of 1-cochains whose derivatives is nonzero. The number of 1-cochains whose derivatives is nonzero is the number of 1-cochains ($|C^1(\mathcal{M}^4;\mathbb{Z}_n)| = n^{N_e}$) divide by $|H^1(\mathcal{M}^4;\mathbb{Z}_n)|$ and by the number of number of 0-cochains whose derivatives is nonzero. The number of 0-cochains whose derivatives is nonzero is the number of 0-cochains ($|C^0(\mathcal{M}^4;\mathbb{Z}_n)| = n^{N_v}$) divide by $|H^0(\mathcal{M}^4;\mathbb{Z}_n)|$. Thus the partition function is

$$\begin{aligned}
&Z(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),0)\\
&= |Z^2(\mathcal{M}^4;\mathbb{Z}_n)|\\
&= |H^2(\mathcal{M}^4;\mathbb{Z}_n)| \frac{|C^1(\mathcal{M}^4;\mathbb{Z}_n)|}{|H^1(\mathcal{M}^4;\mathbb{Z}_n)|} \frac{|H^0(\mathcal{M}^4;\mathbb{Z}_n)|}{|C^0(\mathcal{M}^4;\mathbb{Z}_n)|}\\
&= n^{N_e-N_v} \frac{|H^2(\mathcal{M}^4;\mathbb{Z}_n)||H^0(\mathcal{M}^4;\mathbb{Z}_n)|}{|H^1(\mathcal{M}^4;\mathbb{Z}_n)|},
\end{aligned} \tag{C1}$$

where N_v is the number of vertices and N_e the number of links. The volume-independent topological partition function is given by

$$Z^{\text{top}}(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),0) = \frac{|H^2(\mathcal{M}^4;\mathbb{Z}_n)||H^0(\mathcal{M}^4;\mathbb{Z}_n)|}{|H^1(\mathcal{M}^4;\mathbb{Z}_n)|}. \tag{C2}$$

When $m \neq 0$, the volume-independent topological partition function is given by

$$\begin{aligned}
&Z^{\text{top}}(\mathcal{M}^4;\mathcal{B}(\mathbb{Z}_n,2),0)\\
&= \frac{|H^0(\mathcal{M}^4;\mathbb{Z}_n)|}{|H^1(\mathcal{M}^4;\mathbb{Z}_n)|} \sum_{b\in H^2(M^4;\mathbb{Z}_n)} e^{i2\pi \int_{\mathcal{M}^4} \frac{m}{2n} b^2 + \frac{m}{2} b \underset{1}{\smile} \mathcal{B}b},
\end{aligned} \tag{C3}$$

where $\sum_{b\in H^2(M^4;\mathbb{Z}_n)} e^{i2\pi \int_{\mathcal{M}^4} \frac{m}{2n} b^2 + \frac{m}{2} b \underset{1}{\smile} \mathcal{B}b}$ replaces $|H^2(M^4;\mathbb{Z}_n)|$.

Now, let us compute topological invariants. On $\mathcal{M}^4 = T^4$, the cohomology ring $H^*(T^4;\mathbb{Z}_n)$ is generated by a_I, $I = 1, 2, 3, 4$, where $a_I \in H^1(T^4;\mathbb{Z}_n) = 4\mathbb{Z}_n$. Using the cohomology ring discussed in Ref. [57], we can parametrize $b^{\mathbb{Z}_n}$ as

$$b = \alpha_{IJ} a_I a_J, \quad \alpha_{IJ} = -\alpha_{JI} \in \mathbb{Z}_n. \tag{C4}$$

We also have $\mathcal{B}b \overset{n}{=} 0$. Thus

$$Z(T^4;\mathcal{B}(\mathbb{Z}_n,2),m) = \frac{1}{n^3} \sum_{\alpha_{IJ}\in\mathbb{Z}_n} e^{i2\pi \frac{m}{2n} 2(\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23})}. \tag{C5}$$

Using $\sum_{\alpha_1,\alpha_2\in\mathbb{Z}_n} e^{i2\pi \frac{m}{n}\alpha_1\alpha_2} = \langle m, n\rangle n$, we find that

$$Z^{\text{top}}(T^4;\mathcal{B}(\mathbb{Z}_n,2),m) = \langle m, n\rangle^3. \tag{C6}$$

On $\mathcal{M}^4 = S^2\times T^2$, the cohomology ring $H^*(T^2\times S^2;\mathbb{Z}_n)$ is generated by a_I, $I = 1, 2$ and b, where $a_I \in H^1(T^2\times S^2;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$ and $b_0 \in H^2(T^2\times S^2;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}$. Using the cohomology ring discussed in Ref. [57], we can parametrize b as

$$b = \alpha_1 a_1 a_2 + \alpha_2 b_0, \quad \alpha_1, \alpha_2 \in \mathbb{Z}_n. \tag{C7}$$

Thus

$$Z^{\text{top}}(S^2 \times T^2;\mathcal{B}(\mathbb{Z}_n,2),m) = \frac{1}{n} \sum_{\alpha_1,\alpha_2\in\mathbb{Z}_n} e^{i2\pi \frac{m}{2n} 2\alpha_1\alpha_2} = \langle m, n\rangle. \tag{C8}$$

On $M^4 = S^1\times L^3(p)$, we need to use the cohomology ring $H^*(S^1\times L^3(p);\mathbb{Z}_n)$ calculated in Ref. [57]:

$$\begin{aligned}
&H^1(S^1 \times L^3(p), \mathbb{Z}_n) = \mathbb{Z}_n \oplus \mathbb{Z}_{\langle p,n\rangle} = \{a_1, a_0\},\\
&H^2(S^1 \times L^3(p), \mathbb{Z}_n) = \mathbb{Z}_{\langle p,n\rangle} \oplus \mathbb{Z}_{\langle p,n\rangle} = \{a_1 a_0, b_0\},\\
&H^3(S^1 \times L^3(p), \mathbb{Z}_n) = \mathbb{Z}_n \oplus \mathbb{Z}_{\langle p,n\rangle} = \{c_0, a_1 b_0\},\\
&H^4(S^1 \times L^3(p), \mathbb{Z}_n) = \mathbb{Z}_n = \{a_1 c_0\},
\end{aligned} \tag{C9}$$

where we have also listed the generators. Here, a_1 comes from S^1 and a_0, b_0, c_0 from $L^3(p)$. The cohomology ring $H^*(S^1\times L^3(p), \mathbb{Z}_n)$ is given by

$$a_1^2 = 0, \quad a_0^2 = \frac{n^2 p(p-1)}{2\langle p, n\rangle^2} b_0,$$
$$a_0 b_0 = \frac{n}{\langle p, n\rangle} c_0, \quad b_0^2 = a_0 c_0 = 0. \tag{C10}$$

For $\langle n,p\rangle = 1$, $Z^{\text{top}}(S^1\times L^3(p);\mathcal{B}(\mathbb{Z}_n,2),m) = 1$. For $\langle n,p\rangle \neq 1$, we can parametrize b as

$$b = \alpha_1 a_0 a_1 + \alpha_2 b_0, \quad \alpha_1,\alpha_2 \in \mathbb{Z}_{\langle n,p\rangle}, \tag{C11}$$

which satisfies $\mathcal{B}b = 0$ (see Ref. [57]). Using $a_0a_1b_0 = \frac{n}{\langle n,p\rangle}a_1c_0$ and $(a_0a_1)^2 = b_0^2 = 0$, we find that

$$Z^{\text{top}}(S^1\times L^3(p);\mathcal{B}(\mathbb{Z}_n,2),m) = \frac{1}{\langle n,p\rangle}\sum_{\alpha_1,\alpha_2=0}^{\langle n,p\rangle-1} e^{i2\pi\frac{m}{\langle n,p\rangle}\alpha_1\alpha_2} = \langle m,n,p\rangle. \tag{C12}$$

On $\mathcal{M}^4 = F^4$, we need to use the cohomology ring $H^*(F^4;\mathbb{Z}_n)$ as described in Ref. [57]:

$$H^1(F^4;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}, \quad H^2(F^4;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2},$$
$$H^3(F^4;\mathbb{Z}_n) = \mathbb{Z}_n^{\oplus 2}, \quad H^4(F^4;\mathbb{Z}_n) = \mathbb{Z}_n. \tag{C13}$$

Let a_1, a_2 be the generators of $H^1(F^4;\mathbb{Z}_n)$, b_1, b_2 the generators of $H^2(F^4;\mathbb{Z}_n)$, c_1, c_2 be the generators of $H^3(F^4;\mathbb{Z}_n)$, and v be the generator of $H^4(F^4;\mathbb{Z}_n)$:

$$H^*(F^4;\mathbb{Z}_n) = \{a_1,a_2,b_1,b_2,c_1,c_2,v\}. \tag{C14}$$

We find that the nonzero cup products are given by

$$b_1^2 = -b_2^2 = a_1c_1 = a_2c_2 = v. \tag{C15}$$

All other cup products vanish.

We can parametrize b as

$$b = \alpha_1 b_1 + \alpha_2 b_2, \quad \alpha_1,\alpha_2 \in \mathbb{Z}_n, \tag{C16}$$

where b_1, b_2 are generators of $H^2(F^4;\mathbb{Z}_n)$. Using $b_1^2 = -b_2^2 = v$, $b_1b_2 = 0$, and $\mathcal{B}b_1 = \mathcal{B}b_2 = 0$, we find that

$$Z^{\text{top}}(F^4;\mathcal{B}(\mathbb{Z}_n,2),m) = \frac{1}{n}\sum_{\alpha_1,\alpha_2=0}^{n-1} e^{i2\pi\frac{m}{2n}(\alpha_1^2-\alpha_2^2)} = \begin{cases}\langle m,n\rangle, & \text{if } \frac{mn}{\langle m,n\rangle^2} = \text{even};\\ 0, & \text{if } \frac{mn}{\langle m,n\rangle^2} = \text{odd}.\end{cases} \tag{C17}$$

The above results, plus some previous results from Ref. [57], are summarized in Table I.

APPENDIX D: SIMPLICIAL SETS, KAN CONDITIONS

A simplicial set X is a contravariant functor from the catogory of finiate ordinals to that of sets, $X : \Delta \to \mathsf{Sets}$, where Δ, the category of finite ordinals, is made up by

$$[0] = \{0\}, \quad [1] = \{0,1\}, \quad \ldots, \quad [n] = \{0,1,\ldots,n\}, \quad \ldots,$$

with order-preserving maps, for example,

$$d^i : [n-1] \to [n], \quad \forall j < i, j \mapsto j, \forall j \geqslant i, j \mapsto j+1,$$

that is, to leave i skipped, or

$$s^i : [n] \to [n-1], \quad \forall j < i, j \mapsto j, \forall j \geqslant i, j \mapsto j-1,$$

that is, to leave i-doubly mapped. In fact, all oder-preserving maps are generated by d^i's and s^i's. In another word, X consists of a tower of sets $X_0, X_1, \ldots, X_n$ with face $d_i : X_n \to X_{n-1}$ and degeneracy $s_i : X_{n-1} \to X_n$, which are dual to d^i and s^i. If we take the simplicial decomposition of a topological space $|X|$, and take X_n to be the set of n-simplices, then the collection of X_n for a simplicial set with d_i the natural face maps and s_i the natural degeneracy maps. Thus it is not hard to imagine, in general, for a simplicial set X, d_i and s_i satisfy expected coherence conditions,

$$\begin{aligned} &d_id_j = d_{j-1}d_i \quad \text{if } i<j, \quad s_is_j = s_{j+1}s_i \text{ if } i \leqslant j,\\ &d_is_j = s_{j-1}d_i \quad \text{if } i<j, \quad d_js_j = \text{id} = d_{j+1}s_j,\\ &d_is_j = s_jd_{i-1} \quad \text{if } i > j+1. \end{aligned} \tag{D1}$$

Example D.1 (m-simplex and (m,j)-horn). If we take a geometric n-simplex and take its natural simplicial decomposition, we end up with a simplicial set Δ^n, which can be described in the following combinatoric way,

$$(\Delta^m)_n = \{f : [n] \to [m] \mid f(i) \leqslant f(j) \text{ for all } i \leqslant j\}, \tag{D2}$$

Similarly, we define the simplicial (m,j)-horn as the following (see Fig. 15):

$$(\Lambda_j^m)_n = \{f \in (\Delta^m)_n \mid \{0,\ldots,j-1,j+1,\ldots,m\} \nsubseteq \{f(0),\ldots,f(n)\}\}. \tag{D3}$$

Their geometric realisation is a m-simplex removing the inner and j-th facet.

Clearly, there is an inclusion of simplicial sets $\iota_{m,j} : \Lambda_j^m \to \Delta^m$.

Then the set of simplicial morphisms $\text{Hom}(\Delta^m, X) = X_m$, and $\text{Hom}(\Lambda_j^m, X)$ is usually some sort of product of X_i's and represents horns in X. For example, $\text{Hom}(\Lambda_1^2, X) = X_1 \times_{d_0, X_0, d_1} X_1$.

Definition D.2. A simplicial set X satisfies the Kan condition Kan(m,j) iff the canonical map (i.e., the horn projection)

$$X_m = \text{Hom}(\Delta^m, X) \xrightarrow{\iota_{m,j}^*} \text{Hom}(\Lambda_j^m, X) \tag{D4}$$

is surjective. It satisfies the unique Kan condition Kan!(m,j) iff the canonical map in (D4) is an isomorphism. We call X a Kan simplicial set (or a Kan complex or an ∞-groupoid) iff it satisfies Kan(m,j) for all $m \geqslant 1$, $0 \leqslant j \leqslant m$. X is called an n-groupoid iff it satisfies Kan(m,j) for all $m \geqslant 1$, $0 \leqslant j \leqslant m$ and Kan!(m,j) for all $m \geqslant n+1$, $0 \leqslant j \leqslant m$. X is called an n-group if it is a n-groupoid and X_0 is a point.

For the content of this Appendix, we refer to the standard text books [70,71] for the theory simplicial sets. ∞-groupoid using Kan condition is due to [72], we also refer to Ref. [[73], Sec. 1] for a nice detailed introduction of this topic.

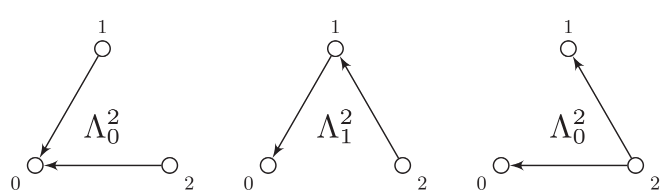

FIG. 15. The horns.

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