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Discussion of critical phenomena for general n-vector models

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We discuss, in the framework of the ϵ expansion, systems with an n-component order parameter and with the most general interaction compatible with the existence of only one length scale. At lowest order, it is found that the only stable fixed point corresponds to an O(n)-symmetric interaction if n is smaller that 4; in that case, two exponents only are sufficient to characterize deviations from scaling. If there is a second nontrivial fixed point, then a third one is always present; among these three fixed points, one and only one is stable.

I. INTRODUCTION

In the study of critical phenomena in $4-\epsilon$ dimension by the renormalization-group methods of Wilson,[1] the interaction is described by an n-component order parameter. In order to write a precise interaction, one must use the symmetry properties of the problem. The model originally studied by Wilson had O(n) symmetry.[2] The effect of cubic anisotropy was then considered by several authors.[3,4] It may drastically change the behavior of the system, leading possibly to a first-order transition[3] or to a different critical behavior.[4]

In order to incorporate other possible anisotropy effects, we have studied here, in the framework of the Wilson-Fisher ϵ expansion,[2] the most general interaction compatible with only one length scale. Most systems studied up to now fall into this class; a nontrivial example is provided by the p-wave superconducting transitions[5] of ^{3}He, in which the order parameter is a 3×3 complex matrix. Several infrared fixed points, i.e., relevant for the critical domain, are in general present. We have discussed the stability of these fixed points with respect to small changes in the interaction. A section is also devoted to the boundedness from below of the free energy. It is shown that renormalization-group arguments lead to an expected result: the boundedness implies that the bare interaction energy should be positive.

Our main results are the following:

(i) At lowest nontrivial order in ϵ, the only stable fixed point for $n<4$ is the O(n)-symmetric one. This is a striking result, since it means that, whatever interactions one takes, the symmetry is dynamically generated near the critical point. The magnitude of the first correction in ϵ to the upper bound on n makes it difficult to predict its actual value for $\epsilon=1$.

(ii) In this domain, where $n<4+\mathcal{O}(\epsilon)$, only two exponents characterize the first deviations to scaling when the lattice spacing increases.

(iii) Conversely, if $n>4+\mathcal{O}(\epsilon)$, the symmetric fixed point is not stable, unless the interaction is O(n) symmetric by itself.

(iv) If, in addition to the symmetric fixed point, a second nontrivial one is present, there is always a third fixed point.

In the linear space of interactions spanned by these three fixed points, one and only one is stable.

The critical exponent η, i.e., the anomalous dimension of the field, can be calculated for each fixed point. The largest of the three values of η always corresponds to the stable fixed point.

At order ϵ, one can perform a global study of the domain of attraction of a fixed point. It is found that this domain does not cover the entire range of coupling constants allowed by the boundedness from below of the free energy, yielding another restriction to universality.

II. THE HAMILTONIAN DENSITY

The model is described by the renormalized Hamiltonian for n massless fields $\varphi_i(x)$, $i=1,\ldots,n$, with a local interaction in $d=4-\epsilon$ dimension[6]:

$$\begin{aligned}\mathcal{H}(x) = {}& \tfrac{1}{2}\partial_\mu\varphi_i\,\partial^\mu\varphi_i + (\mu^\epsilon/4!)\\ &\times[u_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l + (\zeta_{ijkl}-u_{ijkl})\varphi_i\varphi_j\varphi_k\varphi_l]\\ &+\tfrac{1}{2}\partial_\mu\varphi_i(Z_3-1)_{ij}\partial^\mu\varphi_j + \tfrac{1}{2}\varphi_i(\delta m^2)_{ij}\varphi_j\ . \end{aligned}\tag{1}$$

The dimensionless interaction u_{ijkl} is totally symmetrical in the indices, and summation over repeated indices will always be meant.

The counter-terms ζ_{ijkl}, $(Z_3)_{ij}$, and $(\delta m^2)_{ij}$ are fixed by the renormalization conditions given on the inverse propagators $\Gamma^{(2)}_{ij}(p)$

$$\Gamma^{(2)}_{ij}(0)=0\ ,\tag{2a}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}_{ij}(p)\big|_{p^2=\mu^2}=\delta_{ij}\tag{2b}$$

and on the four-point proper vertex function (one-particle irreducible)

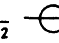
$$\Gamma^{(4)}_{ijkl}\big|_{\text{S.P.}}=\mu^\epsilon u_{ijkl}\ ,\tag{2c}$$

where S.P. stands for the symmetry point

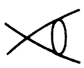
$$p_i\cdot p_j=\mu^2(\delta_{ij}-\tfrac{1}{4})$$

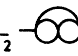
TABLE I. The normalization of the Feynman diagrams is the same as in Ref. 14.

$a =$  $= \frac{1}{\varepsilon}\left(1+\frac{\varepsilon}{2}\right)$

$b = \frac{d}{dp^2}$  $\Big|_{p^2=1} = -\frac{1}{8\varepsilon}\left(1+\frac{5}{4}\varepsilon\right)$

$\bar{c} = c - \frac{a^2}{2} = \frac{1}{4\varepsilon}$

$c =$  $= \frac{1}{2\varepsilon^2}\left(1+\frac{3}{2}\varepsilon\right)$

$d = \frac{d}{dp^2}$  $\Big|_{p^2=1} = -\frac{1}{6\varepsilon^2}(1+2\varepsilon)$

$\bar{d} = d - \frac{4ab}{3} = -\frac{1}{24\varepsilon}$

and μ is an arbitrary renormalization mass.

We have assumed that all the fields are massless, i.e., they are critical all together; in general there is no reason, if the bare masses are different, that by fixing one parameter (the temperature), the renormalized masses should vanish. Therefore the physically interesting case is the one in which the bare masses are the same. This situation will only occur if there is a symmetry group of the interaction which has only one quadratic invariant, namely $\sum_{i=1}^{n}(\varphi_i)^2$.

At lowest order, the condition on the interaction to preserve this symmetry is the "trace condition"

$$u_{iijk} = U\delta_{jk} \ . \tag{3}$$

However, it will not be necessary to impose this restriction in the first part of our calculations.

III. RENORMALIZATION-GROUP EQUATIONS

The method we are using to calculate the fixed points and the corresponding critical exponents has been detailed in Ref. 6.

The relevant diagrams (denoted a, b, c, d, $\bar{c}$, and $\bar{d}$) are listed in Table I, in terms of which the wave function and vertex renormalization constants are

$$(Z_3)_{ij} = \delta_{ij} + \tfrac{1}{6} b\, u_{ipqr} u_{jpqr} + \left[\tfrac{1}{2}(ab) - \tfrac{1}{4} d\right] \times u_{ipqr} u_{qrst} u_{stpj} + O(u^4) \ , \tag{4}$$

$$\zeta_{ijkl} = u_{ijkl} + \tfrac{1}{2} a\left[u_{ijpq} u_{pqkl} + 2\,\mathrm{perm}(ijkl)\right] + \tfrac{1}{4} a^2\left[u_{ijpq} u_{pqrs} u_{rskl} + 2\,\mathrm{perm}(ijkl)\right] - \bar{c}\left[u_{ipqr} u_{sjpr} u_{klqs} + 5\,\mathrm{perm}(ijkl)\right] + O(u^4) \ . \tag{5}$$

The bare coupling constants g^0_{ijkl} are expressed in terms of these quantities by

$$g^0_{ijkl} = \mu^\epsilon \rho_{ijkl} \ ,$$

where the ρ are dimensionless, and

$$\begin{aligned}\rho_{ijkl} &= (Z_3^{-1/2})_{ii'} (Z_3^{-1/2})_{jj'} (Z_3^{-1/2})_{kk'} \times (Z_3^{-1/2})_{ll'} \zeta_{i'j'k'l'} \\ &= \zeta_{ijkl} - \tfrac{1}{12} b\left[u_{i'jkl} u_{i'pqr} u_{ipqr} + 3\,\mathrm{perm}(ijkl)\right] + O(u^4) \ ,\end{aligned} \tag{6}$$

in terms of which the Callan-Symanzik functions β_{ijkl} (Ref. 7) are determined by

$$\beta_{i'j'k'l'} \frac{\partial \rho_{ijkl}}{\partial u_{i'j'k'l'}} = -\epsilon \rho_{ijkl} \ . \tag{7}$$

This gives the result

$$\begin{aligned}\beta_{ijkl} = &-\epsilon u_{ijkl} + \epsilon\, \tfrac{1}{2} a (u_{ijpq} u_{pqkl} + 2\,\mathrm{perm}) \\ &- \epsilon \bar{c}(u_{ipqr} u_{jprs} u_{klqs} + 5\,\mathrm{perm}) \\ &- \epsilon \tfrac{1}{6} b (u_{i'jkl} u_{i'pqr} u_{ipqr} + 3\,\mathrm{perm}) + O(u^4) \ .\end{aligned} \tag{8}$$

The anomalous dimension matrix γ_3 is now given as

$$(\gamma_3)_{ij} = 2\beta_{i'j'k'l'} \left(\frac{\partial Z_3^{1/2}}{\partial u_{i'j'k'l'}} Z_3^{-1/2}\right)_{ij} \ , \tag{9}$$

and thus

$$(\gamma_3)_{ij} = -\epsilon\, \tfrac{1}{3} b\, u_{ipqr} u_{jpqr} + \tfrac{3}{4} \epsilon \bar{d}\, u_{ipqr} u_{qrst} u_{stpj} + O(u^4) \ . \tag{10}$$

Similarly, the insertions of φ^2 fields in Green's functions are renormalized by the constants $(Z_4)_{ij,kl}$, which makes finite the insertion of a $\varphi_i \varphi_j$ pair on a (kl) line. One has

$$\begin{aligned}(Z_4^{-1})_{ij,kl} = &\tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \tfrac{1}{2} a\, u_{ijkl} \\ &+ \tfrac{1}{4}(c - a^2)(u_{ikpq} u_{jlpq} + u_{ilpq} u_{jkpq}) \ .\end{aligned} \tag{11}$$

It is convenient to define

$$\bar{Z}_{ij,kl} = (Z_4)_{ij,mn} (Z_3^{-1/2})_{mk} (Z_3^{-1/2})_{nl} \ , \tag{12}$$

in terms of which the anomalous dimension matrix $\bar{\gamma}$ of the φ^2 fields is

$$\bar{\gamma}_{ij,kl} = \beta_{i'j'k'l'}(u) \frac{\partial \bar{Z}_{ij,mn}}{\partial u_{i'j'k'l'}} (\bar{Z}^{-1})_{mn,kl} \ . \tag{13}$$

At these orders one finds

$$\begin{aligned}\bar{\gamma}_{ij,kl} = &-\epsilon\, \tfrac{1}{2} a\, u_{ijkl} + \tfrac{1}{2} \epsilon \bar{c}(u_{ikmn} u_{jlmn} + u_{ilmn} u_{jkmn}) \\ &+ \epsilon \tfrac{1}{12} b (\delta_{ik} u_{jmnp} u_{lmnp} + 3\,\mathrm{perm}[i \leftrightarrow j,\ k \leftrightarrow l]) \ .\end{aligned} \tag{14}$$

With the help of these functions β, γ_3, and $\bar{\gamma}$, the renormalization-group equations are for the one-particle irreducible correlation functions

$$\Gamma^{(N,L)}((p_\tau, i_\tau); (q_{\tau'}, j_{\tau'} k_{\tau'}); \mu, u_{ijkl}),$$

$$\tau = 1, \ldots, N, \quad \tau' = 1, \ldots, L$$

with $N - \varphi$ fields and $L - \varphi^2$ insertions:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_{i'j'k'l'}\frac{\partial}{\partial u_{i'j'k'l'}}\right)\Gamma^{(N,L)}(i_1,\ldots,i_N;j_1k_1,\ldots,j_Lk_L)-\frac{1}{2}\sum_{n=1}^{N}(\gamma_3)_{i_ni_n'}\Gamma^{(N,L)}(i_1,\ldots,i_n',\ldots,i_N;j_1k_1,\ldots,j_Lk_L)$$
$$-\sum_{l=1}^{L}(\overline{\gamma})_{j_lk_l;j_l'k_l'}\Gamma^{(N,L)}(i_1,\ldots,i_N;j_1k_1,\ldots,j_l'k_l',\ldots,j_Lk_L)=0\ . \qquad (15)$$

Integration of this equation gives

$$\Gamma^{(N,L)}((\lambda p_\tau, i_\tau);(\lambda q_{\tau'},j_{\tau'}k_{\tau'});\mu,u_{ijkl})=\prod_{n=1}^{N}(D_1)_{i_ni_n'}\prod_{l=1}^{L}(D_2)_{j_lk_l;j_l'k_l'}\lambda^{d-N(d-2)/2-2L}$$
$$\times\Gamma^{(N,L)}((p_\tau,i_\tau');(q_{\tau'},j_{\tau'}'k_{\tau'}');\mu,u_{ijkl}(\lambda))\ , \qquad (16)$$

where $u_{ijkl}(\lambda)$ is the solution of

$$\frac{du_{ijkl}(\lambda)}{d\ln\lambda}=\beta_{ijkl}(u(\lambda))\ , \qquad (17)$$

with $u_{ijkl}(1)=u_{ijkl}$, and where the matrices D_1 and D_2 are

$$[D_1(u,\lambda)]_{ij}=[Z_3^{1/2}(u)Z_3^{-1/2}(u(\lambda))]_{ij}\ , \qquad (18)$$

$$[D_2(u,\lambda)]_{ij;kl}=[\overline{Z}(u)\overline{Z}^{-1}(u(\lambda))]_{ij;kl}\ . \qquad (19)$$

We are interested in the critical region where μ is much larger than the momenta, and we therefore look to the solution of Eq. (17) when $\lambda\to 0$.

In this limit a stable fixed point u^*_{ijkl} is therefore the solution of $\beta_{ijkl}(u^*)=0$ with eigenvalues of positive real part for the matrix B defined by

$$B_{ijkl;i'j'k'l'}=\left.\frac{\partial\beta_{ijkl}}{\partial u_{i'j'k'l'}}\right|_{u=u^*}\ . \qquad (20)$$

The definition of γ_3 and $\overline{\gamma}$ implies the matrix equations

$$\lambda\frac{\partial D_1}{\partial\lambda}+\frac{1}{2}\gamma_3(\lambda)D_1(u(\lambda),\lambda)=0\ ,$$
$$\lambda\frac{\partial D_2}{\partial\lambda}+\overline{\gamma}(\lambda)D_2(u(\lambda),\lambda)=0\ . \qquad (21)$$

When λ goes to zero, we assume as usual that the matrices γ_3 and $\overline{\gamma}$ reach finite limits γ_3^* and $\overline{\gamma}^*$. Integration of Eq. (21) for λ small gives

$$D_1\simeq\lambda^{-\gamma_3^*/2}\Delta_1\ ,\quad D_2\simeq\lambda^{-\overline{\gamma}^*}\Delta_2\ , \qquad (22)$$

where Δ_1 and Δ_2 are constant matrices.

Equations (16) and (22) show, therefore, that the existence of a fixed point u^* implies scale invariance with anomalous dimensions related to the eigenvalues of γ_3^* and $\overline{\gamma}^*$.

IV. REMARKS CONCERNING THE POSITIVITY OF ENERGY

The theories that we are considering are asymptotically free[8] (for the ultraviolet limit), since the B matrix of Eq. (20) has only one eigenvalue, which is $-\epsilon$ for the fixed point $u^*_{ijkl}=0$.

The eigenvalues of the B matrix give only a local characterization. Let us call $\mathfrak{D}$ the global domain of (ultraviolet) attraction of the origin, defined by

$$u_{ijkl}\equiv u_{ijkl}(1)\in\mathfrak{D}\ \text{ if }\ \lim_{\lambda\to\infty}u_{ijkl}(\lambda)=0\ . \qquad (23)$$

In this domain, it is possible to show that the stability of the vacuum (i.e., the boundedness from below of the free energy) is related to the sign of the interaction part of the bare Hamiltonian—namely, of

$$g^0_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l\ , \qquad (24)$$

where φ_i is an arbitrary real vector.

The argument is the following: Consider the generating functional of the one-particle irreducible functions:

$$\mathfrak{F}(M_i,\mu,u_{ijkl})=\sum_{N=2}^{\infty}\frac{1}{N!}\int dx_1\ldots dx_N\varphi_{i_1}(x_1)\cdots\varphi_{i_N}(x_N)\Gamma^{(N,0)}_{i_1\cdots i_N}(x_1,\ldots,x_N;\mu,u_{ijkl})\Big|_{\varphi_i(x)=M_i}\ . \qquad (25)$$

It is easy to verify that the renormalization group equation (15) of the vertex functions give for the functional[9]

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_{ijkl}\frac{\partial}{\partial u_{ijkl}}-\frac{1}{2}M_i(\gamma_3)_{ij}\frac{\partial}{\partial M_j}\right)\mathfrak{F}(M_i,\mu,u_{ijkl})=0. \qquad (26)$$

Dimensional analysis yields

$$\mathfrak{F}(M_i,\mu,u_{ijkl})=\mu^{4-\epsilon}F(x_i\equiv M_i\mu^{\epsilon/2-1},u_{ijkl})\ , \qquad (27)$$

and F is solution of

$$\left(4-\epsilon+\beta(u)\frac{\partial}{\partial u}-\frac{x_i}{2}[(\gamma_3)_{ij}+(2-\epsilon)\delta_{ij}]\frac{\partial}{\partial x_j}\right)$$
$$\times F(x_i,u_{ijkl})=0\ . \qquad (28)$$

Any solution of this equation has the property that

$$F(x_i(\lambda), u_{ijkl}) = \lambda^{4-\epsilon} F(x_i, u_{ijkl}(\lambda)) \quad , \tag{29}$$

where $u_{ijkl}(\lambda)$ satisfies Eq. (17), and where

$$\lambda \frac{\partial x_i}{\partial \lambda} = \tfrac{1}{2} x_j \left[(\gamma_3)_{ji}(u(\lambda)) + (2-\epsilon)\,\delta_{ji}\right] \quad . \tag{30}$$

In the limit λ goes to infinity, the $u_{ijkl}(\lambda)$ go to zero as $\lambda^{-\epsilon}$ (asymptotic freedom), and $(\gamma_3)_{ji}(u(\lambda))$, which is of order u^2, is negligible. Therefore in this limit the x_i are proportional to $\lambda^{1-\epsilon/2}$.

Let us examine the consequences for $\mathfrak{F}$: assume all the M_i go to infinity together like $\lambda^{1-\epsilon/2}$ at fixed μ and u_{ijkl}. The x_i go to infinity like $\lambda^{1-\epsilon/2}$. It is equivalent to keep μ and the M_i fixed with the u_{ijkl} replaced by $u_{ijkl}(\lambda)$. These $u_{ijkl}(\lambda)$ go to zero like $\lambda^{-\epsilon}$. Therefore, the large M_i limit of $\mathfrak{F}$ can be obtained from perturbation theory, since

$$\mathfrak{F}(\lambda^{1-\epsilon/2} M_i, \mu, u_{ijkl}) \underset{\lambda\to\infty}{\sim} \lambda^{4-\epsilon}\, \mathfrak{F}(M_i, \mu, u_{ijkl}(\lambda)) \quad . \tag{31}$$

For u small the dominant term of $\mathfrak{F}$ is the Born term of the four-point function, and thus

$$\mathfrak{F}(\lambda^{1-\epsilon/2} M_i, \mu, u_{ijkl}) \underset{\lambda\to\infty}{\sim} \lambda^{4-\epsilon}\left[(\mu^\epsilon/4!)\, u_{ijkl}(\lambda)\, M_i M_j M_k M_l + \mathcal{O}(\lambda^{-2\epsilon})\right] \quad . \tag{32}$$

Let us relate $u_{ijkl}(\lambda)$ to the bare coupling constants through Eq. (6). From the definition of the β functions

$$\beta_{ijkl} = \mu \frac{\partial}{\partial \mu} u_{ijkl}\Big|_{g_0 \text{ fixed}} \quad , \tag{33}$$

we obtain

$$g^0_{ijkl}(u) = \lambda^\epsilon g^0_{ijkl}(u(\lambda)) \quad . \tag{34}$$

Therefore, taking λ large, if u is in $\mathfrak{D}$

$$g^0_{ijkl}(u) \underset{\lambda\to\infty}{\sim} \lambda^\epsilon u_{ijkl}(\lambda) \quad . \tag{35}$$

Consequently, the boundedness from below of the free energy is equivalent to the positivity of the bare quartic form (24). We shall call $\mathfrak{D}^+$ the domain of the u_{ijkl} in $\mathfrak{D}$ for which the bare form (24) is positive.

We shall later see an explicit example of u_{ijkl} belonging to $\mathfrak{D}^+$ but for which the form $u_{ijkl} x_i x_j x_k x_l$ is not positive for some set of x_i.

Another consequence of Eq. (34) is that the domain of attraction of the origin in the variables u coincides with the domain of finiteness of the g_0's. Indeed, assume that the u are such that the $g_0(u)$ are finite; the corresponding $u(\lambda)$ must be such that the $g_0(u(\lambda))$ become infinitesimal for large λ. For small value of g_0, the bare and renormalized coupling constants are equal, and therefore $u(\lambda)$ goes to zero when λ goes to infinity.

In particular, on the boundary of the domain $\mathfrak{D}$ all the g_0 are infinite, and therefore this boundary cannot be crossed. Beyond it the bare coupling constants would be, in general, complex.

All the infrared fixed points, stable or not, are on this boundary, since in view of Eq. (34) they must correspond to g_0 strictly infinite.

Let us note from Eq. (34) that infrared fixed points are obtained when one takes the large g_0 limit: indeed, starting from u such that $g_0(u)$ is finite, $g_0(u(\lambda))$, being equal to $\lambda^{-\epsilon} g_0(u)$, increases when $\lambda \to 0$.

If we stay at order ϵ, it is possible to be a little more precise. Let

$$u(x) \equiv u_{ijkl}\, x_i x_j x_k x_l \quad , \tag{36}$$

where x is an arbitrary vector normalized to $x_i x_i = 1$. From the expression of the β functions, one has

$$\frac{du(x,\lambda)}{d\ln\lambda} = -\epsilon\, u(x,\lambda) + \tfrac{3}{2} x_i x_j u_{ijpq} u_{pqkl} x_k x_l \quad . \tag{37}$$

Schwarz's inequality gives

$$\frac{du(x,\lambda)}{d\ln\lambda} \geq -\epsilon\, u(x,\lambda) + \tfrac{3}{2} u^2(x,\lambda) \quad . \tag{38}$$

If $u(x,1) \equiv u(x) > \frac{2}{3}\epsilon$, the corresponding $u(x,\lambda)$ increases indefinitely. Thus, to be in the domain $\mathfrak{D}$ the u_{ijkl} must fulfill the condition $u(x) < \frac{2}{3}\epsilon$ for any x.

Let us take now a set of u_{ijkl} in $\mathfrak{D}$ and such that $\forall x\, u(x) > 0$. Then, from inequality (38), $u(x,\lambda)$ goes to zero by positive values. Therefore the u_{ijkl} that satisfy $\forall x\, u(x) > 0$ belong to $\mathfrak{D}^+$.

A last general remark concerns the domain of attraction of the infrared fixed points: we have already mentioned that there are u_{ijkl} belonging to $\mathfrak{D}^+$ for which $u(x)$ is not positive. Such points cannot be attracted by the infrared fixed points: indeed, when $\lambda \to 0$, the same inequality (38) gives

$$\frac{du(x,\lambda)}{d\ln(1/\lambda)} \leq \epsilon\, u(x,\lambda) - \tfrac{3}{2} u^2(x,\lambda) \quad .$$

This inequality shows that if for some value λ_0 of λ and some vector x $u(x,\lambda_0) < 0$, then $u(x,\lambda)$ decreases indefinitely when λ goes to zero, and the u_{ijkl} do not converge to any infrared fixed point.[10]

V. CRITICAL EXPONENTS

A. General expressions for η and ν

We shall now specialize for the theories satisfying the trace condition (3).

Let us first show that the matrix γ_3^* is diagonal: All the quantities are expanded in ϵ, and we are only considering the fixed points where the u^* are of order ϵ. From the equations

$$\beta_{ijkk}(u^*) = 0 \quad , \tag{39}$$

$$\beta_{ipqr}(u^*)\, u^*_{jpqr} = 0 \quad , \tag{40}$$

we obtain

$$u^*_{ipqr}u^*_{jpqr}=\delta_{ij}\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right)\left[1+\frac{2b}{3a}U^*+\frac{\bar{c}}{a}\left(\frac{8}{3a}+2U^*\right)\right], \tag{41}$$

and then

$$(\gamma_3^*)_{ij}=\eta\delta_{ij}=\epsilon\delta_{ij}\left(\frac{\bar{d}}{2a}-\frac{b}{3}\right)\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right)\times\left[1+\frac{2b}{3a}U^*+\frac{\bar{c}}{a}\left(\frac{8}{3a}+2U^*\right)\right]+O(\epsilon^4) . \tag{42}$$

Let us note that γ_3^* not only is a multiple of the unit matrix, but also depends only on the value of the trace U^* of the fixed-point interaction.

Considering now the dimensions of the composite operators $\varphi_j\varphi_k$, let us note that derivatives with respect to the temperature involve $\bar{\varphi}^2$ insertions. Therefore a relation between temperature and correlation length will only exist if the $\bar{\gamma}^*_{ij;kl}$ matrix has the vector δ_{kl} as eigenvector. Indeed, the same equations [(39)–(40)] show that

$$(\bar{\gamma}^*)_{ij;kl}\,\delta_{kl}=\left(\frac{1}{\nu}-2\right)\delta_{ij} , \tag{43}$$

with

$$\left(\frac{1}{\nu}-2\right)=-\epsilon\frac{a}{2}U^*+\epsilon\left(\bar{c}+\frac{b}{3}\right)\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right) . \tag{44}$$

Remark

The two exponents η and ν are only functions of the trace U^*. Therefore if we eliminate U^* between their two expressions, we obtain a relation valid *for all the fixed points*, which takes the form

$$\eta=\frac{\epsilon^2}{12}\left(1+\frac{5\epsilon}{6}\right)x(1-x)+O(\epsilon^4) \tag{45}$$

where we have set

$$\left(\frac{1}{\nu}-2\right)=-\epsilon x . \tag{46}$$

Let us stress that this relation (45) is valid for Ising as well as for Heisenberg systems[11] with or without anisotropy. Its comparison with the results for η and ν obtained from high-temperature series in three dimensions is within the error bars.

B. Bounds on U^*, η, and ν

Let us first show that U^* is bounded. Equation (41) summed on $i=j$ implies, using Table I,

$$u^*_{ijkl}u^*_{ijkl}=n\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right)\left(1+\frac{2\epsilon}{3}+\frac{5U^*}{12}\right)+O(\epsilon^4) . \tag{47}$$

Let us define

$$S_{ijkl}\equiv\frac{1}{n+2}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) . \tag{48}$$

Now, if we write

$$u^*_{ijkl}=U^*S_{ijkl}+u'^*_{ijkl} , \tag{49}$$

the u'^*_{ijkl} is traceless and

$$u^*_{ijkl}u^*_{ijkl}=\frac{3n}{n+2}U^{*2}+u'^*_{ijkl}u'^*_{ijkl} . \tag{50}$$

Equations (47) and (50) yield the inequality

$$\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right)\left(1+\frac{2\epsilon}{3}+\frac{5U^*}{12}\right)-\frac{3}{n+2}U^{*2}\geq 0 . \tag{51}$$

The left-hand side is at lowest order a parabola, consequently

$$0\leq U^*\leq U_s^*\leq 2\epsilon , \tag{52}$$

where U_s^* is the O(n)-symmetric solution, given at this order by

$$U_s^*=\frac{2(n+2)}{(n+8)}\epsilon . \tag{53}$$

Next, η, obtained from Eq. (42) as

$$\eta=\frac{1}{24}\left(\frac{U^*}{a}-\frac{U^{*2}}{2}\right)\left(1+\frac{17}{12}\epsilon+\frac{5U^*}{12}\right)+O(\epsilon^4) , \tag{54}$$

appears by inequality (51) to be positive. Its expression has a maximum for

$$U^*_{\max}=\epsilon-\tfrac{7}{24}\epsilon^2+O(\epsilon^3) , \tag{55}$$

which corresponds to a value $n_{\max}$ of n for the symmetric solution

$$U_s^*(n_{\max})=U^*_{\max} , \tag{56}$$

with

$$n_{\max}=4-4\epsilon+O(\epsilon^2) . \tag{57}$$

Therefore one has an upper bound on η:

$$\eta\leq\eta_{\max}=\frac{\epsilon^2}{48}\left(1+\frac{5\epsilon}{6}\right)+O(\epsilon^4) . \tag{58}$$

For $n\leq n_{\max}$, since U^* is always smaller than U_s^* and η is an increasing function of U^* for $0\leq U^*\leq U^*_{\max}$, we obtain the following bounds:

$$\begin{aligned}&n\leq 4-4\epsilon,\\&\eta\leq\eta_{\text{sym}}(n)=\frac{(n+2)}{2(n+8)^2}\epsilon^2\left[1+\epsilon\left(\frac{6(3n+14)}{(n+8)^2}-\frac{1}{4}\right)\right] ,\\&n>4-4\epsilon,\quad \eta\leq\frac{\epsilon^2}{48}\left(1+\frac{5\epsilon}{6}\right) .\end{aligned} \tag{59}$$

A similar result holds for $(1/\nu-2)$, which is essentially proportional to $(-U^*)$, so that

$$\left(\frac{1}{\nu_s}-2\right)=-\epsilon\frac{(n+2)}{(n+8)}\left(1+\frac{\epsilon}{2}\frac{(13n+44)}{(n+8)^2}\right)\leq\left(\frac{1}{\nu}-2\right)\leq 0 . \tag{60}$$

VI. STABILITY CONDITIONS

An infrared stable fixed point corresponds to a matrix B, defined by Eq. (20):

$$B_{ijkl;i'j'k'l'} = \left.\frac{\partial \beta_{ijkl}}{\partial u_{i'j'k'l'}}\right|_{u=u^*} ,$$

with eigenvalues of positive real part.

The trivial fixed point at the origin $u^*_{ijkl} = 0$ is always unstable.

A. Stability of the symmetric solution

Let us now explore the stability of the $O(n)$-symmetric fixed point

$$u^*_{ijkl} = U^*_s(n)\, S_{ijkl} \quad , \tag{61}$$

where, taking into account one more order in ϵ than in Eq. (53),

$$U^*_s(n) = \frac{2(n+2)}{n+8}\,\epsilon\left[1 + \epsilon\left(\frac{3(3n+14)}{(n+8)^2} - \frac{1}{2}\right)\right] + O(\epsilon^3) \quad . \tag{62}$$

In order to linearize the functions β_{ijkl} around the fixed point, let us write

$$u_{ijkl} = [U^*_s(n) + u']\, S_{ijkl} + u'_{ijkl} \quad , \tag{63}$$

where u'_{ijkl} is traceless. Then one gets

$$\beta_{ijkl} \simeq \omega u' S_{ijkl} + \omega_{\text{anis}} u'_{ijkl} \quad , \tag{64}$$

where

$$\omega = \epsilon - \frac{3(3n+14)}{(n+8)^2}\,\epsilon^2 \quad , \tag{65a}$$

$$\omega_{\text{anis}} = \epsilon\frac{4-n}{n+8} - \epsilon^2\frac{5n^2+14n+152}{(n+8)^3} \quad . \tag{65b}$$

The linearized β matrix has thus two eigenvalues which are ω[6,12] and ω_{anis}.[13] Therefore, the symmetric solution is stable (with respect to any kind of quartic interaction) if and only if

$$\omega > 0 \quad \text{and} \quad \omega_{\text{anis}} > 0 \quad . \tag{66}$$

This gives the condition

$$n \leq 4 - 2\epsilon + O(\epsilon^2) \quad . \tag{67}$$

Conversely, we shall show at first order in ϵ that for n smaller than 4 no other fixed point is stable:

Let u^*_{ijkl} be an arbitrary fixed point and B the corresponding matrix (20) of the derivatives of β. The subspace spanned by the two vectors u^*_{ijkl} and S_{ijkl} is invariant under the matrix B which in this subspace reduces to

$$B = \begin{bmatrix} \epsilon & 0 \\ 6 & U^* - \epsilon \end{bmatrix} , \tag{68}$$

where $U^*\delta_{kl} = u^*_{iikl}$. The stability of the fixed point implies $U^* > \epsilon$. Since we know that $U^* < U^*_s(n)$, this nonsymmetric fixed point cannot be stable if n is such that $U^*_s(n) < \epsilon$. This means that no other stable fixed point appears if $n < 4 + O(\epsilon)$.

Remark

Actually one can study the stability of the symmetric fixed point in a more general situation with no restriction on the trace of u_{ijkl}: let us study the vicinity of u^{*s}_{ijkl} with

$$\begin{aligned} \delta u_{ijkl} &= u_{ijkl} - u^{*s}_{ijkl} \\ &= wS_{ijkl} + u'_{ijkl} + \frac{1}{n+4}(v_{kl}\delta_{ij} + 5 \text{ terms}) \ , \end{aligned} \tag{69}$$

where

$$\begin{aligned} &\delta u_{iijj} = nw \ , \quad u'_{iikl} = 0 \ , \\ &v_{kk} = 0 \ , \qquad \delta u_{iijk} = w\delta_{jk} + v_{jk} \ . \end{aligned} \tag{70}$$

The linearization of β_{ijkl} around this point leads to a third eigenvalue ω' in addition to ω and ω_{anis}, with

$$\omega' = \frac{\epsilon}{n+8}\left(8 - \frac{\epsilon}{(n+8)^2}(7n^2+68n+336)\right) + O(\epsilon^3) \ . \tag{71}$$

The physics associated with ω' is a crossover phenomenon from the totally symmetric situation to a region where some components of the order parameter become decoupled.

B. Consequences of the existence of more than one fixed point

The $O(n)$-symmetric fixed point is always present. Let us assume the existence of a second one. We are going to show that this implies the existence of a third infrared fixed point and to study globally, at order ϵ, the resulting situation.

Let u^*_{ijkl} be a second fixed point of order ϵ with a trace

$$u^*_{ijkk} = U^*\delta_{ij} \quad , \tag{72}$$

and let us study the renormalization-group equations for coupling constants being arbitrary linear combination of the symmetric solution and of the second one, i.e.,

$$u_{ijkl} = x\,U^*_s S_{ijkl} + y\,u^*_{ijkl} \quad . \tag{73}$$

The central remark is that, at order ϵ, $\lambda\, du_{ijkl}/d\lambda$ is also a linear combination of the same two quantities. The equations are at lowest order in ϵ

$$\begin{aligned} &\frac{1}{\epsilon}\lambda\frac{dx}{d\lambda} = -x + x^2 + \alpha_2 xy \quad , \\ &\frac{1}{\epsilon}\lambda\frac{dy}{d\lambda} = -y + y^2 + \alpha_1 xy \quad , \end{aligned} \tag{74}$$

where

$$\begin{aligned} &\alpha_1 = 2 - \frac{1}{\epsilon}U^*_s = \frac{12}{n+8} + O(\epsilon) \quad , \\ &\alpha_2 = U^*/\epsilon + O(\epsilon) \quad . \end{aligned} \tag{75}$$

We see that, in addition to the initial fixed points $(x=1, y=0)$ and $(x=0, y=1)$, a third one is always present:

$$x = \frac{1-\alpha_2}{1-\alpha_1\alpha_2}\ ,\quad y = \frac{1-\alpha_1}{1-\alpha_1\alpha_2}\quad . \tag{76}$$

The corresponding u'^*_{ijkl} has a trace given by

$$u'^*_{ijkk} = \epsilon\,\alpha_3\,\delta_{ij}\quad , \tag{77}$$

with

$$\alpha_3 = \frac{2-\alpha_1-\alpha_2}{1-\alpha_1\alpha_2}\quad , \tag{78}$$

or, equivalently,

$$2-(\alpha_1+\alpha_2+\alpha_3)+\alpha_1\alpha_2\alpha_3 = 0\quad . \tag{79}$$

Note that the relation between the fixed points u^* and u'^* is reciprocal.

From inequalities (52) we obtain

$$\alpha_i \geq 0\ ,\quad i = 1, 2, 3\ , \tag{80}$$

and

$$\alpha_1+\alpha_2 \leq 2\ ,\quad \alpha_1+\alpha_3 \leq 2\quad . \tag{81}$$

In addition, from the explicit expression (78), we find

$$\alpha_2+\alpha_3 \leq 2\quad . \tag{82}$$

Thus in the following considerations which depend only on the values of the α_i's, the three fixed points play a completely symmetric role.

Stability condition

To study the stability condition in the space of coupling constants spanned by these three points,

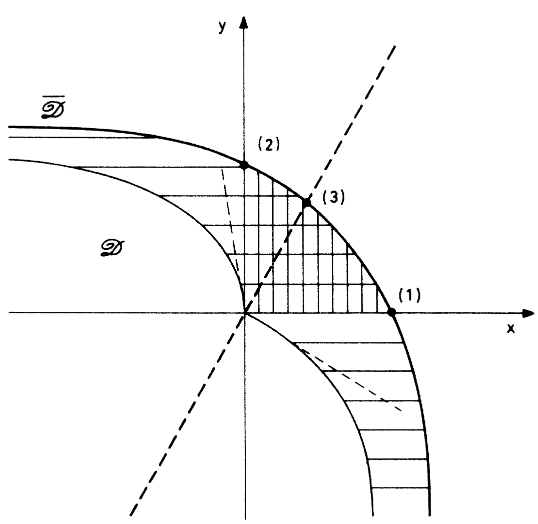


FIG. 1. Plane of the renormalized coupling constants. The domain $\mathfrak{D}$ of ultraviolet attraction of the origin is below its boundary $\bar{\mathfrak{D}}$. The fixed points (1), (2), and (3) are on $\bar{\mathfrak{D}}$; (3) is the stable one. The domain $\mathfrak{D}^+$, domain of positivity of the bare interaction, is depicted by the horizontal hatching. The vertical hatching shows $\mathfrak{D}^*$, domain of attraction of the infrared fixed point (3).

one reads the eigenvalues of the matrix B corresponding to the fixed point $i = 1$, 2, or 3 in Eq. (68): The stability of the ith point is equivalent to the condition

$$\alpha_i > 1\quad . \tag{83}$$

The inequalities $\alpha_i + \alpha_j \leq 2$ show that there is at most one stable fixed point. Furthermore, there certainly exists a stable one: for instance, if α_1 and α_2 are smaller than 1, it is easy to verify that α_3 given by Eq. (78) is larger than 1.

Thus, there is one and only one stable fixed point among the three.

Anomalous dimensions

At this order, the critical exponent η is directly related to the values of the α_i by the equation

$$\eta_i = \tfrac{1}{48}\epsilon^2\,\alpha_i(2-\alpha_i) + O(\epsilon^3)\quad . \tag{84}$$

If, for instance, the third fixed point is the stable one,

$$\alpha_1 < 1\ ,\quad \alpha_2 < 1\ ,\quad (2-\alpha_3) < 1\quad .$$

In the interval $[0, 1]$, $\eta(\alpha)$ is monotonically increasing and the largest η_i corresponds to the largest of the three numbers α_1, α_2, $(2-\alpha_3)$. Inequalities (81) and (82) show that $(2-\alpha_3)$ is the largest one. Therefore the stable fixed point has the largest η.

Trajectories in the plane of the three fixed points

It is easy to obtain the trajectories from Eq. (74). Defining σ by $y = \sigma x$, we find

$$\frac{1}{\epsilon}\,\lambda\,\frac{d\sigma}{d\lambda} = (1-\alpha_2)\,\sigma(\sigma-\sigma_3)\,x(\sigma)\quad , \tag{85}$$

with

$$\sigma_3 \equiv \frac{1-\alpha_1}{1-\alpha_2}\quad ,$$

and

$$\begin{aligned} x(\sigma) &= (1-\alpha_2)^{-1}\,|\sigma|^{-1/(1-\alpha_1)}\,|\sigma-\sigma_3|^{-1/(1-\alpha_3)}\,C(\sigma,\tau)\ , \\ C(\sigma,\tau) &= -\int_\tau^\sigma d\sigma'\,|\sigma'|^{1/(1-\alpha_1)-1} \\ &\quad\times|\sigma'-\sigma_3|^{1/(1-\alpha_3)-1}\,\mathrm{sgn}(\sigma'(\sigma'-\sigma_3))\ . \end{aligned} \tag{86}$$

With the help of these formulas, one can determine the boundary $\bar{\mathfrak{D}}$ of $\mathfrak{D}$, domain of ultraviolet attraction of the origin. The result is depicted in Fig. 1. The infrared domain of attraction of the nontrivial fixed point 3 (chosen as the stable one) is limited by $\bar{\mathfrak{D}}$ and $x > 0$, $y > 0$, and is contained in the domain $\mathfrak{D}^+$ of positivity of the bare interaction.

At this order, one can calculate the bare coupling constant in terms of the renormalized ones: if

$$\mu^{-\epsilon} g^0_{ijkl} = x_0\,U^*_s\,S_{ijkl} + y_0\,u^*_{ijkl}\quad , \tag{87}$$

integration of the renormalization group equations gives

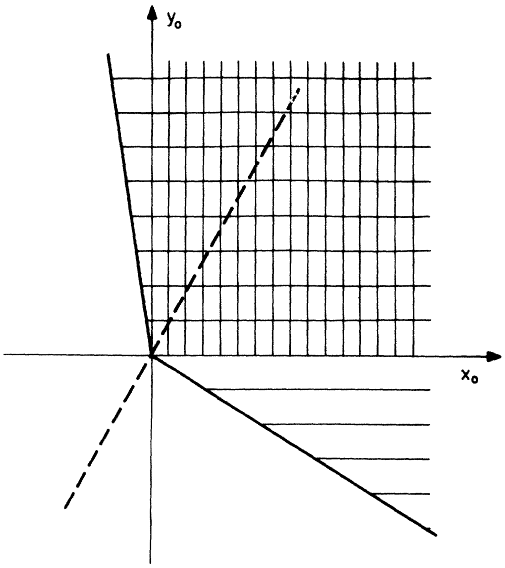


FIG. 2. Plane of the bare coupling constants. $\mathfrak{D}^+$ and $\mathfrak{D}^*$ are depicted with the conventions used in Fig. 1.

$$
\begin{aligned}
x_0(\sigma,\tau) &= x(\sigma,\tau)\left|\frac{\sigma}{\tau}\right|^{1/(1-\alpha_1)}\left|\frac{\sigma-\sigma_3}{\tau-\sigma_3}\right|^{1/(1-\alpha_3)}\ , \\
y_0(\sigma,\tau) &= \tau x_0(\sigma,\tau)\ .
\end{aligned}
\tag{88}
$$

This shows that the domain of attraction of the infrared fixed points in the plane (x_0, y_0) is smaller than the domain $\mathfrak{D}^+$, as depicted in Fig. 2.

This is, in some sense, a restriction to universality: the fixed points depend not only on the number of components of the symmetry group of the interaction, but also on the initial values of the coupling constants. When they are not chosen in the domain of attraction of the infrared fixed point, one cannot obtain the behavior of the correlation functions from perturbation theory.

C. Example

Finally, in order to illustrate the general features discussed above, we shall study a case which generalizes a model studied in Refs. 3 and 4.

The order parameters consist of p vectors, each one having q components:

$$\vec{\varphi}_i = (\varphi_i^1, \ldots, \varphi_i^q)\ ,\quad i = 1, \ldots, p\ . \tag{89}$$

The interaction is

$$\mathcal{H}_{\text{int}}(x) = \frac{\mu^\epsilon}{4!}\left(u_1\sum_{i=1}^{p}(\vec{\varphi}_i^{\,2})^2 + u_2\sum_{i\neq j}\vec{\varphi}_i^{\,2}\vec{\varphi}_j^{\,2}\right) + \text{counter terms.}$$

It satisfies the trace condition

$$u_{ijkk} = U\delta_{ij}\ ,\quad U = \frac{q+2}{3}u_1 + \frac{q(p-1)}{3}u_2\ . \tag{90}$$

More generally it has only one quadratic invariant, namely $\sum_{i=1}^{p}\vec{\varphi}_i^{\,2}$.

At lowest nontrivial order the two β functions are

$$
\begin{aligned}
\beta_1 &= -\epsilon u_1 + \frac{q+8}{6}u_1^2\ \frac{q(p-1)}{6}u_2^2\ , \\
\beta_2 &= -\epsilon u_2 + \frac{q+2}{3}u_1u_2 + \frac{1}{6}[4+q(p-2)]u_2^2\ .
\end{aligned}
\tag{91}
$$

This leads to three different nontrivial fixed points of order ϵ:

(i) the $\mathrm{O}(n)$-symmetric one:

$$u_1^* = u_2^* = \frac{6\epsilon}{n+8}\quad (n\equiv pq)\ ; \tag{92}$$

(ii) the decoupled one:

$$u_1^* = \frac{6\epsilon}{q+8}\ ,\quad u_2^* = 0\ ; \tag{93}$$

(iii) a "mixed" one

$$
\begin{aligned}
u_1^* &= \frac{6q(p-1)\epsilon}{pq(q+8)-16(q-1)}\ , \\
u_2^* &= \frac{6(4-q)\epsilon}{pq(q+8)-16(q-1)}\ ,
\end{aligned}
\tag{94}
$$

where the denominator is always positive when $p, q \geq 1$.

The B matrix for an arbitrary fixed point is

$$B = \begin{bmatrix} -\epsilon + \dfrac{q+8}{3}u_1^* & \dfrac{q(p-1)}{3}u_2^* \\ \dfrac{q+2}{3}u_2^* & -\epsilon + \dfrac{q+2}{3}u_1^* + \dfrac{1}{3}[4+q(p-2)]u_2^* \end{bmatrix}. \tag{95}$$

We are now in position to study the stability of the three fixed points:

(i) The symmetric one is stable, at this order in ϵ, if and only if

$$n = pq < 4\ ,$$

in agreement with the general proof given above.

(ii) The decoupled one is stable when

$$q > 4\ .$$

(iii) The third one is stable in the complementary domain

$$4 < n < 4p\ .$$

Notice that for any value of p and q, there is effectively one and only one stable fixed point (except perhaps for the limiting cases $n=4$ or $q=4$).

At this order, the critical exponent η is related directly to the trace U^*, since from Eq. (54),

$$\eta = \tfrac{1}{24}[\epsilon U^* - \tfrac{1}{2}(U^*)^2] + O(\epsilon^3)\ . \tag{96}$$

The values for η are for the three fixed points, respectively:

$$\eta_{\mathrm{sym}}=\frac{n+2}{2(n+8)^2}\epsilon^2 \;;$$

$$\eta_{\mathrm{decoupled}}=\frac{q+2}{2(q+8)^2}\epsilon^2 \;; \tag{97}$$

$$\eta_{\mathrm{mixed}}=\frac{q(p-1)\,[pq(q+2)-10q+16]}{2[pq(q+8)-16(q-1)]^2}\epsilon^2 \;.$$

These values of η can be continued for arbitrary p and q, which are not necessarily in the domain of stability of the corresponding fixed points. As shown above, for any value of p and q the stable fixed point has the largest value for η.

This leads to the intriguing conjecture that the stable fixed point could in general correspond to the largest value of η.

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