Theory of layered Ising models: Thermodynamics

Helen Au-Yang
Baker Laboratory, Cornell University, Ithaca, New York 14850

Barry M. McCoy*†
Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts 02138‡

(Received 10 January 1974)

We consider a two-dimensional Ising model whose vertical interaction energies $E_2(j)$ between row j and row $j+1$ are allowed to be arbitrary for $1 \le j \le n$. This set of bonds is then repeated indefinitely to make up an infinite lattice. For any set of the n energies $E_2(j)$ we show that the specific heat has a logarithmic divergence as $T \to T_c$ and we derive an explicit formula for the amplitude of $\ln|1 - T/T_c|$. From this result we demonstrate that if the $E_2(j)$ are considered to be independent random variables with a distribution function $P(E_2)$ which is not a δ function, then, for almost all lattices constructed from $P(E_2)$, the amplitude of the logarithmic singularity vanishes as $n \to \infty$.

I. INTRODUCTION

Several years ago McCoy and Wu[1] (MW I) studied the influence of frozen-in random impurities on ferromagnetic phase transitions by introducing a two-dimensional Ising model specified by the interaction energy

$$\mathcal{E} = -E_1 \sum_{j=1}^{\mathfrak{M}} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{j,k}\sigma_{j,k+1} - \sum_{j=1}^{\mathfrak{M}-1} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} E_2(j)\sigma_{j,k}\sigma_{j+1,k} , \tag{1.1}$$

where $E_2(j)$ are taken to be independent random variables with the probability distribution function $P(E_2)$, and the first (second) index on σ specifies the row (column) of the lattice. It was found that[2] if $P(E_2)$ is sharply peaked then the specific heat [which is the same for almost all lattices constructed from $P(E_2)$] possesses [at least to leading order in the width of $P(E_2)$] an infinitely differentiable essential singularity at the critical temperature T_c, where T_c is determined as the solution of

$$\int_{-\infty}^{\infty} dE_2\, P(E_2) \ln|\tanh\beta_c E_2| = -2\beta_c |E_1| . \tag{1.2}$$

Later[3] this behavior at T_c was shown to hold for any arbitrary $P(E_2)$ (not equal to a δ function).

This very smooth behavior of the specific heat near T_c is qualitatively different from the behavior of the specific heat of Onsager's lattice[4] [$E_2(j) = E_2$ independent of j], which near T_c behaves as

$$c/k = -\beta_c^2 2\pi^{-1}[E_1^2 \sinh 2\beta_c E_2 + 2E_1E_2 + E_2^2 \sinh 2\beta_c E_1] \ln|1 - T/T_c| + O(1), \tag{1.3}$$

and because the difference between the impure and the pure lattice is so striking, it seems worthwhile to study the relation between these two extreme cases in detail. To do this, we investigate in this series of papers the properties of nth-order layered Ising models[5] which are specified by

$$\mathcal{E} = -E_1 \sum_{j=1}^{n\mathfrak{M}+1} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{j,k}\sigma_{j,k+1} - \sum_{j=0}^{\mathfrak{M}-1} \sum_{l=1}^{n} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} E_2(l)\sigma_{nj+l,k}\sigma_{nj+l+1,k} . \tag{1.4}$$

Loosely speaking, this layered Ising model consists of $\mathfrak{M}$ strips, each one of which is made up of n layers. The n constants $E_2(j)$ between the n layers are chosen at will, but once they are specified the basic strip is repeated $\mathfrak{M}$ times, where the bonds $E_2(n)$ of the nth row of one strip connect with the first row of the next strip. We are of course interested in the thermodynamic limit $\mathfrak{M} \to \infty$ and $\mathfrak{N} \to \infty$.

In this paper we study the thermodynamics of nth-order layered Ising models. In Sec. II we derive a general formula for the free energy for arbitrary n. This is, as expected, not a very transparent formula and therefore in Sec. III we concentrate on that part of the specific heat that diverges as $T \to T_c$. We find for any finite n and any set $\{E_2(j)\}$ that when $T \to T_c$

$$c/k = -A(n; \{E_2\}) \ln|1 - T/T_c| + O(1) , \tag{1.5}$$

where $A(n; \{E_2\})$ is explicitly given by (3.11).

We find from this explicit expression that if n is large then $A(n; \{E_2\})$ depends strongly on the spacial arrangement of the energies $E_2(j)$ in the sense that a mere permutation of the $E_2(j)$ can cause $A(n; \{E_2\})$ to vary from being on the order of 1 to being exponentially small in n. We conclude by demonstrating that if the energies $\{E_2\}$ are treated as independent random variables with the probability distribution function $P(E_2)$ (not equal to a δ function) then as $n\to\infty$, $A(n;\{E_2\})$ will vanish for almost all lattices constructed from $P(E_2)$.

II. FREE ENERGY

The lattice specified by (1.4) is taken to have cyclic boundary conditions in the horizontal direction and free boundary conditions in the vertical direction. With these boundary conditions the derivation presented in MW I for the partition function of the random lattice specified by (1.1) may be carried over word for word to the n-layer model specified by (1.4). Calling Z_n the partition function for the n-layer model we find

$$Z_n^2=(2\cosh\beta E_1)^{4(n\mathfrak{M}+1)\mathfrak{N}}\left(\prod_{j=1}^{n}\cosh E_2(j)\right)^{4\mathfrak{M}\mathfrak{N}}\times\prod_\theta\left(|1+z_1e^{i\theta}|^{2(n\mathfrak{M}+1)}\det C^{(n)}(\theta)\right), \tag{2.1}$$

where $\prod_\theta$ is the product over

$$\theta=\pi(2l-1)/2\mathfrak{N},\quad l=1,2,\ldots,2\mathfrak{N} \tag{2.2}$$

and we use the notation

$$z_1=\tanh\beta E_1, \tag{2.3a}$$

$$z_2(j)=\tanh\beta E_2(j), \tag{2.3b}$$

with $\beta=(kT)^{-1}$. The $2(n\mathfrak{M}+1)\times2(n\mathfrak{M}+1)$ matrix $C^{(n)}(\theta)$ is defined by

$$C_{j,j}^{(n)}=\begin{bmatrix} ia & b\\ -b & ia\end{bmatrix}, \tag{2.4a}$$

$$C_{j,j+1}^{(n)}=-C_{j+1,j}^{(n)T}=\begin{bmatrix}0&0\\ z_2(j)&0\end{bmatrix}, \tag{2.4b}$$

with all other $C_{j,j'}^{(n)}$ zero. Here we use the further notation

$$a=-2z_1\sin\theta|1+z_1e^{i\theta}|^{-2}, \tag{2.5a}$$

$$b=(1-z_1^2)|1+z_1e^{i\theta}|^{-2}. \tag{2.5b}$$

The free energy per spin F_n is then given by

$$\begin{aligned}-\beta F_n&=\lim_{\substack{\mathfrak{M}\to\infty,\\ \mathfrak{N}\to\infty}}[4\mathfrak{N}(n\mathfrak{M}+1)]^{-1}\ln Z_n^2\\ &=\ln(2\cosh\beta E_1)+n^{-1}\sum_{l=1}^{n}\ln\cosh E_2(l)\\ &\quad+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\left(\ln|1+z_1e^{i\theta}|^2+\lim_{\mathfrak{M}\to\infty}(n\mathfrak{M}+1)^{-1}\ln\det C^{(n)}(\theta)\right).\end{aligned} \tag{2.6}$$

Following MW I we define $C_l^{(n)}$ to be the determinant of the $2l\times 2l$ matrix whose nonzero elements are given by (2.4) and define $iD_l^{(n)}$ to be the $(2l-1)\times(2l-1)$ determinant obtained from $C_l^{(n)}$ by omitting the last row and column. Then

$$\det C^{(n)}(\theta)=C_{n\mathfrak{M}+1}^{(n)}(\theta) \tag{2.7}$$

and we have the recurrence relation for $l\ge 1$

$$\begin{bmatrix}C_{l+1}^{(n)}(\theta)\\ D_{l+1}^{(n)}(\theta)\end{bmatrix}=A(l)\begin{bmatrix}C_l^{(n)}(\theta)\\ D_l^{(n)}(\theta)\end{bmatrix}, \tag{2.8}$$

where

$$A(l)=\begin{bmatrix}a^2+b^2 & a\\ a & 1\end{bmatrix}\begin{bmatrix}1&0\\0&z_2^2(l)\end{bmatrix}=\begin{bmatrix}a^2+b^2 & az_2^2(l)\\ a & z_2^2(l)\end{bmatrix}, \tag{2.9}$$

with the initial condition

$$C_1^{(n)}(\theta)=a^2+b^2, \tag{2.10a}$$

$$D_1^{(n)}(\theta)=a. \tag{2.10b}$$

We now make use of the periodicity of the $z_2^2(l)$ by iterating (2.8) $n-1$ times. Therefore

$$\begin{bmatrix}C_{l+n}^{(n)}(\theta)\\ D_{l+n}^{(n)}(\theta)\end{bmatrix}=T(n)\begin{bmatrix}C_l^{(n)}(\theta)\\ D_l^{(n)}(\theta)\end{bmatrix}, \tag{2.11}$$

where

$$T(n)=A(n)A(n-1)\cdots A(1). \tag{2.12}$$

Note that $T(n)$ depends on n only and not on l. Therefore we may compute F_n by precisely the same method used[6] for the case $n=1$.

Let the two eigenvectors of $T(n)$ be $\vec{v}_n$ and $\vec{v}_n'$ and let the corresponding eigenvalues be λ_n and λ_n', where we define $|\lambda_n|\ge|\lambda_n'|$. In general, since $T(n)$ is not Hermitian, $\vec{v}_n$ and $\vec{v}_n'$ will not be orthogonal. We use the normalization condition

$$\vec{v}_n^2=[\vec{v}_n'-(\vec{v}_n\cdot\vec{v}_n')\vec{v}_n]^2=1. \tag{2.13}$$

Then for any vector $\vec{v}^0$

$$T^{\mathfrak{M}}(n)\vec{v}^0=\lambda_n^{\mathfrak{M}}(\vec{v}^0\cdot\vec{v}_n)\vec{v}_n+\vec{v}^0\cdot[\vec{v}'-(\vec{v}_n\cdot\vec{v}_n')\vec{v}_n]\times[\lambda_n'^{\mathfrak{M}}\vec{v}_n'-(\vec{v}_n\cdot\vec{v}_n')\lambda_n^{\mathfrak{M}}\vec{v}_n]. \tag{2.14}$$

Moreover, the eigenvalue λ_n is explicitly obtained in terms of $T(n)$ as

$$\lambda_n=\tfrac{1}{2}(\mathrm{Tr}T(n)+\{[\mathrm{Tr}T(n)]^2-4\det T(n)\}^{1/2}). \tag{2.15}$$

Substituting (2.7), (2.14), and (2.15) into (2.6) with $\vec{v}^0$ given by (2.10) we obtain the explicit result

$$-\beta F_n=\ln(2\cosh\beta E_1)+n^{-1}\times\sum_{l=1}^{n}\ln\cosh\beta E_2(l)+(4\pi n)^{-1}\int_{-\pi}^{\pi}d\theta\times\ln[\tfrac{1}{2}|1+z_1e^{i\theta}|^{2n}(\mathrm{Tr}T(n)+\{[\mathrm{Tr}T(n)]^2-4\det T(n)\}^{1/2})]. \tag{2.16}$$

The quantity $\det T(n)$ is easily evaluated from (2.9) and (2.12) as

$$\det T(n)=\prod_{l=1}^{n}\det A(l)=b^{2n}\prod_{l=1}^{n}z_2^2(l)\ . \qquad (2.17)$$

The only difficulty in explicitly evaluating (2.16), therefore, lies in the evaluation of $\mathrm{Tr}T(n)$. For small values of n the trace may readily be computed and we find

$$\mathrm{Tr}T(1)=a^2+b^2+z_2^2(1)\ , \qquad (2.18a)$$

$$\mathrm{Tr}T(2)=(a^2+b^2)^2+a^2[z_2^2(1)+z_2^2(2)]+z_2^2(1)z_2^2(2)\ , \qquad (2.18b)$$

$$\mathrm{Tr}T(3)=(a^2+b^2)^3+(a^2+b^2)a^2[z_2^2(1)+z_2^2(2)+z_2^2(3)]+a^2[z_2^2(1)z_2^2(2)+z_2^2(2)z_2^2(3)+z_2^2(3)z_2^2(1)]+z_2^2(1)z_2^2(2)z_2^2(3)\ , \qquad (2.18c)$$

$$\mathrm{Tr}T(4)=(a^2+b^2)^4+(a^2+b^2)^2a^2\sum_{l=1}^{4}z_2^2(l)+(a^2+b^2)a^2\sum_{l=1}^{4}z_2^2(l)z_2^2(l+1)+a^2$$
$$\times\sum_{l=1}^{4}z_2^2(l)z_2^2(l+1)z_2^2(l+2)+\prod_{l=1}^{n}z_2^2(l)+a^4[z_2^2(2)z_2^2(4)+z_2^2(1)z_2^2(3)]\ . \qquad (2.18d)$$

In (2.18d) recall that $z_2^2(l+n)=z_2^2(l)$.

When $a=0$

$$\mathrm{Tr}T(n)=b^{2n}(0)+\prod_{l=1}^{n}z_2^2(l)$$
$$=[(1-z_1)/(1+z_1)]^{2n}+\prod_{l=1}^{n}z_2^2(l)\ . \qquad (2.19)$$

Therefore at $a=0$ the argument of the square root in (2.16) is

$$\left(b^{2n}(0)-\prod_{l=1}^{n}z_2^2(l)\right)^2\ . \qquad (2.20)$$

The vanishing of this expression is the condition for $T=T_c$.

When E_1 and all $E_2(j)$ are non-negative, it may be verified that if (2.20) does not vanish then the argument of the square root in (2.16) never vanishes for $0\le\theta\le 2\pi$. Therefore we conclude that if $T\ne T_c$, then F_n is an analytic function of T for all positive T.

III. SINGULARITY IN THE SPECIFIC HEAT AS $T\to T_c$

The free energies obtained by substituting (2.17) and (2.18) into (2.16), while explicit, are awkward to handle and, moreover, as n increases they become increasingly cumbersome. Therefore, we will restrict our attention to that term in the specific heat which diverges as $T\to T_c$. For this purpose we only need to keep terms in $\mathrm{Tr}T(n)$ which are either quadratic in a or independent of a as $a\to 0$ [we restrict our attention to the ferromagnetic case $E_1\ge 0$, $E_2(j)\ge 0$].

We may easily compute $\mathrm{Tr}T(n)$ correct to order a^2 if we note that the following expressions for $T_{ij}(n)$ satisfy the recursion relation

$$T(n+1)=\begin{bmatrix}a^2+b^2 & z_2^2(n+1)\\ a & z_2^2(n+1)\end{bmatrix}T(n) \qquad (3.1)$$

to order a^2 and reduce correctly to $A(1)$ when $n=1$:

$$T_{11}(n)=(a^2+b^2)^n+a^2\sum_{m=0}^{n-2}(a^2+b^2)^{n-2-m}$$
$$\times\sum_{l=2+m}^{n}\prod_{j=0}^{m}z_2^2(l-j)\ , \qquad (3.2a)$$

$$T_{21}(n)=a\left((a^2+b^2)^{n-1}+\sum_{m=0}^{n-2}(a^2+b^2)^{n-2-m}\right.$$
$$\left.\times\prod_{j=0}^{m}z_2^2(n-j)\right)\ , \qquad (3.2b)$$

$$T_{12}(n)=a\sum_{l=1}^{n}(a^2+b^2)^{n-l}\prod_{j=1}^{l}z_2^2(j)\ , \qquad (3.2c)$$

$$T_{22}(n)=\prod_{l=1}^{n}z_2^2(l)+a^2\sum_{m=0}^{n-2}(a^2+b^2)^{n-2-m}$$
$$\times\sum_{l=0}^{m}\prod_{j=1}^{m+1}z_2^2(j-l)\ , \qquad (3.2d)$$

where by definition $z_2^2(-l)=z_2^2(n-l)$. Thus we obtain to order a^2

$$\mathrm{Tr}T(n)=(a^2+b^2)^n+\prod_{l=1}^{n}z_2^2(l)+a^2$$
$$\times\sum_{m=0}^{n-2}(a^2+b^2)^{n-2-m}\sum_{l=1}^{n}\prod_{j=0}^{m}z_2^2(l+j)\ . \qquad (3.3)$$

Note in particular that (3.3) reduces, correct to order a^2, to (2.18) when $n=1, 2, 3$, or 4.

Call $c_n^{(S)}$ that part of the specific heat (at zero magnetic field) which diverges as $T\to T_c$. Then, defining $\doteq$ to mean that the terms which diverges as $T\to T_c$ are the same on both sides, we substitute (2.17) and (3.3) into (2.16) to find

$$\frac{c_n^{(S)}}{k} \doteq \beta_c^2 \frac{\partial^2}{\partial\beta^2}(-\beta F_n) \doteq (4\pi n)^{-1}\beta_c^2 \frac{\partial^2}{\partial\beta^2}\int_{-\epsilon}^{\epsilon} d\theta \ln b^{2n} \times\left(1+\left\{\frac{1}{4}\left[1-\prod_{l=1}^{n}\left(\frac{1+z_1}{1-z_1}\right)^2 z_2^2(l)\right]^2+\theta^2 A_2^2\right\}^{1/2}\right), \tag{3.4}$$

where

$$A_2^2 = 4z_{1c}^2(1-z_{1c}^2)^{-2}\left[n+\sum_{m=0}^{n-2}\sum_{l=1}^{n}\prod_{j=0}^{m}\left(\frac{1+z_{1c}}{1-z_{1c}}\right)^2 z_{2c}^2(l+j)\right]$$
$$=4z_{1c}^2(1-z_{1c}^2)^{-2}\sum_{m=0}^{n-1}\sum_{l=1}^{n}\prod_{j=0}^{m}\left(\frac{1+z_{1c}}{1-z_{1c}}\right)^2 z_{2c}^2(l+j), \tag{3.5}$$

where the subscript c means $T=T_c$ and in the last line of (3.5) the T_c condition

$$\left(\frac{1-z_{1c}}{1+z_{1c}}\right)^{2n}=\prod_{l=1}^{n} z_{2c}^2(l) \tag{3.6}$$

has been used. The first term in the argument of the square root in (3.4) may be expanded near T_c as

$$1-\prod_{l=1}^{n}\left(\frac{1+z_1}{1-z_1}\right)^2 z_2^2(l) \sim -2(\beta-\beta_c)\left\{2nE_1+\sum_{l=1}^{n} E_2(l)\left[z_{2c}^{-1}(l)-z_{2c}(l)\right]\right\}$$
$$=-2(\beta-\beta_c)A_1, \tag{3.7}$$

where the last line defines A_1. Therefore

$$\frac{c_n^{(S)}}{k} \doteq (4\pi n)^{-1}\beta_c^2\frac{\partial^2}{\partial\beta^2}\int_{-\epsilon}^{\epsilon} d\theta \ln \times\{1+[(\beta-\beta_c)^2A_1^2+\theta^2A_2^2]^{1/2}\}$$
$$\doteq (2\pi n)^{-1}\beta_c^2\frac{\partial}{\partial\beta}\int_0^{\epsilon} d\theta\,(\beta-\beta_c)A_1^2 \times\{(\beta-\beta_c)^2A_1^2+\theta^2A_2^2\}^{-1/2}. \tag{3.8}$$

As $\beta\to\beta_c$

$$\int_0^{\epsilon} d\theta\,[(\beta-\beta_c)^2A_1^2+\theta^2A_2^2]^{-1/2} = A_2^{-1}\ln|\beta-\beta_c|+O(1). \tag{3.9}$$

Thus we have the desired result that as $T\to T_c$

$$c_n/k = -A(n;\{E_2\})\ln|1-T/T_c|+O(1), \tag{3.10}$$

where

$$A(n;\{E_2\})=\beta_c^2(2\pi n)^{-1}A_1^2A_2^{-1} = \beta_c^2(4\pi n)^{-1}(z_{1c}^{-1}-z_{1c})\left(2nE_1+\sum_{l=1}^{n}E_2(l)\times[z_{2c}^{-1}(l)-z_{2c}(l)]\right)^2 B^{-1}(n;\{E_2\}), \tag{3.11}$$

with

$$B^2(n;\{E_2\})=\sum_{m=0}^{n-1}\sum_{l=1}^{n}\prod_{j=0}^{m}\left(\frac{1+z_{1c}}{1-z_{1c}}\right)^2 z_{2c}^2(l+j). \tag{3.12}$$

There are several special cases of (3.12) which should be noted:

If $n=1$, then $B(1)=1$ and, calling $E_2(1)=E_2$, we have

$$A(1;E_2)=\beta_2^2(4\pi)^{-1}(z_{1c}^{-1}-z_{1c})\{2E_1+E_2[z_{2c}^{-1}-z_{2c}]\}^2 = \beta_2^2 2\pi^{-1}\{E_1^2\sinh 2\beta_cE_2+2E_1E_2 + E_2^2\sinh 2\beta_cE_1\}, \tag{3.13}$$

which agrees with Onsager's result (1.3). Similarly, if $E_2(j)=E_2$ for all j, then

$$B^2(n)=n^2 \tag{3.14}$$

and $A(n;\{E_2\})$ reduces to the right-hand side of (3.13) independent of n.

If $n=2$,

$$B^2(2)=[z_{2c}(1)z_{2c}(2)]^{-1}[z_{2c}(1)+z_{2c}(2)]^2 = (1+z_{1c})^2(1-z_{1c})^{-2}[z_{2c}(1)+z_{2c}(2)]^2, \tag{3.15}$$

$$A(2;E_2(1),E_2(2))=\beta_c^2(8\pi)^{-1}(1-z_{1c})^2z_{1c}^{-1} \times[z_{2c}(1)+z_{2c}(2)]^{-1}\{4E_1+E_2(1) \times[z_{2c}^{-1}(1)-z_{2c}(1)]+E_2(2)[z_{2c}^{-1}(2)-z_{2c}(2)]\}^2. \tag{3.16}$$

In particular we note that as $x=E_2(1)/E_2(2)\to 0$

$$A(2;E_2(1),E_2(2))\sim\beta_c^2\pi^{-1}\sinh 2\beta_cE_{2\infty} \times(2E_1+E_{2\infty}\sinh 4\beta_cE_1+2x^{-1}e^{-2\beta_cE_{2\infty}/x})^2, \tag{3.17}$$

where $E_{2\infty}$ is determined from β_c and E_1 by

$$\sinh 2\beta_cE_{2\infty}=1/\sinh 4\beta_cE_1. \tag{3.18}$$

A much more interesting special case is obtained if we consider n to be even and let $\frac{1}{2}n$ of the $E_2(j)$ take on the value $E_2^{(1)}$ while the other $\frac{1}{2}n$ $E_2(j)$ have the value $E_2^{(2)}$. The amplitude $A(n;\{E_2\})$ will still depend on the spacial arrangement of $E_2^{(1)}$ and $E_2^{(2)}$. To illustrate the possibilities we consider the following two extreme cases:

(i)

$$E_2(2l)=E_2^{(1)}$$
$$E_2(2l-1)=E_2^{(2)},\quad 1\le l\le\tfrac{1}{2}n \tag{3.19a}$$

(ii)

$$E_2(l)=E_2^{(1)},\quad 1\le l\le\tfrac{1}{2}n$$
$$E_2(l)=E_2^{(2)},\quad \tfrac{1}{2}n+1\le l\le n. \tag{3.19b}$$

Case (i) is, of course, the same as the case $n=2$ considered above and it is easily verified that when (3.19a) holds (3.11) reduces to (3.16).

Case (ii) is, however, drastically different. If we call

$$(1+z_{1c})^2(1-z_{1c})^{-2}z_{2c}^{(1)2}=z_{2c}^{(1)}/z_{2c}^{(2)}=r, \tag{3.20}$$

then a bit of algebra shows that

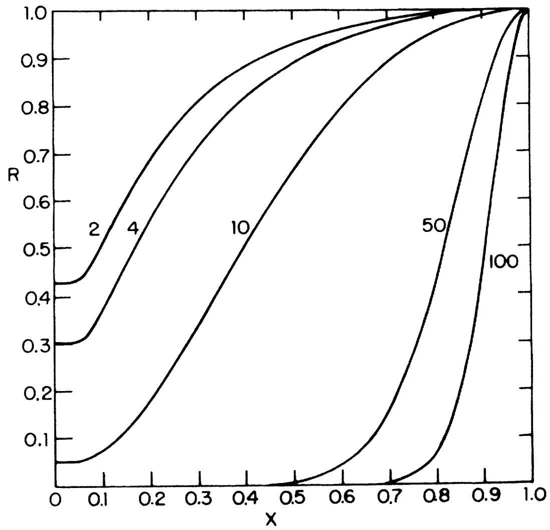

FIG. 1. Plot of the ratio $R(n;\ E_2^{(1)},\ E_2^{(2)})=A(n;\ E_2^{(1)},\ E_2^{(2)})/A(1;\ E_1,\ E_1)$ for fixed E_1 and β_c and for $n=2$, 4, 10, 50, and 100 as a function of $x=E_2^{(1)}/E_2^{(2)}$, where the spacial configuration of bonds is that of case (ii) of the text.

$$B^2(n)=(r+1)(r^{-1}+1)+r\left(\frac{1+r}{1-r}\right)^2(r^{n/2-1}-1)$$
$$+r^{-1}\left(\frac{1+r^{-1}}{1-r^{-1}}\right)^2(r^{-(n/2-1)}-1)\ . \qquad (3.21)$$

Thus for case (ii)

$$A=\beta_c^2(4\pi)^{-1}n(z_{1c}^{-1}-z_{1c})\{2E_1+E_2^{(1)}\ \mathrm{csch}2\beta_c E_2^{(1)}$$
$$+E_2^{(2)}\ \mathrm{csch}\beta_c E_2^{(2)}\}^2 B^{-1}(n). \qquad (3.22)$$

If $E_2^{(1)}=E_2^{(2)}$, then (3.22) reduces to (3.13); if $n=2$, then (3.22) reduces to (3.16); and if $n=4$,

$$A=\beta_c^2\pi^{-1}(z_{1c}^{-1}-z_{1c})z_{2c}^{(1)}z_{2c}^{(2)}[z_{2c}^{(1)}+z_{2c}^{(2)}]^{-2}$$
$$\times\{2E_1+E_2^{(1)}\ \mathrm{csch}2\beta_c E_2^{(1)}+E_2^{(2)}\ \mathrm{csch}2\beta_c E_2^{(2)}\}^2\ . \qquad (3.23)$$

In Fig. 1 we plot for case (ii) the ratio

$$R(n;\ E_2^{(1)},\ E_2^{(2)})=A(n;\ E_2^{(1)};\ E_2^{(2)})/A(n;\ E_1,\ E_1) \qquad (3.24)$$

as a function of $x=E_2^{(1)}/E_2^{(2)}$ for E_1 and β_c fixed and various values of n.

When $n\to\infty$ and $r\neq 1$, (3.21) and (3.22) show that the coefficient of $-\ln|1-T/T_c|$ vanishes exponentially rapidly. The physical reason for the vanishing of this amplitude must be related to the fact that locally (i.e., in the interior of the strips which have $n/2$ bonds of strength $E_2^{(1)}$ or $E_2^{(2)}$) the lattice is not at T_c. It is only globally that the lattice is at T_c. As n lets larger and larger, the local effect tends to dominate more and more over the global effect and hence the singularity in the specific heat is increasingly depressed.

IV. $n\to\infty$ RANDOM LIMIT

The most interesting aspect of the amplitude (3.11) is its behavior when $n\to\infty$ and the $E_2(j)$ are treated as independent random variables with the probability distribution $P(E_2)$. To examine the behavior of $A(n;\ \{E_2\})$ we first study the behavior of $B^2(n;\ \{E_2\})$ by using the T_c condition (3.6) to write

$$B^2(n;\ \{E_2\})=\sum_{m=0}^{n-1}\sum_{l=1}^{n}\left(\prod_{j=1}^{m} z_{2c}^{2[1-(m+1)/n]}(l+j)\right)$$
$$\times\left(\prod_{j=m+1}^{n-1} z_{2c}^{-2(m+1)/n}(l+j)\right)\ . \qquad (4.1)$$

We now consider $A(n;\ \{E_2\})$ and hence $B^2(n;\ \{E_2\})$ as functions of the $n+1$ variables E_1 and $\{E_2\}$. Only n of these energies are independent because T_c is fixed. Therefore we will consider the $E_2(j)$ as independent and E_1 as the dependent variable. Then, since (4.1) does not contain E_1, we may average each $E_2(j)$ independently over $P(E_2)$ and obtain

$$\langle B^2(n)\rangle_{E_2}=n\sum_{m=0}^{n-1}\langle z_2^{2[1-(m+1)/n]}\rangle_{E_2}^{m+1}$$
$$\times\langle z_2^{-2(m+1)/n}\rangle_{E_2}^{n-m-1}\ . \qquad (4.2)$$

Furthermore, we write the summand as

$$\langle z_2^{2[1-(m+1)/n]}\rangle_{E_2}^{m+1}\langle z_2^{-2(m+1)/n}\rangle_{E_2}^{n-m-1}=Q^n(n,\ m)\ , \qquad (4.3)$$

where, calling

$$z_2^2=\lambda \qquad (4.4a)$$

and

$$dE_2\,P(E_2)=d\lambda\,p(\lambda)\ , \qquad (4.4b)$$

$$Q(n,\ m)=\left[\int_0^1 d\lambda\,p(\lambda)\lambda^{1-(m+1)/n}\right]^{(m+1)/n}$$
$$\times\left[\int_0^1 d\lambda\,p(\lambda)\lambda^{-(m+1)/n}\right]^{1-(m+1)/n}\ . \qquad (4.5)$$

Then if we use Hölders inequality (for positive f and g)

$$\int_0^1 d\lambda f(\lambda)g(\lambda)\le\left[\int_0^1 d\lambda f^p(\lambda)\right]^{1/p}\left[\int_0^1 d\lambda\, g^q(\lambda)\right]^{1/q}, \qquad (4.6)$$

with

$$\frac{1}{p}=\frac{m+1}{n},\quad \frac{1}{q}=\frac{n-m-1}{n}\ ,$$
$$f^p(\lambda)=p(\lambda)\lambda^{1/q} \qquad (4.7a)$$

and

$$g^q(\lambda)=p(\lambda)\lambda^{-1/p}\ , \qquad (4.7b)$$

we find that

$$Q(m,\ n)\ge 1\ . \qquad (4.8)$$

Moreover, if we consider terms in the sum (4.2) where m/n approaches a limit different from 0 or 1 as $n\to\infty$ and if $p(\lambda)$ is not a δ function then

$$\lim_{n\to\infty} Q(m,n) > 1 . \tag{4.9}$$

Therefore we conclude that if $P(E_2)$ is not a δ function then $\langle B^2(n)\rangle_{E_2}$ diverges exponentially rapidly as $n\to\infty$.

Since $n^{-2}\langle B^2(n)\rangle_{E_2}\to\infty$ as $n\to\infty$, we know that, for at least a finite fraction of the lattices constructed with $P(E_2)$, $A(n,\{E_2\})\to 0$. But we recall[7] that in the random limit the specific heat is the same for almost all sets $\{E_2\}$. Therefore we conclude that for almost all sets $\{E_2\}$, $\lim_{n\to\infty} A(n,\{E_2\})=0$.

We may now make contact with the work of MW I. In that paper the distribution

$$p(\lambda)=N\lambda_0^{-N}\lambda^{N-1} \tag{4.10}$$

was considered and

$$N\gg 1 . \tag{4.11}$$

For (4.10)

$$\langle B^2(n)\rangle_{E_2}=n\sum_{m=0}^{n-1}\left(\frac{N}{N+(n-m-1)/n}\right)^{m+1} \times\left(\frac{N}{N-(m+1)/n}\right)^{n-m-1} . \tag{4.12}$$

The largest term in the sum is at $m=\frac{1}{2}n-1$. Therefore

$$n[1-(2N)^{-2}]^{-n/2}<\langle B^2(n)\rangle_{E_2}<n^2[1-(2N)^{-2}]^{n/2} , \tag{4.13}$$

so, in order for $\langle B^2(n)\rangle_{E_2}$ to be much larger than n^2 we need

$$n\gg N^2 . \tag{4.14}$$

Since N^{-1} is the width of the distribution function $p(\lambda)$, (4.13) says that for $n^{-2}\langle B^2(n)\rangle_{E_2}$ to be large [and hence for $A(n;\{E_2\})$ to be small], the width of the strip (n) must be much larger than the mean-square fluctuation of the bond strengths. For n much smaller than (4.14) the lattice behaves essentially as though it were pure.

When both (4.14) and (4.11) hold, there are three temperature scales:

(i) When

$$|T-T_c|\gg N^{-2} , \tag{4.15}$$

then the work of MW I implies that the specific heat deviates only to order N^{-1} from a pure lattice. For this temperature range we observe a logarithmic divergence in the specific heat with the amplitude given by (1.3).

(ii) When

$$|T-T_c|N^2=O(1) , \tag{4.16}$$

the results of MW I are applicable and we observe a smooth specific heat.

(iii) Finally, where $T-T_c$ becomes so small that

$$-A(n;\{E_2\})\ln|1-T/T_c|\gg\ln N^2 , \tag{4.17}$$

a logarithmic divergence in the specific heat is again observed but now its amplitude is given by (3.11), which is very much smaller than the Onsager amplitude (1.3). Only in this third temperature scale will the geometrical arrangement of the bonds have any appreciable effect on the specific heat.

---

*Alfred P. Sloan Fellow.

†Supported in part by Grant No. H037758 of the National Science Foundation.

‡Permanent address: State University of New York, Stony Brook, New York, 11790.

[1] B. M. McCoy and T. T. Wu, Phys. Rev. 176, 631 (1968). This paper will be referred to as MW I.

[2] In Ref. 1 a particular $P(E_2)$ of narrow width was considered. The more general case of any $P(E_2)$ with a narrow width was treated by B. McCoy, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2.

[3] B. M. McCoy, Phys. Rev. B 2, 795 (1970).

[4] L. Onsager, Phys. Rev. 65, 117 (1944).

[5] Layered Ising models for n finite were first introduced by M. E. Fisher, J. Phys. Soc. Jap. Suppl. 26, 87 (1968).

[6] B. M. McCoy and T. T. Wu, Phys. Rev. 164, 719 (1967).

[7] See Ref. 1. See also R. Griffiths and J. Lebowitz, J. Math. Phys. 9, 1284 (1968).