Charge carriers with fractional exclusion statistics in cuprates

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We show that in the $SU(2) \times U(1)$ spin-charge gauge approach we developed earlier one can attribute consistently an exclusion statistics with parameter $1/2$ to the spinless charge carriers of the t-J model in two dimensions, as it occurs in one dimension. Like the one-dimensional case, the no-double occupation constraint is at the origin of this fractional exclusion statistics. With this statistics we recover a large Fermi volume of holes at high dopings, close to that of the tight binding approximation. Furthermore, the composite nature of the hole, made of charge and spin carriers only weakly bounded, can provide a natural explanation of many unusual experimental features of the hole-doped cuprates.

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I. INTRODUCTION

Despite continuous experimental advances an agreement has not yet been obtained on the interpretation of the low-energy physics of cuprates; see, e.g., Ref. [1] for an excellent, state-of-the-art review. A general consensus has been achieved, however, that most of the relevant phenomena in hole-doped materials can be derived from modeling the CuO planes in these materials in terms of a two-dimensional (2D) t-J model on the copper sites with the Hamiltonian:

$$H_{t-J} = \sum_{\langle i,j\rangle} P_G[-t c^*_{i\alpha} c_{j\alpha} + J\vec{S}_i \cdot \vec{S}_j]P_G, \tag{1.1}$$

where $\langle i, j\rangle$ denotes the nearest-neighbor (NN) sites, $c_{i\alpha}$ the hole field operator with spin index α on site i, P_G the Gutzwiller projection eliminating double occupation, and summation over repeated spin (and vector) indices is understood hereafter. The Gutzwiller projected holes describe the Zhang-Rice singlets [2] of cuprates.

A way of implementing the Gutzwiller projection is to apply to fermions of the model a spin-charge decomposition formalism. It has been pioneered by Anderson [3] and Kivelson [4] and it is suggested for cuprates by the rather different response of charge and spin degrees of freedom in many experiments: One rewrites the fermion field c_α as a product of a spinless holon field h carrying the charge degree of freedom and a spin-1/2 spinon field s_α carrying the spin degree of freedom, imposing on them a constraint reproducing the Gutzwiller projection. Due to this decomposition an emergent (slave-particle) $U(1)$ gauge symmetry appears, since holon and spinon fields can be multiplied by factors with opposite phases leaving the original fermion field unchanged. For this reason only slave-particle gauge-invariant fields are physical and therefore neither the holon nor the spinon by themselves are physical and they are strongly coupled by gauge fluctuations. However, by gauge fixing this gauge symmetry one can consider gauge-dependent fields, which may be convenient in the description of at least some momentum-energy range, as gluons and quarks in high-energy QCD, in spite of the fact that only mesons and baryons are physical in the strict sense. It makes sense then to discuss the statistics of holon and spinon fields and of the quasiparticle excitations, in this generalized sense, that they might generate in the low-energy limit. This is a key issue of this paper. Notice that the spin-charge decomposition formalism does not prohibit *a priori* that the fermion field c_α describes an elementary excitation without a composite structure, as it happens, e.g., for the meson field written in terms of quark fields in lattice QCD in the superconfining phase, as discussed in Ref. [5]. Whether the fermion excitation described by c_α is composite or not is a dynamical question, it does not have a purely kinematical character as, on the contrary, the spin-charge decomposition does. Somewhat related considerations on the kinematical character of the spin-charge decompositions versus the

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dynamical character of their low-energy excitations can be found in Ref. [6].

As rigorously shown in Ref. [7], spin-charge decomposition can be achieved in the Lagrangian formalism by coupling the original fermions with suitably chosen Chern-Simons gauge fields. Furthermore, one may change the braid statistics of holons and spinons, while still keeping the Fermi statistics of the original holes. Different choices of Chern-Simons actions precisely reflect these different braid statistics.

In one and two dimensions Abelian braid statistics of particles (or of field operators that create them) can be characterized by the phase factor $e^{\pm i(1-\alpha)\pi}$ acquired by their many-body wave function (or the product of equal-time field operators) when one performs an oriented exchange among two of them, with $\alpha \in [0, 2)$ and $\pm$ referring to the two orientations (see, e.g., Ref. [8]). Fermions(bosons) correspond to $\alpha = 0$ (1), while excitations with $\alpha = 1/2$ are called semions.

All appropriate choices of holon and spinon statistics reproduce exactly the correlation functions of the original fermion fields, as shown in the specific examples of slave bosons, slave fermions and slave semions in Ref. [7], and using techniques developed there many more schemes can be employed, e.g., the slave anyons considered in Ref. [9] or a variant of the slave semion approach considered in Ref. [10], on which our subsequent discussion is based. Slightly different roads follow, e.g., the approach of Ref. [11], as commented in this spirit in Ref. [12].

Although all Chern-Simons choices are completely equivalent if implemented exactly, as soon as one makes some mean-field-like approximation they give rather different results [12]. It is then crucial to understand which choice is better to perform mean-field treatments. As discussed below, a key issue is the area (2D volume) of the Fermi surface and that issue is in turn closely linked to another form of statistics for the elementary excitations of the model, i.e., the exclusion statistics.

The exclusion statistics was introduced by Haldane [13] to generalize the Pauli exclusion principle. It can be characterized at finite density by the average occupation of momenta at $T = 0$ as follows [14]: Consider a Fermi gas with fixed volume (in two dimensions later on called area) enclosed by the Fermi surface and let n_0 be the corresponding fermion density; we say that a particle obeys exclusion statistics with parameter g if the particle density with the same Fermi volume, denoted by n_g, satisfies

$$n_0 = (1 - g)n_g. \tag{1.2}$$

This implies that at a fixed momentum (neglecting other internal degrees of freedom) a particle with exclusion statistics 1/2 can have an occupation number twice that of a free fermion, so that the volume of its Fermi surface is half of that of a Fermi gas with the same density. In simpler terms, one might say that a semion in momentum space behaves like one-half of a fermion.

For the statistics of quasiparticle excitations of the 2D t-J model and the related Fermi surface area we have two sources of suggestions: The solvable one-dimensional t-J model and, under the assumption initially made, the experiments on cuprates.

Concerning the 1D model the answer for the braid statistics is unique: Both holon and spinon fields and the related low-energy quasiparticles should be semions, i.e., with braid statistics parameter 1/2, to reproduce in a suitable mean-field treatment the correct scaling limit of correlation functions obtained via Bethe ansatz or conformal field theory methods [15,16].

Furthermore, the holon in one dimension has exclusion statistics parameter $g = 1/2$. As shown in Ref. [17], in general there is no relation between the braid and exclusion statistics in one dimension. In fact one can introduce the braid statistics coupling 1D spinless fermions to a Chern-Simons field and then perform its dimensional reduction. The physical gauge-invariant field is obtained by adding to the fermion field a gauge string and it obeys a braid statistics consistent with the Chern-Simons coefficient. The Fermi points are shifted by the gauge string but the Fermi 1D volume remains constant, hence the corresponding low-energy excitations still obey an exclusion Fermi statistics. A nontrivial exclusion statistics emerges if the fermionic fields have a Luttinger interaction, a connection previously clearly stated in Ref. [18]. In the approach of Ref. [16] to the 1D t-J model, neglecting at first the coupling with spinons, the holon at large scale behaves as a free $U(1)$ semion; spinons are described by Gutzwiller projected fermions in a squeezed chain obtained by omitting the holon sites and at large scale they form a semion gas with exclusion parameter $g = 1/2$ described effectively by a Luttinger liquid theory. We then perform a field redefinition, eliminating the Fermi surface for the spinon fields by suitably stripping away their gauge strings and adding them to the holon in the correlation functions of the physical hole. As result the 1/2 exclusion statistics of spinons is transferred to holons; holons then have $g = \alpha = 1/2$ statistics, so that the Fermi momentum of the $U(1)$ semionic holon equals the Fermi momentum of the original spin-1/2 fermion treated in the tight binding approximation, in agreement with the exact solution of the model. In fact in such exact solution the Fermi points of the hole are in the position expected for a spin-1/2 fermion with the standard Pauli principle, consistently with the Luttinger theorem, as extended to 1D Luttinger liquids in Ref. [20]. Since in the spin-charge decomposition spinons with the above redefinition do not have a Fermi surface while holons are spinless, that result is correctly recovered by the 1/2 exclusion statistics of holons.

We now turn to the suggestion coming from experiments on cuprates. In overdoped materials the Fermi surface seen in ARPES is close to that obtained in a tight-binding approximation of a t-t'-J model and satisfies the standard Luttinger theorem. (The introduction of a next-nearest-neighbor (NNN) hopping parameter t', and possibly a NNNN t'', in the formalism discussed here is straightforward and it does not change the qualitative features, so it will not be elaborated anymore). To reproduce this result in the spin-charge decomposition formalism we have two natural options: Either the spin-1/2 spinon is fermionic with Fermi surface and the holon is a hard-core boson, as in the slave-boson approach (see, e.g.. Ref. [6]), or the spinless holon has Fermi surface with exclusion statistics parameter 1/2, hence with the same Fermi surface of spinons of the slave-boson approach, while the spinon has no Fermi surface.

We see that, if both spinon and holon are semions, the second case would be a close analog to what happens in one dimension. We remarked above that in one dimension the natural condition for the appearance of exclusion statistics is the Luttinger interaction; analogously on general grounds we proved in Ref. [19] that in two dimensions we have $g = \alpha$ if the original fermionic system without Chern-Simons coupling has Hall conductivity $1/2\pi$ and is incompressible. The main goal of this paper is to show that indeed these conditions can be satisfied for the holon in the 2D *t-J* model and the second approach considered above to get the correct Fermi area can be consistently implemented, sketching also a derivation from it of some consequences for cuprates.

II. $SU(2) \times U(1)$ SPIN-CHARGE GAUGE APPROACH

To implement a semionic decomposition of the hole in the 2D *t-J* model we start by making use of the following theorem [7,16].

Theorem. We embed the lattice of the 2D *t-J* model in a three-dimensional space, denoting by $x = (x^0, x^1, x^2)$ coordinates of the corresponding 2+1 space time, x^0 being the euclidean time. We couple fermions of the *t-J* model to a $U(1)$ gauge field, B^μ, gauging the global charge symmetry, and to an $SU(2)$ gauge field, V^μ, gauging the global spin symmetry of the model, and we assume that the dynamics of the gauge fields is described by the Chern-Simons actions $-2S_{c.s.}^{U(1)}(B) + S_{c.s.}^{SU(2)}(V)$ with:

$$S_{c.s.}^{U(1)}(B) = \frac{1}{4\pi i}\int d^3x \epsilon_{\mu\nu\rho} B^\mu \partial^\nu B^\rho(x),$$
$$S_{c.s.}^{SU(2)}(V) = \frac{1}{4\pi i}\int d^3x \mathrm{Tr}\epsilon_{\mu\nu\rho}\left[V^\mu\partial^\nu V^\rho + \frac{2}{3}V^\mu V^\nu V^\rho\right](x), \tag{2.1}$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita antisymmetric tensor in three dimensions. Then the spin-charge [or $SU(2) \times U(1)$] gauged model so obtained is exactly equivalent to the original *t-J* model. In particular the spin and charge invariant correlation functions of the fermion fields $c_{j\alpha}$ of the *t-J* model are exactly equal to the correlation functions of the fields $\exp(-i\int_{\gamma_j} B)P[\exp(i\int_{\gamma_j} V)]_{\alpha\beta}c_{j\beta}$, where c denotes now the fermion field of the gauged model, γ_j a string at constant Euclidean time connecting the point j to infinity and $P(\cdot)$ the path ordering, which amounts to the usual time ordering $T(\cdot)$, when time is used to parametrize the curve along which one integrates. One can view the result of this theorem as an analog of the construction of composite fermions in Jain's approach to the quantum Hall effect [21]. In that case magnetic vortices with even quantum flux (depending on the filling) are bound to the electron and the resulting composite entity is still a fermion, dubbed composite fermion; in the present case the electron of the *t-J* model is bound to a charge-vortex of flux $-1/2$ and a spin-vortex of flux $1/2$, while the resulting entity still being a fermion. Notice that, contrary to what one might naively think, although the Chern-Simons actions individually explicitly break the parity and time-reversal symmetries, the particular combination considered above still preserves explicitly these two symmetries.

We now rewrite the hole field c of the gauged model as a product of a charge 1 spinless fermion field h and a neutral spin-1/2 boson field $\tilde{s}_\alpha$: $c_\alpha = h^*\tilde{s}_\alpha$. Then we identify $\exp[i\int_{\gamma_j} B]h_j$ as the holon and $P(\exp[i\int_{\gamma_j} V])_{\alpha\beta}\tilde{s}_{j\beta}$ as the spinon fields. The Chern-Simons coupling automatically ensures that both corresponding field operators obey semionic braid statistics. The holon h being spinless implements exactly the Gutzwiller constraint due to the Pauli principle. Furthermore, if the constraint $(P(\exp[i\int_{\gamma_j} V])_{\alpha\beta}\tilde{s}_{j\beta})^\dagger(P(\exp[i\int_{\gamma_j} V])_{\alpha\beta'}\tilde{s}_{j\beta'}) = \tilde{s}^*_\alpha\tilde{s}_\alpha = 1$ is imposed, as $c^*_\alpha c_\alpha = 1 - h^*h$ we see that $(\exp[i\int_{\gamma_j} B]h_j)^*$ $(\exp[i\int_{\gamma_j} B]h_j) = h^*_j h_j$ is just the density of empty sites in the model, corresponding to the Zhang-Rice singlets.

The charge-flux associated to the electrons produces a π-flux phase for every plaquette, plus vortices centered on the empty sites, i.e., on the holon positions. More precisely, introducing a Coulomb gauge fixing $\partial_\mu B^\mu = 0$ for the $U(1)$ charge gauge symmetry, one finds

$$\bar{b}^\mu(z) = \frac{1}{2}\left[\sum_j \partial^\mu \arg(\vec{z} - j)(h^*_j h_j)(z^0)\right] \tag{2.2}$$

and we can choose

$$\bar{B}_{\langle i,i\pm\vec{e}_x\rangle} = \pm\pi/4, \bar{B}_{\langle i,i\pm\vec{e}_y\rangle} = \mp\pi/4, \tag{2.3}$$

where i is a site of the even Néel sublattice and $\vec{e}_x, \vec{e}_y$ the two unit vectors along the link directions. We recognize $\partial^\mu \arg(\vec{z} - j)$ as the vector potential of a vortex centered at the holon position j, i.e., centered at an empty site of the *t-J* model. Vortices in Eq. (2.2) appear in the charge $U(1)$ group and are responsible for the semionic nature of holons.

Neglecting at first these charge vortices, through Hofstadter mechanism the π flux converts the spinless holon field h into a pair of Dirac fields in the magnetic Brillouin zone (BZ), with pseudospin indices corresponding to the two Néel sublattices and two small FS centered at $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. If we reinsert the charge-vortices, and assume for the corresponding semionic holons an exclusion statistics 1/2, then these holons have the same FS of the fermionic spinons of the slave-boson approach in the π-flux phase [6], and the same dispersion: $\omega_h \sim 2t[\sqrt{\cos^2 k_x + \cos^2 k_y} - \delta]$, where δ denotes the density of empty sites, corresponding in cuprates to the in-plane doping concentration. When these holons are coupled to spinons through the self-generated slave-particle gauge field, as a consequence of the Dirac structure of the holons the resulting holon-spinon bound state generated in the low-energy limit exhibits Fermi arcs qualitatively consistent with those found in ARPES experiments in the pseudogap phase in cuprates [22]. The underlying FS for the hole is modified *w.r.t.* the holon FS by the spinon gap proportional to $\delta^{1/2}$ discussed later. Furthermore, using the techniques of Refs. [22,23] and assuming a doping independent renormalization of the spinon gap, the fraction of BZ enclosed can be for all dopings approximately $\delta/2$, where the factor 1/2 comes precisely from the 1/2 exclusion statistics [24].

We can now use the $SU(2)$ gauge freedom to rotate the spinons $\tilde{s}$ to a configuration $\tilde{s}^m$, depending on the holon configuration, optimizing on average the holon-partition function in that spinon background, in a Born-Oppenheimer approximation. In this configuration spinons are antiferromagnetically ordered along the magnetization direction of the undoped model, which we arbitrarily fix along z. There is in addition a spin flip on the sites where holons are present, also for the final site of a hopping link of holons, at the time of hopping. Above a crossover temperature T^* we find that $\tilde{s}^m$ involves also a phase factor canceling the contribution of $\bar{B}$ in the loops of hopping links of holons, so that the hopping holons feel an approximately zero flux [25]. Assuming the exclusion statistics with parameter 1/2 for the holon (to be proved in the next section), the disappearance of the π flux implies that the Hofstadter mechanism does not hold anymore and above T^* one recovers for holons the large FS of the tight-binding approximation. In particular the fraction of BZ enclosed is of order $(1+\delta)/2$ and the factor 1/2 comes precisely again from the 1/2 exclusion statistics. This FS will be inherited by the physical hole as a holon-spinon bound state [25], and, with the addition of a t' term and a renormalization of the spinon gap, it is in approximate agreement with the FS observed in ARPES in the strange-metal region of the phase diagram of the cuprates [24]. For this reason we call pseudogap (PG) for the t-J model in our approach the region below T^* and strange metal (SM) the region above it. Actually, although only qualitatively, the above results on the FS have been proved in Refs. [22,25] in a rough approximation in which the 1/2 Haldane statistics for the holon was assumed, but, somewhat inconsistently, the semionic nature of the holon field was not taken into account, keeping, on the other hand, the treatment of the spinon consistent with this approximation. As discussed in the introduction the proof of the Haldane statistics of the holon is the main aim of this paper, but this proof needs some more details on the spin-charge approach that we now provide. At the end of the paper some results obtained with this approach, including a non-BCS mechanism for superconductivity, are outlined, making also contact with experiments in cuprates.

Having used the $SU(2)$ gauge freedom to rotate spinons to the optimal configuration $\tilde{s}^m$, we need to integrate the $SU(2)$ gauge field V over all its configurations. Therefore, we split the integration over V into an integration over a field $\bar{V}$, satisfying the Coulomb gauge fixing $\partial_\nu \bar{V}^\nu = 0$ with $\nu = 1, 2$ and its gauge transformations expressed in terms of an $SU(2)$-valued scalar field U, *i.e.*, $V^\mu = U^\dagger \bar{V}^\mu U + U^\dagger \partial^\mu U$. Notice that $P[\exp(i\int_x^y V)] = U_y^\dagger P[\exp(i\int_x^y \bar{V})]U_x$. U describes fluctuations of spinons around the "optimal" configuration and can be written as:

$$U = \begin{pmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{pmatrix}, \tag{2.4}$$

with s satisfying the constraint:

$$s^*_{\alpha j} s_{\alpha j} = 1. \tag{2.5}$$

We will call in the following s_α again spinons. Up to now no approximation has been made and the model in terms of h, s, and $\bar{V}$ is still equivalent to the original t-J model. However, due to the optimization procedure on the spinon, we expect that the configurations of U are dominated by small fluctuations around identity. As discussed in Ref. [12], the spin flip due to the $SU(2)$ gauge freedom in $\tilde{s}^m$ allows a simultaneous optimization in terms of the spinon s both of the t and the J term in the Born-Oppenheimer approximation considered above, because (neglecting $\bar{V}$) s appears in the form $s^*_{\alpha i} s_{\alpha j}$ on links in the t term and as $|\epsilon_{\alpha\beta} s_{\alpha i} s_{\beta j}|^2$ and the identity $|s^*_{\alpha i} s_{\alpha j}|^2 + |\epsilon_{\alpha\beta} s_{\alpha i} s_{\beta j}|^2 = 1$ holds. This phenomenon occurs also in one dimension and might be the origin of the good mean-field approximation for the semionic statistics.

We now briefly discuss a mean-field approximation essentially based on the conjecture that the fluctuations U are small. If we neglect fluctuations U in the calculation of $\bar{V}$ (up to an irrelevant field-independent term) one gets (with $\mu = 1, 2$):

$$\bar{V}^\mu(z) = -\frac{1}{2}\sum_j (-1)^{|j|} \partial^\mu \arg(\vec{z} - j) h_j^* h_j(z^0) \sigma_z. \tag{2.6}$$

We recognize in the term $(-1)^{|j|}\partial^\mu \arg(\vec{z} - j)$ of Eq. (2.6) the vector potential of a vortex centered at the holon position j, with vorticity (or chirality) depending on the parity of $|j| = j_x + j_y$. We call these vortices antiferromagnetic (AF) spin vortices, since they record in their vorticity the Néel structure of the lattice. Hence they are still a peculiar manifestation of the AF interaction, such as the more standard AF spin waves. As one can see, they are the topological excitations of the $U(1)$ subgroup of the original $SU(2)$ spin group unbroken in the AF phase, along the spin direction z of the magnetization.

These vortices are of purely quantum origin, since, as in the Aharonov-Bohm effect, they induce a topological effect far away from the position of the holon itself, where their classically visible field strength is supported. Hence in this approach the empty sites of the 2D t-J model, mimicking the Zhang-Rice singlets and corresponding to the holon positions, are cores of the AF spin vortices, quantum distortions of the AF spin background. These vortices have no analog in the slave-boson approach and in our approach are responsible for both short-range AF order, which we now outline since it will be used in the proof of Haldane statistics, and a new pairing mechanism leading to superconductivity, sketched in the final section, referring to Ref. [26] for details. Semionic holons dressed by AF spin vortices are similar to semionic holons in one dimension with attached a spinon-derived spin string; the role of kinks as topological defects in one dimension is replaced by vortices in two dimensions.

We can write the total action of the system as a sum of a spinon action S_s and a holon action S_h. For our purpose of the spinon action it is enough to know [10] that in the long wavelength continuum limit it is given by a $O(3)$ nonlinear σ model (in CP^1 form) for spinons describing the continuum limit of the undoped Heisenberg model, with an additional coupling between spinons s and the AF spin vortices:

$$\int d^3x (\bar{V}^\mu \bar{V}_\mu)(x) s^*_\alpha s_\alpha(x). \tag{2.7}$$

A quenched average, $\langle\cdot\rangle$, over positions of centers for spin-vortices yields the following estimate [10]: $\langle \bar{V}^\mu \bar{V}_\mu\rangle \approx \delta|\log\delta|$. Hence the term (2.7) provides a mass-gap to spinons, converting the long-range AF order of the Heisenberg model, corresponding to zero doping, to short-range AF order at finite dopings; therefore, spinons s have no FS. The spinon system behaves as a spin liquid since spinon confinement is avoided by the interaction with the gapless holons. However, in spite of the presence at lattice level of a Chern-Simons term, which turns spinons into semions, in the mean-field long-wavelength limit considered involving the coupling to the holons via AF spin vortices, it is not a chiral spin liquid and spinons s in the low-energy limit can be considered as spin-1/2 hard-core boson quasiparticles excitations. Although not confined, in the entire system spinons are weakly bound to holons and antispinons by slave-particle gauge fluctuations to form the physical composite holes and magnons, respectively.

By making in Eq. (2.7) a mean-field approximation for $s^*_\alpha s_\alpha(x)$, instead of what was previously considered for $\bar{V}^\mu \bar{V}_\mu(x)$, we obtain the term

$$\langle s^*_\alpha s_\alpha \rangle \sum_{i,j} (-1)^{|i|+|j|} \Delta^{-1}(i-j) h^*_i h_i h^*_j h_j, \tag{2.8}$$

where Δ is the 2D Laplacian. In the static approximation for holons Eq. (2.8) describes a 2D lattice Coulomb gas with charges ± 1 depending on the Néel sublattices. In particular the interaction is attractive between holons in opposite Néel sublattices, with maximal strength for nearest-neighbor sites, along the lattice directions with a d-wave symmetry. Putting back coefficients one finds that the coupling constant of this interaction is $J_{\text{eff}} = J(1-2\delta)\langle s^*_\alpha s_\alpha\rangle$, which decreases with increasing doping. For 2D Coulomb gases with the above parameters, pairing appears below a temperature $T_{ph} \sim J_{\text{eff}}$. Hence the charge pairing originates from the attraction between AF spin vortices with opposite chirality, eventually leading to superconductivity as sketched in the final section.

We write now explicitly the holon action, S_h, since its expression is needed in the proof of exclusion statistics. In PG region S_h can be written as:

$$\begin{aligned} S_h = \int dx^0 \sum_j [h^*_j[\partial_0 - ib_0(j) - 2t\delta] h_j] \\ - h^*_j h_j (\sigma_x^{|j|}[U_j^\dagger \partial_0 U_j + iv_0\sigma_z]\sigma_x^{|j|})_{11} \\ - \sum_{\langle i,j\rangle} t h^*_j \exp\left(i\bar{B}_{\langle i,j\rangle} - i\int_i^j b\right) \\ \times h_i \left[\sigma_x^{|i|} U_i^\dagger \exp\left(i\int_i^j v\sigma_z\right) U_j \sigma_x^{|i|}\right]_{11} - 2S^{U(1)}_{c.s.}(b) \\ + 2S^{U(1)}_{c.s.}(v). \end{aligned} \tag{2.9}$$

The fact that σ_x has the same power at both ends of a hopping link is due to the spin-flip generated by $\tilde{s}^m$. b is a gauge field of the $U(1)$-charge group and v is a gauge field of the $U(1)$ subgroup of the spin group $SU(2)$ previously selected by choosing the directions of $\tilde{s}^m$. The factor 2 in the Chern-Simons action is due to a normalization needed passing from $SU(2)$ to its $U(1)$ subgroup. Integrating over b_0 and v_0 one reproduces the previous description in terms of $\bar{b}_\mu$ and $\bar{V}_\mu$. Notice that since coefficients of the Chern-Simons terms for b and v have opposite sign, at this stage the parity (P) and time-reversal (T) symmetries are still explicitly preserved. Formally this can be seen by rewriting the gauge fields in the combinations $b\mathbf{1} + v\sigma_3$ and $b\mathbf{1} - v\sigma_3$, where $\mathbf{1}$ is the 2×2 identity matrix. Holons are coupled only to $b\mathbf{1} - v\sigma_3$, so integrating $b\mathbf{1} + v\sigma_3$ from the Chern-Simons one gets a δ function for the field strength of $b\mathbf{1} - v\sigma_3$ which is P and T invariant. We argue, however, that the continuum limit should not be taken considering simultaneously the coupling of holons to b and v, but firstly only to b, to make the holon field $U(1)$ charge-gauge invariant and to enforce the semionic statistics, and only afterwards introducing the coupling with the spin degrees of freedom.

To summarize, in PG S_h describes fermionic lattice holons in the presence of π flux per plaquette with attached charge-vortex generated by b that turn them into semions, interacting with spinons s and the AF spin vortices described by $\bar{V}$. In SM the π flux in the hopping is suppressed.

III. 1/2 EXCLUSION STATISTICS OF HOLONS

Having explained the relevance for self-consistency of the exclusion statistics 1/2 for the holon in the spin-charge gauge approach to cuprates, in this section we turn to its proof.

A. Braid-exclusion statistics relation

A key ingredient of the proof is the result contained in Ref. [19], connecting braid and exclusion statistics under some conditions, that we now sketch. Consider a planar Hall system consisting of fermions in a thermodynamically large domain with a boundary; it is well known that there are chiral edge modes on the sample boundary leading to a boundary current. We then couple the system to a Chern-Simons field b_μ, defined in the whole space time, with coupling strength α, while keeping fixed the chemical potential μ. We denote by $N(\mu,\alpha)$ the number of particles contained in the considered domain with Chern-Simons coupling α. The Lagrangian (in real time) reads

$$\mathcal{L} = \mathcal{L}_M - b_\mu J^\mu + \frac{1}{4\pi\alpha}\int d^3x \epsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda, \tag{3.1}$$

where J^μ is the current density. The exact form of the Lagrangian $\mathcal{L}_M$ of fermions is not so important, and it is only required to provide a nonvanishing Hall conductance σ_H.

By differentiating Eq. (3.1) $w.r.t.$ $b_\nu(\nu = 1,2)$, one obtains the following relation between the current and the electric field: $\vec{E} \equiv \partial_0\vec{b} - \vec{\nabla} b_0 = 2\pi\alpha(\hat{z}\times\vec{J})$, where $\hat{z}$ is the unit vector perpendicular to the plane of the system. Thus a current $I(\mu)$ flowing on the boundary leads to an electric field normal to the boundary.

In the scaling limit the fermion Hall system contributes a Chern-Simons term to the gauge-effective action with coefficient $-\sigma_H/2$ in the bulk (plus a term localized at the boundary by gauge invariance). Taking into account this contribution in random phase approximation (RPA) leads to an effective Chern-Simons coupling for the b field: $\tilde{\alpha} \equiv \alpha/(1-2\pi\sigma_H\alpha)$. The above electric field generated by the boundary current implies a jump $\Delta V = -2\pi\tilde{\alpha} I(\mu)$ of the scalar potential across the sample boundary. Then the change of the free energy $\mathcal{F}$ due to the Chern-Simons coupling reads

$$\mathcal{F}(\mu,\alpha) = \mathcal{F}(\mu,0) - 2\pi\tilde{\alpha} I(\mu) N(\mu,0). \tag{3.2}$$

Differentiating both sides of Eq. (3.2) with respect to μ, we obtain

$$N(\mu,\alpha) = N(\mu,0) + 2\pi\tilde{\alpha}\frac{\partial I}{\partial\mu}N(\mu,0) + 2\pi\tilde{\alpha} I(\mu)\frac{\partial N(\mu,0)}{\partial\mu}, \tag{3.3}$$

where $\partial I/\partial\mu = \sigma_H$ and $\partial N/\partial\mu$ is proportional to the compressibility. For an incompressible liquid, one obtains

$$N(\mu, 0) = (1 - 2\pi\sigma_H\alpha)N(\mu, \alpha). \quad (3.4)$$

Since μ was kept invariant, by comparing Eq. (3.4) with Eq. (1.2) one concludes that anyons of the system described in Eq. (3.1) obey an exclusion statistics with parameter $g = 2\pi\sigma_H\alpha$. In particular, if $2\pi\sigma_H = 1$ we have $g = \alpha$.

Let us now come back to our holon system. The above argument shows that if holons without Chern-Simons coupling to b have a Hall conductivity $1/2\pi$ and the system is incompressible, then the semionic holons obtained by coupling with b obey exclusion statistics $1/2$, as we would like to prove.

B. Free holons

In the holon action Eq. (2.9) if no further approximations are made the holon density remains δ, since the Gutzwiller projection is still exactly implemented by $U \in SU(2)$ with the constraint Eq. (2.5) being satisfied. In particular when $\delta = 0$ the holon density vanishes, correctly reproducing the vanishing density of Zhang-Rice singlets at half-filling.

However, in the large-scale continuum limit we have seen that, thanks to the interaction with AF vortices, the spinon $s(x)$ is gapped. It implies that in this limit the constraint Eq. (2.5) is not fully satisfied, as the spinon mass gap is incompatible with it, so the Gutwiller projection is not anymore exactly implemented. To understand the situation let us first consider the free holons without coupling to spinons and the Chern-Simons fields b and v, while still keeping fixed the chemical potential. The corresponding action is given by

$$S_h^0 = \int dx^0 \sum_j h_j^*(\partial_0 - 2t\delta)h_j - \sum_{\langle i,j\rangle} t h_j^* \exp[i\bar{B}_{\langle i,j\rangle}]h_i. \quad (3.5)$$

Due to the staggered π flux implemented by $\bar{B}_{\langle i,j\rangle}$, we divide the square lattice into two sublattices, A (even sites) and B (odd sites). On these sublattices, the annihilation operators of holons are denoted by h^a and h^b, respectively. Let us choose a unit cell with A and B sites along the x direction, then the Hamiltonian corresponding to the free holon action Eq. (3.5) can be recast in a quadratic form, with a matrix in the momentum space given by:

$$H(\vec{k}) = 2t\begin{pmatrix} 0 & \cos k_x + i\cos k_y \\ \cos k_x - i\cos k_y & 0 \end{pmatrix}. \quad (3.6)$$

In Eq. (3.6), the momentum k only takes values in the range $[-\pi, \pi] \times [-\pi/2, \pi/2]$, which is a half of the original BZ. One can easily see that it describes two massless Dirac double cones with vertices at $(\pm\pi/2, \pi/2)$. Shifting the two Dirac nodes to the origin in k space, inserting the chemical potential $\mu = 2t\delta$, and taking the continuum limit, we see that the corresponding continuum fields are described by massless Dirac fields with two flavors, corresponding to the two double cones. The two upper bands of the double cones are filled up to energy $2t\delta$, hence even at $\delta = 0$ the lower bands of the two Dirac double cones are filled, so that the holon density no more vanishes even in the half-filling case. The lower bands are thus an artifact produced by the violation of the constraint Eq. (2.5) introduced when we treat in mean field Eq. (2.7).

Since spinons are gapped, the Gutzwiller constraint is relaxed in the large-scale continuum limit. Going back to the lattice model with no-double-occupation constraint ignored temporarily, one expects that at half-filling with $\delta = 0$ the number of holons equals the number of unprojected holes, hence one expects the holon number is 1 per site on average. If these holons were fermions obeying Fermi statistics, both upper and lower bands would be completely filled, which is at odds with the previous half-filling result obtained from the free holon Lagrangian.

This would lead to an inconsistency in the above continuum limit. However, if holons satisfy the semionic exclusion statistics with $g = 1/2$, at half-filling they fill the lower bands leaving the upper bands empty to give a density 1 on average. Since these semionic holons in the lower bands are a result of relaxing the Gutzwiller projection, they are spurious and describe the singly occupied sites in the original unprojected lattice model. When the doping holes are introduced in the t-J model, the corresponding physical holons partially fill the upper bands and are responsible for the low-energy physics. Although the spurious lower band holons are not directly relevant to the low-energy physics in the scaling limit, they are responsible for the $1/2$ exclusion statistics when coupled to the $U(1)$ statistical field b, making the theory self-consistent, as we prove below. Before closing this section, we emphasize that one should be careful not introducing an unphysical coupling of the spurious holons in the lower bands with spinons, so that the density of physical holons coupled to spinons still correctly vanishes at $\delta = 0$.

C. Hall conductivity of spurious holons

According to the strategy outlined in Sec. III A we now compute the Hall conductivity of the holon system without Chern-Simons couplings. If we look at the corresponding holon action in Eq. (2.9), we see that for sites in the A sublattice and links starting from the A sublattice the coupling with spinons and the v field is of the form $(U_j^\dagger\partial_0 U_j + iv_0\sigma_z)_{11}$ and $[U_i^\dagger \exp(i\int_i^j v\sigma_z)U_j]_{11}$, whereas for the B sublattice the corresponding terms are $(U_j^\dagger\partial_0 U_j + iv_0\sigma_z)_{22}$ and $[U_i^\dagger \exp(i\int_i^j v\sigma_z)U_j]_{22}$. Since $(U_j^\dagger\partial_0 U_j + iv_0\sigma_z)_{22} = (U_j^\dagger\partial_0 U_j + iv_0\sigma_z)_{11}^*$ and $[U_i^\dagger \exp(i\int_i^j v\sigma_z)U_j]_{22} = [U_i^\dagger \exp(i\int_i^j v\sigma_z) U_j]_{11}^*$, the action is not invariant under time reversal, but is invariant under time reversal combined with interchange of the two Néel sublattices realized by parity transformation with respect to a line in the dual lattice. This can be intuitively understood since the time-reversal operation reverses the chirality of the spin vortices described by v, but an exchange of the Néel sublattices also does the same job.

As is well known [27], to compute the Hall conductivity of massless Dirac fields we need to introduce an infrared regulator (such as a mass) with a parameter m, respecting the symmetry of the system; at the end of the computation one takes the limit $m \to 0$. The reason for introducing a regulator is that, due to the parity anomaly one cannot consistently define a gauge-invariant coupling for massless Dirac fermions in two dimensions. The mass regulator breaks parity and even after it is sent to zero, in the gauge-effective action its remnant is still there, keeping the information of the mass sign in the coefficient of the generated Chern-Simons action. However, for our system one cannot take as regulator simply a mass term in the lattice as in the standard systems, since it would preserve the time-reversal symmetry, broken in our case.

A regularized free Hamiltonian for the field $(h^a(\vec{k}), h^b(\vec{k}))^t$ maintaining the above discussed symmetry has a matrix form in the momentum space given by:

$$H(\vec{k}) = \begin{pmatrix} m\cos(k_x+k_y) & 2t(\cos k_x + i\cos k_y) \\ 2t(\cos k_x - i\cos k_y) & m\cos(k_x-k_y) \end{pmatrix}. \tag{3.7}$$

One can check directly that this Hamiltonian with the regulator added respects the combined symmetry of time reversal and the exchange of Néel lattice of the original Hamiltonian. To be specific the time-reversal operation is implemented by complex conjugation and $\vec{k} \to -\vec{k}$, while the interchange of Néel sublattices corresponds to $k_y \to -k_y$ (a mirror reflection about the x axis) followed by a similarity transformation implemented by σ^x. In our units the Hall conductivity of the lower bands is given by $c_1/(2\pi)$, where c_1 is the Chern number of the corresponding bands. For a two-dimensional $H(\vec{k})$ as ours, c_1 can be computed as follows (see, e.g., Ref. [28]): We write $H(\vec{k})$ in terms of $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$, with $\mu = 0, 1, 2, 3$ and $\vec{\sigma}$ the Pauli matrices: $H(\vec{k}) = \sum_\mu H_\mu(\vec{k})\sigma^\mu$. We call D the set of points in the BZ where $H_1 = H_2 = 0$, which are called Dirac points, then

$$c_1 = \frac{1}{2}\sum_{x\in D} \text{sign}[H_3(x)]\text{sign}\left[\epsilon_{3ij}\frac{\partial H_i}{\partial k_1}\frac{\partial H_j}{\partial k_2}(x)\right]. \tag{3.8}$$

If we compute the Chern number c_1 of the lower bands of $H(\vec{k})$, describing spurious holons as discussed above, one then finds 1, since at the two Dirac points (becoming for $m = 0$ the Dirac nodes) the regulator term has opposite sign: $H_3(-\pi/2, \pi/2) = m$, $H_3(\pi/2, \pi/2) = -m$ and the second sign in Eq. (3.8) is also opposite at those points. We then see that the lower bands in our holon system contribute $\text{sgn}(m)/(2\pi)$ to the Hall conductivity.

In order to discuss the spinon coupling to the physical upper bands taking into account the gap of spinons, as done in the next section, one needs to go to the long-wavelength continuum limit. In that limit the lower bands of the above Hamiltonian are just the lower bands of two Dirac double cones regularized with the same mass m. Since every cone contributes to the Hall conductivity with $m/(4\pi|m|)$ [27], we see again that the lower bands in our system contribute $\text{sgn}(m)/(2\pi)$.

D. Hall conductivity of physical holons

In the absence of the spinon coupling, for free Dirac holons the partially filled upper bands would contribute exactly the opposite Hall conductivity of the lower bands, since in case of partial filling the Hall conductivity of the free system is zero. Introducing a mass, we get a nonvanishing result only if the chemical potential is in the mass gap, which is not our case. However, the coupling of the upper band to spinons changes the situation.

To discuss the effect of spinon coupling we need to extract an effective action and compute the coefficient of the corresponding Chern-Simons term for the b field. This calculation involves a mixing of the upper physical and lower spurious bands. In order to minimize such mixing, a careful treatment is needed. In fact, we need only consider the mixing in an infinitesimal neighborhood of the Dirac nodes following the procedure outlined below; more details are deferred to the Appendix.

We consider one partially filled Dirac double cone, while the other one can be treated in the same way. To identify the Green's function of the continuum fields associated with the two bands we start with rewriting the relevant free Dirac propagator G with chemical potential μ_F at $T = 0$ in the following form: let γ^μ with $\mu = 0, 1, 2$ denote the 2+1 Dirac γ matrices and $k^\mu = \omega, \vec{k}$ the three-momentum.

Then we find [29]:

$$G = (\not{k} - m)\left[\Theta(-k^0)\frac{1}{k^2 - m^2 + i\epsilon} + \Theta(k^0)\left(\frac{\Theta(k^0 - \mu_F)}{k^2 - m^2 + i\epsilon} + \frac{\Theta(\mu_F - k^0)}{k^2 - m^2 - i\epsilon}\right)\right]. \tag{3.9}$$

Naively the first term corresponds to the lower band, but to take into account the problem of mixing quoted above we extend the first term up to a small cutoff η with $\mu_F \gg \eta \gg |m|$ to include the bottom of the upper band, replacing $\Theta(-k^0)$ by $\Theta(-k^0 + \eta)$ and subtracting the corresponding contribution in the second term of Eq. (3.9). Note that η is eventually sent to zero, after the limit $m \to 0$ has been taken. Hence the introduction of the small cutoff η takes into account only the contribution from the conduction band edge. Since the relevant contribution for physical holons at large scales comes only from the region near the Fermi surface, it is unmodified by the above operation. According to the previous discussion we then insert in the modified second term (assumed to describe the physical holons) the minimal coupling to b, to spinons s and to v, whereas we insert only the minimal coupling to b in the modified first term describing the spurious holons appeared with the violation of the Gutzwiller projection.

We have already calculated above the Hall conductivity of the first term, $\sigma_H = 1/(2\pi)$; correspondingly the leading contribution in the long wavelength continuum limit of the effective action is given by the Chern-Simons term $S^{U(1)}_{c.s.}(b)$. We now discuss the Hall conductivity of the second term.

The long wavelength continuum limit of the spinon interaction is just the minimal coupling of holons to the slave-particle gauge field $A_\mu(x) \sim s^*(x)\partial_\mu s(x)$. This is the gauge field of the CP^1 representation of the $O(3)$ spinon σ model, implementing in the continuum the slave-particle gauge invariance. Then the leading term of the effective action due to physical holons turns out to be $-S^{U(1)}_{c.s.}(b + A) - S^{U(1)}_{c.s.}(v)$, as shown in the Appendix.

We now need to integrate A to find both Hall conductivity and, as required by Eq. (3.3), the compressibility of the holon system. The compressibility is proportional to the scalar polarization bubble evaluated at zero energy in the limit of zero momenta.

Since the upper band is partially filled, the leading contribution comes from a region near the Fermi surface. Then at $\omega = 0$ in the limit $k \to 0$ in the Coulomb gauge its polarization bubble matrix is given by:

$$\pi^h(\vec{k}) = \begin{pmatrix} \chi_0^h & -k_2\sigma_H^h & k_1\sigma_H^h \\ k_2\sigma_H^h & \vec{k}^2\chi_\perp^h & 0 \\ -k_1\sigma_H^h & 0 & \vec{k}^2\chi_\perp^h \end{pmatrix}, \tag{3.10}$$

where χ_0^h, $\chi_\perp^h$, $\sigma_H^h = -1/(2\pi)$ are the density of states at the Fermi energy, the diamagnetic susceptibility and the Hall conductivity, respectively.

As spinons are gapped, integrating them out one obtains a Maxwell effective action for the slave-particle gauge field A_μ; the spinon polarization bubble matrix at $\omega = 0$ is then given by the diagonal matrix

$$\pi^s(\vec{k}) = \mathrm{diag}\big(\chi_0^s k^2, \vec{k}^2\chi_\perp^s, \vec{k}^2\chi_\perp^s\big), \tag{3.11}$$

with χ_0^s, $\chi_\perp^s$ the electric and the diamagnetic susceptibility of the spinon system. The scaling analysis presented in Ref. [30] based upon a tomographic representation of fermion Green's functions suggests that the RPA approximation gives the leading term in the scaling limit of the polarization bubbles of holons in the presence of the Maxwell interaction originated from spinons, described by $(\pi^s)^{-1}$, since such interaction is of long range. The polarization bubble of holons dressed by the spinon interaction in RPA in the small k limit is then given by

$$\Pi^h(\vec{k}) = \pi^h[1 + (\pi^s)^{-1}\pi^h]^{-1}, \tag{3.12}$$

where the scalar component (corresponding to compressibility) reads

$$\Pi_0^h(\vec{k}) = \chi_0^s k^2, \tag{3.13}$$

while the Hall polarization bubble by

$$\sigma_H(\vec{k}) = \frac{\sigma_H^h k^2 \chi_0^s \chi_\perp^s}{\left(\sigma_H^h\right)^2 + \chi_0^h(\chi_\perp^s + \chi_\perp^h)}; \tag{3.14}$$

therefore both vanish at $k = 0$, implying that the upper bands of physical holons are incompressible and do not contribute to the Hall conductivity for b. The origin of incompressibility can be traced back to the unscreened long-range 2D Coulomb repulsion generated by the slave-particle gauge field, due to the spinon gap. Hence it is a consequence of the Gutzwiller constraint, origin of the gauge field, and of the destruction of the Néel order due to the AF vortices introduced by doping.

Since the lower holon bands are completely filled it then turns out that the total holon system before it is coupled to the Chern-Simons b field is incompressible and provides Hall conductivity $1/(2\pi)$. Hence, according to the result stated at the beginning of the section, after coupling with b the resulting semionic holons have exclusion statistics parameter $1/2$, as we would like to prove.

As seen in the proof, the role of the slave-particle gauge field, as a direct consequence of the no-double occupation constraint, is crucial to obtain the $1/2$ exclusion statistics, exactly as in the 1D case, where the constraint is also crucial to realize the $1/2$ exclusion statistics by producing a Luttinger-type interaction. Notice that incompressibility of the holon liquid does not imply incompressibility of the hole liquid, because the polarization bubble of the hole involves also the renormalization of the gauge propagators due to the holons.

Furthermore, since the Chern-Simons term of v does not contain a coupling to A one finds that the total Chern-Simons contribution to the effective action of (both physical and spurious) holons in the bulk is given by $S_{c.s.}^{U(1)}(b) - S_{c.s.}^{U(1)}(v)$. (Also a gauged Wess-Zumino-Novikov-Witten boundary term is generated by gauge invariance [31,32].) Therefore, although broken by the system of semionic holons alone, parity and time-reversal symmetries are still explicitly preserved in the holon system coupled to the spinon-inherited v field. In fact, the bands of spurious holons produce a chiral structure, due to b, but the bands of physical holons produce an opposite chiral structure, due to v. The induced additional Chern-Simons terms change, however, the braid statistics of the low-energy holon quasiparticle excitations (in Landau's sense) near the ground state of the semionic holon liquid. Since the coefficients of the total, original plus induced, Chern-Simon actions for b and v are -1 and $+1$, respectively, we expect that the statistics of such quasiparticles is fermionic, as indeed suggested by preliminary calculations based on an approximate explicit expression for the low-energy behavior of the holon Green's function that will be presented in a separate paper.

This change of statistics from the fields to the Landau quasiparticle excitations is somewhat analogous to what happens in the composite fermion theory of the Fractional quantum Hall effect [21], where the quasiparticle excitations near the composite fermion ground state are anyons.

Above we discussed the situation in the PG phase, now we add a brief comment for the SM phase. As stated in Sec. II, the optimal spinon configuration $\tilde{s}^m$, around which we expand the spinon fluctuations described by s, acquires for hopping holons $-\pi$-flux phase factors in the SM phase that cancel the original π-flux phase factors. The spinon coupling occurs only for the upper band and this additional phase factor modifies the dispersion of the upper band so that close to the Fermi surface it is turned into that of the t-J model in the tight-binding approximation; both compressibility and the Hall conductivity of the upper band still vanish. Since the lower band is not involved in the modification its Hall conductivity remains $1/(2\pi)$, hence even in SM phase the hopping holons have exclusion statistics $1/2$.

On the basis of the result on the Fermi surface discussed in Sec. III and the generalization of the Luttinger theorem discussed in Ref. [33], we suspect that the physical hole system, at least in the PG phase, possesses a $\mathbf{Z_2}$ topological order of the kind considered in the above quoted references [33].

IV. CONCLUSIONS

Let us summarize our results. The Gutzwiller projection and the low dimensionality (1D or 2D), allow a gauging of the $U(1)$-charge and $SU(2)$-spin symmetries of the t-J model leaving its physics completely unmodified. As a result at the lattice level the charge degrees of freedom, described by spinless holons, and the spin degrees of freedom, described by spinons, of the t-J model acquire a semionic braid statistics both in two dimensions, and in one dimension, where this statistics holds also for the corresponding low-energy quasiparticles and can be explicitly checked comparing with the exact solution. The additional freedom provided by the $SU(2)$ gauging allows a better simultaneous optimization of both t

and J terms, and in two dimensions it introduces a novel kind of excitations, i.e., the AF spin vortices. These are quantum distortions of the AF spin background in the $U(1)$ subgroup of the $SU(2)$-spin group unbroken by antiferromagnetism. Their cores are located on the empty sites of the 2D t-J model, mimicking the Zhang-Rice singlets of cuprates, and they record in their vorticity the Néel structure of the lattice. In a mean-field treatment at large scales their interaction with the spin degrees of freedom turn the long-range AF order of the model at half-filling into a short-range AF order above a critical doping and this implies a relaxation of the Gutzwiller constraint at large scales. Although with the Gutzwiller projection exactly implemented the charge degrees of freedom have physical bands empty at half-filling, at mean-field level the constraint is relaxed and spurious filled lower holon bands appear, describing the unprojected holes at half-filling. With a proper regularization, the role of these spurious holon bands is to give an approximate but self-consistent description of the Gutzwiller projection on holons, providing an effective vacuum in which the physical hopping holons move. But this vacuum is topologically nontrivial, as shown by its nonvanishing Chern number or Hall conductivity, and this nontriviality deeply affects the physical upper bands, in particular forcing a $1/2$ exclusion statistics for their semionic holons, as in one dimension. The slave-particle gauge attraction between holon and spinon then produces a Fermi surface for holes, as holon-spinon low-energy bound states, with the addition of a t' term, approximately consistent with the ARPES experiments in hole-doped cuprates [24]. As in one dimension the parity and time-reversal symmetry are broken separately for the holon and spinon subsystems, due to their semionic nature, but for physical slave-particle gauge-invariant quantities they are restored.

To conclude, we remark that the spin-charge gauge approach allows us to recover, sometimes even semiquantitatively, many unusual experimental features of hole-doped cuprates, and for completeness we now briefly mention the most relevant results. We believe that the most interesting feature of the approach is that holes are composites made of only weakly bound holons and spinons, so that some physical responses are dominated by the spin carriers, in totally non-Fermi-liquid manner. As noticed above, previously this approach was implemented in the approximation in which the $1/2$ Haldane statistics for the holon was assumed, but, somewhat inconsistently, its semionic nature was not taken into account, consistently neglecting, however, the influence of AF spin vortices on holon hopping. The expected fermionic nature of the Landau holon quasiparticle discussed in the previous section nevertheless suggests that the approximation made was already not unreasonable. Anyway a careful account of the two neglected effects mentioned above cannot change significantly the physical responses dominated by spinons, which are responsible for results that we now sketch in words, referring to the original papers for explicit formulas and plots.

A phenomenon naturally explained by this approach is the metal-insulator crossover (MIC) found decreasing T in the in-plane resistivity of underdoped cuprates (see, e.g., Ref. [34]). Although it is often attributed to disorder-induced localization, that interpretation is at odds with the fact that, depending on materials and dopings, MIC occurs from far below to far above the Ioffe-Regel limit. That interpretation is also at odds with the existence in a large range of temperatures, including the MIC, of a universal curve [35] for a normalized resistivity as a function of T/T^*, where T^* can be identified as an inflection point in the in-plane resistivity. For these reasons we believe that the MIC is intrinsic, although disorder-induced localization may play a role at lower temperatures where, in fact, universality breaks down. In the spin-charge gauge approach the MIC can be easily explained: Due to the slave-particle gauge string binding spinon to holon, the velocity of the hole-bound state is determined by the slowest among spinon and holon (Ioffe-Larkin rule [36]). The holon has a metallic behavior with a FS, whereas, due to the AF gap, at low T the spinon can only move by thermal diffusion leading to a semiconducting behavior. However, at higher temperatures its dynamics is dominated by the dissipation growing with T induced by slave-particle gauge fluctuations, leading to a metallic behavior. The universality is explained by the spinon dominance [12], leading to insensitivity to details of the FS, and even quantitatively the universal curve can be well reproduced [37]. We call T^* the low-pseudogap temperature at which the resistivity curve exhibits an inflection point and, on the basis of the comparison between the experimental and the theoretically derived resistivity curve, we identify it with the crossover from PG to SM in our approach. A similar crossover with increasing T from a AF-gap-dominated region to a gauge-induced dissipation dominated region can explain the peak in the spin-lattice relaxation rate ${}^{63}(1/T_1T)$ in underdoped cuprates [38].

The spin-charge gauge approach provides a three-step mechanism for superconductivity that might explain several crossovers appearing in the phase diagram of cuprates. First, at a temperature that we denote by T_{ph}, the attraction mediated by the AF-spin vortices described in Eq. (2.8) produces charge pairing, the spin degrees of freedom being still unpaired. Since the formation of charge pairs induces a reduction of the spectral weight on the FS of holons, inherited then by holes, we identify in cuprates T_{ph} as the temperature below which a pseudogap appears in the spectral weight of the hole (even well above T^* in the t-t'-J model [23]), and we call this temperature high pseudogap. Qualitatively many features of this high pseudogap in the hole spectral weight derived in Ref. [23] are consistent with experimental data, but, according to the result of the present paper the influence of the semionic nature of the holon field should be reconsidered.

At a lower temperature, T_{ps}, the slave-particle gauge attraction between holon and spinon induces the formation of short-range spin-singlet (RVB) spinon pairs, in a sense, using the holon pairs as the source of attraction, thus leading to a finite density of incoherent hole pairs. Comparing the behavior of spinon-pair density [26,39] with the intensity of the Nernst signal [40] seen in cuprates, we identify this crossover as the onset of the diamagnetic/Nernst signals induced by magnetic vortices.

Finally, at an even lower temperature, the superconducting transition temperature T_c, the hole pairs become coherent and a d-wave hole condensate appears, leading to superconductivity. The presence of three crossover temperatures T^*, T_{ph}, T_{ps} is typical of this approach and finds a reasonable correspondence in the experimental phase diagram of cuprates [39].

In particular, for the same reason given for in-plane resistivity, the superfluid density satisfies Ioffe-Larkin rule and is dominated by spinons in the underdoped region. Below T_{ps} the low-energy effective action obtained integrating out the massive spinons is a Maxwell-gauged 3D XY model, where the angle field of the XY model is the phase of the long-wave limit of the hole-pair field and the gauge field is the slave-particle gauge field. This explains the 3D XY critical exponent of the superfluid density found in experiments. Furthermore, as in the case of resistivity, the spinon dominance explains the experimental observation of a universal curve for the normalized superfluid density as a function of T/T_c [41], which can be well reproduced even quantitatively by the spin-charge gauge approach [37].

Let us end this paper by remarking that we are computing some physical response dominated by holons, to check the effect of the semionic nature of the holon field in our approach, in comparison with the experimental data of cuprates. Preliminary calculations suggest that the main effect with respect to the previous approximate treatment is a modification of the wave-function renormalization constant of the hole, which becomes temperature independent allowing, for example, a recovery of the experimentally observed Fermi-liquid behavior of the Knight shift at high T in the strange-metal phase of hole-doped cuprates (see, e.g., Ref. [42]).

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APPENDIX

In this Appendix we outline the calculation of the Chern-Simons term for the upper band of holons in one of the Dirac double cones. For simplicity we consider only the coupling to $a \equiv A + b$, the coupling with v can be done in a similar way. The two terms do not mix due to the σ^z factor in the v coupling arising from the fact that the two components of the Dirac field arise from different Néel sublattices, hence with opposite charge for v. Following Ref. [27] we consider a coupling to a field a of constant field strength $f_{\mu\nu}$, calculate the expectation of the induced current

$$\langle J^\mu \rangle = \mathrm{Tr}[\gamma^\mu G_a(x, x')]_{x\to x'} \tag{A1}$$

with G_a the gauge invariantly regularized Green's function in the presence of a, and we keep only the term proportional to $\epsilon^{\mu\nu\rho} f_{\nu\rho}$. Its coefficient, which we denote by J_{CS}, multiplied by 4π is the coefficient of the Chern-Simons action. We define $K_\mu = k_\mu + a_\mu$ and consider the case $a_0 = 0$. According to the discussion in Sec. III D, the Green's functions for free physical holons in momentum space is given by

$$G = (\not{k} - m)\Theta(k^0)\left[\left(\frac{\Theta(k^0 - \mu_F)}{k^2 - m^2 + i\epsilon} + \frac{\Theta(\mu_F - k^0)}{k^2 - m^2 - i\epsilon}\right) - \Theta(\eta - k^0)\frac{1}{k^2 - m^2 + i\epsilon}\right], \tag{A2}$$

with $\mu_F \gg \eta \gg |m|$. Then, using Schwinger's proper time formalism, the Green's function G_a for the physical holons in the limit $x \to x'$ can be represented as:

$$\begin{aligned}[G_a(x, x')]_{x\to x'} = &-i\int \frac{d^3k}{(2\pi)^3}(\not{K} - m)\int_0^\infty ds\Theta(k^0)\big[\Theta(k^0 - \mu_F)e^{is(k_0^2 - m^2)}e^{-is\vec{K}^2}e^{is[\gamma^\nu,\gamma^\rho]f_{\nu\rho}/2} \\ &- \Theta(\mu_F - k^0)e^{-is(k_0^2 - m^2)}e^{is\vec{K}^2}e^{-is[\gamma^\nu,\gamma^\rho]f_{\nu\rho}/2} - \Theta(\eta - k^0)e^{is(k_0^2 - m^2)}e^{-is\vec{K}^2}e^{is[\gamma^\nu,\gamma^\rho]f_{\nu\rho}/2}\big].\end{aligned} \tag{A3}$$

The relevant term for the Chern-Simons action in $\mathrm{Tr}(\gamma^\mu(\not{K} - m)e^{is[\gamma^\nu,\gamma^\rho]f_{\nu\rho}/2})$ is given by $ms\epsilon^{\mu\nu\rho}f_{\nu\rho}$. Inserting this term in (A1), (A3) and performing the integral over spatial momenta one finds

$$\begin{aligned}J_{CS} &= \frac{m}{8\pi^2}\int dk^0\int_0^\infty ds\Theta(k^0)\big[\Theta(k^0 - \mu_F)e^{is(k_0^2 - m^2)} - \Theta(\mu_F - k^0)e^{-is(k_0^2 - m^2)} - \Theta(\eta - k^0)e^{is(k_0^2 - m^2)}\big] \\ &= \frac{1}{16\pi}\frac{m}{|m|}\left[\Theta(|m| - \mu_F) - \Theta(\mu_F - |m|) - \Theta(\eta - |m|) + \frac{2i}{\pi}\int_0^{\eta/|m|} P\left(\frac{1}{x^2 - 1}\right)dx\right],\end{aligned} \tag{A4}$$

where $P(\cdot)$ denotes the principal value. In the limit $m \to 0$ the imaginary term disappears and one recovers the result $-\frac{1}{8\pi}\frac{m}{|m|}$. A similar calculation can be done for the band of spurious holons with free Green's function defined, as discussed in Sec. III D, by

$$G = (\not{k} - m)\left[\Theta(\eta - k^0)\frac{1}{k^2 - m^2 + i\epsilon}\right].$$

Analogously one obtains, besides an imaginary term vanishing in the limit $m \to 0$,

$$J_{CS} = \frac{1}{16\pi}\frac{m}{|m|}[\Theta(\eta - |m|) + \Theta(\eta + |m|)] = \frac{1}{8\pi}\frac{m}{|m|},$$

proving that indeed the lower band of spurious holons has the Hall conductance calculated in Sec. III C.

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