Fermionic quantum spin liquids: Exact results for parametric pumping

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The time dependence of the response of the quantum spin liquid with fermionic excitations to parametric pumping has been calculated exactly for the special case of periodic pumping, the steplike periodic field, without using the resonance approximation, and for any value of the pumping magnitude. In the closed regime each mode (related to each eigenstate of the system) oscillates with time. The oscillations persist about the steady-state values, which are determined by the parameters of the pumping. The magnitude of the oscillations is finite for any value of the magnitude of the pumping. Those features drastically differ from the well-known resonance parametric pumping of magnons in magnetically ordered system, where the number of resonance magnons grows exponentially in time. The interference of infinitely many modes generically yields only decaying-in-time modulated oscillations of the total number of quasiparticles per mode even in the closed regime. For the fermionic system only one mode is in resonance, while others are out of resonance. It is totally different from the bosonic behavior of magnons under parametric pumping, where all modes can exist in resonance. The inclusion of the linear relaxation in the open regime for that quantum spin liquid produces the decay of oscillations to the initial state.

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I. INTRODUCTION

Parametric excitation of the simple harmonic oscillator (parametric oscillator) without relaxation [1] is usually studied using the Mathieu equation [2]

$$\frac{d^2y}{dt^2} + [a - \nu\cos(\omega t)]y = 0 \tag{1}$$

[notice that the Mathieu equation is the single-harmonic case of the Hill differential equation $(d^2y/dt^2) + f(t)y = 0$ with the periodic function $f(t)$]. The parametric pumping of the oscillator by the harmonic field is usually performed using Floquet's theorem [3] (see also Bloch's theorem [4]). The latter states that for fixed values of a and ν the Mathieu equation admits the complex solution. For those regions of the plane a-ν, which asymptotically (at $|a| + |\nu| \gg 1$) are the regions of unstable solutions, the increment is asymptotically proportional to the magnitude of the pumping. Hence, in that case the system becomes parametrically unstable even in the presence of linear damping if the magnitude of the pumping is large enough. It is believed that the parametric instability is the property of not only a single-harmonic case but also any periodic-in-time function $f(t)$ for the Hill equation.

For many-body systems the complete analogy is known between the behavior of the parametric oscillator and the behavior of the magnon (bosonic) system, e.g., under the action of the ac magnetic field, parallel to the dc one. The low-temperature behavior of a magnetically ordered system can be described by the gas of weakly interacting bosonic quasiparticles, known as magnons. The Hamiltonian $\mathcal{H}_m$ of the magnon system of the magnetically ordered system (up to an independent operator term) under the action of the ac magnetic field, parallel to the dc one, has the form [5]

$$\mathcal{H}_m = \sum_k ([\epsilon_{\mathbf{k}} + h_t(\mathcal{A}_k/\epsilon_k)]c_k^\dagger c_k + (h_t\mathcal{B}_k/2\epsilon_k) \times [c_k c_{-k} + \text{H.c.}]), \tag{2}$$

where $\epsilon_k = \sqrt{\mathcal{A}_k^2 - \mathcal{B}_k^2}$ is the energy of a magnon, c_k and $c_k^\dagger$ destroy and create the magnon, and h_t is the ac field. The coefficient $\mathcal{A}_k$ is determined by the dc field and the exchange interaction, while the one for $\mathcal{B}_k$, which does not conserve the number of magnons, is usually of a relativistic nature. Suppose $h_t = h\cos(\omega t)$. Within the so-called resonance approximation [6], one can consider exactly terms, explicitly dependent on time, which produce the nonzero contribution to the linear response. The remaining terms with an explicit time dependence can be omitted due to the smallness of the magnitude of the ac magnetic field $h \ll \epsilon_k, \omega$. (Their contribution can be, in principle, calculated in the framework of the perturbation theory.) For the Hamiltonian $\mathcal{H}_m$ the resonance approximation means that only terms like $[\exp(i\omega t)c_k c_{-k} + \text{H.c.}]$ are kept. Then, after well-known algebra [5], the increment (decrement) of the time dependence of c_k and $c_{-k}^\dagger$ (and therefore of the average of the number of magnons $\langle c_k^\dagger c_k\rangle$ with the density matrix or with the ground-state wave function) depends on whether the value $-[\epsilon_k - (\omega/2)]^2 + |h\mathcal{B}_k/\epsilon_k|^2$ is larger (smaller) than $(\gamma_k)^2$, where γ_k is the linear relaxation rate. Here and below we use units in which $\hbar = 1$ for simplicity. In resonance we can neglect the term $[\epsilon_k - (\omega/2)]^2$. Then for any γ_k there exists a threshold value of the magnitude of the ac field h_c: For a magnitude of the ac field larger than that threshold value,

$h > h_c$, the number of magnons in the system grows with time exponentially (and the linear relaxation cannot limit that growth). Such a parametric instability [5] is observed in many magnetic systems [7].

In quantum spin liquids the magnetic order is suppressed down to the lowest temperatures due to the frustration of spin-spin interactions and/or enhanced quantum fluctuations in low-dimensional systems [8,9]. In many quantum spin liquids emergent magnetic excitations are fermions (as a rule they carry fractionalized spin) instead of magnons (bosons, which carry spin 1) for ordered magnetic systems. The problem of parametric pumping in several quantum spin liquids was studied [10]. It was shown that instead of parametric instability, i.e., instead of exponential growth with time of the number of magnons, parametric pumping in those quantum spin liquid systems yields an oscillation of the number of (fermionic) emergent magnetic excitations. However, in Refs. [10] the resonance approximation was used. Therefore, the question remains whether the dropped nonresonance terms, which depend on time explicitly, can produce the parametric instability for quantum spin liquids. It constitutes the aim of the present work: to study the effect of parametric pumping in the fermionic model of the quantum spin liquid exactly. In what follows we solve the problem for periodic parametric pumping. It has a periodic steplike structure. For such periodic pumping the problem of the dynamics of the considered quantum system can be solved exactly, without any approximations (for noninteracting fermions).

On the other hand, theoretical investigations of sharp changes in applied fields are very important in the context of experiments in high magnetic pulse fields [11], with ultrafast (e.g., terahertz) pulses [12] performed in solids [13] and in ultracold gases in optical traps [14]. High magnetic fields imply that the condition that the magnitude of the ac field is smaller than the characteristic energies of the considered system, $h \ll \epsilon_k$, is not satisfied. Hence, the other goal of the present work is to study the response of the considered system to parametric pumping, for which the magnitude is unlimited by the applied method of consideration.

II. RESPONSE TO THE PERIODIC STEPLIKE PUMPING

Let us consider the Hamiltonian of the studied system is the quadratic form of spinless Fermi operators:

$$\mathcal{H}_0 = \sum_k \left[A_k a_k^\dagger a_k + \frac{1}{2}(B_k a_k a_{-k} + B_k^* a_{-k}^\dagger a_k^\dagger)\right], \quad (3)$$

where $a_k^\dagger$ (a_k) creates (destroys) the fermion of mode k. We can present the complex value $B_k = |B_k|\exp(i\psi_k)$. Then, going to the new operators $a_k \to a_k \exp(-i\psi_k/2)$, etc., we get the Hamiltonian (3) with the real $|B_k|$. In what follows we will use the notation $|B_k| \to B_k$ for simplicity. With the help of the standard Bogoliubov transformation the Hamiltonian $\mathcal{H}_0$ can be diagonalized; that is, it takes the form $\mathcal{H}_0 = \sum_{k>0} \varepsilon_k (b_k^\dagger b_k - 1/2)$, where $b_k^\dagger$ (b_k) are Fermi operators for creating (destroying) the true excitations of the considered system and $\varepsilon_k = \sqrt{A_k^2 + B_k^2}$ is the energy of those excitations.

Let us study the dynamics of the system with the Hamiltonian $\mathcal{H}_0$ under the action of periodic pumping $\mathcal{H} = \mathcal{H}_0 + h_t \sum_k a_k^\dagger a_k$, where h_t is the periodic-in-time t function. Rewriting the Hamiltonian in terms of Fermi operators for true eigenstates, we obtain

$$\mathcal{H} = E_0 + \sum_k \left[\left(\varepsilon_k + h_t \frac{A_k}{\varepsilon_k}\right) b_k^\dagger b_k + \frac{h_t B_k}{2\varepsilon_k}(b_k b_{-k} + b_{-k}^\dagger b_k^\dagger)\right], \quad (4)$$

where E_0 does not contain operators. The time dependence of the operators b_k and $b_k^\dagger$ can be written via the solution of the Heisenberg equations of motion:

$$i\left(\frac{\partial}{\partial t} + \gamma_k\right) b_{-k}^\dagger = -\left(\varepsilon_k + h_t \frac{A_k}{\varepsilon_k}\right) b_{-k}^\dagger + \frac{h_t B_k}{\varepsilon_k} b_k,$$
$$i\left(\frac{\partial}{\partial t} + \gamma_k\right) b_k = \left(\varepsilon_k + h_t \frac{A_k}{\varepsilon_k}\right) b_k + \frac{h_t B_k}{\varepsilon_k} b_{-k}^\dagger, \quad (5)$$

where we have introduced the linear relaxation for each mode k with the rate γ_k. For $t = 0$ we suppose that the Heisenberg operators coincide with the Schrödinger ones. For the spinors $f_k^\dagger \equiv (b_k^\dagger\ b_{-k})$ we can rewrite Eqs. (5) as

$$i\left(\frac{\partial}{\partial t} + \gamma_k\right) f_k = L_k f_k, \quad (6)$$

where $L_k = [\varepsilon_k + h_t(A_k/\varepsilon_k)]\sigma^z + h_t(B_k/\varepsilon_k)\sigma^x$, with the Pauli matrices $\sigma^{x,y,z}$. Introducing $\tilde{f}_k = \exp(-\gamma_k t) f_k$, we can remove the relaxation terms from the equations of motion.

Let us consider the periodic function $h_t = h$ for $n(\tau_1 + \tau_2) \leqslant t \leqslant n(\tau_1 + \tau_2) + \tau_1$, where n is a non-negative integer and $h_t = 0$ for $n(\tau_1 + \tau_2) + \tau_1 \leqslant t \leqslant (n+1)(\tau_1 + \tau_2)$; that is, we consider the pumping with the magnitude $h \geqslant 0$ and the period $\tau_1 + \tau_2$. We can introduce the angular frequency of the pumping $\omega = 2\pi/(\tau_1 + \tau_2)$. The pumping is periodic, and it can be written as a series of harmonic modes. In what follows we consider $t = n(\tau_1 + \tau_2)$, which is reasonable for large values of $n \gg 1$. Using the piecewise constancy of the pumping, the solution for the equations can be written as $\tilde{f}_k(t) = R_k^n \tilde{f}_k(0)$, where

$$R_k = \exp(-iL_{k2}\tau_2)\exp(-iL_{k1}\tau_1). \quad (7)$$

We can write

$$e^{-iL_{kj}\tau_j} = \cos(\varepsilon_{k,j}\tau_j) - i\frac{A_{k,j}\sigma^z + B_{k,j}\sigma^x}{\varepsilon_{k,j}}\sin(\varepsilon_{k,j}\tau_j), \quad (8)$$

where $j = 1, 2$, $A_{k,j} = \varepsilon_k + h_j(A_k/\varepsilon_k)$, $B_{k,j} = h_j(B_k/\varepsilon_k)$, $\varepsilon_{k,j} = \sqrt{A_{k,j}^2 + B_{k,j}^2}$, with $h_1 = h$ and $h_2 = 0$. We see that $A_{k,1} = \varepsilon_k + h(A_k/\varepsilon_k)$, $A_{k,2} = \varepsilon_k$ and $B_{k.1} = h(B_k/\varepsilon_k)$, $B_{k,2} = 0$, with $\varepsilon_{k,1} = \sqrt{\varepsilon_k^2 + 2A_k h + h^2}$ and $\varepsilon_{k,2} = \varepsilon_k$ for our choice of the periodic pumping. The operator R_k can be written as

$$R_k = \rho + \rho_x\sigma^x + \rho_y\sigma^y + \rho_z\sigma^z, \quad (9)$$

with

$$\rho = \cos(\varepsilon_k\tau_2)\cos(\varepsilon_{k,1}\tau_1) - \frac{\varepsilon_k^2 + hA_k}{\varepsilon_k\varepsilon_{k,1}}\sin(\varepsilon_k\tau_2)\sin(\varepsilon_{k,1}\tau_1),$$
$$\rho_z = -i\left[\sin(\varepsilon_k\tau_2)\cos(\varepsilon_{k,1}\tau_1) + \frac{\varepsilon_k^2 + hA_k}{\varepsilon_k\varepsilon_{k,1}}\cos(\varepsilon_k\tau_2) \times \sin(\varepsilon_{k,1}\tau_1)\right], \quad (10)$$

$$\rho_x = -i\frac{hB_k}{\varepsilon_k\varepsilon_{k,1}}\cos(\varepsilon_k\tau_2)\sin(\varepsilon_{k,1}\tau_1),$$
$$\rho_y = i\frac{hB_k}{\varepsilon_k\varepsilon_{k,1}}\sin(\varepsilon_k\tau_2)\sin(\varepsilon_{k,1}\tau_1). \qquad (11)$$

We can see that

$$\rho^2-\rho_z^2 = \cos^2(\varepsilon_1\tau_1)+\frac{\left(\varepsilon_k^2+hA_k\right)^2}{\varepsilon_k^2\varepsilon_{k,1}^2}\sin^2(\varepsilon_{k,1}\tau_1),$$
$$\rho_x^2+\rho_y^2 = -\frac{h^2B_k^2}{\varepsilon_k^2\varepsilon_{k,1}^2}\sin^2(\varepsilon_{k,1}\tau_1); \qquad (12)$$

that is, those combinations do not depend on τ_2. We also see that $\rho_x^2+\rho_y^2+\rho_z^2=\rho^2-1$. Let us define $\rho=\cos(\phi_k)$, so that $\rho_x^2+\rho_y^2+\rho_z^2=-\sin^2\phi_k$. It is possible because $|\rho|\leqslant 1$. To check the latter, we can use the following property: $[(\varepsilon_k+hA_k)/\varepsilon_k\varepsilon_{k,1}]\leqslant 1$ for any $h\neq 0$ and $B_k\neq 0$. Then $R_k=\Pi D\Pi^{-1}$, where $D=\exp(i\phi_k\sigma^z)$, and $\Pi=\exp(-i\theta\sigma^z/2)\exp(-i\theta'\sigma^x/2)$, with $\cos\theta=\rho_x/\sqrt{\rho_x^2+\rho_y^2}$ and $\cos(\theta')=\rho_z/\sqrt{\rho^2-1}$. The solution of the Heisenberg equations of motion is then

$$\tilde{f}_k(t)=\Pi D^n\Pi^{-1}\tilde{f}_k(0). \qquad (13)$$

Combining all expressions, we obtain, after some algebra, for $n_k(t)\equiv(1/2)\langle[b_k^\dagger(t)b_k(t)+b_{-k}^\dagger(t)b_{-k}(t)]\rangle$,

$$n_k(t)\exp(2\gamma_k t)=n_k(0)+\sin^2\theta'\sin^2(n\phi_k)\tanh(\varepsilon_k/2T), \qquad (14)$$

where T is the temperature in energy units, $n_k(0)=[\exp(\varepsilon_k/T)+1]^{-1}$ is the Fermi distribution function, and we use the property $1-2n_k(0)=\tanh(\varepsilon_k/2T)$. We can use the definition of θ', obtaining

$$n_k(t)e^{2\gamma_k t}=n_k(0)+\tanh(\varepsilon_k/2T)\frac{h^2B_k^2}{\varepsilon_k^2\varepsilon_{k,1}^2}\sin^2(\varepsilon_{k,1}\tau_1)\times\frac{\sin^2[t\phi_k/(\tau_1+\tau_2)]}{\sin^2\phi_k}. \qquad (15)$$

Notice that due to the diagonalization of the Hamiltonian $\mathcal{H}_0$ the pumping is coupled to the combination of operators $(A_k/2\varepsilon_k)[b_k^\dagger(t)b_k(t)+b_{-k}^\dagger(t)b_{-k}(t)]+(B_k/2\varepsilon_k)[b_k(t)b_{-k}(t)+b_{-k}^\dagger(t)b_k^\dagger(t)]$. Hence, let us define $\alpha_k(t)=(1/2)\langle[b_k(t)b_{-k}(t)+b_{-k}^\dagger(t)b_k^\dagger(t)]\rangle$. The solution for $\alpha_k(t)$ is

$$\alpha_k(t)e^{2\gamma_k t}=\tfrac{1}{2}\tanh(\varepsilon_k/2T)[\cos\theta\sin 2\theta'\sin^2(n\phi_k)-\sin\theta\sin\theta'\sin(2n\phi_k)]. \qquad (16)$$

Using the definitions of θ and θ', we obtain

$$\alpha_k(t)e^{2\gamma_k t}=\tanh(\varepsilon_k/2T)\frac{hB_k}{\varepsilon_k\varepsilon_{k,1}}\sin(\varepsilon_{k,1}\tau_1)\frac{\sin[t\phi_k/(\tau_1+\tau_2)]}{\sin\phi_k}\left\{\frac{\sin[t\phi_k/(\tau_1+\tau_2)]}{\sin\phi_k}\left[\frac{1}{2}\cos(\varepsilon_{k,1}\tau_1)\sin(2\varepsilon_k\tau_2)+\frac{\varepsilon_k^2+hA_k}{\varepsilon_k\varepsilon_{k,1}}\sin(\varepsilon_{k,1}\tau_1)\cos^2(\varepsilon_k\tau_2)\right]-\sin(\varepsilon_k\tau_2)\cos[t\phi_k/(\tau_1+\tau_2)]\right\}. \qquad (17)$$

It turns out that it is, probably, physically more appropriate to consider the relaxation to the state, defined by the Hamiltonian $\mathcal{H}_0$. (In general, the relaxation can exist, e.g., for a zero value of n_k or for some other value [15]). It implies that only the second term on the right-hand side of Eq. (15) has to have the damping multiplier $\exp(-2\gamma_k t)$. For the closed system (in which there are no relaxation processes) the results are formally obtained by taking the limit $\gamma_k\to 0$ in Eqs. (15) and (17).

Equations (15) and (17) are the main result of our paper. It must be pointed out that those results are *exact*; that is, we do not drop any terms and do not consider any parameter to be small when studying the dynamics of the system with the Hamiltonian $\mathcal{H}_0$ under the action of pumping.

III. ANALYSIS OF THE RESULTS

Let us analyze the obtained results. First, we can check that for $B_k=0$, as well as for $h=0$, or for $\tau_1=0$ we get $n_k(t)=n_k(0)$, as it must be. (For $B_k=0$ the pumping term commutes with $\mathcal{H}_0$.) The temperature dependence is totally determined by the initial state with the Hamiltonian $\mathcal{H}_0$. In the ground state we can replace $\tanh(\varepsilon_k/2T)\to\varepsilon_k/|\varepsilon_k|$, as usual. Hence, the dynamical renormalization is maximal in the ground state, and it decreases with the growth of the temperature. The time-dependent oscillating parts of $n_k(t)$ and $\alpha_k(t)$ can be written as a function of $\omega t\phi_k/2\pi$. We also see that the renormalization of n_k and α_k is finite with the growth of h for both the open and closed systems, as it must be. For general values of the magnitude of the pumping, the correlation functions oscillate with h as well as with τ_1 and τ_2. For small values of the magnitude of the pumping, $h\ll A_k, B_k$, the main contribution comes from the anomalous correlation function $\alpha_k(t)$, namely,

$$\alpha_k(t)\approx-\tanh(\varepsilon_k/2T)e^{-2\gamma_k t}\frac{hB_k}{\varepsilon_k^2}\sin(\varepsilon_k\tau_1)\times\left\{\frac{\sin^2(\varepsilon_k t)\cos(\varepsilon_k\tau_2)}{\sin^2[\varepsilon_k(\tau_1+\tau_2)]}[\cos(\varepsilon_k\tau_1)\sin(\varepsilon_k\tau_2)+\sin(\varepsilon_k\tau_1)]-\frac{\sin(2\varepsilon_k t)\sin(\varepsilon_k\tau_2)}{2\sin[\varepsilon_k(\tau_1+\tau_2)]}\right\}, \qquad (18)$$

while for small h the value of $n_k(t)$ is proportional to h^2, namely,

$$n_k(t)\approx n_k(0)+\tanh(\varepsilon_k/2T)e^{-2\gamma_k t}\frac{h^2B_k^2}{\varepsilon_k^4}\frac{\sin^2(\varepsilon_k\tau_1)}{\sin^2[\varepsilon_k(\tau_1+\tau_2)]}\times\sin^2(\varepsilon_k t). \qquad (19)$$

The resonance condition corresponds to $\sin\phi_k = 0$, i.e., $\rho = \pm 1$. One can check that it corresponds to $\varepsilon_k \tau_2 = \pi m$ and $\varepsilon_{k,1}\tau_1 = \pi m$, i.e., $\varepsilon_k\tau_2 + \varepsilon_{k,1}\tau_1 = 2\pi m$, where $m \geqslant 0$ is a non-negative integer. For small h and $m = 1$ it implies $\varepsilon_k = \omega/2$, i.e., the standard condition of the parametric resonance for many-body systems [5,10].

In resonance the terms proportional to $\sin[t\phi_k/(\tau_1 + \tau_2)]/\sin\phi_k$ grow linearly with t [notice that the main contribution to $n_k(t)$ and $\alpha_k(t)$ for the closed system grows as t^2]. It happens for a single mode k, defined by the resonance condition. Other modes are out of resonance due to the fermionic nature of eigenstates. However, the linear relaxation for the open system limits that growth because in resonance the relaxation rate γ_k for systems with stable eigenstates is much less than $\omega \sim 2\varepsilon_k$. This means that unlike the usual situation for the parametric resonance, e.g., for the linear oscillator, the growth of the deviations from the steady state in resonance is nonexponential in time in our case and can be limited by the inclusion of the linear relaxation.

For $\tau_1 = \tau_2 = \tau$ the angular frequency is $\omega = \pi/\tau$. For small h we get

$$\alpha_k(t) \approx -\tanh(\varepsilon_k/2T)e^{-2\gamma_k t}\frac{hB_k}{\varepsilon_k^2}\left\{\frac{\sin^2(\varepsilon_k t)}{4\cos[\pi\varepsilon_k/\omega]}\times[\cos(\pi\varepsilon_k/\omega)+1] - \frac{\sin(2\varepsilon_k t)}{4}\tan[\pi\varepsilon_k/\omega]\right\} \quad (20)$$

and

$$n_k(t) \approx n_k(0) + \tanh(\varepsilon_k/2T)e^{-2\gamma_k t}\frac{h^2B_k^2}{\varepsilon_k^4}\frac{1}{4\cos^2[\pi\varepsilon_k/\omega]}\times\sin^2(\varepsilon_k t). \quad (21)$$

For the resonance mode we have $\varepsilon_k = \omega/2$; hence,

$$\alpha_k^{\rm res}(t) \approx -\tanh(\omega/4T)e^{-2\gamma_k t}\frac{2hB_k}{\omega^2}\sin^2(\omega t/2), \quad (22)$$

and

$$n_k^{\rm res}(t) \approx n_k(0) + \tanh(\omega/4T)e^{-2\gamma_k t}\frac{4h^2B_k^2}{\omega^4}\sin^2(\omega t/2). \quad (23)$$

We can write the expressions for the quantities which can be measured in experiments with parametric pumping. First, the expression for the time dependence of the average total number of quasiparticles per site coupled to the pumping is

$$n_{av}(t) = \frac{1}{L}\sum_{k>0}\left[n_k(t)\frac{A_k}{\varepsilon_k} + \alpha_k(t)\frac{B_k}{\varepsilon_k}\right] \equiv n_{av}(0) + \Delta n_{av}(t), \quad (24)$$

where L is the number of sites of the system. In the thermodynamic limit $L \to \infty$ the summation can be replaced by the integration. The average total number of quasiparticles per site oscillates with time. Naturally, those oscillations decay with time due to nonzero γ_k in the open regime. In the closed regime in our many-body system each mode produces an oscillation with its own frequency; hence, the average total number must manifest the complicated interference of (infinitely) many oscillating modes. The interference generically causes the decay of the magnitude of those oscillations to some steady-state value even for the closed regime. According to the Riemann-Lebesque lemma if $\int_a^b f(x)dx$ exists and $f(x)$ has a limited total variation in the range (a, b), then one has $\lim_{y\to\infty}\int_a^b f(x)\sin(yx)dx = \lim_{y\to\infty}\int_a^b f(x)\cos(yx)dx = 0$. This means that in the closed regime the average total number of quasiparticles oscillates about the steady-state value determined from

$$\Delta n_{av}(st) = -\frac{1}{L}\sum_{k>0}\tanh(\varepsilon_k/2T)\frac{hB_k^2}{\varepsilon_k^2\varepsilon_{k,1}\sin^2\phi_k}\times\left(\frac{1}{2}\cos(\varepsilon_{k,1}\tau_1)\sin(2\varepsilon_k\tau_2) + \frac{\sin(\varepsilon_{k,1}\tau_1)}{\varepsilon_k\varepsilon_{k,1}}\times\left\{\varepsilon_k^2\cos^2(\varepsilon_k\tau_2) + hA_k[\sin(\varepsilon_{k,1}\tau_1) + \cos^2(\varepsilon_k\tau_2)]\right\}\right). \quad (25)$$

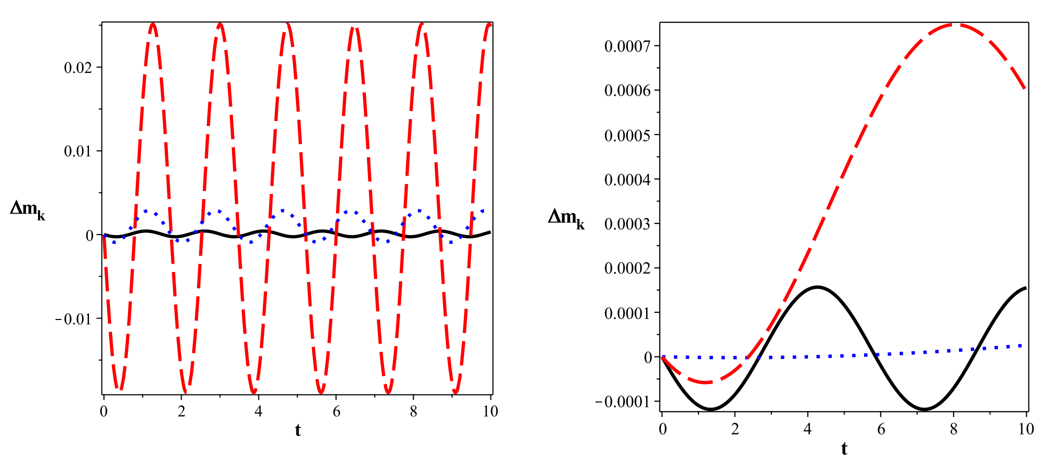


FIG. 1. Left: The time dependence of Δm_k for the honeycomb Kitaev model in the gapless case $J_x = J_z = 1$, $J_y = 0.5$ for $h = 0.01$ in resonance $\tau_1 = \pi/\varepsilon_{k,1}$ and $\tau_2 = \pi/\varepsilon_k$ in the closed regime. The solid black line shows $k_x = 0$, $k_y = \pi/2$; the dotted blue line shows $k_x = \pi/2$ and $k_y = 0$, and the dashed red line shows $k_x = k_y = \pi/2$. Right: The same as in the left panel, but for the gapped case $J_x = 1$, $J_y = 0.5$, and $J_z = 2$.

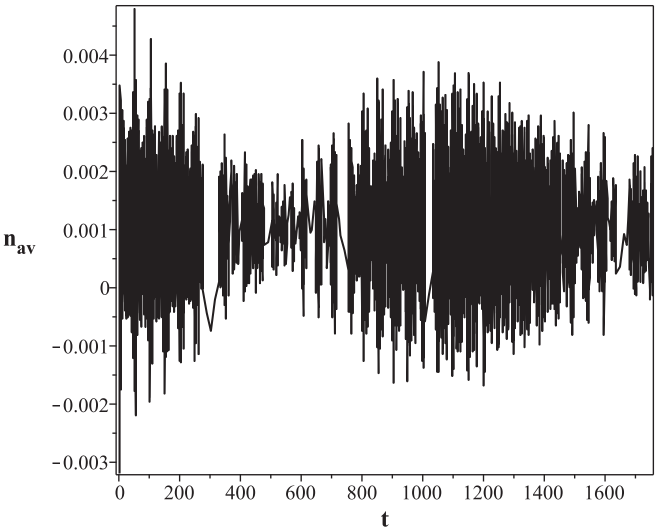


FIG. 2. The time dependence of n_{av} for the finite honeycomb Kitaev model in the gapless case out of resonance in the closed regime.

We see that in the closed regime the steady-state value is determined not only by the initial value $n_k(0)(A_k/\varepsilon_k)$ but also by the parameters of the pumping. Obviously, for the open regime, the relaxation yields the damping of the oscillating total number of quasiparticles to its initial value. The average value oscillates with τ_1 and τ_2 and also with the magnitude of the pumping h via $\varepsilon_{k,1}$.

The expression for the quantum work, associated with the considered time-dependent pumping, can be written as $W = (1/L)\langle \mathcal{H} - \mathcal{H}_0 \rangle$; that is, it is equal to $h_t \Delta n_{av}(t)$. For small h that work is proportional to h^2, as it must be. The averaged-in-time quantum work in the open regime, according to the Riemann-Lebesque lemma, is equal to $h\Delta n_{av}(st)$. On the other hand, in the closed regime the quantum work tends to zero.

IV. APPLICATION TO SOME QUANTUM SPIN LIQUIDS

Let us consider the realization of the Hamiltonian $\mathcal{H}_0$ in quantum spin liquids. First, for the XY spin-1/2 chain we have $A_k = H - (J_x + J_y)\cos k/2$ and $B_k = (J_x - J_y)\sin k/2$ [9,16]. Here, $J_{x,y}$ are exchange integrals, and H (we use units in which the effective magneton is equal to 1) is the dc external magnetic field, directed transverse to the spin-spin couplings. Notice that the well-known Ising chain in the transverse magnetic field [17] corresponds to either $J_x = 0$ or $J_y = 0$. The other example of the quantum spin liquid is the honeycomb Kitaev spin-1/2 model [18], for which one uses $A_k = \pm J_z + J_x \cos k_x + J_y \cos k_y$ and $B_k = J_x \sin k_x + J_y \sin k_y$, where $J_{x,y,z}$ are exchange integrals. Notice that for the Kitaev model our consideration is limited by the ground state (where fluxes, characteristic for the Kitaev model, are absent [19]). One more realization of the Hamiltonian $\mathcal{H}_0$ is the mean-field spinon Hamiltonian [20], used to describe the properties of the highly frustrated $YbMgGaO_4$. The latter is believed to be a quantum spin liquid; for example, the inelastic neutron scattering experiments manifested the broad continuum of spin excitations associated with the fractionalized quantum spin liquid with the Fermi surface of spinons [21]. In all of the above-mentioned examples h_t describes the periodic magnetic field [10,20]. Other realizations of the systems described by the Hamiltonian $\mathcal{H}_0$ are the Kitaev chain [22], the edge states of topological insulators [23], and the quantum wires situated in the vicinity of a conventional superconductor [24]. For the last two examples, parametric pumping in the resonance approximation was studied in [25]. For those realizations h_t describes the periodic applied potential. Our results can also be applied to type-I inversion-symmetric Weyl semimetals [26], which can also be described by the Hamiltonian $\mathcal{H}_0$. Notice that the interaction between fermions can be taken into account within the applied model, too (the interaction can be taken into account in the mean-field-like or random-phase approximation, leading to the same quadratic Hamiltonian for the fermions) [27].

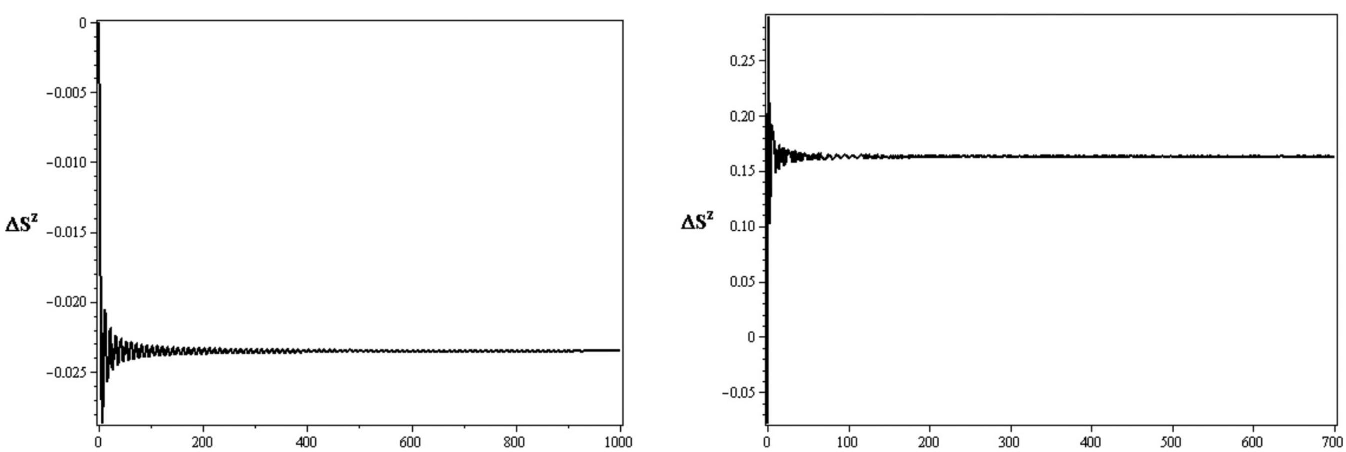


FIG. 3. Left: The time dependence of the renormalization of the z component of the averaged per site total spin of the Ising chain (with $J_x = 2$) per site in the transverse field h_t at $T = 0.1$ in resonance (with $\tau_1 = \tau_2 = \pi$) for $h = 0.1$. Right: The same as in the left panel, but for $h = 1.5$.

Let us start with the case of the Kitaev honeycomb spin model. Notice that our approach is different from the one considered in [28]. There dynamical structure factors [for $k = 0$ related to the electron spin resonance (ESR)] were obtained for the Kitaev model. However, the geometry of [28] is related to the ESR, in which the ac magnetic field is polarized perpendicular to the ac field. It yields the conservation law for each elementary act of the pumping in resonance $\omega = \varepsilon_k$. In our work, instead, we consider parametric pumping with $\omega = 2\varepsilon_k$ in resonance. The difference is drastic. For magnetically ordered systems the standard ESR geometry leads to the standard response caused by the periodic driving force. The growth of the amplitude is limited by the linear relaxation. On the other hand, in our approach the periodic pumping is *parametric*, and for magnetically ordered systems it causes the parametric instability (exponential growth of the magnitude, which cannot be limited by the linear relaxation). Here, we limit ourselves to the ground state (see above; as shown in Ref. [29], disordered thermally excited fluxes become important at a temperature of about 1% of the Kitaev energy scale). Notice that the temperature is determined by the initial state and does not depend on the pumping. It is also important that homogeneous pumping (which is connected to the projection of the total spin) can change all signs of flux quantum numbers only homogeneously [10]. Define $\Delta m_k \equiv \langle a_k^\dagger(t)a_k(t) - a_k^\dagger(0)a_k(0)\rangle$. Consider the time dependence of Δm_k for several values of k_x and k_y. Obviously, for $k_x = k_y = 0$ we have $\Delta m_k = 0$. Figure 1 shows the time dependence of Δm_k for several values of k_x and k_y for the gapless case in resonance and in the gapped case out of resonance (left and right panels, respectively). We see that time-dependent oscillations differ from each other for different modes. The period of single-mode oscillation in resonance is smaller in resonance than out of resonance (for both gapped and gapless cases). Notice the different scales in the panels of Fig. 1.

Unfortunately, the integration of the total number of modes cannot be performed explicitly. It is only possible to sum over the finite number of modes, and the results of those summations support our conclusions. See, e.g., Fig. 2, where $n_{av}(t)$ is shown as a function of time for the Kitaev honeycomb model for the lattice 100×100 sites (similar behavior can be seen, e.g., for the case 500×500 sites) in the gapless case $J_x = J_z = 1$ and $J_y = 0.5$ for $h = 0.01$ out of resonance $\tau_1 = \pi/2.501$, and $\tau_2 = \pi/2.5$ for the finite closed system. One can see modulated decaying oscillations of the average number of fermions.

The case of the Ising chain in the transverse field permits us to obtain all integral values explicitly, i.e., to observe the interference of infinitely many modes. For example, the renormalization of the z component of total spin per site of the Ising chain in the transverse field h_t is related to Δn_{av} via $\Delta S^z \equiv (1/L)\sum_n [S_n^z(t) - S_n^z(0)] = -\Delta n_{av}$. Figure 3 shows the behavior of that renormalization as a function of time for the Ising chain $J_x = 2$ for $H = 0$ at $T = 0.1$ in resonance (for $\tau_1 = \tau_2 = \pi$) for several values of the magnitude of the pumping h in the closed regime. We see that the average spin projection oscillates with time, decaying to some steady-state value, determined by the parameters of the spin chain and the pumping. When the magnitude of the pumping h grows, the oscillations persist (however, with different magnitudes and frequencies), also decaying to the steady-state values. That case $h = 0.01$ is shown in Fig. 4 (top panel), where the decaying oscillations of the z projection of the averaged per site total spin are also clearly seen. Figure 4 (bottom panel) shows the above-mentioned oscillations for the same Ising chain out of resonance with $\tau_1 = \pi/4$ and $\tau_2 = \pi$. We see that the decaying oscillations persist, however with smaller period and with modulations characteristic of the closed regime of the parametric pumping (see [10]).

In Fig. 5 for comparison, the resonance ($\tau_1 = \tau_2 = \pi$) and nonresonance ($\tau_1 = \pi/2$ and $\tau_2 = \pi$) oscillations are shown.

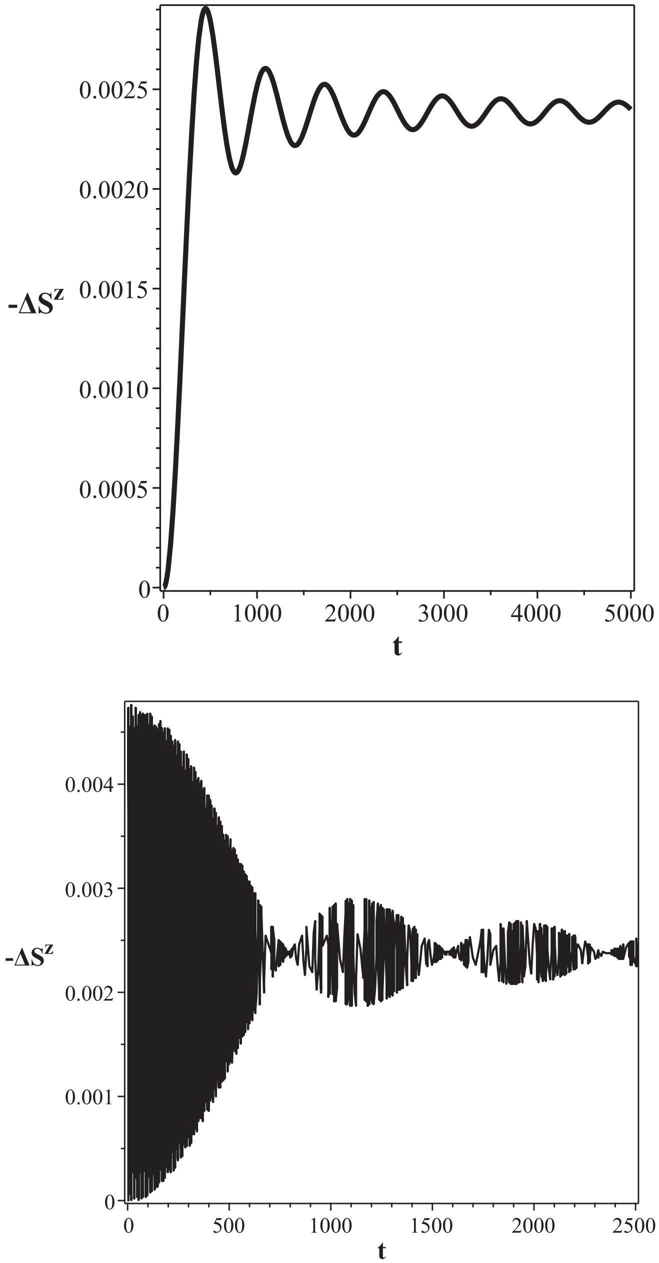


FIG. 4. Top: The time dependence of the renormalization of the z component of the averaged per site total spin of the Ising chain (with $J_x = 2$) per site in the transverse field h_t at $T = 0.1$ in resonance (with $\tau_1 = \tau_2 = \pi$) at $h = 0.01$. Bottom: The same as in the top panel, but out of resonance (with $\tau_1 = \pi/4$ and $\tau_2 = \pi$) at $h = 0.01$.

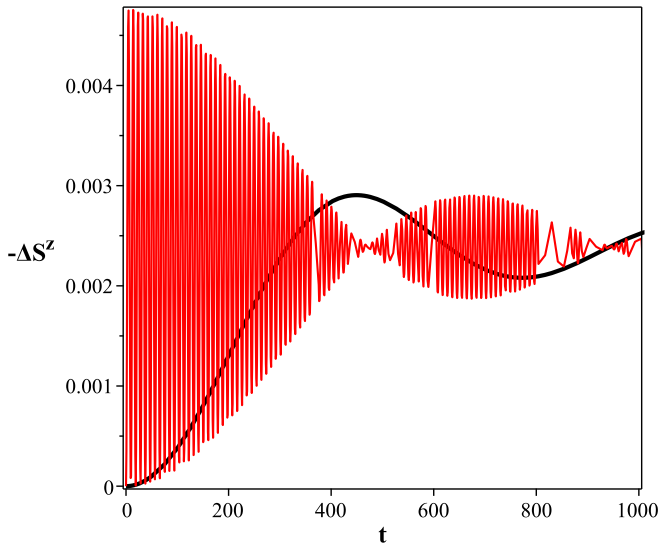


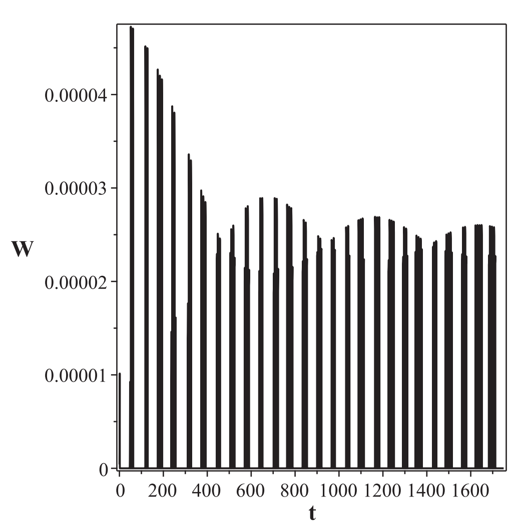
FIG. 5. The time dependence of the renormalization of the z component of the averaged per site total spin of the Ising chain (with $J_x = 2$) per site in the transverse field h_t at $T = 0.1$ in resonance (black line, $\tau_1 = \pi$ and $\tau_2 = \pi$) and out of resonance (red line, with $\tau_1 = \pi/2$ and $\tau_2 = \pi$) at $h = 0.01$.

We see that in resonance the frequency of the oscillations becomes much smaller.

Then, Fig. 6 shows the time dependence of the quantum work associated with the considered time-dependent pumping for the Ising chain (with the same parameters as in Figs. 3–5) out of resonance (in the left panel, $\tau_1 = \pi/2$ and $\tau_2 = \pi$) and in resonance (in the right panel, $\tau_1 = \tau_2 = \pi$). In resonance the area under the curve is almost filled by lines because of the time dependence of h_t with the period $\tau_1 + \tau_2$, much smaller than for Δn_{av} (we show those oscillations in this scale to manifest the modulations). On the other hand, out of resonance one can distinguish separate peaks related to the periodic steplike form of the pumping.

Finally, Fig. 7 (showing the in-resonance and out-of-resonance behaviors in the left and right panels, respectively) manifests the normalized by h^2 power $Q = W/th^2$, associated with the quantum work. In Fig. 7 in the right panel separate peaks are clearly seen in resonance, too.

The solution is obtained for the steplike periodic pumping. For the purpose of this work (whether the nonresonance terms can yield the parametric instability) the most important fact is the periodic modulation of the parameter of the Hamiltonian; that is, the presence of parametric pumping does not matter regardless of the form of the periodic field. However, it is important to understand whether that form can also be essential. Generally speaking, the answer to that question is very complicated and deserves special consideration. It is possible to answer that question only in some limiting case [30]. In brief, the answer is that following [31], we can consider the solution for harmonic parametric pumping, e.g., for the case $B_k \ll A_k$. In [31] a similar model (the quadratic form of the creation and destruction operators) with harmonic parametric pumping ($h_t = h[1 - \cos(\omega t)]$) was studied, however for bosons. Adapting the results of [31] for the fermionic case, it is possible to obtain the expression describing the dynamics of spinors f_k for harmonic pumping in the second order with respect to the small parameter B_k/A_k. For that case the operator R_k in the diagonal form can also be written as $R_k = \rho \pm i\sqrt{1-\rho^2}$ (i.e., similar to the steplike shape of the pumping), with the different definition of $\rho = \cos[2\pi(\alpha_k + \delta)] - (|\beta_k|^2/2)\{\kappa \exp[2\pi(\alpha_k + \delta)] + \text{c.c.}\}$, where $\alpha_k = A_k/\omega$, $\beta_k = B_k/\omega$, $\delta = h/\omega$, and $\kappa_k = \int_0^{2\pi} dx \int_0^x dy \exp(2i\{\delta[\sin(x) - \sin(y)] - (\alpha_k + \delta)(y - x)\})$. Then it is possible to prove that $\rho \leqslant 1$ for fermions for any δ. Hence, the fermionic system is stable with respect to the parametric action of the harmonic periodic field, too, for the case $B_k \ll A_K$ without using the resonance approximation. For general values of B_k/A_k it is impossible to obtain the solution (it is similar to the problem of obtaining the general solution for the Hill equation). Numerical solutions are, unfortunately, limited to the used choice of parameters.



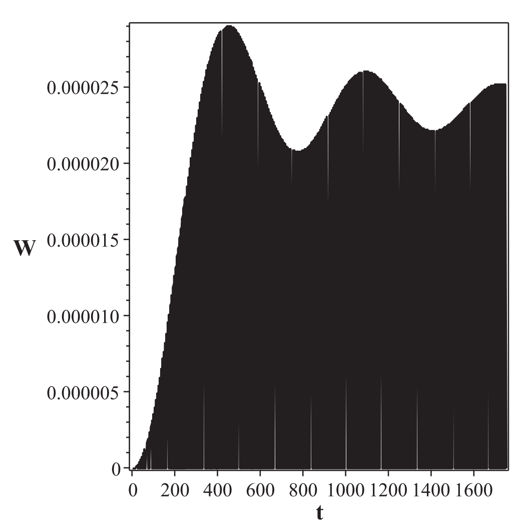


FIG. 6. Left: The quantum work associated with the considered time-dependent pumping for the Ising chain as a function of time out of resonance $\tau_1 = \pi/2$ and $\tau_2 = \pi$. Right: The same as in the left panel, but in resonance $\tau_1 = \tau_2 = \pi$.

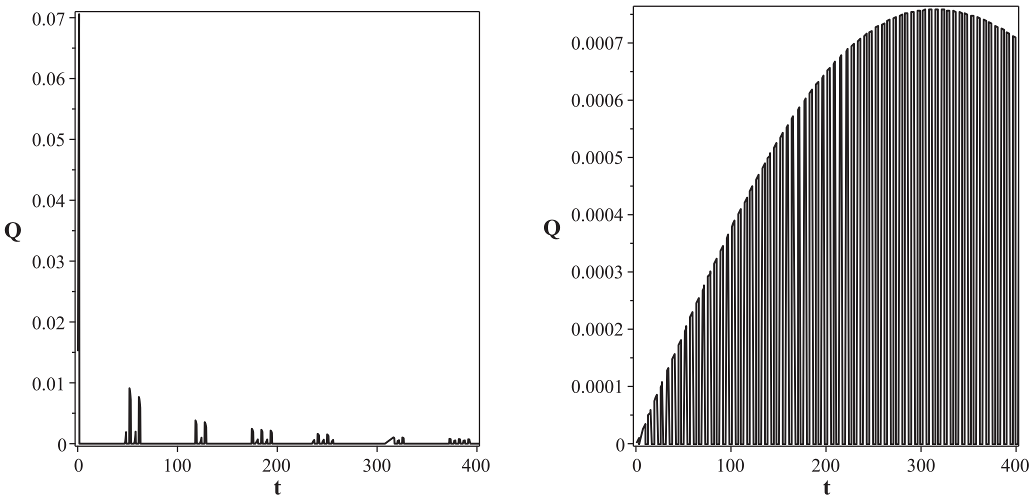


FIG. 7. Left: The normalized power associated with the considered time-dependent pumping for the Ising chain as a function of time out of resonance $\tau_1 = \pi/2$ and $\tau_2 = \pi$. Right: The same as in the left panel, but in resonance $\tau_1 = \tau_2 = \pi$.

V. SUMMARY

In summary, the time dependence of the response of the quantum spin liquid with fermionic excitations on the parametric periodic-in-time pumping of the steplike form has been calculated exactly. The result has been obtained without using the resonance approximation and for any value of the pumping magnitude. In the closed regime each mode (related to each eigenstate of the system) oscillates with time. The oscillations persist about the steady-state values. The latter is determined not only by the initial state but also by the parameters of the pumping (the magnitude and the frequency). The magnitude of the oscillations is finite for any value of the magnitude of the pumping. It is very different from the well-known exponential-in-time growth of the magnitude of the parametric oscillator in resonance and from the resonance parametric pumping of magnons in magnetically ordered systems, where the number of resonance magnons grows exponentially with time. The interference of infinitely many modes (in the thermodynamic limit) generically yields only decaying-in-time modulated oscillations of the total number of quasiparticles per mode even in the closed regime. For the fermionic system only one mode is in resonance, while the others are out of resonance. It is totally different from the bosonic behavior of magnons under parametric pumping, where all modes can exist in resonance. The inclusion of the linear relaxation in the open regime for the quantum spin liquid produces the decay of oscillations to the initial state. In resonance the frequency of the oscillations of the total number of quasiparticles becomes much smaller than out of resonance. Other characteristics, like the quantum work associated with parametric pumping, have also been calculated. The obtained results can be applied directly to many models of quantum spin liquids with fermionic excitations (mostly with gapped excitations), like the Kitaev honeycomb spin model, the spin XY or Ising chains in the transverse field, and the mean-field spinon liquid, used to describe the properties of the highly frustrated $YbMgGaO_4$. On the other hand, the results can be applied to other fermionic systems, like the Kitaev chain, edge states of topological insulators, and quantum wires in the vicinity of a superconductor, and to type-I inversion-symmetric Weyl semimetals.

ACKNOWLEDGMENT

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