

One-loop functional renormalization group study for the dimensional reduction and its breakdown in the long-range random field $O(N)$ spin model near lower critical dimension

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We consider the random-field $O(N)$ spin model with long-range exchange interactions which decay with distance r between spins as $r^{-d-\sigma}$ and/or random fields which correlate with distance r as $r^{-d+\rho}$, and reexamine the critical phenomena near the lower critical dimension by use of the perturbative functional renormalization group. We compute the analytic fixed points in the one-loop beta functions, and study their stability. We also calculate the critical exponents at the analytical fixed points. We show that the analytic fixed point which governs the phase transition in the system with the long-range correlations of random fields can be destabilized by the nonanalytic perturbation in both cases where the exchange interactions between spins are short ranged and long ranged. For the system with the long-range exchange interactions and uncorrelated random fields, we show that the $d \rightarrow d - \sigma$ dimensional reduction at the leading order of the $d - 2\sigma$ expansion holds only for $N > 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$. Our investigation into the system with the long-range exchange interactions and uncorrelated random fields also gives the value of the boundary between critical behaviors in systems with long-range and short-range exchange interactions, which is identical to that predicted by Sak [Phys. Rev. B **8**, 281 (1973)]. For the system with the long-range exchange interactions and the long-range correlated random fields, we show that the $d \rightarrow d - \sigma - \rho$ dimensional reduction does not hold within the present framework, as far as N is finite.

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I. INTRODUCTION

The random-field $O(N)$ spin model is the model in which nonrandom exchange interactions between spins are ferromagnetic and external magnetic fields are random. To clarify the critical phenomena in this model is one of the fundamental problems in the disordered spin system, and there are a lot of intensive studies on this [1,2]. The $d \rightarrow d - \theta$ dimensional reduction gives an important clue to clarify the nature of this model. The $d \rightarrow d - \theta$ dimensional reduction means that the effect of random fields reduces the spatial dimension by θ ; namely, the critical phenomena in d -dimensional random-field system is equivalent to that in the $(d - \theta)$ -dimensional corresponding pure system. Here θ denotes the exponent describing that the flow of the renormalized temperature goes to zero under the renormalization-group iteration. If the $d \rightarrow d - \theta$ dimensional-reduction prediction is correct, all critical exponents in the d -dimensional random-field system should be the same as those in the corresponding pure system in θ dimensions less.

In the spin system with the short-range ferromagnetic exchange interactions and the uncorrelated random fields (SR), the $d \rightarrow d - 2$ dimensional reduction and its breakdown are one of the central issues. This conjecture was obtained by the perturbation theory [3–5] and the supersymmetry argument [6]. Rigorous proofs have shown that the $d \rightarrow d - 2$ dimensional-reduction prediction is incorrect below four dimensions in the case of the random-field Ising model ($N = 1$ case) [7,8]. The $d \rightarrow d - 2$ dimensional reduction

and its breakdown for the random-field $O(N)$ spin model above four dimensions have been intensively studied. Fisher studied the critical phenomena in $4 + \epsilon$ dimensions by use of the $O(N)$ nonlinear-sigma model [9]. He showed that all possible higher-rank random anisotropies which are all relevant operators are generated by the perturbative functional renormalization-group iteration of the $O(N)$ nonlinear-sigma model with only the random field term. Then he treated the $O(N)$ nonlinear-sigma model including the random-field and all the random-anisotropy terms, and derived the one-loop beta function in $4 + \epsilon$ dimensions. He showed that there is no singly unstable fixed point corresponding to the $d \rightarrow d - 2$ dimensional reduction at $O(\epsilon)$, and concluded that the $d \rightarrow d - 2$ dimensional-reduction prediction is incorrect near four dimensions. The one-loop beta function obtained by Fisher and the two-loop beta function extended by Le Doussal and Wiese [10] and Tissier and Tarjus [11] have been examined carefully [10–15]. The breakdown of the dimensional reduction is characterized by a nonanalyticity which emerges in the first derivative of the function including the random-field and all the random-anisotropy terms. Namely, the nonanalyticity forms a cusp in the first derivative of the function including the random-field and all the random-anisotropy terms, which causes the breakdown of the dimensional reduction. The singly unstable fixed point corresponding to the $d \rightarrow d - 2$ dimensional reduction exists for $N > 18 - (49/5)\epsilon$, although it has the weak nonanalyticity which does not change the value of the fixed point. However, it is unstable with respect to the perturbation with nonanalyticity for $N < 2(4 + 3\sqrt{3}) - [3(2 + 3\sqrt{3})/2]\epsilon$. Thus, the $d \rightarrow d - 2$ dimensional reduction holds for $N > 2(4 + 3\sqrt{3}) - [3(2 + 3\sqrt{3})/2]\epsilon$, and the critical exponents of the connected and the disconnected

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correlation functions η and $\bar{\eta}$ satisfy $\bar{\eta} = \eta$. Whereas in $N < 2(4 + 3\sqrt{3}) - [3(2 + 3\sqrt{3})/2]\epsilon$, the critical phenomena is governed by the fixed point with the nonanalyticity, and thus the $d \rightarrow d - 2$ dimensional reduction is broken. Moreover, a complete theoretical explanation of the $d \rightarrow d - 2$ dimensional reduction and its breakdown has been provided through the nonperturbative functional renormalization group [16–19].

In the case where the ferromagnetic exchange interactions are short ranged and the random fields are correlated over the distance r as $r^{-d+\rho}$ (LRF), the $d \rightarrow d - 2 - \rho$ dimensional reduction and its breakdown are still under debate. The symbol ρ denotes the exponent which characterizes the range of the random-field correlations. To study the long-range effect of the random-field correlations in the system, we consider the case $0 \leq \rho < d - 2d_p$, where d_p is the lower critical dimension of the corresponding pure system. Kardar, McClain, and Taylor performed the renormalization-group calculation near the upper critical dimension $d_u = 6 + \rho$, and concluded that the $d \rightarrow d - 2 - \rho$ dimensional reduction is broken at $O(\epsilon^2)$ in $\epsilon = d_u - d$ [20]. Bray pointed out an error in Kardar, McClain, and Taylor's result but their conclusion still holds [21]. Chang and Abrahams carried out the one-loop renormalization-group calculation for the $O(N)$ nonlinear-sigma model near the lower critical dimension $d_l = 4 + \rho$, and showed that the $d \rightarrow d - 2 - \rho$ dimensional reduction is broken at $O(\epsilon)$ in $\epsilon = d - d_l$ and for $N > 3$ [22]. Fedorenko and Kühnel [23] examined the one-loop beta functions of the $O(N)$ nonlinear-sigma model including not only the uncorrelated and the long-range correlated random fields but also all the uncorrelated and the long-range correlated random anisotropies which are missed in the work by Chang and Abrahams. They showed that the correlation length exponent ν and the phase diagram obtained by Chang and Abrahams are incorrect, and the exponents η and $\bar{\eta}$ are correct only in a region controlled by the singly unstable fixed point with the weaker nonanalyticity.

In the case where the long-range ferromagnetic exchange interactions decay with distance r between spins as $r^{-d-\sigma}$ and random fields are uncorrelated (LRE), the $d \rightarrow d - \sigma$ dimensional reduction and its breakdown are still under debate. The symbol σ denotes the exponent which controls the range of the exchange interactions. It should be positive to ensure that the energy density stays finite in the thermodynamic limit. To study the long-range character of the exchange interactions in the system, we consider the case $0 < \sigma < 2$. Young performed the renormalization-group calculation near the upper critical dimension $d_u = 3\sigma$, and concluded that the $d \rightarrow d - \sigma$ dimensional reduction is broken at $O(\epsilon^2)$ in $\epsilon = d_u - d$ [5]. Bray pointed out an error in Young's result but the conclusion still holds [21]. Chang and Abrahams carried out the one-loop renormalization-group calculation for the $O(N)$ nonlinear-sigma model near the lower critical dimension $d_l = 2\sigma$, and showed that the $d \rightarrow d - \sigma$ dimensional reduction holds at $O(\epsilon)$ in $\epsilon = d - d_l$ and for $N > 1$ [24]. However, the $O(N)$ nonlinear-sigma model studied by Chang and Abrahams does not contain an infinite number of relevant operators which should be included in the model. Recently, Balog, Tarjus, and Tissier studied the critical phenomena of a one-dimensional random-field Ising model with the long-range exchange interactions and uncorrelated random fields by use

of the nonperturbative renormalization group, and found that there are two distinct regimes characterized by the presence or absence of the nonanalyticity in the region of $\frac{1}{3} \leq \sigma < \frac{1}{2}$ where the critical exponents take nonclassical values [25].

In the spin system with the long-range ferromagnetic exchange interactions and the long-range correlated random fields (LREF) with $0 < \sigma < 2$ and $0 \leq \rho < d - 2d_p$, the $d \rightarrow d - \sigma - \rho$ dimensional reduction and its breakdown are still under debate. Bray used the renormalization-group scaling theory, and showed $\eta = 2 - \sigma$, $2\eta - \bar{\eta} = \rho$, and $\theta = \sigma + \rho$. Recently, we put $\rho = 2 - \sigma$, and studied the critical phenomena in the three-dimensional long-range random-field Ising model in the region of $1/2 < \sigma < 1$ by using the nonperturbative functional renormalization group combined with the supersymmetric formalism [26]. We showed that the $d \rightarrow d - 2$ dimensional reduction holds for $1/2 < \sigma < \sigma_{\text{DR}} \approx 0.71$, and its breakdown is observed in the exponent ν for $\sigma_{\text{DR}} < \sigma < 1$.

In contrast to the case $0 < \sigma < 2$ in which the long-range feature of the exchange interactions is dominant, the phase transition for large σ belongs to the short-range universality class. As the exponent σ decreases from large σ , the universality class of the phase transition crosses over from the short-range one to the long-range one at a critical value σ_* . In spite of theoretical and numerical studies over 40 years, the critical behavior in the vicinity of $\sigma = \sigma_*$ is still an ongoing problem. There are a lot of studies on this problem in the pure system [27–38]. In Refs. [27,29] it was shown that the exponent η changes discontinuously from the value in the corresponding short-range system to $2 - \sigma$ at $\sigma = \sigma_* = 2$, as σ decreases from large σ . In Refs. [28,30,31,34,36–38] it was shown that the effect of the long-range exchange interactions is relevant for $\sigma < \sigma_* = 2 - \eta_{\text{sr}}$, where η_{sr} denotes the exponent of the connected correlation function in the corresponding short-range system. Then the exponent η is continuous at $\sigma_* = 2 - \eta_{\text{sr}}$, whose value takes η_{sr} for $\sigma \geq \sigma_*$, and $2 - \sigma$ for $\sigma < \sigma_*$. Moreover, the presence of a logarithmic correction to the connected correlation function at $\sigma_* = 2 - \eta_{\text{sr}}$ was reported in Ref. [35]. In Refs. [32,33] it was shown that the discontinuity of the exponent η at $\sigma = 2$ does not occur, and the value of η is interpolated smoothly from η_{sr} to $2 - \sigma$, as σ decreases from $\sigma = 2$. In the random-field spin system, Bray showed $\sigma_* = 2 - \eta_{\text{sr}}$ by using the renormalization-group scaling theory [21].

As stated above, the phase transitions in this model are classified into four universality classes (SR, LRF, LRE, and LREF), according to whether the exchange interactions and/or the random-field correlations in the system are short ranged or long ranged. However, most studies of the critical phenomena in the random-field $O(N)$ spin model have been dedicated to the SR case. In this paper we consider all four cases. We study the critical phenomena near the lower critical dimension with the use of the $O(N)$ nonlinear-sigma model combined with the replica formalism. The model treated in this paper contains not only the uncorrelated and the correlated random-field terms but also all the uncorrelated and the correlated random-anisotropy terms. We employ the perturbative functional renormalization group in order to obtain the one-loop beta functions. We examine the properties of the fixed point functions, and investigate the stability of

the analytic fixed points on the basis of the argument by Baczyk, Tarjus, Tissier, and Balog [15]. Then we calculate the critical exponents η , $\bar{\eta}$, and ν at each of four analytic fixed points, and discuss the critical properties of the system for each universality class. We show that the destabilization of the analytic fixed point controlling the critical behavior in the system with the long-range correlations of random fields can be caused by the perturbation with nonanalyticity in both cases where the exchange interactions between spins are short ranged and long ranged. In the system with LRE, we find that the analytic fixed point of $O(\epsilon)$ in $\epsilon = d - d_l$ which controls the critical behavior is singly unstable not for $N > 1$ but for $N > 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$. We show that the validity of the $d \rightarrow d - \sigma$ dimensional reduction at the leading order of the $d - 2\sigma$ expansion is confirmed only for $N > 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$. Moreover, by investigating the relation between the critical exponents η and $\bar{\eta}$, we also obtain the critical value $\sigma_* = 2 - \eta_{\text{sr}}$, which is the same as that obtained in Refs. [21,28,30,31,34,36–38]. In the system with LREF, we find that the analytic fixed point which is singly unstable exists under a certain condition. However, we show that the $d \rightarrow d - \rho - \sigma$ dimensional reduction does not hold within the present framework, as far as N is finite.

The organization of this paper is as follows. In Sec. II we study the systems with the short-range exchange interactions, namely the SR and LRF cases. We perform the one-loop functional renormalization group analysis. We show that the analytic fixed point which governs the phase transition in the system with LRF can be destabilized by the perturbation with nonanalyticity. In Sec. III we study the systems with the long-range exchange interactions, namely the LRE and LREF cases. We treat the one-loop beta functions, and carefully analyze the properties of the analytic fixed points and their stability. It is shown that the analytic fixed point controlling the critical behavior in the system with LRE becomes unstable against the perturbation with nonanalyticity for $N < 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$, which is the same as the case of the system with SR. We also show that the destabilization of the analytic fixed point which governs the phase transition in the system with LREF can occur due to the nonanalytic perturbation. As a result, we obtain a certain region in the plane of the parameters N and ρ where the analytic fixed points are singly unstable. In Sec. IV we calculate the critical exponents η , $\bar{\eta}$, and ν at the analytic fixed point which controls the critical behavior in the system with LRE. We reconsider the validity of the $d \rightarrow d - \sigma$ dimensional reduction. We also present the result for the critical value σ_* . In Sec. V we calculate the critical exponents η , $\bar{\eta}$, and ν at the analytic fixed point which controls the critical behavior in the system with LREF. We show that $d \rightarrow d - \sigma - \rho$ dimensional reduction breaks down within the present framework, as far as N is finite. Section VI summarizes our results.

II. CRITICAL PHENOMENA AT ZERO TEMPERATURE OF LONG-RANGE CORRELATED RANDOM FIELD $O(N)$ SPIN MODEL WITH SHORT-RANGE EXCHANGE INTERACTIONS IN $4 + \epsilon$ DIMENSIONS

This section is intended as a reexamination of the critical phenomena at zero temperature of the long-range correlated

random field $O(N)$ spin model with short-range exchange interactions in $4 + \epsilon$ dimensions. We discuss the nature of analytic fixed points and their stability. And we calculate the critical exponents at the analytic fixed point which controls the critical behavior in the system with SR and with LRF.

A. Model

Let us consider an N -component vector spin system where an N -component vector spin $\mathbf{S}(x)$ with a fixed-length constraint $\mathbf{S}(x)^2 = 1$ couples to a random field. In order to carry out the average over the random field, we use the replica method. The critical phenomena of the long-range correlated random field $O(N)$ spin model with the short-range exchange interactions near lower critical dimension is described by the $O(N)$ nonlinear-sigma model of the following replica partition function \mathcal{Z} and effective action βH_{rep} :

$$\begin{aligned} \mathcal{Z} &= \int \prod_{\alpha=1}^n \mathcal{D}\mathbf{S}^\alpha \delta[\mathbf{S}^\alpha(x)^2 - 1] e^{-\beta H_{\text{rep}}}, \\ \beta H_{\text{rep}} &= \frac{a^{2-d}}{2T} \int_x \sum_{\alpha=1}^n \mathbf{S}^\alpha(x) \cdot (-\partial^2) \mathbf{S}^\alpha(x) \\ &\quad - \frac{a^{-d}}{2T^2} \int_x \sum_{\alpha,\beta} R_1[\mathbf{S}^\alpha(x) \cdot \mathbf{S}^\beta(x)] \\ &\quad - \frac{a^{-d-\rho}}{2T^2} \int_{x,x'} \sum_{\alpha,\beta} g(x-x') R_2[\mathbf{S}^\alpha(x) \cdot \mathbf{S}^\beta(x')], \quad (1) \end{aligned}$$

where a is the ultraviolet cutoff, and $\int_x := \int d^d x$. The replica indices denoted by Greek indices take values $\alpha, \beta, \dots = 1, \dots, n$. The first term in the action (1) is the kinetic term which corresponds to the short-range exchange interactions between spins. The parameter T is the dimensionless temperature. The function $R_i(\mathbf{S}^\alpha \cdot \mathbf{S}^\beta)$ ($i = 1, 2$) represents the random field and all the random anisotropies, and is given by

$$R_i(\mathbf{S}^\alpha \cdot \mathbf{S}^\beta) = \sum_{r=1}^{\infty} \Delta_{i,r} (\mathbf{S}^\alpha \cdot \mathbf{S}^\beta)^r. \quad (2)$$

Here $\Delta_{i,r}$ denotes the strength of the random field and the r th rank random anisotropy ($r = 1$ is the random field, and $r = 2$ is the random second-rank anisotropy). The subscript $i = 1$ corresponds to the uncorrelated random fields and random anisotropies, and the subscript $i = 2$ corresponds to the long-range correlated random fields and random anisotropies with $g(x-x') \sim |x-x'|^{-d+\rho}$. The lower critical dimension of this model is $d_l = 4 + \rho$. In the present study we consider the case of $0 \leq \rho < \epsilon$.

B. One-loop beta functions and the zero-temperature fixed points

To perform the renormalization group transformation, we put each replicated vector spin $\mathbf{S}^\alpha(x)$ as a combination of a slow field $\mathbf{n}_0^\alpha(x)$ of the unit length and fast fields $\varphi_i^\alpha(x)$, $i = 1, \dots, N - 1$ such that

$$\mathbf{S}^\alpha(x) = \mathbf{n}_0^\alpha(x)\sqrt{1 - \varphi^\alpha(x)^2} + \varphi^\alpha(x) \simeq \mathbf{n}_0^\alpha(x) - \frac{1}{2}[\varphi^\alpha(x)^2]\mathbf{n}_0^\alpha(x) + \varphi^\alpha(x), \quad (3)$$

$$\varphi^\alpha(x) = \sum_{i=1}^{N-1} \varphi_i^\alpha(x)\mathbf{e}_i^\alpha(x), \quad (4)$$

where the unit vectors $\mathbf{e}_i^\alpha(x)$ are perpendicular to each other and also to the vector $\mathbf{n}_0^\alpha(x)$. Integrating out the fast fields $\varphi_i^\alpha(x)$, and calculating the new replicated action $\beta H'_{\text{rep}}$ up to the second order of the perturbation expansion, we get the one-loop beta functions for T , R_1 , and R_2 , which have been obtained by Fedorenko and Kühnel [23]. The one-loop beta function for the temperature T is

$$\partial_t T = -(d-2)T + (N-2)T[T + R'_1(1) + R'_2(1)], \quad (5)$$

where ∂_t denotes a derivative with respect to $t = \log l$ with l being the length-scale parameter which increases toward the infrared direction. Here we have rescaled T , R_1 , and R_2 by $2/[(4\pi)^{d/2}\Gamma(d/2)]$. We find that $T = 0$ is the fixed point, at which the parameter T is irrelevant for $d > 2$. The one-loop beta functions at $T = 0$ for R_1 and R_2 are

$$\begin{aligned} \partial_t R_1(z) = & -\epsilon R_1(z) + 2(N-2)[R'_1(1) + R'_2(1)]R_1(z) - (N-1)z[R'_1(1) + R'_2(1)]R'_1(z) + (1-z^2)[R'_1(1) + R'_2(1)]R'_1(z) \\ & + \frac{1}{2}(N-2+z^2)[R'_1(z) + R'_2(z)]^2 - z(1-z^2)[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] + \frac{1}{2}(1-z^2)^2[R''_1(z) + R''_2(z)]^2, \quad (6) \end{aligned}$$

$$\begin{aligned} \partial_t R_2(\zeta) = & -(\epsilon - \rho)R_2(\zeta) + 2(N-2)[R'_1(1) + R'_2(1)]R_2(\zeta) - (N-1)\zeta[R'_1(1) \\ & + R'_2(1)]R'_2(\zeta) + (1-\zeta^2)[R'_1(1) + R'_2(1)]R''_2(\zeta), \quad (7) \end{aligned}$$

where $z = \mathbf{n}_0^\alpha(x) \cdot \mathbf{n}_0^\beta(x)$, $\zeta = \mathbf{n}_0^\alpha(x) \cdot \mathbf{n}_0^\beta(x')$. Here we have put $d = 4 + \epsilon$. Practically, the beta functions for the first and second derivatives of R_1 and R_2 play a central role in the critical phenomena at zero temperature near the lower critical dimension. The beta functions at $T = 0$ for R'_1 , R''_1 , R'_2 , and R''_2 in $d = 4 + \epsilon$ are

$$\begin{aligned} \partial_t R'_1(z) = & -\epsilon R'_1(z) + (N-3)[R'_1(1) + R'_2(1)]R'_1(z) - (N+1)z[R'_1(1) + R'_2(1)]R'_1(z) + (1-z^2)[R'_1(1) + R'_2(1)]R''_1(z) \\ & + z[R'_1(z) + R'_2(z)]^2 + (N-3+4z^2)[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] - z(1-z^2)[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] \\ & - 3z(1-z^2)[R''_1(z) + R''_2(z)]^2 + (1-z^2)^2[R''_1(z) + R''_2(z)][R'''_1(z) + R'''_2(z)], \quad (8) \end{aligned}$$

$$\begin{aligned} \partial_t R''_1(z) = & -\epsilon R''_1(z) - 4[R'_1(1) + R'_2(1)]R''_1(z) - (N+3)z[R'_1(1) + R'_2(1)]R''_1(z) + (1-z^2)[R'_1(1) + R'_2(1)]R^{(IV)}_1(z) \\ & + [R'_1(z) + R'_2(z)]^2 + 10z[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] + (N-4+7z^2)[R'_1(z) + R'_2(z)][R'''_1(z) + R'''_2(z)] \\ & - z(1-z^2)[R'_1(z) + R'_2(z)][R^{(IV)}_1(z) + R^{(IV)}_2(z)] + (N-6+13z^2)[R''_1(z) + R''_2(z)]^2 - 11z(1-z^2)[R'_1(z) \\ & + R'_2(z)][R''_1(z) + R''_2(z)] + (1-z^2)^2[R''_1(z) + R''_2(z)][R^{(IV)}_1(z) + R^{(IV)}_2(z)] + (1-z^2)^2[R'''_1(z) + R'''_2(z)]^2, \quad (9) \end{aligned}$$

$$\begin{aligned} \partial_t R'_2(\zeta) = & -(\epsilon - \rho)R'_2(\zeta) + (N-3)[R'_1(1) + R'_2(1)]R'_2(\zeta) - (N+1)\zeta[R'_1(1) \\ & + R'_2(1)]R''_2(\zeta) + (1-\zeta^2)[R'_1(1) + R'_2(1)]R''_2(\zeta), \quad (10) \end{aligned}$$

$$\begin{aligned} \partial_t R''_2(\zeta) = & -(\epsilon - \rho)R''_2(\zeta) - 4[R'_1(1) + R'_2(1)]R''_2(\zeta) - (N+3)\zeta[R'_1(1) + R'_2(1)]R''_2(\zeta) + (1-\zeta^2)[R'_1(1) + R'_2(1)]R^{(IV)}_2(\zeta). \quad (11) \end{aligned}$$

The properties of the fixed point solution $[R'_1(z)^*, R'_2(\zeta)^*]$ are determined under the condition that $|R'_1(z)|$ and $|R'_2(\zeta)|$ remain finite during the renormalization group flows. We discuss the properties of the fixed point solution $[R'_1(z)^*, R'_2(\zeta)^*]$. Equation (10) is linear in the function $R'_2(\zeta)$, which can be solved analytically. Solving the fixed point equation $\partial_t R'_2(\zeta)^* = 0$, we can find that the fixed point solution $R'_2(\zeta)^*$ is analytic on ζ . Next, we assume that the functions $R'_1(z)$ and $R'_2(\zeta)$ take the following form:

$$R'_1(z) = R'_1(1) - R''_1(1)(1-z) + \cdots + a_l(1-z)^\alpha + \cdots, \quad (12)$$

$$R'_2(\zeta) = R'_2(1) - R''_2(1)(1-\zeta) + \frac{R'''_2(1)}{2}(1-\zeta)^2 + \cdots, \quad (13)$$

with $\alpha > 0$. To keep $|R'_1(z)|$ and $|R'_2(\zeta)|$ finite, the following condition on the function (12) is required:

$$\alpha = \frac{1}{2} \quad \text{or} \quad \alpha \geq 1. \quad (14)$$

Thus, the fixed point function $R_1(z)^*$ also has the same behavior of $(1-z)^{\alpha^*}$ with $\alpha^* = 1/2$ or $\alpha^* \geq 1$. Only in the case of $\alpha^* = 1/2$, $R''_1(1)^*$ diverges. We use the term ‘‘cuspy’’ on a function with $(1-z)^{1/2}$ and ‘‘cusplless’’ if the first and the second derivatives of a function are finite.

C. Stability of fixed points and critical exponents η , $\bar{\eta}$, and ν

The critical exponents η and $\bar{\eta}$ of the connected and disconnected correlation functions are expressed by use of $R'_1(1)^*$ and $R'_2(1)^*$ which are the values of $R'_1(1)$ and $R'_2(1)$ at the fixed point:

$$\eta = R'_1(1)^* + R'_2(1)^*, \quad (15)$$

$$\bar{\eta} = (N - 1)[R'_1(1)^* + R'_2(1)^*] - \epsilon. \quad (16)$$

The critical exponent ν of the correlation length is given by the inverse of the maximal eigenvalue of the scaling matrix at the fixed point. Then we find the fixed points by solving $\partial_t R'_1(1)^* = 0$, $\partial_t R''_1(1)^* = 0$, $\partial_t R'_2(1)^* = 0$, and $\partial_t R''_2(1)^* = 0$, study their stability, and calculate the critical exponents η , $\bar{\eta}$, and ν in the following.

The fixed points are

$$[R'_1(1)^*, R'_2(1)^*, R''_{1+}(1)^*, R''_2(1)^*] = \left(\frac{\epsilon}{N-2}, 0, \frac{\epsilon[N-8+\sqrt{(N-2)(N-18)}]}{2(N+7)(N-2)}, 0 \right), \quad (17)$$

$$[R'_1(1)^*, R'_2(1)^*, R''_{1-}(1)^*, R''_2(1)^*] = \left(\frac{\epsilon}{N-2}, 0, \frac{\epsilon[N-8-\sqrt{(N-2)(N-18)}]}{2(N+7)(N-2)}, 0 \right), \quad (18)$$

$$[R'_1(1)^*, R'_2(1)^*, R''_{1+}(1)^*, R''_2(1)^*] = \left(\frac{(\epsilon - \rho)^2}{(N-3)^2 \rho}, \frac{(\epsilon - \rho)^2}{(N-3)^2 \rho} \{(N-3)\hat{\epsilon} - (N-2)\}, \right. \\ \left. \frac{(\epsilon - \rho)[(N-3)\hat{\epsilon} - 6 + \sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)}]}{2(N+7)(N-3)}, 0 \right), \quad (19)$$

$$[R'_1(1)^*, R'_2(1)^*, R''_{1-}(1)^*, R''_2(1)^*] = \left(\frac{(\epsilon - \rho)^2}{(N-3)^2 \rho}, \frac{(\epsilon - \rho)^2}{(N-3)^2 \rho} \{(N-3)\hat{\epsilon} - (N-2)\}, \right. \\ \left. \frac{(\epsilon - \rho)[(N-3)\hat{\epsilon} - 6 - \sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)}]}{2(N+7)(N-3)}, 0 \right). \quad (20)$$

Here we have introduced the reduced variable $\hat{\epsilon}$:

$$\hat{\epsilon} = \frac{\epsilon}{\epsilon - \rho}. \quad (21)$$

The stability of the cusplless fixed points with respect to the cusplless perturbation can be investigated by calculating eigenvalues of the 4×4 scaling matrix whose elements are the first derivatives of the beta functions $\partial_t R'_1(1)$, $\partial_t R''_1(1)$, $\partial_t R'_2(1)$, and $\partial_t R''_2(1)$ at the cusplless fixed points.

The cusplless fixed points (17) and (18) exist for $N \geq 18$. The eigenvalues $\lambda_1, \dots, \lambda_4$ of the scaling matrix at the cusplless fixed points (17) and (18) are given by

$$\lambda_1 = \epsilon, \quad (22)$$

$$\lambda_2 = (\epsilon - \rho) \frac{(N-3)\hat{\epsilon} - (N-2)}{N-2}, \quad (23)$$

$$\lambda_3^\pm = \pm \epsilon \sqrt{\frac{N-18}{N-2}}, \quad (24)$$

$$\lambda_4 = -\left(\epsilon - \rho + \frac{4\epsilon}{N-2} \right). \quad (25)$$

Thus, the cusplless fixed point (17) is multiply unstable. If $1 < \hat{\epsilon} < (N-2)/(N-3)$, namely $\lambda_2 < 0$, the cusplless fixed point (18) is singly unstable. Due to $R'_2(1)^* = 0$, the long-range correlations of random fields and random anisotropies are irrelevant, and thus the cusplless fixed point (18) governs the phase transition in the system with SR. The critical exponents η_{SR} of the connected correlation function and $\bar{\eta}_{\text{SR}}$ of the

disconnected correlation function at the cusplless fixed point (18) are

$$\eta_{\text{SR}} = \frac{\epsilon}{N-2}, \quad (26)$$

$$\bar{\eta}_{\text{SR}} = \frac{\epsilon}{N-2}. \quad (27)$$

And the critical exponent ν_{SR} which characterizes the divergence of the correlation length in the vicinity of transition is

$$\nu_{\text{SR}} = \frac{1}{\epsilon}. \quad (28)$$

Whereas in $N \leq 18$ the cusplless fixed points (17) and (18) merge and annihilate, and thus the beta functions have no cusplless fixed point of $O(\epsilon)$.

The cusplless fixed points (19) and (20) exist for $N > 3$ and

$$\hat{\epsilon} \geq \hat{\epsilon}_{\text{cusplless}} = \frac{6 + 2\sqrt{N+7}}{N-3}. \quad (29)$$

The eigenvalues $\lambda_1, \dots, \lambda_4$ of the scaling matrix at the fixed points (19) and (20) are given by

$$\lambda_1 = (\epsilon - \rho) \frac{N-2}{N-3} - \frac{\epsilon}{2} + \frac{\epsilon}{2} \sqrt{1 + \frac{4[N-2-\hat{\epsilon}(N-3)]}{\hat{\epsilon}^2(N-3)^2}}, \quad (30)$$

$$\lambda_2 = (\epsilon - \rho) \frac{N-2}{N-3} - \frac{\epsilon}{2} - \frac{\epsilon}{2} \sqrt{1 + \frac{4[N-2-\hat{\epsilon}(N-3)]}{\hat{\epsilon}^2(N-3)^2}}, \quad (31)$$

$$\lambda_3^\pm = \pm(\epsilon - \rho) \frac{\sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)}}{N-3}, \quad (32)$$

$$\lambda_4 = -(\epsilon - \rho) \frac{N+1}{N-3}. \quad (33)$$

Thus, the cusplless fixed point (19) is multiply unstable. If $\hat{\epsilon} > (N-2)/(N-3)$, namely $\lambda_2 < 0$, the cusplless fixed point (20) is singly unstable. Due to $R'_2(1)^* > 0$, the effect of the long-range correlation of random fields and random anisotropies appears, and then the cusplless fixed point (20) governs the phase transition in the system with LRF. Thus, the critical exponents η_{LRF} and $\bar{\eta}_{\text{LRF}}$ at the cusplless fixed point (20) are

$$\eta_{\text{LRF}} = \frac{\epsilon - \rho}{N-3}, \quad (34)$$

$$\bar{\eta}_{\text{LRF}} = \frac{2\epsilon - (N-1)\rho}{N-3}. \quad (35)$$

These exponents satisfy the Schwartz-Soffer inequality $\bar{\eta}_{\text{LRF}} \leq 2\eta_{\text{LRF}}$ [39], and saturate the generalized Schwartz-Soffer inequality $\bar{\eta}_{\text{LRF}} \leq 2\eta_{\text{LRF}} - \rho$ [40]. And the inverse of the exponent ν_{LRF} is

$$\nu_{\text{LRF}}^{-1} = \frac{(N-2)(\epsilon - \rho)}{N-3} \left[1 - \frac{(N-3)\hat{\epsilon}}{2(N-2)} + \frac{(N-3)\hat{\epsilon}}{2(N-2)} \sqrt{1 + \frac{4[N-2 - \hat{\epsilon}(N-3)]}{\hat{\epsilon}^2(N-3)^2}} \right]. \quad (36)$$

Whereas in $\hat{\epsilon} \leq (6 + 2\sqrt{N+7})/(N-3)$ the cusplless fixed points (19) and (20) merge and annihilate, and thus the beta functions have no cusplless fixed point of $O(\epsilon)$.

As Tissier and Tarjus (TT) and co-workers argued in Refs. [11,14–17], the cusplless fixed points (18) and (20) have weaker nonanalyticities $(1-z)^{\alpha^*}$ with a noninteger $\alpha^* \geq 1$. The weaker nonanalyticity is called ‘‘subcusp.’’ We refer to the cusplless fixed points (18) and (20) as ‘‘SR TT FP’’ and ‘‘LRF TT FP,’’ respectively. The weaker nonanalyticity does not alter the flow equations for $R'_1(1)$ and $R'_2(1)$. The power α^* is obtained as follows. Calculating the flow of a_l in Eq. (12), we have

$$\partial_t a_l = a_l \Lambda_{\alpha+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*], \quad (37)$$

$$\begin{aligned} \Lambda_{\alpha+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] &= 2\{R'_1(1)^* + R'_2(1)^* + 3[R''_1(1)^* + R''_2(1)^*]\alpha^2 \\ &\quad - \{(N-5)[R'_1(1)^* + R'_2(1)^*] \\ &\quad - (N+7)[R''_1(1)^* + R''_2(1)^*]\alpha \\ &\quad + (N-1)[R'_1(1)^* + R'_2(1)^*] \\ &\quad + (N+1)[R''_1(1)^* + R''_2(1)^*] - \epsilon. \end{aligned} \quad (38)$$

The power α^* is determined from

$$\Lambda_{\alpha^*+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] = 0. \quad (39)$$

Substituting the SR TT FP (18) and the LRF TT FP (20) into the above equation, we have explicit expressions for α^* , respectively. Here we treat only the LRF case (see Ref. [14]

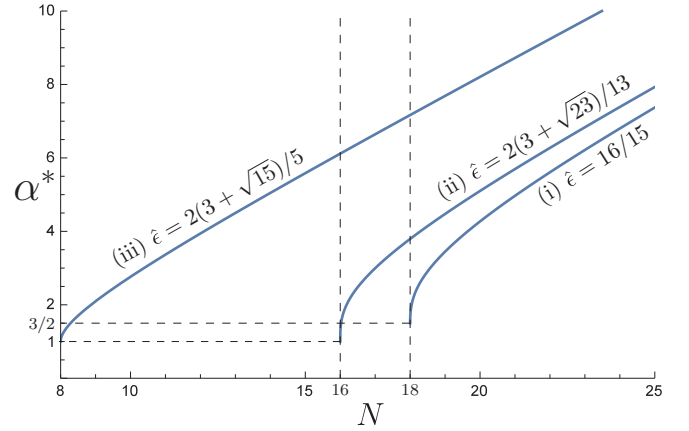


FIG. 1. Exponent $\alpha^* = \alpha(N, \hat{\epsilon})^*$ characterizing the nonanalyticity $(1-z)^{\alpha^*}$ of the LRF TT FP (20) for three values of $\hat{\epsilon}$. The ordinate is α^* , and the abscissa is N . (i) For $\hat{\epsilon} = 16/15$, the value N of the lower boundary above which the LRF TT FP are singly unstable with respect to the cusplless perturbation is $N = 18$. $\alpha(N \searrow 18, \hat{\epsilon})^* = 3/2$. (ii) For $\hat{\epsilon} = 2(3 + \sqrt{23})/13$, $N = 16$. $\alpha(N \searrow 16, \hat{\epsilon})^* = 1$. (iii) For $\hat{\epsilon} = 2(3 + \sqrt{15})/5$, $N = 8$. $\alpha(N \searrow 8, \hat{\epsilon})^* = 1$.

for the SR case). From Eqs. (38) and (39) we obtain the following quadratic equation for α^* :

$$\begin{aligned} &\left(2 + \frac{3}{N+7}[(N-3)\hat{\epsilon} - 6] \right. \\ &\quad \left. - \sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)} \right) \alpha^{*2} \\ &\quad - \left(N-5 - \frac{1}{2}[(N-3)\hat{\epsilon} - 6] \right. \\ &\quad \left. - \sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)} \right) \alpha^* \\ &\quad + N-1 - (N-3)\hat{\epsilon} + \frac{N+1}{2(N+7)}[(N-3)\hat{\epsilon} - 6] \\ &\quad \left. - \sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)} \right) = 0. \end{aligned} \quad (40)$$

Solving the above quadratic equation, we obtain the solution α^* as a function of N and $\hat{\epsilon}$. It goes to $N/2 + O(1)$ at large N . The graphs of $\alpha^* = \alpha(N, \hat{\epsilon})^*$ for some values of $\hat{\epsilon}$ are depicted in Fig. 1.

We proceed to investigate the stability of the cusplless fixed points with respect to the cuspy perturbation, following the work by Baczyk, Tarjus, Tissier, and Balog [15]. The eigenvalue λ relating to the cuspy deformation from the cusplless fixed points is given by

$$\lambda = \Lambda_{3/2}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*]. \quad (41)$$

Substituting the SR TT FP (18) and the LRF TT FP (20) into the above equation, we have explicit expressions for λ , respectively. The eigenvalues λ_{SR} for the system with SR and λ_{LRF} for the system with LRF are as follows:

$$\lambda_{\text{SR}} = -\frac{\epsilon}{4(N+7)} \left[3(N+4) \sqrt{\frac{N-18}{N-2}} - N + 8 \right], \quad (42)$$

$$\lambda_{\text{LRF}} = -\frac{\epsilon - \rho}{4(N-3)(N+7)} \times [3(N+4)\sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)} + (N+16)\{(N-3)\hat{\epsilon} - 6\} - 2(N+7)(N-8)]. \quad (43)$$

In the case of the system with SR, we find that, below $N = N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \simeq 18.3923\dots$, the eigenvalue λ_{SR} takes a positive value. Thus the cuspy perturbation becomes relevant for $N < N_{\text{cusp}}$, where the SR TT FP (18) is multiply unstable with respect to the cuspy perturbation. Whereas it remains singly unstable with respect to the perturbation with and without the cuspy behavior for $N > N_{\text{cusp}}$. As shown in Ref. [15], there exists a singly unstable cuspy fixed point below $N = N_{\text{cusp}}$. As N decreases from sufficiently large N , the fixed point which governs the phase transition in the system continuously changes from the SR TT FP to the singly unstable cuspy fixed point at $N = N_{\text{cusp}}$ before N reaches to $N = 18$. Accordingly, the values of the critical exponents η_{SR} and $\bar{\eta}_{\text{SR}}$ deviate from the dimensional-reduction results (26) and (27) below $N = N_{\text{cusp}}$.

In the case of the system with LRF, the eigenvalue λ_{LRF} takes a positive value below

$$\hat{\epsilon} = \hat{\epsilon}_{\text{cusp}} = \frac{3(N+4)\sqrt{N^2 - 8N + 48} - (N+4)(N-20)}{4(N-2)(N-3)}. \quad (44)$$

Since $\hat{\epsilon}_{\text{cusp}} \geq \hat{\epsilon}_{\text{cusplless}}$ for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$, the LRF TT FP (20) is destabilized by the cuspy perturbation for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\hat{\epsilon} < \hat{\epsilon}_{\text{cusp}}$. Even in this case, a singly unstable cuspy fixed point which governs the phase transition in the system is considered to exist for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\hat{\epsilon} < \hat{\epsilon}_{\text{cusp}}$.

Finally, we calculate the eigenfunction which belongs to the eigenvalue (41). Solving the eigenvalue equation, we obtain two solutions. One takes the form of $(1-z)^{\alpha_-(\lambda)}$ with $\alpha_-(\lambda) = 1/2$ when $z \rightarrow 1$, and the other takes the form of $(1-z)^{\alpha_+(\lambda)}$. Both solutions individually diverge in $z = -1$. The physical eigenfunction is represented as a linear combination of two solutions of the eigenvalue equation, in which the coefficients should be chosen to eliminate the singularities at $z = -1$. The power $\alpha_+(\lambda)$ of the function $(1-z)^{\alpha_+(\lambda)}$ can be obtained by imposing

$$\Lambda_{\alpha_+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] = \Lambda_{3/2}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*]. \quad (45)$$

In the case of the system with SR, substituting the SR TT FP (18) into Eq. (45), we have

$$\alpha_+(\lambda_{\text{SR}}) = \frac{1}{4}(N - 10 + \sqrt{(N-2)(N-18)}). \quad (46)$$

For $N \geq 18$, $\alpha_+(\lambda_{\text{SR}})$ takes $\alpha_+(\lambda_{\text{SR}}) \geq 2$.

In the case of the system with LRF, substituting the LRF TT FP (20) into Eq. (45), we have

$$\alpha_+(\lambda_{\text{LRF}}) = \frac{(N-14)(N-3)\hat{\epsilon} + (N-2)(N-6)}{2[3(N-3)\hat{\epsilon} + N-2]} + \frac{\sqrt{\{(N-3)\hat{\epsilon} - 6\}^2 - 4(N+7)}}{3(N-3)\hat{\epsilon} + N-2}. \quad (47)$$

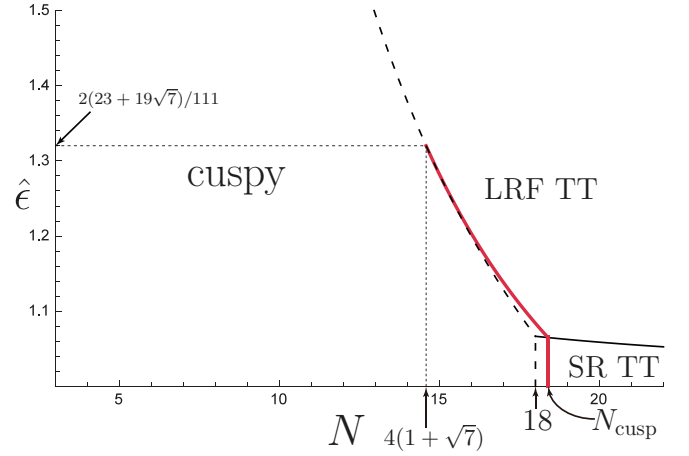


FIG. 2. The regions where various FPs are singly unstable. The ordinate is $\hat{\epsilon} [= \epsilon / (\epsilon - \rho)]$, and the abscissa is N . The broken line (black) denotes the lower boundary above which the SR TT and the LRF TT FPs are singly unstable against the cusplless perturbation. The border line (black line) between the SR TT and LRF TT FPs is given by $\hat{\epsilon} = (N-2)/(N-3)$. The solid line (red line) denotes the lower boundary above which the SR TT and LRF TT FPs are singly unstable against the cuspy perturbation.

The power $\alpha_+(\lambda_{\text{LRF}})$ takes $\alpha_+(\lambda_{\text{LRF}}) \geq 1 + \sqrt{3}$ for $N \geq N_{\text{cusp}}$ and $\hat{\epsilon} \geq (N-2)/(N-3)$, and $\alpha_+(\lambda_{\text{LRF}}) \geq 1$ for $4(1 + \sqrt{7}) \leq N < N_{\text{cusp}}$ and $\hat{\epsilon} \geq \hat{\epsilon}_{\text{cusplless}}$. However, we should note that, for $N < 4(1 + \sqrt{7})$ and in the region of

$$\hat{\epsilon}_{\text{cusplless}} \leq \hat{\epsilon} < \frac{(N-2)(N^2 + 32)}{(N-3)(N-8)(N+16)}, \quad (48)$$

$\alpha_+(\lambda_{\text{LRF}}) < 1$, which is in contradiction with the condition (14). Thus, for $N < 4(1 + \sqrt{7})$ and in the region (48), the cuspy deformation from the LRF TT FP (20) is unphysical. Then the destabilization of the LRF TT FP (20) by the cuspy perturbation does not occur for $\hat{\epsilon} > 2(23 + 19\sqrt{7})/111$.

The regions where the various fixed points are singly unstable are depicted in Fig. 2. Outside the areas where the SR TT and the LRF TT FPs are singly unstable, the cuspy fixed point is considered to control the critical behavior in the system. Particularly, in the region of $1 \leq \hat{\epsilon} < 2(23 + 19\sqrt{7})/111 \simeq 1.32$, the destabilization of the SR TT and the LRF TT FPs by the cuspy perturbation is caused at N_{cusp} for the SR TT FP, and at $\hat{\epsilon}_{\text{cusp}}$ for the LRF TT FP, respectively.

III. FIXED POINTS AND THEIR STABILITY IN THE RENORMALIZATION GROUP OF LONG-RANGE CORRELATED RANDOM FIELD $O(N)$ SPIN MODEL WITH LONG-RANGE EXCHANGE INTERACTIONS IN $2\sigma + \rho + \epsilon$ DIMENSIONS

We now study the critical phenomena at zero temperature of the long-range correlated random field $O(N)$ spin model with the long-range exchange interactions in $2\sigma + \rho + \epsilon$ dimensions by use of the renormalization group. The critical phenomena at zero temperature of the long-range correlated random field $O(N)$ spin model with the long-range exchange interactions near lower critical dimension is described by

the $O(N)$ nonlinear-sigma model. In this section we investigate the fixed points and their stability of the one-loop beta functions in the $O(N)$ nonlinear-sigma model. The critical phenomena are carefully discussed in the subsequent sections.

A. Model

We start from the $O(N)$ nonlinear-sigma model with the replica effective action

$$\begin{aligned} \beta H_{\text{rep}} = & \frac{a^{\sigma-d}}{2T} \int_x \sum_{\alpha=1}^n \mathbf{S}^\alpha(x) \cdot (-\partial^2)^{\sigma/2} \mathbf{S}^\alpha(x) \\ & - \frac{a^{-d}}{2T^2} \int_x \sum_{\alpha,\beta} R_1[\mathbf{S}^\alpha(x) \cdot \mathbf{S}^\beta(x)] \\ & - \frac{a^{-d-\rho}}{2T^2} \int_{x,x'} \sum_{\alpha,\beta} g(x-x') R_2[\mathbf{S}^\alpha(x) \cdot \mathbf{S}^\beta(x')]. \end{aligned} \quad (49)$$

The first term in the action (49) is the kinetic term which corresponds to the long-range exchange interactions between spins. The operator $(-\partial^2)^{\sigma/2}$ denotes the fractional Laplacian in the Euclidean space. In the present study we consider the case of $0 < \sigma < 2$. The parameter T denotes the dimensionless temperature. The function $R_i(z)$ ($i = 1, 2$) represents the random field and all the random anisotropies, which is defined by Eq. (2). The lower critical dimension of this model is $d_l = 2\sigma + \rho$. In the present study, we consider the case of $\rho \geq 0$.

B. One-loop beta functions

To carry out the renormalization group transformation, it is convenient to use the momentum representation. The fractional Laplacian $(-\partial^2)^{\sigma/2}$ is written by its Fourier transformation:

$$(-\partial^2)^{\sigma/2} f(x) = \int_k k^\sigma \tilde{f}(k) e^{ikx}, \quad (50)$$

where $kx = k^{(1)}x^{(1)} + \dots + k^{(d)}x^{(d)}$, $k^\sigma = (k^{(1)2} + \dots + k^{(d)2})^{\sigma/2}$, and $\int_k \equiv \int \frac{d^d k}{(2\pi)^d}$. The correlation of the random fields $g(x-x')$ is written as

$$g(x-x') \sim \frac{1}{|x-x'|^{d-\rho}} = \int_k k^{-\rho} e^{ik(x-x')}, \quad (51)$$

in the momentum representation. The N -component replicated vector spin $\mathbf{S}^\alpha(x)$ of the magnetization (3) is rewritten in the momentum representation as follows:

$$\begin{aligned} \mathbf{S}^\alpha(x) \simeq & \mathbf{n}_0^\alpha(x) - \frac{1}{2} [\boldsymbol{\varphi}^\alpha(x)^2] \mathbf{n}_0^\alpha(x) + \boldsymbol{\varphi}^\alpha(x) = \int_k \tilde{\mathbf{n}}_0^\alpha(k) e^{ikx} - \frac{1}{2} \int_{k,k_1,\dots,k_4} \left(\sum_{i,j}^{N-1} \tilde{\varphi}_i^\alpha(k_1) \tilde{\varphi}_j^\alpha(k_2) \tilde{\boldsymbol{\epsilon}}_i^\alpha(k_3) \cdot \tilde{\boldsymbol{\epsilon}}_j^\alpha(k_4) \right) \tilde{\mathbf{n}}_0^\alpha(k) e^{i(k_1+\dots+k_4)x} \\ & + \int_{k_1,k_2} \left(\sum_{i=1}^{N-1} \tilde{\varphi}_i^\alpha(k_1) \tilde{\boldsymbol{\epsilon}}_i^\alpha(k_2) \right) e^{i(k_1+k_2)x}. \end{aligned} \quad (52)$$

We integrate out the fast fields $\tilde{\varphi}_i^\alpha(k)$, and calculate the new replicated action $\beta H'_{\text{rep}}$ up to the second order of the perturbation expansion. After rewriting $\beta H'_{\text{rep}}$ in the coordinate representation again, we can then obtain the one-loop beta functions for T , R_1 , and R_2 . The one-loop beta function for the temperature T is

$$\partial_t T = -(d-\sigma)T + (N-1)T[T + R'_1(1) + R'_2(1)]. \quad (53)$$

Here we have rescaled T , R_1 , and R_2 by $2/[(4\pi)^{d/2}\Gamma(d/2)]$. For $d > \sigma$, we find that $T = 0$ is the fixed point, at which the parameter T is irrelevant. The one-loop beta functions at $T = 0$ for R_1 and R_2 are

$$\begin{aligned} \partial_t R_1(z) = & -(\epsilon + \rho)R_1(z) + 2(N-1)[R'_1(1) + R'_2(1)]R_1(z) - (N-1)z[R'_1(1) + R'_2(1)]R'_1(z) \\ & + (1-z^2)[R'_1(1) + R'_2(1)]R'_1(z) + \frac{1}{2}(N-2+z^2)[R'_1(z) + R'_2(z)]^2 \\ & - z(1-z^2)[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] + \frac{1}{2}(1-z^2)^2[R''_1(z) + R''_2(z)]^2, \end{aligned} \quad (54)$$

$$\partial_t R_2(\zeta) = -\epsilon R_2(\zeta) + 2(N-1)[R'_1(1) + R'_2(1)]R_2(\zeta) - (N-1)\zeta[R'_1(1) + R'_2(1)]R'_2(\zeta) + (1-\zeta^2)[R'_1(1) + R'_2(1)]R''_2(\zeta). \quad (55)$$

Here we have put $d = 2\sigma + \rho + \epsilon$. To study the fixed points and their stability, we consider the beta functions for their derivative. Differentiating Eqs. (54) and (55) with respect to z and ζ , respectively, we obtain the one-loop beta functions for $R'_1(z)$, $R''_1(z)$, $R'_2(\zeta)$, and $R''_2(\zeta)$:

$$\begin{aligned} \partial_t R'_1(z) = & -(\epsilon + \rho)R'_1(z) + (N-1)[R'_1(1) + R'_2(1)]R'_1(z) - (N+1)z[R'_1(1) + R'_2(1)]R''_1(z) + (1-z^2)[R'_1(1) + R'_2(1)]R'''_1(z) \\ & + z[R'_1(z) + R'_2(z)]^2 + (N-3+4z^2)[R'_1(z) + R'_2(z)][R''_1(z) + R''_2(z)] - z(1-z^2)[R'_1(z) + R'_2(z)][R'''_1(z) + R'''_2(z)] \\ & - 3z(1-z^2)[R''_1(z) + R''_2(z)]^2 + (1-z^2)^2[R''_1(z) + R''_2(z)][R'''_1(z) + R'''_2(z)], \end{aligned} \quad (56)$$

$$\begin{aligned} \partial_t R_1''(z) = & -(\epsilon + \rho)R_1''(z) - 2[R_1'(1) + R_2'(1)]R_1''(z) - (N + 3)z[R_1'(1) + R_2'(1)]R_1'''(z) + (1 - z^2)[R_1'(1) + R_2'(1)]R_1^{(IV)}(z) \\ & + [R_1'(z) + R_2'(z)]^2 + 10z[R_1'(z) + R_2'(z)][R_1''(z) + R_2''(z)] + (N - 4 + 7z^2)[R_1'(z) + R_2'(z)][R_1'''(z) + R_2'''(z)] \\ & - z(1 - z^2)[R_1'(z) + R_2'(z)][R_1^{(IV)}(z) + R_2^{(IV)}(z)] + (N - 6 + 13z^2)[R_1''(z) + R_2''(z)]^2 - 11z(1 - z^2)[R_1''(z) \\ & + R_2''(z)][R_1'''(z) + R_2'''(z)] + (1 - z^2)^2[R_1''(z) + R_2''(z)][R_1^{(IV)}(z) + R_2^{(IV)}(z)] + (1 - z^2)^2[R_1'''(z) + R_2'''(z)]^2, \end{aligned} \quad (57)$$

$$\partial_t R_2'(\zeta) = -\epsilon R_2'(\zeta) + (N - 1)[R_1'(1) + R_2'(1)]R_2'(\zeta) - (N + 1)\zeta[R_1'(1) + R_2'(1)]R_2''(\zeta) + (1 - \zeta^2)[R_1'(1) + R_2'(1)]R_2'''(\zeta), \quad (58)$$

$$\partial_t R_2''(\zeta) = -\epsilon R_2''(\zeta) - 2[R_1'(1) + R_2'(1)]R_2''(\zeta) - (N + 3)\zeta[R_1'(1) + R_2'(1)]R_2'''(\zeta) + (1 - \zeta^2)[R_1'(1) + R_2'(1)]R_2^{(IV)}(\zeta). \quad (59)$$

We discuss the properties of the fixed point solution $[R_1'(z)^*, R_2'(\zeta)^*]$. First, we investigate the fixed point solution $R_2'(\zeta)^*$ for Eq. (58). Since Eq. (58) is linear in the function $R_2'(\zeta)$, the fixed point equation $\partial_t R_2'(\zeta)^* = 0$ can be solved analytically. The fixed point equation $\partial_t R_2''(\zeta)^* = 0$ takes the form

$$\begin{aligned} (1 - \zeta^2)[R_1'(1)^* + R_2'(1)^*]R_2'''(\zeta)^* \\ - (N + 1)\zeta[R_1'(1)^* + R_2'(1)^*]R_2''(\zeta)^* \\ + [(N - 1)[R_1'(1)^* + R_2'(1)^*] - \epsilon]R_2'(\zeta)^* = 0. \end{aligned} \quad (60)$$

The solutions of this equation have regular singular points at $\zeta = \pm 1$ for the interval $-1 \leq \zeta \leq 1$. Under the condition of $|R_2'(\zeta)^*| < \infty$ on the interval $-1 \leq \zeta \leq 1$, the solutions of Eq. (60) can be expressed in terms of the Gaussian hypergeometric function:

$$R_2'(\zeta)^* = \begin{cases} C_2 F_1[x_1, x_2; y; (1 - \zeta)/2] & \text{around } \zeta = 1, \\ C'_2 F_1[x_1, x_2; y; (1 + \zeta)/2] & \text{around } \zeta = -1, \end{cases} \quad (61)$$

where C and C' are constants fulfilling the condition $|R_2'(\zeta)^*| < \infty$. Here the generalized hypergeometric function is defined by the following series expansion:

$$\begin{aligned} {}_mF_n(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n; z) \\ \equiv \sum_{k=0}^{\infty} \frac{(x_1)_k (x_2)_k \dots (x_m)_k z^k}{(y_1)_k (y_2)_k \dots (y_n)_k k!}, \end{aligned} \quad (62)$$

$$(x)_k = \Gamma(x + k)/\Gamma(x). \quad (63)$$

And x_1, x_2 , and y are

$$x_1, x_2 = \frac{1}{2} \left[N \pm \sqrt{N^2 + 4 \left\{ N - 1 - \frac{\epsilon}{R_1'(1)^* + R_2'(1)^*} \right\}} \right], \quad (64)$$

$$y = \frac{N + 1}{2}. \quad (65)$$

Thus the fixed point solution $R_2'(\zeta)^*$ is an analytic function on ζ . Next we examine the renormalization group flow of $R_1'(z)$. We assume that the functions $R_1'(z)$ and $R_2'(\zeta)$ take the forms given by Eqs. (12) and (13) with $\alpha > 0$. To keep $|R_1'(z)|$ and $|R_2'(\zeta)|$ finite, the following condition on the function (12) is required:

$$\alpha = \frac{1}{2} \quad \text{or} \quad \alpha \geq 1. \quad (66)$$

The fixed point solution $R_1'(z)^*$ also has the same singularity. Only in the case of $\alpha = 1/2$, $R''(1)^*$ diverges.

C. Stability of cusplless fixed points

The critical exponents η and $\bar{\eta}$ are expressed by use of $R_1'(1)^*$ and $R_2'(1)^*$:

$$\eta = 2 - \sigma, \quad (67)$$

$$\bar{\eta} = (N - 1)[R_1'(1)^* + R_2'(1)^*] - (2\sigma + \rho + \epsilon - 4). \quad (68)$$

The critical exponent ν is determined from the inverse of the maximum eigenvalue of the 4×4 scaling matrix at the fixed point. Then we find the fixed points by solving $\partial_t R_1'(1)^* = 0$, $\partial_t R_1''(1)^* = 0$, $\partial_t R_2'(1)^* = 0$, and $\partial_t R_2''(1)^* = 0$, and study their stability.

The cusplless fixed points are

$$[R_1'(1)^*, R_2'(1)^*, R_1''(1)^*, R_2''(1)^*] = \left(\frac{\epsilon + \rho}{N}, 0, \frac{(\epsilon + \rho)[N - 8 + \sqrt{(N - 2)(N - 18)}}{2N(N + 7)}, 0 \right), \quad (69)$$

$$[R_1'(1)^*, R_2'(1)^*, R_1''(1)^*, R_2''(1)^*] = \left(\frac{\epsilon + \rho}{N}, 0, \frac{(\epsilon + \rho)[N - 8 - \sqrt{(N - 2)(N - 18)}}{2N(N + 7)}, 0 \right), \quad (70)$$

$$[R_1'(1)^*, R_2'(1)^*, R_1''(1)^*, R_2''(1)^*] = \left(\frac{\epsilon^2}{(N - 1)^2 \rho}, \frac{\epsilon^2[(N - 1)\bar{\epsilon} - N]}{(N - 1)^2 \rho}, \frac{\epsilon[(N - 1)\bar{\epsilon} - 8 + \sqrt{\{(N - 1)\bar{\epsilon} - 8\}^2 - 4(N + 7)}}{2(N + 7)(N - 1)}, 0 \right), \quad (71)$$

$$[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] = \left(\frac{\epsilon^2}{(N-1)^2\rho}, \frac{\epsilon^2[(N-1)\tilde{\epsilon} - N]}{(N-1)^2\rho}, \frac{\epsilon[(N-1)\tilde{\epsilon} - 8 - \sqrt{\{(N-1)\tilde{\epsilon} - 8\}^2 - 4(N+7)}}{2(N+7)(N-1)}, 0 \right). \quad (72)$$

Here we have introduced the reduced variable $\tilde{\epsilon}$:

$$\tilde{\epsilon} = \frac{\epsilon + \rho}{\epsilon}. \quad (73)$$

The stability of the cusplless fixed points with respect to the cusplless perturbation can be investigated by calculating eigenvalues of the 4×4 scaling matrix whose elements are the first derivatives of the beta functions $\partial_t R'_1(1)$, $\partial_t R''_1(1)$, $\partial_t R'_2(1)$, and $\partial_t R''_2(1)$ at the cusplless fixed points.

The cusplless fixed points (69) and (70) exist for $N \geq 18$. The eigenvalues $\lambda_1, \dots, \lambda_4$ of the scaling matrix at the cusplless fixed points (69) and (70) are given by

$$\lambda_1 = \epsilon + \rho, \quad (74)$$

$$\lambda_2 = -\epsilon \left(1 - \frac{N-1}{N} \tilde{\epsilon} \right), \quad (75)$$

$$\lambda_3^\pm = \pm(\epsilon + \rho) \frac{\sqrt{(N-2)(N-18)}}{N}, \quad (76)$$

$$\lambda_4 = -\epsilon \left(1 + \frac{2}{N} \tilde{\epsilon} \right). \quad (77)$$

Thus the fixed point (69) is multiply unstable. If $1 \leq \tilde{\epsilon} < N/(N-1)$ or $0 \leq \rho < \epsilon/(N-1)$, the fixed point (70) is singly unstable. Due to $R'_2(1)^* = 0$, the long-range correlations of random fields and random anisotropies are irrelevant, and thus the fixed point (70) governs the phase transition in the system with LRE. Whereas, in $N \leq 18$, the cusplless fixed points (69) and (70) merge and annihilate, and thus the beta functions have no cusplless fixed point of $O(\epsilon)$.

The cusplless fixed points (71) and (72) exist for $N > 1$ and

$$\tilde{\epsilon} \geq \tilde{\epsilon}_{\text{cusplless}} = \frac{8 + 2\sqrt{N+7}}{N-1}. \quad (78)$$

The eigenvalues $\lambda_1, \dots, \lambda_4$ of the scaling matrix at the cusplless fixed points (71) and (72) are given by

$$\lambda_1 = \epsilon \left[\frac{N}{N-1} - \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} \sqrt{1 + \frac{4[N - \tilde{\epsilon}(N-1)]}{\tilde{\epsilon}^2(N-1)^2}} \right], \quad (79)$$

$$\lambda_2 = \epsilon \left[\frac{N}{N-1} - \frac{\tilde{\epsilon}}{2} - \frac{\tilde{\epsilon}}{2} \sqrt{1 + \frac{4[N - \tilde{\epsilon}(N-1)]}{\tilde{\epsilon}^2(N-1)^2}} \right], \quad (80)$$

$$\lambda_3^\pm = \pm(\epsilon + \rho) \sqrt{1 - \frac{4[N - 9 + 4\tilde{\epsilon}(N-1)]}{\tilde{\epsilon}^2(N-1)^2}}, \quad (81)$$

$$\lambda_4 = -\epsilon \frac{N+1}{N-1}. \quad (82)$$

Thus the fixed point (71) is multiply unstable. If $\tilde{\epsilon} > N/(N-1)$ or $\rho > \epsilon/(N-1)$, the fixed point (72) is singly unstable. Due to $R'_2(1)^* > 0$, the effect of the long-range correlation of random fields and random anisotropies appears, and then the fixed point (72) governs the phase transition in system with LREF.

The cusplless fixed points (70) and (72) have subcusplless singularities $(1-z)^{\alpha_*}$ with a noninteger $\alpha_* \geq 1$. Then we call the singly unstable fixed points (70) and (72) as the ‘‘LRE TT FP’’ and the ‘‘LREF TT FP,’’ respectively. The power α_* is obtained as follows. Calculating the flow of a_l in Eq. (12) we have

$$\partial_t a_l = a_l \Lambda_{\alpha+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*], \quad (83)$$

$$\begin{aligned} \Lambda_{\alpha+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] \\ = 2\{R'_1(1)^* + R'_2(1)^* + 3[R''_1(1)^* + R''_2(1)^*]\} \alpha^2 \\ - \{(N-5)[R'_1(1)^* + R'_2(1)^*] - (N+7)[R''_1(1)^* \\ + R''_2(1)^*]\} \alpha + (N+1)[R'_1(1)^* + R'_2(1)^* \\ + R''_1(1)^* + R''_2(1)^*] - (\epsilon + \rho). \end{aligned} \quad (84)$$

The power α^* is determined from

$$\Lambda_{\alpha^*+1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] = 0. \quad (85)$$

Substituting the LRE TT FP (70) and the LREF TT FP (72) into the above equation, we have explicit expressions for α^* , respectively. First, substituting the LRE TT FP (70) into Eq. (85), we obtain the following quadratic equation for α^* :

$$\begin{aligned} \left(2 + \frac{3}{N+7} [N - 8 - \sqrt{(N-2)(N-18)}] \right) \alpha^{*2} \\ - \left(N - 5 - \frac{1}{2} [N - 8 - \sqrt{(N-2)(N-18)}] \right) \alpha^* \\ + 1 + \frac{N+1}{2(N+7)} [N - 8 - \sqrt{(N-2)(N-18)}] = 0. \end{aligned} \quad (86)$$

Solving the above equation, we have the solution α^* as a function of N . It goes to $N/2 + O(1)$ at large N . The solution is the same as that of the system with SR [14,17]. The graph of $\alpha^* = \alpha^*(N)$ is depicted in Fig. 3. Next, substituting the LREF TT FP (72) into Eq. (85), we obtain the following quadratic equation for α^* :

$$\begin{aligned} \left(2 + \frac{3}{N+7} [(N-1)\tilde{\epsilon} - 8 \right. \\ \left. - \sqrt{\{(N-1)\tilde{\epsilon} - 8\}^2 - 4(N+7)}] \right) \alpha^{*2} \\ - \left(N - 5 - \frac{1}{2} [(N-1)\tilde{\epsilon} - 8 \right. \\ \left. - \sqrt{\{(N-1)\tilde{\epsilon} - 8\}^2 - 4(N+7)}] \right) \alpha^* \\ + N + 1 - (N-1)\tilde{\epsilon} + \frac{N+1}{2(N+7)} [(N-1)\tilde{\epsilon} - 8 \\ \left. - \sqrt{\{(N-1)\tilde{\epsilon} - 8\}^2 - 4(N+7)}] = 0. \end{aligned} \quad (87)$$

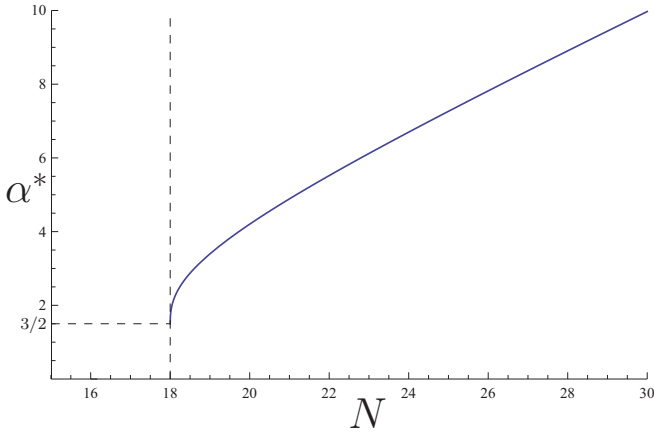


FIG. 3. Exponent $\alpha^* = \alpha(N)^*$ characterizing the subcuspy singularity $(1-z)^{\alpha^*}$ of the LRE TT FP (70). The ordinate is α^* , and the abscissa is N . $\alpha(N \searrow 18)^* = 3/2$.

Solving the above equation, we obtain the solution α^* as a function of N and $\tilde{\epsilon}$. It goes to $N/2 + O(1)$ at large N . The graphs of $\alpha^* = \alpha(N, \tilde{\epsilon})^*$ for some values of $\tilde{\epsilon}$ are depicted in Fig. 4.

We investigate the stability of the TT fixed points with respect to the cuspy perturbation. It can be done by calculating the eigenvalue λ relating to the cuspy deformation from the TT fixed point, which is given by

$$\lambda = \Lambda_{3/2}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*]. \quad (88)$$

Substituting the LRE TT FP (70) and the LREF TT FP (72) into the above equation, we obtain explicit expressions for λ ,

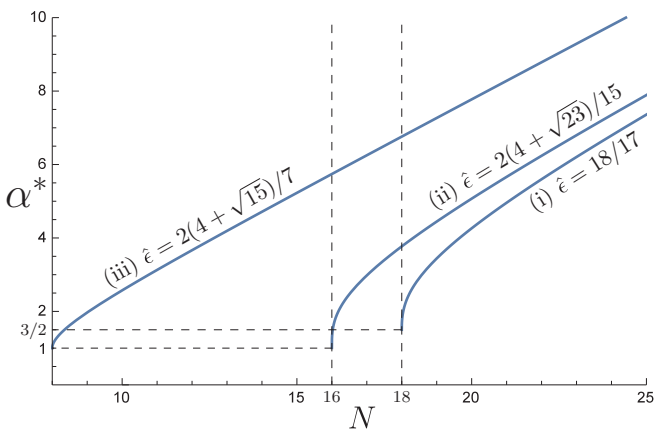


FIG. 4. Exponent $\alpha^* = \alpha(N, \tilde{\epsilon})^*$ characterizing the subcuspy singularity $(1-z)^{\alpha^*}$ of the LREF TT FP (72) for three values of $\tilde{\epsilon}$. The ordinate is α^* , and the abscissa is N . (i) For $\tilde{\epsilon} = 18/17$, the value N of the lower boundary above which the LREF TT FP are singly unstable with respect to the cusplless perturbation is $N = 18$. $\alpha(N \searrow 18, \tilde{\epsilon})^* = 3/2$. (ii) For $\tilde{\epsilon} = 2(4 + \sqrt{23})/15$, $N = 16$. $\alpha(N \searrow 16, \tilde{\epsilon})^* = 1$. (iii) For $\tilde{\epsilon} = 2(4 + \sqrt{15})/7$, $N = 8$. $\alpha(N \searrow 8, \tilde{\epsilon})^* = 1$.

respectively. The eigenvalues λ_{LRE} for the system with LRE and λ_{LREF} for the system with LREF are as follows:

$$\lambda_{\text{LRE}} = -\frac{(\epsilon + \rho)(N-2)}{4N(N+7)} \left[3(N+4)\sqrt{\frac{N-18}{N-2}} - N + 8 \right]. \quad (89)$$

$$\lambda_{\text{LREF}} = -\frac{\epsilon}{4(N-1)(N+7)} \times [3(N+4)\sqrt{[(N-1)\tilde{\epsilon} - 8]^2 - 4(N+7)} - 2(N+7)(N-8) + (N+16)[(N-1)\tilde{\epsilon} - 8]]. \quad (90)$$

In the case of the system with LRE, we find that, below $N = N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$, the eigenvalue λ_{LRE} takes a positive value. Thus the cuspy perturbation becomes relevant for $N < N_{\text{cusp}}$, where the LRE TT FP (70) is multiply unstable with respect to the cuspy perturbation. Whereas it remains singly stable with respect to the perturbation with and without the cuspy behavior for $N > N_{\text{cusp}}$.

In the case of the system with LREF, the eigenvalue λ_{LREF} takes a positive value below

$$\tilde{\epsilon} = \tilde{\epsilon}_{\text{cusp}} = \frac{3(N+4)\sqrt{N^2 - 8N + 48} - (N^2 - 24N - 64)}{4(N-1)(N-2)}. \quad (91)$$

Since $\tilde{\epsilon}_{\text{cusp}} \geq \tilde{\epsilon}_{\text{cusplless}}$ for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$, the LREF TT FP (72) is destabilized by the cuspy perturbation for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\tilde{\epsilon} < \tilde{\epsilon}_{\text{cusp}}$. Even in this case, a singly unstable cuspy fixed point which governs the phase transition in the system is considered to exist for $4(1 + \sqrt{7}) \leq N \leq 2(4 + 3\sqrt{3})$ and $\tilde{\epsilon} < \tilde{\epsilon}_{\text{cusp}}$.

Finally, we calculate the eigenfunction which belongs to the eigenvalue (88). Solving the eigenvalue equation, we obtain two solutions. One takes the form of $(1-z)^{\alpha_-(\lambda)}$ with $\alpha_-(\lambda) = 1/2$ when $z \rightarrow 1$, and the other takes the form of $(1-z)^{\alpha_+(\lambda)}$. Both solutions individually diverge in $z = -1$. The physical eigenfunction is represented as a linear combination of two solutions of the eigenvalue equation, in which the coefficients should be chosen to eliminate the singularities at $z = -1$. The power $\alpha_+(\lambda)$ of the function $(1-z)^{\alpha_+(\lambda)}$ can be obtained by imposing

$$\Lambda_{\alpha_++1}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*] = \Lambda_{3/2}[R'_1(1)^*, R'_2(1)^*, R''_1(1)^*, R''_2(1)^*]. \quad (92)$$

In the case of the system with LRE, substituting the LRE TT FP (70) into Eq. (92), we have

$$\alpha_+(\lambda_{\text{LRE}}) = \frac{1}{4}[N - 10 + \sqrt{(N-2)(N-18)}]. \quad (93)$$

For $N \geq 18$, $\alpha_+(\lambda_{\text{LRE}})$ takes $\alpha_+(\lambda_{\text{LRE}}) \geq 2$.

In the case of the system with LREF, substituting the LREF TT FP (72) into Eq. (92), we have

$$\alpha_+(\lambda_{\text{LREF}}) = \frac{(N-14)(N-1)\tilde{\epsilon} + N^2 - 10N + 40}{2[3(N-1)\tilde{\epsilon} + N - 8]} + \frac{(N-2)\sqrt{[(N-1)\tilde{\epsilon} - 8]^2 - 4(N+7)}}{3(N-1)\tilde{\epsilon} + N - 8}. \quad (94)$$

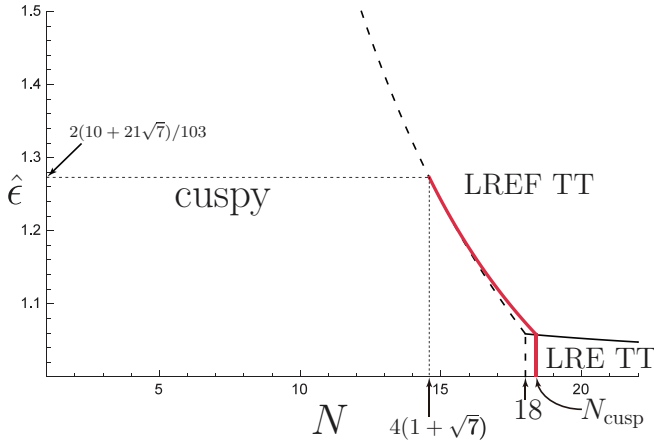


FIG. 5. The regions where various FPs are singly unstable. The ordinate is $\hat{\epsilon} = (\epsilon + \rho)/\epsilon$, and the abscissa is N . The broken line (black) denotes the lower boundary above which the LRE TT and the LREF TT FPs are singly unstable against the cuspy perturbation. The border line (black line) between the LRE TT and LREF TT FPs is given by $\hat{\epsilon} = N/(N - 1)$. The solid line (red line) denotes the lower boundary above which the LRE TT and LREF TT FPs are singly unstable against the cuspy perturbation.

The power $\alpha_+(\lambda_{\text{LREF}})$ takes $\alpha_+(\lambda_{\text{LREF}}) \geq 1 + \sqrt{3}$ for $N \geq N_{\text{cusp}}$ and $\tilde{\epsilon} \geq N/(N - 1)$, and $\alpha_+(\lambda_{\text{LREF}}) \geq 1$ for $4(1 + \sqrt{7}) \leq N < N_{\text{cusp}}$ and $\tilde{\epsilon} \geq \tilde{\epsilon}_{\text{cusplless}}$. However, we should note that, for $N < 4(1 + \sqrt{7})$ and in the region of

$$\tilde{\epsilon}_{\text{cusplless}} \leq \tilde{\epsilon} < \frac{N^3 + 48N - 320}{(N - 1)(N - 8)(N + 16)}, \quad (95)$$

$\alpha_+(\lambda_{\text{LREF}}) < 1$, which is in contradiction with the condition (66). Thus, for $N < 4(1 + \sqrt{7})$ and in the region (95), the cuspy deformation from the LREF TT FP (72) is unphysical. Then the destabilization of the LREF TT FP (72) by the cuspy perturbation does not occur for $\tilde{\epsilon} > 2(10 + 21\sqrt{7})/103$.

The regions where the various fixed points are singly unstable are depicted in Fig. 5. Outside the areas where the LRE TT and the LREF TT FPs are singly unstable, the cuspy fixed point is considered to control the critical behavior in the system. Particularly, in the region of $1 \leq \tilde{\epsilon} < 2(10 + 21\sqrt{7})/103 \simeq 1.27$, the destabilization of the LRE TT and the LREF TT FPs by the cuspy perturbation is caused at N_{cusp} for LRE TT FP, and at $\tilde{\epsilon}_{\text{cusp}}$ for the LREF TT FP, respectively.

In the following sections, we carefully examine the critical phenomena governed by the LRE TT FP (70) and the LREF TT FP (72).

IV. CRITICAL PHENOMENA IN THE SYSTEM WITH LRE IN $2\sigma + \rho + \epsilon$ DIMENSIONS

In this section we study the critical phenomena controlled by the LRE TT FP (70). We calculate the critical exponents η , $\bar{\eta}$, and ν at $O(\epsilon)$. We put $\rho = 0$, and investigate the $d \rightarrow d - \sigma$ dimensional reduction. We also discuss the relations between η and $\bar{\eta}$, and present the result for the critical value σ_* .

Substituting the LRE TT FP (70) into Eqs. (67) and (68), we obtain the critical exponents η_{LRE} and $\bar{\eta}_{\text{LRE}}$ at the LRE TT fixed point (70):

$$\eta_{\text{LRE}} = 2 - \sigma, \quad (96)$$

$$\bar{\eta}_{\text{LRE}} = 4 - 2\sigma - \frac{\epsilon + \rho}{N}. \quad (97)$$

These exponents satisfy $\eta_{\text{LRE}} \geq (4 - d)/2$, $\bar{\eta}_{\text{LRE}} \geq 4 - d$, and the Schwartz-Soffer inequality $\bar{\eta}_{\text{LRE}} \leq 2\eta_{\text{LRE}}$. In the large N limit, the relation between η_{LRE} and $\bar{\eta}_{\text{LRE}}$ satisfies $\bar{\eta}_{\text{LRE}} = 2\eta_{\text{LRE}}$, which is identical to the result of the previous study for the critical properties of the random field spherical model by Vojta and Schreiber [41]. For finite N but $N > N_{\text{cusp}}$, the relation between η_{LRE} and $\bar{\eta}_{\text{LRE}}$ satisfies $2\eta_{\text{LRE}} - \bar{\eta}_{\text{LRE}} = (\epsilon + \rho)/N$ for $\epsilon = d - 2\sigma - \rho$. Our result is consistent with the result of $1/N$ expansion study by Bray [21]. He showed $2\eta_{\text{LRE}} - \bar{\eta}_{\text{LRE}} = \epsilon/N$ for $\epsilon = d - 2\sigma$ by the use of the $1/N$ expansion. Thus, the relation $2\eta_{\text{LRE}} - \bar{\eta}_{\text{LRE}} = (d - 2\sigma)/N$ holds in the region where the scaling behavior in the system is controlled by the LRE TT fixed point.

We turn to compute the exponent ν_{LRE} of the correlation length. The critical exponent ν_{LRE} is determined from the inverse of the maximal eigenvalue given by Eq. (74). Thus we obtain the inverse of the critical exponent ν_{LRE} as

$$\nu_{\text{LRE}}^{-1} = \epsilon + \rho. \quad (98)$$

If we put $\rho = 0$, the spatial dimension in the present system becomes $d = 2\sigma + \epsilon$, and then ν_{LRE}^{-1} is

$$\nu_{\text{LRE}}^{-1} = \epsilon, \quad (99)$$

which is in agreement with that in the pure long-range system in σ dimensions less [42]. Therefore, the $d \rightarrow d - \sigma$ dimensional reduction holds at $O(\epsilon)$, and its validity is recognized only for $N > N_{\text{cusp}}$.

The relation between η_{LRE} and $\bar{\eta}_{\text{LRE}}$ is classified on the basis of the value of σ as follows:

$$1. \quad \sigma < 2 - \frac{\epsilon + \rho}{N} : \bar{\eta}_{\text{LRE}} > \eta_{\text{LRE}}, \quad (100)$$

$$2. \quad \sigma = 2 - \frac{\epsilon + \rho}{N} : \bar{\eta}_{\text{LRE}} = \eta_{\text{LRE}}, \quad (101)$$

$$3. \quad \sigma > 2 - \frac{\epsilon + \rho}{N} : \bar{\eta}_{\text{LRE}} < \eta_{\text{LRE}}, \quad (102)$$

for $N > N_{\text{cusp}}$. Since $\eta_{\text{LRE}} \leq \bar{\eta}_{\text{LRE}} \leq 2\eta_{\text{LRE}}$, case 3 is unphysical. Thus, the critical value $\sigma = \sigma_*$ which separates between the long-range and the short-range exchange regimes of the theory is

$$\sigma_* = 2 - \frac{\epsilon + \rho}{N}. \quad (103)$$

Here we comment on the critical value σ_* . If $\sigma > 2 - (\epsilon + \rho)/2$, the spatial dimension in the present system is above four. Then we put $d = 2\sigma + \rho + \epsilon = 4 + \epsilon'$ ($0 < \epsilon' \ll 1$). The critical value σ_* is rewritten in terms of ϵ' as follows:

$$\sigma_* = 2 - \frac{\epsilon'}{N - 2}. \quad (104)$$

Since the exponent η of the random field $O(N)$ spin model with SR in $4 + \epsilon'$ dimensions is $\eta = \eta_{\text{SR}} = \epsilon'/(N - 2)$ at

$O(\epsilon')$ and for $N > N_{\text{cusp}}$, our result confirms that the critical value σ_* which separates between the long-range and the short-range exchange regimes of the theory is

$$\sigma_* = 2 - \eta_{\text{SR}}. \quad (105)$$

V. CRITICAL PHENOMENA IN SYSTEM WITH LREF IN $2\sigma + \rho + \epsilon$ DIMENSIONS

In this section we study the critical phenomena controlled by the LREF TT fixed point (72). We calculate the critical exponents η , $\bar{\eta}$, and ν , and investigate the $d \rightarrow d - \sigma - \rho$ dimensional reduction and the $d \rightarrow d - 2$ dimensional reduction.

Substituting the LREF TT FP (72) into Eqs. (67) and (68), we obtain the critical exponents η_{LREF} and $\bar{\eta}_{\text{LREF}}$ at LREF TT fixed point (72):

$$\eta_{\text{LREF}} = 2 - \sigma, \quad (106)$$

$$\bar{\eta}_{\text{LREF}} = 4 - 2\sigma - \rho. \quad (107)$$

These exponents satisfy the Schwartz-Soffer inequality $\bar{\eta}_{\text{LREF}} \leq 2\eta_{\text{LREF}}$ [39], and saturate the generalized Schwartz-Soffer inequality $\bar{\eta}_{\text{LREF}} \leq 2\eta_{\text{LREF}} - \rho$ [40]. And the inverse of the critical exponent ν_{LREF} is

$$\nu_{\text{LREF}}^{-1} = \epsilon \left[\frac{N}{N-1} - \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} \sqrt{1 + \frac{4[N - \tilde{\epsilon}(N-1)]}{\tilde{\epsilon}^2(N-1)^2}} \right]. \quad (108)$$

Since $\lim_{N \rightarrow \infty} \nu_{\text{LREF}}^{-1} = \epsilon$, in the large N limit, the exponent ν_{LREF} agrees with that of the pure long-range system in $\sigma + \rho$ dimensions less [42]. However, as long as N is finite, $\nu_{\text{LREF}}^{-1} \neq \epsilon$. Thus, the $d \rightarrow d - \sigma - \rho$ dimensional reduction is broken for finite N . Hence, the $d \rightarrow d - 2$ dimensional reduction in the case of $\rho = 2 - \sigma$ is also broken for finite N , although the exponents η_{LREF} and $\bar{\eta}_{\text{LREF}}$ satisfy $\bar{\eta}_{\text{LREF}} = \eta_{\text{LREF}} = 2 - \sigma$ for $N > 1$ and $\sigma > 2 - \epsilon/(N-1)$. The graphs of $(\epsilon \nu_{\text{LREF}})^{-1}$ for some values of N are depicted in Fig. 6. It shows that $(\epsilon \nu_{\text{LREF}})^{-1}$ tends to draw to 1 as the value of the parameter σ decreases. Then one expects that $(\epsilon \nu_{\text{LREF}})^{-1}$ reaches 1 if the value of the parameter σ decreases even further. However, it is impossible to study within the present framework, since the nontrivial fixed point of $O(\epsilon)$ disappears.

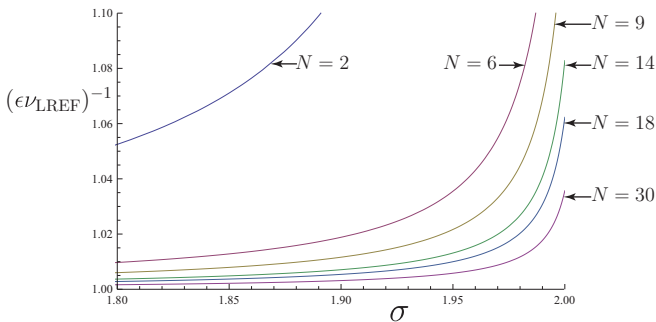


FIG. 6. The graphs of Eq. (108) in $\sigma \in [1.80, 2.00]$. Here we have put $\rho = 2 - \sigma$ in Eq. (108). Then the reduced variable $\tilde{\epsilon}$ takes $\tilde{\epsilon} = (\epsilon + 2 - \sigma)/\epsilon$. The ordinate is $(\epsilon \nu_{\text{LREF}})^{-1}$, and the abscissa is σ . Here we have set $\epsilon = 0.01$. Then $\tilde{\epsilon} = (2.01 - \sigma)/0.01$.

VI. SUMMARY

In this paper we have reexamined the critical phenomena of the long-range random field $O(N)$ spin model near the lower critical dimension by using the $O(N)$ nonlinear-sigma model with the random fields and all possible higher-rank random anisotropies. By the use of the perturbative functional renormalization group, we have investigated the stability of the analytic fixed points in the one-loop beta functions. Also, we have calculated the critical exponents η , $\bar{\eta}$, and ν at the analytic fixed point controlling the critical behavior in the system.

We have shown that the analytic fixed point controlling the critical behavior in the system with the long-range correlations of the random fields has the subcusp, and that it can be destabilized by the cuspy perturbation in both cases where the exchange interactions between spins are short ranged and long ranged.

We have studied the critical phenomena in the spin system with LRE. We have investigated the $d \rightarrow d - \sigma$ dimensional reduction. We have found that there exists the once-unstable analytic fixed point corresponding to the $d \rightarrow d - \sigma$ dimensional reduction for $N > N_{\text{cusp}} = 2(4 + 3\sqrt{3}) \simeq 18.3923 \dots$. Although it has the subcusp, the weaker nonanalyticity does not change the value of the fixed point. Then the critical exponents η_{LRE} and $\bar{\eta}_{\text{LRE}}$ evaluated at the once-unstable analytic fixed point are $\eta_{\text{LRE}} = 2 - \sigma$ and $\bar{\eta}_{\text{LRE}} = 4 - 2\sigma - (d - 2\sigma)/N$, respectively, and satisfy the relation $2\eta_{\text{LRE}} - \bar{\eta}_{\text{LRE}} = (d - 2\sigma)/N$. The inverse of the exponent ν_{LRE} takes $\nu_{\text{LRE}}^{-1} = \epsilon$ at $O(\epsilon)$ in $\epsilon = d - 2\sigma$. Therefore, we conclude that the $d \rightarrow d - \sigma$ dimensional reduction at the leading order of the $d - 2\sigma$ expansion holds only for $N > N_{\text{cusp}}$. For $N < N_{\text{cusp}}$, the nonanalyticity occurring by the appearance of the linear cusp breaks down the $d \rightarrow d - \sigma$ dimensional reduction. This is considered to violate the simple relation between the exponents. Thus, one expects that the critical scaling behavior in the spin system with LRE is described by three independent exponents [19]. Moreover, we have also obtained the critical value $\sigma_* = 2 - \eta_{\text{SR}}$ on the basis of the condition $\eta \leq \bar{\eta} \leq 2\eta$. Thus, our result supports the prediction that the crossover between the long-range and the short-range exchange regimes of the theory occurs at $\sigma_* = 2 - \eta_{\text{sr}}$ [21,28,30,31,34–38].

We have studied the critical phenomena in the spin system with LREF. We have investigated the $d \rightarrow d - \sigma - \rho$ dimensional reduction and the $d \rightarrow d - 2$ dimensional reduction. We have found the once-unstable analytic fixed point controlling the critical behavior. Although it has the subcusp, the weaker nonanalyticity does not change the value of the fixed point. Then the critical exponents η_{LREF} and $\bar{\eta}_{\text{LREF}}$ evaluated at the once-unstable analytic fixed point are $\eta_{\text{LREF}} = 2 - \sigma$ and $\bar{\eta}_{\text{LREF}} = 4 - 2\sigma - \rho$, respectively, and satisfy $2\bar{\eta}_{\text{LREF}} - \eta_{\text{LREF}} = \rho$. However, we have shown that the $d \rightarrow d - \sigma - \rho$ dimensional reduction does not hold within the present analysis, as far as N is finite; the exponent ν_{LREF} does not coincide with that of the pure long-range system in $\sigma + \rho$ dimensions less. Thus, the $d \rightarrow d - 2$ dimensional reduction in the case of $\rho = 2 - \sigma$ is also broken for finite N . The result does not contradict that in our previous study for the three-dimensional long-range random field Ising model [26]. Since our present study by the use of the perturbative renormalization group

has been restricted to $\epsilon + \rho = \epsilon + 2 - \sigma \sim O(\epsilon)$, only the breakdown of the $d \rightarrow d - 2$ dimensional reduction has been observed. Then, to study the $d \rightarrow d - 2$ dimensional reduction and its breakdown in the $(\sigma + 2 + \epsilon)$ -dimensional long-range random field $O(N)$ spin model, the nonperturbative analysis are needed.

Finally, we comment on the validity of the $d \rightarrow d - \sigma$ dimensional reduction in the system with LRE near the lower critical dimension and for $N > N_{\text{cusp}}$. As shown in the previous works by Young [5] and Bray [21], the value of ν_{LRE}^{-1} coincides with that of the pure long-range system in σ dimensions less at the leading order in $\epsilon = d_u - d$ near

the upper critical dimension $d_u = 3\sigma$. However, it fails at $O(\epsilon^2)$. Thus, although we have shown that the $d \rightarrow d - \sigma$ dimensional reduction holds at the leading order in $\epsilon = d - d_l$ near the lower critical dimension $d_l = 2\sigma$ and for $N > N_{\text{cusp}}$ in the present work, there is room for doubt whether it holds beyond one loop, even if $N > N_{\text{cusp}}$. Further studies by using the higher-loop calculation should shed light on this problem.

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