AC anomalous Hall effect in topological insulator Josephson junctions

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A nonstationary anomalous Hall current is calculated for a voltage biased Josephson junction, which is composed of two s-wave superconducting contacts deposited on the top of a three-dimensional topological insulator (TI). A homogeneous Zeeman field was assumed at the surface of the TI. The problem has been considered within the ballistic approximation and on the assumption that tunneling of electrons between contacts and the surface of the TI is weak. In this regime the Josephson current has no features of the 4π-periodic topological effect which is associated with Andreev bound states. It is shown that the Hall current oscillates in time. The phase of these oscillations is shifted by $\pi/2$ with respect to the Josephson current and their amplitude linearly decreases with the electric potential difference between contacts. It is also shown that the Hall current cannot be induced by a stationary phase difference of the contact's order parameters.

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I. INTRODUCTION

Anomalous Hall effect (AHE), as well as the more conventional Hall effect, were first observed by Hall more than a century ago [1]. It has been named "anomalous" because in some magnetic materials this effect was detected in the absence of an external magnetic field. In magnetic systems AHE may be explained by the presence of topologically nontrivial magnetic textures [2–5]. Other theories of AHE do not rely on topological magnetic textures, but rather on a strong spin-orbit coupling (SOC) of electrons which, in combination with a homogeneous magnetic order, can lead to AHE [6]. Recently, great interest in AHE was attracted by the discovery of the quantum anomalous Hall effect [7], which can be observed in one-dimensional quantized transport of electrons along edges of topological insulators (TIs) [8,9]. This quantized effect has recently been observed in various systems [10–12].

In addition to AHE in normal metals, the anomalous Hall transport in superconducting systems is fundamental to many practical applications. In some cases this effect may be realized in topologically nontrivial materials via the quantized electron transport of Majorana fermions along chiral edge channels [13–16]. The nontopological AHE was also considered [17–19]. On the other hand, the latter effect has not been addressed sufficiently, while it can extend considerable functionality to superconducting quantum circuits. Important elements of such circuits are Josephson junctions. A great interest is attracted to junctions where the weak link is represented by a two-dimensional electron gas on the conducting surface of a three-dimensional topological insulator [20–30]. Since such systems are characterized by the strong SOC, it is of fundamental interest to find out whether the AHE can be observed there together with the Josephson effect. In contrast to the latter, which manifests itself in a supercurrent between superconducting contacts, the anomalous Hall supercurrent should be directed parallel to the gap separating the contacts. The AHE may be observed only in systems with the broken time inversion symmetry, for example, in the presence of a magnetic order, which adds a mass term into the Dirac Hamiltonian of electron states on the surface of the TI. Such a magnetic TI can be created by a magnetic impurity doping [10–12], in antiferromagnetic TIs [31,32], or in TI magnetic insulator heterostructures [33–37].

Our goal is to study the nontopological AHE in Josephson contacts with a magnetic TI taken as a weak link. The Josephson current in nontopological junctions has a conventional 2π periodicity, as a function of the phase difference between order parameters of superconducting contacts. In contrast, the topological Josephson effect is characterized by the 4π periodicity [20]. From the experimental point of view such a nontopological AHE is of special interest, because it does not require special experimental conditions. In particular, it is not necessary to provide a large proximity induced superconducting gap on the surface of the TI, in order to guarantee that Andreev bound states will dominate the electron transport between contacts. We will consider a junction under the voltage bias, because the analysis below shows that AHE cannot be driven by a static phase difference between superconducting terminals. Therefore, the Hall (super)current oscillates in time, as does the Josephson current of Cooper pairs. The model system is shown in Fig. 1. It will be assumed that the superconducting contacts are weakly coupled to the TI surface. Therefore, the superconducting proximity effect, which is induced in the TI by the contacts, may be taken into account perturbatively. In this situation the role of Andreev bound states of TI electrons is not important, because the proximity induced minigap under contacts is small. It is of the order of the tunneling rate Γ between a contact and TI. This rate is assumed to be much less than the superconducting gap Δ in both contacts. Hence, the transport of electrons between superconductors mostly occurs by means of quasiparticle states outside the minigap.

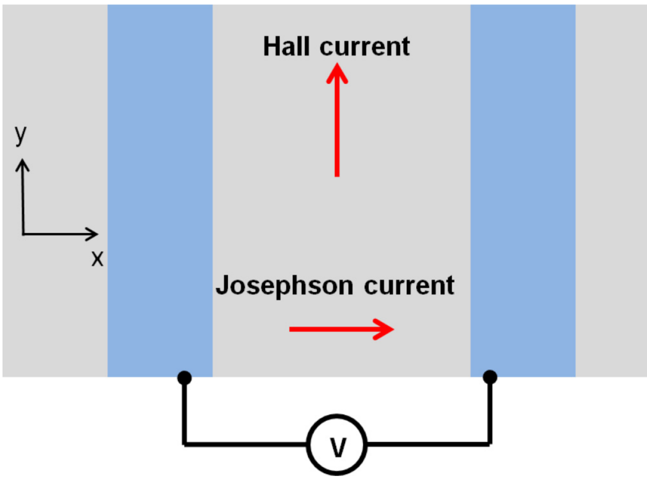

FIG. 1. Two superconducting contacts are placed on the top of a three-dimensional magnetic topological insulator. The voltage bias V is applied to the contacts. Under this bias the oscillating in time anomalous Hall current of Cooper pairs is induced perpendicular to the Josephson current.

Even within the perturbation theory an analysis of the considered problem poses serious difficulties because it is beyond a conventional semiclassical approach [38]. Moreover, for this reason one cannot use the Born approximation for an analysis of impurity scattering effects. Therefore, it will be assumed that the transport of electrons between contacts is ballistic. It requires the sufficiently small distance L between contacts. For example, highly ballistic TI junctions were reported in Ref. [27] with $L \simeq 100$ nm.

The paper is organized in the following way. The formalism employed in this paper will be presented in Sec. II. In Sec. III two situations will be considered, depending on the position of the chemical potential with respect to the energy gap, which is induced by the Zeeman field. A discussion of results is presented in Sec. IV.

II. FORMALISM

The unperturbed Hamiltonian of two-dimensional electron gas on the TI surface is given by [14] $H_0 = \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \mathcal{H}_{0\mathbf{k}} \psi_{\mathbf{k}}$, where $\psi_{\mathbf{k}}$ are the electron field operators, which are defined in the Nambu basis as $\psi = (\psi_\uparrow, \psi_\downarrow, \psi^\dagger_\downarrow, -\psi^\dagger_\uparrow)$, and the one-particle Hamiltonian $\mathcal{H}_{0\mathbf{k}}$ is given by

$$\mathcal{H}_{0\mathbf{k}} = \tau_3(v\mathbf{k} \times \boldsymbol{\sigma} - \mu) + \tau_0 M \sigma_z. \quad (1)$$

Here, μ is the chemical potential, M is the exchange field produced by the magnetic order, and σ^j denote Pauli matrices ($j = x, y, z$). The Pauli matrices τ_i, $i = 0, 1, 2, 3$ operate in the Nambu space, where τ_0 is the unit matrix. We assume a weak tunneling coupling between the TI and superconducting contacts. The corresponding tunneling Hamiltonians H_L and H_R for the left and right contacts can be written in the form

$$H_{L(R)} = \sum_{\mathbf{k},\mathbf{k}'} \left(\psi^\dagger_{\mathbf{k}} t_{L(R)\mathbf{k},\mathbf{k}'} \tau_3 \psi^S_{L(R)\mathbf{k}'} + \text{H.c.}\right), \quad (2)$$

where $\psi^S_{L(R)\mathbf{k}'}$ are electron field operators in the left and right contacts. Generally, the tunneling parameters $t_{L(R)\mathbf{k},\mathbf{k}'}$ are spin dependent. Let us consider, as an example, time-reversal-symmetric TI belonging to the Bi_2Se_3 family. In these materials in the leading $\mathbf{k}\cdot\mathbf{p}$ expansion the tunneling parameters are diagonal in spin space, but are different for opposite spin projections. This is dictated by the form of Bloch functions which are associated with surface states near the Dirac point. According to Refs. [14,39], the degenerate pair of such functions has the form $(\psi_1 + i\psi_2)|\uparrow\rangle$ and $(\psi_1 - i\psi_2)|\downarrow\rangle$, where the arrows denote the spin projection and real functions ψ_1 and ψ_2 are composed from p_z atomic orbitals. Within the tight-binding approximation the tunneling parameters are determined by the respective overlap integrals a_1 and a_2 of these functions, which are adjacent to TI atomic orbitals of a contact material (a superconductor or a spacer). The latter are assumed spin independent. Therefore, the tunneling parameters in Eq. (2) are proportional to $a_1 + ia_2$ and $a_1 - ia_2$ for up- and down-spin projections, respectively. A $\mathbf{k}\cdot\mathbf{p}$ expansion near the Γ point may result in small spin-dependent corrections, which will be ignored below. One should take into account that, since the contact size in the x direction is finite, the in-plane component of the wave vector of a tunneling particle is not conserved. Therefore, $t_{L(R)\mathbf{k},\mathbf{k}'} \propto \delta_{\mathbf{k}-\mathbf{k}',\mathbf{q}}$, where $\mathbf{q}$ is the Fourier wave vector of a function which describes the contact shape.

The Hall current in the junction is directed parallel to the y axis. The corresponding one-particle current operator is given by $v\sigma_x$. Therefore, the Hall current density J^y_H may be expressed in terms of the Keldysh Green's function as

$$J^y_H(\mathbf{r}, t) = -\frac{ive}{4}\text{Tr}[\sigma^x G^K(t, \mathbf{r}; t, \mathbf{r})]. \quad (3)$$

The Josephson current and anomalous Hall current are given by the fourth order in the expansion of G^K with respect to the tunneling parameter. Such a perturbative approach was previously employed for calculation of the Josephson current [40] and the spin-Hall current in voltage biased Josephson junctions [41]. Each superconducting contact gives rise to the self-energy $\Sigma_{L(R)}$, which may be written in the form

$$\Sigma_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(t, t') = \sum_{\mathbf{k}'} t_{L(R)\mathbf{k},\mathbf{k}'} t_{L(R)\mathbf{k}',\mathbf{k}+\mathbf{q}} G^S_{\mathbf{k}'}(t, t'). \quad (4)$$

Because of the electric potentials $V_{L/R}/e = \pm V/2e$ on contacts, $\Sigma_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(t, t')$ takes the form

$$\Sigma_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(t, t') = e^{i\tau_3 V_{L(R)} t} \Sigma_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(t - t') e^{-i\tau_3 V_{L(R)} t'}. \quad (5)$$

Although $t_{L(R)\mathbf{k},\mathbf{k}'}$ depends on spin, the self-energy in Eq. (4) is a spin-independent function. It is guaranteed by a spin-singlet structure of the superconductor Green's function G^S in Eq. (4) and by a form of the spin dependence of tunneling parameters. Indeed, since the spin-dependent part of $t_{\mathbf{k},\mathbf{k}'}$ has the form $a_1 \pm ia_2$ with real a_1 and a_2, it is easy to see that for both spin orientations the self-energy will be proportional to $a_1^2 + a_2^2$ and, hence, the self-energy is spin independent.

As was discussed above, the only important wave-vector dependence of $t_{L(R)\mathbf{k},\mathbf{k}'}$ is associated with the finite size of contacts in the x direction. The shape of the contacts may be taken into account by multiplying the self-energies $\Sigma_{L(R)}$ by the functions $s_L(x) = \theta(x - x_{L2})\theta(x_{L1} - x)$ and $s_R(x) = \theta(x - x_{R1}\theta(x_{R2} - x)$ where the distance between the contacts is $L = x_{R1} - x_{L1}$. The Fourier components of these functions will be denoted as $s_{L(R)}(\mathbf{q})$. By integrating G^S in Eq. (4) over $\mathbf{k}'$ one may obtain a simple expression for temporal Fourier components of the retarded (r) and advanced (a) self-energies $\Sigma^{r/a}_{L(R)}(t-t')$ in the form

$$\Sigma^{r/a}_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(\omega) = \Gamma s_{L(R)}(\mathbf{q})\frac{\tau_1\Delta}{\sqrt{(\omega\pm i\delta)^2-\Delta^2}}, \tag{6}$$

where Δ is the superconducting order parameter and Γ can be expressed through the resistance R_b of the superconductor-normal metal interface, as $\Gamma = 1/4e^2N_FR_b$, with N_F denoting the state density at the Fermi energy. Only nondiagonal matrix elements of Σ in the Nambu space are taken into account, because only these terms contribute to the Josephson current in the tunneling regime. In the following it will be convenient to write Nambu matrix elements of $\Sigma^{r/a}_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(\omega)$ in Eq. (6) as

$$\left(\Sigma^{r/a}_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(\omega)\right)_{12} = \left(\Sigma^{r/a}_{L(R)\mathbf{k},\mathbf{k}+\mathbf{q}}(\omega)\right)_{21} = \Sigma^{r/a}(\omega)s_{m\mathbf{q}}, \tag{7}$$

where the subscript m takes the values 1 and -1 for Σ_R and Σ_L, respectively, and $\Sigma^{r/a}(\omega) = \Gamma\Delta/\sqrt{(\omega\pm i\delta)^2-\Delta^2}$.

We are interested in the total electric current in the y direction. Therefore, the current density in Eq. (3) should be integrated over x. In the second order with respect to Γ, Fourier components of the total current can be expressed from Eq. (3) in the form

$$J^y_H(\pm 2\Omega) = ie\frac{v}{4}\int\frac{d\omega}{2\pi}dx dx' \sum_{\mathbf{k},\mathbf{q},m=\pm\tau} s_m(x)s_{-m}(x') \times \mathrm{Tr}\left[\sigma_x G_{\tau,\mathbf{k}+\mathbf{q}}(\omega+m\tau\Omega)\Sigma\left(\omega+\frac{m\tau\Omega}{2}\right)G_{-\tau,\mathbf{k}}(\omega)\Sigma\left(\omega-\frac{m\tau\Omega}{2}\right)G_{\tau,\mathbf{k}+\mathbf{q}}(\omega-m\tau\Omega)\right]^K e^{iq(x'-x)}, \tag{8}$$

where the superscript K denotes the Keldysh component of the matrix product in square brackets, $\Omega = V$, $q \equiv q_x$, and the trace is taken over spin variables. The unperturbed Green's functions $G_{\tau,\mathbf{k}}(\omega)$ of TI electrons in Eq. (8) may be obtained from Eq. (1), where $\tau_3 \to \tau = \pm 1$. Therefore, the corresponding retarded, advanced, and Keldysh functions are given by

$$G^{r(a)}_{\tau,\mathbf{k}}(\omega) = (\omega - \tau v\mathbf{k}\times\boldsymbol{\sigma} + \tau\mu - M\sigma_z \pm i\delta)^{-1},$$
$$G^K_{\tau,\mathbf{k}}(\omega) = \left[G^r_{\tau,\mathbf{k}}(\omega) - G^a_{\tau,\mathbf{k}}(\omega)\right]\tanh\frac{\omega}{2k_BT}, \tag{9}$$

where T is the temperature. Since the system is uniform in the y direction, one may set $k_y = 0$. Therefore, the Green's functions in Eq. (9) can be written as

$$G^{r(a)}_{\tau,\mathbf{k}}(\omega) = \frac{\omega + \tau v k_x\sigma_y + \tau\mu + M\sigma_z}{(\omega+\tau\mu\pm i\delta)^2 - E^2_{\mathbf{k}}}, \tag{10}$$

where $E_{\mathbf{k}} = \sqrt{v^2k_x^2 + M^2}$ and $\delta \to 0$.

The sum over $\mathbf{k}$ and $\mathbf{q}$ in Eq. (8) involves a product of three Green's functions. By taking the Keldysh component of the matrix in Eq. (8) one obtains various combinations of the retarded and advanced Green's functions. These combinations are summed up over $\mathbf{k}$, $\mathbf{q}$, and spin variables. As a result, we obtain a set of the functions $b^{\mathrm{abc}}_{m,\tau}(x-x')$, which are given by

$$b^{\mathrm{abc}}_{m,\tau} = \sum_{\mathbf{k},\mathbf{q}}\mathrm{Tr}\big[\sigma_x G^{\mathrm{a}}_{\tau,\mathbf{k}+\mathbf{q}}(\omega+m\tau\Omega) \times G^{\mathrm{b}}_{-\tau,\mathbf{k}}(\omega)G^{\mathrm{c}}_{\tau,\mathbf{k}+\mathbf{q}}(\omega-m\tau\Omega)\big]e^{iq(x'-x)}. \tag{11}$$

Each of the symbols a, b, and c takes the value r or a. By taking the trace in Eq. (11), the latter can be transformed to

$$b^{\mathrm{abc}}_{m,\tau} = -4im\Omega vM\sum_{\mathbf{k},\mathbf{q}}(2k_x+q)D^{\mathrm{a}}_{\tau,\mathbf{k}+\mathbf{q}}(\omega+m\tau\Omega) \times D^{\mathrm{b}}_{-\tau,\mathbf{k}}(\omega)D^{\mathrm{c}}_{\tau,\mathbf{k}+\mathbf{q}}(\omega-m\tau\Omega)e^{iq(x'-x)}, \tag{12}$$

where the functions D^r and D^a are given by $D^{r/a}_{\tau,\mathbf{k}}(\omega) = [(\omega+\tau\mu\pm i\delta)^2 - E^2_{\mathbf{k}}]^{-1}$. It turned out that the functions $b^{\mathrm{abc}}_{m,\tau}$ are proportional to Ω. This means that within the considered model the Hall current cannot be induced by a "phase" bias, provided by a static phase difference of order parameters in contacts. It follows from Eq. (12) that the functions b^{abc} satisfy the equations

$$b^{\mathrm{abc}}_{m,\tau}(\omega,x) = -b^{\mathrm{abc}}_{m,\tau}(\omega,-x),$$
$$b^{\mathrm{abc}}_{1,-1}(\omega,x) = b^{*\mathrm{abc}}_{1,1}(-\omega,x). \tag{13}$$

By calculating the Keldysh projection of the matrix in Eq. (8) and taking into account Eq. (13) the Hall current $J^y_H(2\Omega)$ may be written in terms of the functions b^{abc}, as

$$J^y_H(2\Omega) = ie\frac{v}{4}\int\frac{d\omega}{2\pi}dx dx' s_R(x)s_L(x')(S_1+S_2+S_3), \tag{14}$$

where

$$S_1 = b^{rrr}\Sigma^r_+\Sigma^r_-\tanh\frac{\omega-\Omega}{2k_BT} - b^{aaa}\Sigma^a_+\Sigma^a_-\tanh\frac{\omega+\Omega}{2k_BT},$$
$$S_2 = b^{rra}\Sigma^r_+\Sigma^r_-\left(\tanh\frac{\omega}{2k_BT} - \tanh\frac{\omega-\Omega}{2k_BT}\right) + b^{raa}\Sigma^a_+\Sigma^a_-\left(\tanh\frac{\omega+\Omega}{2k_BT} - \tanh\frac{\omega}{2k_BT}\right), \tag{15}$$

and

$$S_3 = b^{rra}\Sigma^r_+(\Sigma^r_- - \Sigma^a_-)\left(\tanh\frac{2\omega-\Omega}{4k_BT} - \tanh\frac{\omega}{2k_BT}\right) + b^{raa}(\Sigma^r_+ - \Sigma^a_+)\Sigma^a_-\left(\tanh\frac{2\omega+\Omega}{4k_BT} - \tanh\frac{\omega}{2k_BT}\right), \tag{16}$$

where $\Sigma^{r/a}_\pm = \Sigma^{r/a}(\omega\pm\frac{\Omega}{2})$ and $b^{\mathrm{abc}} = b^{\mathrm{abc}}_{1,1}(x-x') + b^{\mathrm{abc}}_{-1,-1}(x'-x)$.

In its turn, the time dependence of the Hall current is given by

$$J_H^y(t) = J_H^y(2\Omega)e^{2i\Omega t} + J_H^{*y}(2\Omega)e^{-2i\Omega t}. \tag{17}$$

III. LIMITING CASES

A. The chemical potential outside the mass gap

The exchange interaction M gives rise to a gap in the spectrum of electron states, as can be seen from the poles of the Green's function in Eq. (10). In doped TIs the Fermi level can be outside the gap. Let us consider the case when $\mu > M > 0$. It will also be assumed that $V \ll \Delta \ll \mu$ and $|q| \ll k_F$, where $vk_F = \sqrt{\mu^2 - M^2}$, with the Fermi velocity v_F given by $v_F = v\sqrt{\mu^2 - M^2}/\mu$. At these assumptions the functions $b^{\rm abc}$ can be calculated analytically from Eqs. (11) and (12), by linearizing the denominators of Green's functions Eq. (10) near the Fermi energy. As a result, we obtain for $m = \tau = 1$ and $x - x' > 0$

$$\begin{aligned}
b_{1,1}^{rrr}(\omega) &= \gamma \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \exp\frac{2i\omega(x-x')}{v_x} \sin\frac{\Omega(x-x')}{v_x},\\
b_{1,1}^{aaa}(\omega) &= b_{1,1}^{rrr*}(\omega),\\
b_{1,1}^{rra}(\omega) &= -i\frac{\gamma}{2}\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \exp\frac{2i\omega(x-x')}{v_x} \exp\frac{i\Omega(x-x')}{v_x},\\
b_{1,1}^{raa}(\omega) &= b_{1,1}^{rra}(-\omega),\, b_{1,1}^{\rm abc}(x-x') = b_{-1,-1}^{\rm abc}(x'-x),
\end{aligned} \tag{18}$$

where $\gamma = 2M/\mu v^3$ and $v_x = v_F\cos\phi > 0$.

According to Eq. (6), at $\Omega \ll \Delta$ the function $\Sigma_\pm^r - \Sigma_\pm^a$ in Eq. (16) is finite only at $\omega \gtrsim \Delta$. In this range, however, the temperature-dependent factors $[\tanh(\omega/2k_BT \pm \Omega/2k_BT) - \tanh(\omega/2k_BT)]$ are exponentially small, as $\exp(-2\Delta/k_BT)$ at $k_BT \ll \Delta$. Therefore, one may ignore S_3 in Eq. (14). An important parameter range is determined by the distance L between contacts and by their width w. In the case of $w \sim L$ the characteristic flight time $T_f = L/v$. For a typical TI (for example, Bi_2S_3) with $v = 5\times 10^5$ m/s and $L = 100$ nm one obtains $1/T_f = 3$ meV. Therefore, for such a ballistic junction both $1/T_f \gg \Omega$ and $\Delta \gg \Omega$. In this case a contribution to Eq. (14) given by S_2 can be easily calculated, because, according to Eq. (15), only small $\omega \sim \Omega$ contribute to S_2. Hence, one may set $\omega = 0$ in b^{rra} and b^{raa} in Eq. (18), as well as in $\Sigma^{r/a}$ given by Eqs. (6) and (7). In this case $\Sigma_\pm^{r/a} = i\Gamma$ in Eq. (15). As a result, the integration over x, x', ω, and ϕ in Eqs. (14) and (18) gives for the current $J_{H2}^y(2\Omega)$, which is associated with the second term in Eq. (14), the following expression:

$$J_{H2}^y(2\Omega) = \frac{e}{4\pi}\Omega\Gamma^2 w^2 \frac{M}{\mu v^2}. \tag{19}$$

The total current, which includes the first two terms in Eq. (14) and ignores the small third one, can be written in the form

$$J_H^y(2\Omega) = J_{H2}^y(2\Omega)R(L,w), \tag{20}$$

where the function R is plotted as a function of L in Fig. 2, at various w.

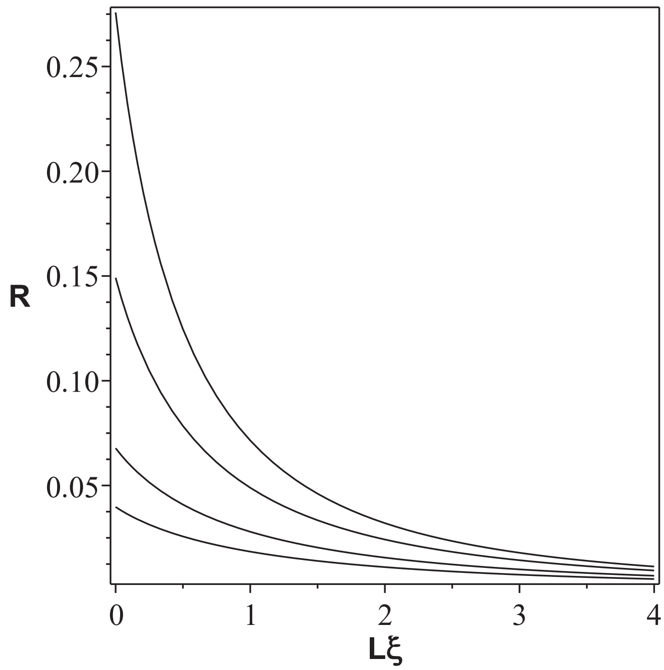

FIG. 2. The anomalous Hall current as a function of the distance between contacts [see Eq. (20)]. Curves from the top to the bottom: $w\xi$=0.5, $w\xi$=1, $w\xi$=2, and $w\xi$=3, where $\xi = \Delta/v$

B. The chemical potential inside the mass gap

In this section we will assume $\mu = 0$. In this case the magnetic gap M prevents penetration of Cooper pairs into the TI. Therefore, the distance between superconducting contacts must be small enough so that $MT_f \lesssim 1$. In the same way as in Eq. (14), the anomalous Hall current can be expressed in terms of the functions $b^{\rm abc}$, that are given by Eq. (12). In contrast to Sec. IIIA, however, the factors b^{rra} and b^{raa} are proportional to Ω at $\Omega \ll M$. Therefore, by taking into account the temperature-dependent statistical factor in Eq. (15), which is $\sim\Omega$, we arrive at $S_2 \sim \Omega^2$. Hence, the leading contribution to J_H is given by S_1. Since $b^{rrr}\Sigma_+^r\Sigma_-^r$ and $b^{aaa}\Sigma_+^a\Sigma_-^a$ are analytical functions of ω in the upper and lower complex semiplanes, respectively, it is convenient to transform the integration over ω in Eq. (14) into the sum over Matsubara frequencies $\Omega_n = \pi(2n+1)$. As a result, in the leading approximation with respect to Ω Eq. (14) gives

$$\begin{aligned}
J_H^y(2\Omega) &= ie2\Omega v^2 MT\Gamma^2 \int dx dx' s_R(x)s_L(x')\\
&\quad\times \sum_{\omega_n,\mathbf{k},q} \frac{2k_x - q}{(\omega_n^2 + E_\mathbf{k}^2)^2}\frac{\exp[iq(x'-x)]}{\omega_n^2 + E_{\mathbf{k}-\mathbf{q}}^2}\frac{\Delta^2}{\omega_n^2+\Delta^2}
\end{aligned} \tag{21}$$

where $\mathbf{q} = q\mathbf{n}_x$. Analytical expressions for the Hall current may be obtained, by assuming that the distance between contacts $L \ll v/M$. Since at $v/M < L$ the current decreases fast, this limiting case gives the upper bound on the current. In order to analyze main qualitative trends, it is sufficient to consider the cases of the wide ($w \gg v/M$) and narrow ($w \ll v/M$) contacts. At small temperatures $T \ll \Delta$ Eq. (21)

gives

$$J_H^y(2\Omega) = e\frac{\Omega}{2\pi}\frac{\Delta}{24}\frac{\Gamma^2}{M^2}\frac{\Delta+2M}{(\Delta+M)^2} \tag{22}$$

for wide contacts and

$$J_H^y(2\Omega) = e\frac{\Omega}{2\pi}\frac{w^2}{4}\frac{\Gamma^2}{v^2}\frac{\Delta}{\Delta+M} \tag{23}$$

for narrow contacts.

IV. DISCUSSION

The anomalous Hall current has been calculated for a ballistic Josephson junction in two regimes, depending on a position of the chemical potential with respect to the mass gap, which in turn is induced by an exchange field. In both cases, according to Eqs. (19), (22), and (23), the Hall current is proportional to $V\cos 2Vt$. Hence, its oscillation amplitude vanishes at $V \to 0$. This result also signals that the Hall effect cannot be observed in a stationary state, where the Josephson current is induced by the phase difference between superconducting contacts. This effect has no relevance to the topological 4π Josephson tunneling [20]. The latter requires a good contact of superconductors with the TI surface, so that a sufficiently large proximity induced energy gap might be formed under contacts. Such a gap can support the bound Andreev states which are involved in the 4π Josephson current. In the case considered here, however, such a proximity gap is equal to the small tunneling rate Γ and can be ignored.

Above, the Josephson current was calculated within the ballistic approximation. It is important to understand a possible influence of disorder on this current. At first sight it seems that this influence is weak when the mean free path l of electrons in the TI is much larger than the distance between contacts, as well as their size in the x direction. The situation, however, is more complicated in the case when $\mu > M$. This becomes evident from an analysis of the Hall current distribution in the x direction. As shown in Appendix A, the current which is associated with S_2 in Eq. (14) extends far outside the contacts over the distance $\sim v/\Omega$. At small Ω the latter can exceed l and the ballistic approximation becomes invalid. It is reasonable to expect that at such small frequencies the Hall current is able to penetrate only over the distance which is less than l. As a result, at $\sim v/\Omega \gg l$ the Hall current J_{H2}^y should be proportional to $\sim\Omega^2$, rather than Ω in Eq. (19). At the same time, as shown in Appendix A, the Hall current which is associated with S_1 is distributed only in the region between and under contacts. Therefore, the effect of a disorder is not so destructive on this current, as long as the contacts are close to each other and are not too wide.

In the case of $\mu < M$ the mass gap restricts the distance over which the current propagates outside the contacts. Hence, the scattering effects are not important, as long as $M \gg v/l$.

In the model considered here the contacts are infinitely extended in the y direction. A restriction of their size in this direction would lead to charge accumulation and to an electric potential buildup near contact ends. Therefore, the presence of low ohmic normal contacts is assumed at $y = \pm\infty$. Also, one may consider TI wires, or ribbons, which are coated with superconducting films. In this case the Hall current will circulate around the wire.

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APPENDIX: SPATIAL DISTRIBUTION OF THE HALL CURRENT DENSITY

The above analysis has been focused on a calculation of the total anomalous Hall current, which is given by an integral of the current density over the x coordinate. On the other hand, the x dependence of this density allows one to better understand a physics of the considered Hall effect. The current density $J_H^y(\pm 2\Omega, x)$ has a more complicated structure in comparison with Eq. (8) and is given by

$$J_H^y(\pm 2\Omega, x) = \frac{iev}{4}\int \frac{d\omega}{2\pi} dx' dx'' \sum_{m=\pm\tau} s_m(x') s_{-m}(x'') \sum_{\mathbf{k},\mathbf{q}_1,\mathbf{q}_2} e^{iq_1(x-x')} e^{iq_2(x-x'')}$$
$$\times \operatorname{Tr}\left[\sigma_x G_{\tau,\mathbf{k}+\mathbf{q}_1}(\omega + m\tau\Omega)\Sigma\left(\omega + \frac{m\tau\Omega}{2}\right) G_{-\tau,\mathbf{k}}(\omega)\Sigma\left(\omega - \frac{m\tau\Omega}{2}\right) G_{\tau,\mathbf{k}-\mathbf{q}_2}(\omega - m\tau\Omega)\right]^K. \tag{A1}$$

Instead of the functions $b^{\mathrm{abc}}_{m,\tau}$, which are defined by Eq. (11), one can introduce the x-dependent functions $b^{\mathrm{abc}}_{m,\tau}(x)$. In the range of parameters considered in Sec. III A they have the form

$$b^{\mathrm{abc}}_{m,\tau}(x) = -8im\Omega vM \sum_{\mathbf{k},\mathbf{q}_1,\mathbf{q}_2} k_x e^{iq_1(x-x')} e^{iq_2(x-x'')} \times D^{\mathrm{a}}_{\tau,\mathbf{k}+\mathbf{q}_1}(\omega + m\tau\Omega) D^{\mathrm{b}}_{-\tau,\mathbf{k}}(\omega) D^{\mathrm{c}}_{\tau,\mathbf{k}-\mathbf{q}_2}(\omega - m\tau\Omega). \tag{A2}$$

From this equation one obtains

$$\begin{aligned}
b^{rrr}_{1,1}(\omega) &= \gamma\theta(x'-x)\theta(x-x'')\Omega\Phi(x),\\
b^{rra}_{1,1}(\omega) &= -\gamma\theta(x'-x'')\theta(x''-x)\Omega\Phi(x),\\
b^{raa}_{1,1}(\omega) &= -\gamma\theta(x'-x'')\theta(x-x')\Omega\Phi^*(x),\\
b^{aaa}_{1,1}(\omega) &= b^{rrr*}_{1,1}(\omega),
\end{aligned} \tag{A3}$$

where $\theta(x)$ is the Heaviside step function and

$$\Phi(x) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{2i\widetilde{\omega}(x'-x'')} e^{-2i\widetilde{\Omega}x} e^{i\widetilde{\Omega}(x'+x'')}. \qquad (A4)$$

In these equations $\widetilde{\Omega} = \Omega/v_x$ and $\widetilde{\omega} = \omega/v_x$. Other functions, such as $b^{\rm abc}_{1,-1}$, depend on x in a similar way.

It is easy to see that the integration of Eq. (A3) over x from $-\infty$ to $+\infty$ results in Eq. (18). However, the convergence of integrals for the functions b^{rrr} and b^{aaa} is much better in comparison with the convergence of b^{rra} and b^{raa}. In the former case x is confined due to theta functions between x' and x'', which are coordinates belonging to the contacts, while in the latter case x is free to vary either to $+\infty$ (for b^{raa}) or to $-\infty$ (for b^{rra}). Therefore, in this case the corresponding integrals are converging only at finite Ω. This is the consequence of the ballistic approximation used in this paper. As a result, there are physical restrictions on Ω. Namely, this frequency must be large enough with respect to various competing parameters which may restrict the distance of the ballistic propagation of electrons. For example, it must be large in comparison with the elastic-scattering rate. Since, according to Eq. (15), b^{rra} and b^{raa} contribute to J_{H2} in Eq. (19), this current must depend strongly on the impurity scattering, as discussed in Sec. IV.

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