Cosine and other terms in the Josephson tunneling current

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To second order in perturbation theory the tunneling current between two superconductors can be expressed as follows:

$$I(V,T)=I_{J1}(V,T)\sin\varphi+I_{J2}(V,T)\cos\varphi+I_{\rm qp}(V,T),$$

where V is a constant voltage across the tunneling barrier, T is the temperature, and $\varphi=-2eVt/\hbar+\varphi_0$ is the difference in the phases of the wave functions of the superconductors on each side of the barrier. Numerical evaluations of each of the terms are presented as functions of voltage for several temperatures. For the second term we find a different sign from that found in previous numerical work. When the superconductors on each side of the tunneling barrier are different, structure occurs at a voltage corresponding to the difference in the energy gaps. For the first two terms this structure was previously unrecognized. In addition, it is shown that the term in $\cos\varphi$ has no effect upon rf-induced steps in the time-averaged current-voltage curve for a tunneling junction biased by a voltage source. Finally a relation is discussed between tunneling and other experiments such as far-infrared absorption and acoustic attenuation in superconductors. It is shown that tunneling can be thought of in terms of a slight generalization of the coherence effects which dominate the other kinds of experiments.

I. INTRODUCTION

Josephson[1] first showed that the current through a tunneling junction between superconductors can be written as follows:

$$I(V,T)=I_{J1}(V,T)\sin\varphi+I_{J2}(V,T)\cos\varphi+I_{\rm qp}(V,T), \quad (1)$$

where V is a constant voltage across the junction, T is the temperature, and φ is the difference between the phases of the wave functions of the superconductors on each side of the tunneling barrier. It is emphasized that this equation is valid only when the voltage V is constant. In this case $\varphi=-2eVt/\hbar+\varphi_0$,[2] where e is the magnitude of the electronic charge (a positive number), $\hbar$ is Planck's constant divided by 2π, t is the time, and φ_0 is a constant. The first two terms are often referred to as the sine and cosine terms, respectively. The usual Josephson effect is associated with the sine term which describes the tunneling of paired ground-state electrons. It gives rise to both lossless dc currents at zero voltage and oscillating currents at nonzero voltage. For many years the cosine term was quite correctly approximated as zero for low voltages and temperatures. Recently, however, experimental observation of the cosine term by Pederson, Finnegan, and Langenberg[3] has prompted renewed interest in it, and a description of this term is a major subject of this paper. It is found to have a sign opposite that apparently measured. The last term in Eq. (1) describes the tunneling of quasiparticles. We include this term only for completeness as it has been considered in detail previously.[4]

In Sec. II we review the theoretical formulations of the three terms and the numerical evaluations of them as well. All three terms are presented in integral form in Sec. III and evaluated at $T=0$. Numerical evaluations of the three terms for nonzero temperature are given in Sec. IV for identical superconductors on each side of the tunneling barrier. Results for different superconductors are given in Sec. V. In Sec. VI it is shown for a constant voltage across the junction that the cosine term has no effect on the amplitude of rf-induced steps in the current-voltage curve. It is shown in Sec. VII that tunneling can be thought of in terms of a slight generalization of the coherence effects which dominate other kinds of experiments on superconductors. Section VIII contains a summary and a brief discussion of the range of applicability of existing tunneling theory.

II. PREVIOUS WORK

Since both theoretical and experimental work on the Josephson effect often deal exclusively with the sine term, we wish to discuss here those papers in which we have found mention of the cosine term as well. In many theoretical treatments, the cosine term arises as a natural consequence of the process used to obtain the sine term. On the other hand, the presence of this term in the result is often obscured because the author has emphasized the sine term. In addition, some authors immediately notice that for low frequencies the cosine term can be approximated as zero, and then neglect it in the remainder of their work. Today, however, when Josephson junctions are being considered for use as very-high-frequency microwave detectors[5] and as high-speed switching devices,[6] it may be essential to include the cosine term in quantitative analyses of Josephson-effect devices.

Josephson included all three terms of the tunneling current in his original work.[1] His discussion of the dependence of the sine and cosine terms on voltage and temperature was limited to mentioning that the sine term is the only nonzero one at zero temperature and voltage. Ambegaokar and Baratoff[7] produced expressions for all three terms but examined their dependence on temperature only at zero voltage where the cosine and quasiparticle terms vanish. In later papers on the topic,[8,9] Josephson pointed out that the cosine term can be considered as a phase-dependent part of the quasiparticle current. Riedel[10] studied the voltage dependence of the sine term at zero temperature although his formulation was more general and also included the cosine term. The logarithmic singularity in the amplitude of the sine term is referred to as the "Riedel peak." This singularity has now been experimentally observed.[11,12] Rickayzen[13] and later Svidzinskii and Slyusarev[14] also produced expressions for all three terms but they did not call attention to the presence of the cosine term. Werthamer[15] produced expressions for all three terms, analytic formulas for them at zero temperature, and actually plotted them as functions of voltage for zero temperature. His mention of the cosine term may have attracted little attention because the cosine term was the imaginary part of a complex-valued integral of which the real part was the sine term that commanded more interest.[16] Shortly afterwards, Larkin and Ovchinnikov produced integral formulas[17] for all three terms and gave many asymptotic forms for them as well. Their work is particularly useful because they reduced the sine term from the usual double integral to a single integral. This makes the numerical work required for finite temperature much simpler. They also applied their result to superconductors containing magnetic impurities. Nam[18] later gave a general integral formula for all three terms in an appendix to a paper on the electromagnetic properties of superconductors. His approach is valid for superconductors having strong electron-phonon coupling. Wilkins[19] was the first to point out explicitly that the sine and cosine terms are the Kramers-Kronig transforms of each other, although Werthamer probably realized this connection between them.

The theoretical work discussed above goes back to 1962, but it was not until 1972 that Poulsen[20] for the first time numerically evaluated the cosine term at finite temperature. His initial work was limited to low voltages. (His work has now been extended to more general cases.) Even more recent numerical work by Schlup[21] gave all three terms graphically over a wide range of temperature and voltage. This was the first evaluation of the sine term at both nonzero temperature and voltage. Each of these authors gave results for identical superconductors on each side of the tunneling barrier. Harris[22] presented the cosine term as a function of temperature and voltage for tunnel junctions composed of different superconductors. The present paper is an extension of that work and presents the sine and quasiparticle terms as well.

Experimentally, Dahm, Denenstein, Finnegan, Langenberg, and Scalapino[23] first mentioned that a Josephson plasma-resonance experiment would produce evidence of the cosine term. More recently Pederson, Finnegan, and Langenberg[3] completed this experiment. Their measured value for the average amplitude of the cosine term at small voltages is apparently equal, within experimental error, to that subsequently calculated by Poulsen but the experimental sign is opposite from the theoretical one. The disagreement is not understood at this time. No experiments have yet been done which measure the cosine term as a function

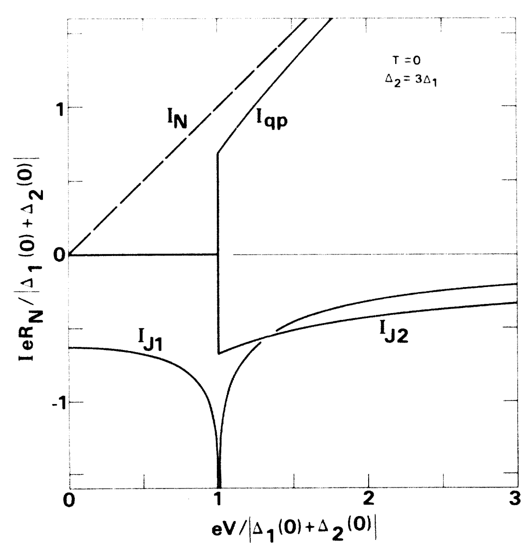

FIG. 1. Three terms in the Josephson tunneling current and the normal-state tunneling current for zero temperature and different superconductors.

of voltage.

After the majority of this paper was completed three additional papers appeared. First by interpreting the effects of noise on the dc current-voltage characteristics of proximity-effect bridges, Falco, Parker, and Trullinger[24] have deduced the existence of a cosine term having the same sign and similar magnitude to that found in Ref. 3. Hansma[25] has shown theoretically that the cosine term may produce an observable effect on superconducting-quantum-interference devices (SQUIDs). Vincent and Deaver[26] have analyzed their data on a toroidal point-contact SQUID and concluded that there is a cosine term. The two experimental papers conclude that the cosine term has sign opposite that predicted by the microscopic theory of tunnel junctions discussed in this work. On the other hand, neither experiment involved a tunnel junction so the disagreement does not necessarily have great significance for tunnel junctions.

Langenberg[27] has very recently reviewed both experiments and theories related to the cosine term.

III. SINE, COSINE, AND QUASIPARTICLE TERMS

A. Integral formulas for tunneling current

The basis for the numerical evaluation of the three terms in the tunneling current is the set of integral formulas presented below. They are valid for junctions composed of different superconductors on each side of the tunneling barrier. The superconductors are assumed to be described by the BCS theory,[28] to have weak electron-phonon coupling, and to contain no magnetic impurities. These formulas have been given in nearly the same form by Larkin and Ovchinnikov[17] but we include them to indicate changes we have made and to indicate clearly the signs we have used:

$$I_{J1}=-\frac{\Delta_1\Delta_2}{2eR_N}\int_{-\infty}^{\infty}\left(\frac{\theta(\Delta_1-|\omega-eV|)\theta(|\omega|-\Delta_2)}{[\Delta_1^2-(\omega-eV)^2]^{1/2}(\omega^2-\Delta_2^2)^{1/2}}+\frac{\theta(|\omega|-\Delta_1)\theta(\Delta_2-|\omega+eV|)}{(\omega^2-\Delta_1)^{1/2}[\Delta_2-(\omega+eV)^2]^{1/2}}\right)[1-2f(|\omega|)]\,d\omega\,, \tag{2a}$$

$$I_{J2}=-\frac{\Delta_1\Delta_2}{eR_N}\int_{-\infty}^{\infty}[f(\omega+eV)-f(\omega)]\frac{(\mathrm{sgn}\omega)[\mathrm{sgn}(\omega+eV)]\theta(|\omega|-\Delta_1)\theta(|\omega+eV|-\Delta_2)}{(\omega^2-\Delta_1^2)^{1/2}[(\omega+eV)^2-\Delta_2^2]^{1/2}}\,d\omega\,, \tag{2b}$$

$$I_{\mathrm{qp}}=\frac{1}{eR_N}\int_{-\infty}^{\infty}[f(\omega)-f(\omega+eV)]|\omega||\omega+eV|\frac{\theta(|\omega|-\Delta_1)\theta(|\omega+eV|-\Delta_2)}{(\omega^2-\Delta_1^2)^{1/2}[(\omega+eV)^2-\Delta_2^2]^{1/2}}\,d\omega\,, \tag{2c}$$

where

$$\mathrm{sgn}\omega=+1 \text{ for } \omega>0$$
$$=-1 \text{ for } \omega<0,$$
$$\theta(\omega)=0 \text{ for } \omega<0$$
$$=1 \text{ for } \omega>0,$$

and

$$f(\omega)=[1+\exp(\omega/k_BT)]^{-1}\,.$$

The physical quantities Δ_1 and Δ_2 are the energy gaps of the two superconductors at temperature T; R_N is the normal-state resistance of the junction; e is the absolute value of the electronic charge; and k_B is Boltzmann's constant. The current is defined as a conventional current of positive charges from side 1 to side 2. The voltage difference V is given by the voltage on side 1 minus that on side 2. Using these conventions we have $I=V/R_N$ for the normal state. In this paper the phrase "amplitude of the cosine term" means I_{J2} while "magnitude of the cosine term" means the absolute value of I_{J2}. Similar definitions apply to the sine term.

The integral formulas [Eqs. (2)] for the three terms are not in agreement with all theoretical papers we have found. They do agree, however, with the expressions obtained by Ambegaokar and Baratoff, Svidzinskii and Slyusarev, Werthamer (for $T\neq 0$), Josephson,[29] and Schrieffer.[30] Differences between the signs given in the present paper and those of additional authors are noted in the references.

In the case of the sine term a single integral similar to Eq. (2a) is given only by Larkin and Ovchinnikov. The integral given here differs from theirs, however. The essential difference is that we have used the same sign for both terms in the integrand, while their signs differ. This change is necessary to obtain agreement (except for an overall sign) with Werthamer's analytic expression[15] at $T=0$. This change also makes I_{J1} and I_{J2} as given in Eqs. (2) consistent with the Kramers-Kronig relations between them as discussed later in this section.

As can be readily seen the integrand of each integral contains a number of square-root singularities. These singularities are integrable but transformations must be made to remove them before numerical integrations can be performed. The general approach to these transformations is given by Shapiro *et al.*[4] A complete discussion of the transformations for this problem and of the computer programs used for the numerical integrations is available from the author.

In Fig. 1 we have plotted I_{J1}, I_{J2}, I_{qp}, and $I_N = V/R_N$, the last being the normal-state current. This graph is for zero temperature and is similar to that given by Werthamer except that we have used different energy gaps for the superconductors on each side of the tunneling barrier ($\Delta_2 = 3\Delta_1$) and I_{J1} and I_{J2} have signs opposite from Werthamer's corresponding quantities.[15] We have plotted the voltage and currents in dimensionless units in which a unit voltage corresponds to the sum of the energy gaps and a unit current is the normal-state current for unit dimensionless voltage.

The sine term, $I_{J1}\sin\varphi$, describes the tunneling of ground-state pairs. This process can occur at zero voltage as indicated by the nonzero value of I_{J1} at zero voltage. The function I_{J1} falls with voltage to a logarithmic singularity at a voltage corresponding to the sum of the energy gaps and then rises gradually toward zero at large voltage. The singularity, the Riedel peak, evidently describes a resonance between the pair tunneling and quasiparticle tunneling.

The sign of I_{J1} is shown in the figure to be negative for all values of voltage V. However, if the wave functions of the two superconductors are in phase at time zero, $\varphi = -2eVt/\hbar$, the argument of the sine function which multiplies I_{J1} is also negative for positive time t. As the sine is an odd function, the negative sign of its argument cancels the negative sign of I_{J1}. That is, for an initially positive voltage ($0 < 2eVt/\hbar < \frac{1}{2}\pi$), one obtains an initially positive contribution to the current. For longer times this term oscillates between $\pm I_{J1}$ because φ continues to decrease with time.

It is also useful to note that I_{J1} is an even function of the voltage difference V. If one reverses the voltage, $I_{J1}(-V, T) = +I_{J1}(+V, T)$, but the sine function changes sign because its argument changes sign. Thus the contribution to the current is reversed for negative voltage.

The quasiparticle term I_{qp} at zero temperature is zero for voltages below the sum of the gap voltages, rises abruptly, and gradually approaches the normal-state current. The abrupt rise occurs because at that voltage enough energy can be gained by a pair to split into two quasiparticles. The onset is abrupt because of the singularity in the density of quasiparticle states at the gap. The function $I_{qp}(V, T)$ is an odd function of V so that, like the sine term, it too reverses sign when the sign of the voltage is reversed.

The cosine term is similar to the quasiparticle term in that it is also zero for low voltage, but then drops abruptly at the sum of the gap voltages, and rises gradually toward zero with increasing voltage. Josephson[8,9] has described this term essentially as a phase-dependent part of the quasiparticle current. The net quasiparticle current is thus the sum of the quasiparticle and cosine terms. It is therefore not surprising that the cosine term is zero below the sum of the gaps because quasiparticles cannot be created with energy less than this at $T = 0$. It is also not surprising that the net quasiparticle current can never become negative for positive voltage because the magnitude of the cosine term is always less than or equal to the quasiparticle current. This can be seen [Eqs. (2b) and (2c)] to be true for any temperature or voltage. That the cosine term may be important in understanding how a tunnel junction operates is indicated by the fact that the cosine term has magnitude larger than the sine term for voltages above about $1.3 \times (\Delta_1 + \Delta_2)$. For $\Delta_1 = \Delta_2 = \Delta$ the cosine term becomes bigger than the sine term at about $1.4 \times 2\Delta$.

To see how the cosine term affects the total current through a tunneling junction we recall again that $\varphi = -2eVt/\hbar$. Thus the phase decreases linearly with time t and both the sine and cosine terms oscillate sinusoidally in time. Unlike the sine term, for initially positive voltage V, the cosine term gives an initially negative (or zero) contribution to the current at zero temperature. For nonzero temperature it is shown in Secs. IV and V that the cosine term gives an initially positive current for $V < (\Delta_1 + \Delta_2)/e$, but an initially negative current for $V > (\Delta_1 + \Delta_2)/e$. As with the sine and quasiparticle terms, reversing the voltage causes the contribution to the current from the cosine term to reverse because $I_{J2}(V, T)$ is an odd function of V while the cosine is an even one.

In Fig. 2 we show graphically the composition

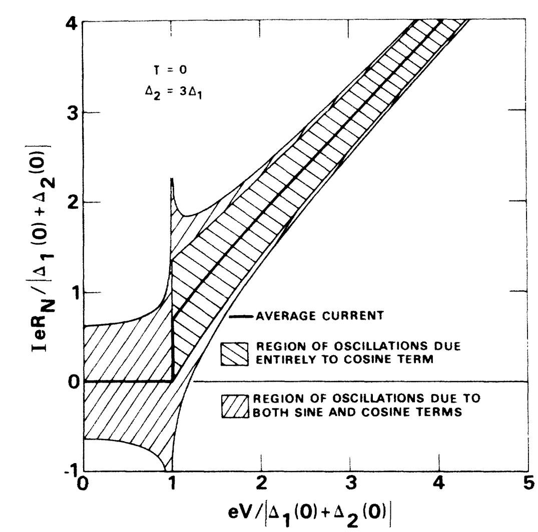

FIG. 2. Composition of the total tunneling current.

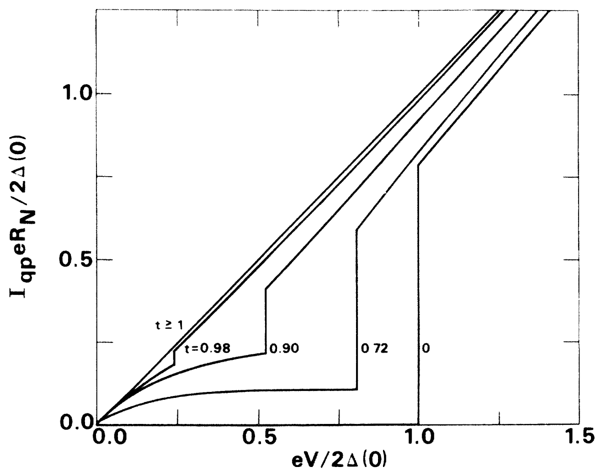

FIG. 3. Quasiparticle current I_{qp} for identical superconductors.

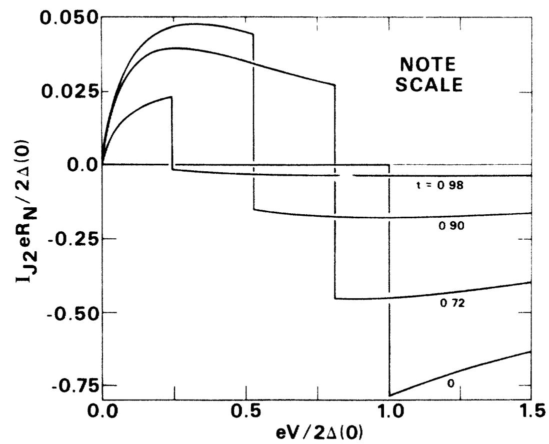

FIG. 4. Amplitude of the cosine term I_{J2} for identical superconductors. Note that the scale for negative currents is ten times finer than that for positive currents.

of the total current. The heavy line gives the constant part of the quasiparticle current I_{qp} as a function of voltage. The net quasiparticle current, which is the sum of the quasiparticle and cosine terms, is bounded by $I_{\mathrm{qp}} \pm |I_{J2}|$ because the cosine varies between +1 and −1. This region is indicated by the area having hatch marks with negative slope. As mentioned before, this region always represents nonnegative currents. The total current is bounded by $I_{\mathrm{qp}} \pm [I_{J1}^2 + I_{J2}^2]^{1/2}$. This region is the entire hatched area. Notice that above the sum of the gaps, the sine term adds very little to the amplitude of the oscillations. On the other hand, as we shall see in Sec. VI, the sine term is the only one which contributes to the rf-induced steps.

B. Kramers-Kronig relations

Wilkins[19] was the first to point out that I_{J1} and I_{J2} are related by the Kramers-Kronig relations.[31] This is especially easy to see from the form in which Rickayzen writes the two quantities. He shows that they arise from the real and imaginary parts, respectively, of the sum of four complex-valued integrals each having the following form:

$$\lim_{\eta\to 0} \int\int \frac{g(\omega, \omega')\, d\omega\, d\omega'}{\omega' - \omega - \omega_0 - i\eta}$$

$$= \mathcal{P} \int\int \frac{g(\omega, \omega')\, d\omega\, d\omega'}{\omega' - \omega - \omega_0} + \pi i \int g(\omega, \omega + \omega_0)\, d\omega\, .$$

The sine term comes from the real, principal part of each of the integrals while the cosine term comes from the imaginary, single integral part. It is simple to show that the Kramers-Kronig relations follow:

$$I_{J1}(\omega_0) = \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega I_{J2}(\omega)\, d\omega}{\omega^2 - \omega_0^2}\, , \tag{3a}$$

$$I_{J2}(\omega_0) = \frac{2\omega_0}{\pi} \mathcal{P} \int_0^\infty \frac{I_{J1}(\omega)}{\omega_0^2 - \omega^2}\, d\omega\, . \tag{3b}$$

We can use this result to prove that we have used the correct relative signs for the sine and cosine terms at zero temperature. We assume that the cosine term has been correctly evaluated except possibly for an overall sign. Recall that the cosine term vanishes for voltage less than the sum of the gap voltages. Then it follows using Eq. (3a) that

$$I_{J1}(\omega_0 < \Delta_1 + \Delta_2) = \frac{2}{\pi} \int_{\Delta_1+\Delta_2}^\infty \frac{\omega I_{J2}(\omega)}{\omega^2 - \omega_0^2}\, d\omega\, .$$

Since we restricted ourselves to evaluating the integral for values of ω_0 less than the sum of the gap voltages, the denominator is positive over the entire range of integration. It follows that the sine term for voltages less than the sum of the

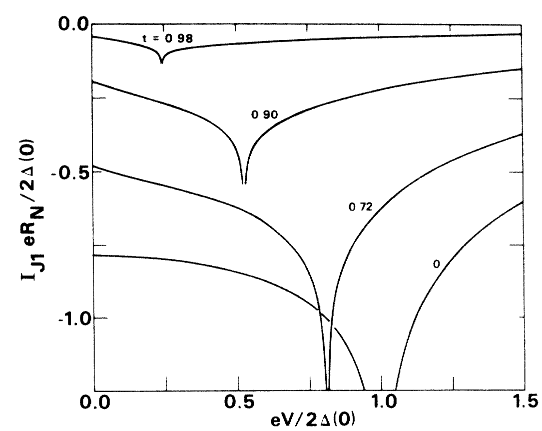

FIG. 5. Amplitude of the sine term I_{J1} for identical superconductors.

gaps has the same sign as the cosine term has at higher voltages. Since the amplitude of the sine term is negative we can conclude that for zero temperature the cosine term must also be negative.

At finite temperature the cosine term is nonzero for voltages less than the sum of the gap voltages. This simple argument is thus no longer useful. In fact, the cosine term is positive in this range, a result which seems to contradict the experimental measurements. To determine that we have obtained the correct relative signs for the cosine term above and below the sum of the gaps we have evaluated the sine term both from Eq. (2a) and numerically from the Kramers-Kronig relation Eq. (3a). The results were in agreement within numerical errors. When the sign of the cosine term below the sum of the gap voltages was changed, its Kramers-Kronig transform was inconsistent with the sine term calculated directly.

IV. NUMERICAL RESULTS FOR IDENTICAL SUPERCONDUCTORS

In Figs. 3–5 the numerical results for I_{qp}, I_{J1}, and I_{J2} are given for identical superconductors on each side of the tunneling barrier. The temperature dependence of $\Delta(T)$[32] has been included in the calculations. The reduced temperature t is given by $t = T/T_c$, where T_c is the critical temperature of the superconductors. For a BCS superconductor, $t = 1.764 k_B T/\Delta(0)$.

The quasiparticle current (Fig. 3) increases with temperature everywhere and especially below $2\Delta(T)$ as increasing numbers of thermally excited quasiparticles become available for tunneling at higher temperature. In addition, at zero voltage this current becomes zero with infinite slope, although this may not be apparent given the scale of this figure.

The cosine term (Fig. 4) is *positive* below 2Δ, *negative* above 2Δ, and vanishes with infinite slope at zero voltage. Note that the scale for positive values of current is ten times finer than that for negative currents. One can see that the cosine term has smaller magnitude than the quasiparticle term for voltages below 2Δ. Above the energy gap it is also clear that increasing temperature depresses the cosine term.

The sine term (Fig. 5) has qualitatively the same shape for all temperatures. However, at finite temperatures it approaches zero voltage with finite slope in contrast to zero slope at zero temperature. This is connected through the Kramers-Kronig relations with the infinite slope of the cosine term at zero voltage and finite temperature.

Although we have plotted these three terms as currents, Josephson, and the authors of papers giving numerical evaluations, Poulsen and Schlup, present these results as conductivities:

$$I = V[\sigma_0(V, T) + \sigma_1(V, T)\cos\varphi] + I_{J1}(V, T)\sin\varphi .$$

When multiplied by the voltage these conductivities are identical to our currents: $V\sigma_0(V, T) = I_{qp}(V, T)$ and $V\sigma_1(V, T) = I_{J2}(V, T)$. Note that Schlup gives the opposite sign for σ_1. Because of the infinite slope of I_{qp} and I_{J2} at zero voltage and finite temperature the conductivities corresponding to these currents have infinite value at zero voltage. In this case the infinite conductivity corresponds to zero current. Because of the possibility that this last feature might be overlooked we have chosen to discuss all three terms as currents.

V. NUMERICAL RESULTS FOR DIFFERENT SUPERCONDUCTORS

Figures 6–8 give the numerical results for tunnel junctions composed of different superconductors on each side of the tunneling barrier. We have chosen the hypothetical example in which one of the superconductors has an energy gap at zero temperature which is three times that of the other: $\Delta_2(0) = 3\Delta_1(0)$. In calculating the finite-temperature curves, the temperature dependence of each energy gap has been included. The voltage has been normalized by $\Delta_1(0) + \Delta_2(0)$ so that the sum of the energy gaps appears at unity for zero temperature. The difference in the energy gaps appears at 0.5 for zero temperature. The reduced temperature t_1 corresponds to the superconductor which has the smaller energy gap, Δ_1.

Figure 6 gives I_{qp} and reveals the well-known logarithmic singularity at the difference in the energy gaps. With different superconductors the infinite slope at zero voltage is no longer present.

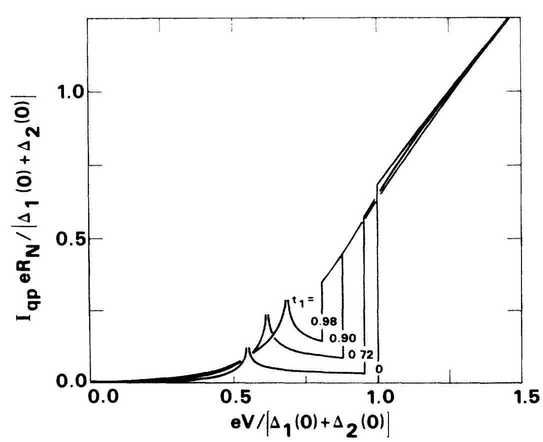

FIG. 6. Quasiparticle current I_{qp} for different superconductors.

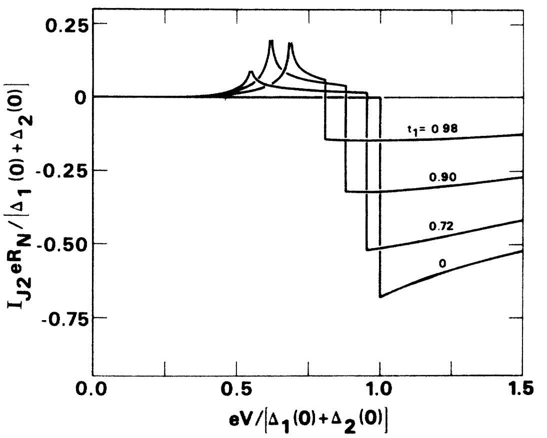

FIG. 7. Amplitude of the cosine term I_{J2} for different superconductors.

Figure 7 gives the amplitude of the cosine term I_{J2} and reveals a similar logarithmic singularity at the difference in the energy gaps. This singularity has the same sign and strength as that in the quasiparticle term.

Figure 8 gives the amplitude of the sine term I_{J1}. We see a step at the difference in the gaps when the temperature is nonzero. This step is connected through the Kramers-Kronig relation with the logarithmic singularity in the cosine term. This step represents a change in the amplitude of the sine term in some cases of almost a factor of 2. This is only slightly smaller than the measured change[11,12] of this term near the Riedel peak which is presumably reduced in size by anisotropy in the sample. In contrast, anisotropy should merely broaden the step rather than reduce its height. The demonstrated ability to measure a change in I_{J1}[11,12] of less than a factor of 2 near the Riedel peak implies that the step change at the difference in the energy gaps should be measurable by the same techniques.

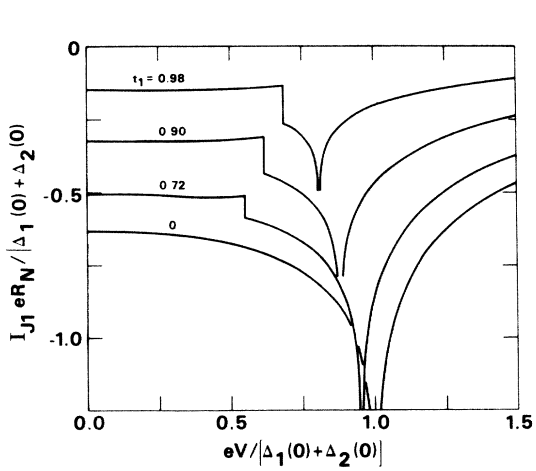

FIG. 8. Amplitude of the sine term I_{J1} for different superconductors.

In Fig. 9 we have plotted the ratio of the cosine term to the quasiparticle term. It has previously been plotted (with the opposite sign) for identical superconductors by Schlup. We see that this ratio is +1 at the difference in the gaps but at low temperatures approaches −1 at the sum of the gaps. It is clearly bounded by ±1.

VI. EFFECT OF COSINE TERM ON RF-INDUCED STEPS

To our knowledge the effect of the cosine term on rf-induced steps in tunnel junctions has never been explicitly considered. We show here that when one carries through to completion a calculation by Werthamer,[15] there is no effect due to the cosine term on the amplitude of the rf-induced steps in the special case of a junction biased by a constant dc voltage.

Werthamer assumes that the rf field acts like a voltage source. His Eq. (11) describes the resulting time-dependent current. Our Eq. (1) is a special case of this result for zero rf field. Werthamer shows that the magnitude of the dc component of the current on the Nth rf-induced step is given by the following:

$$I^{\mathrm{dc}} = \sum_{n=-\infty}^{\infty} \{J_n^2(\gamma) I_{\mathrm{qp}}[(n-\tfrac{1}{2}N)h\nu] + J_{N-n}(\gamma)J_n(\gamma)I_{J1}[(n-\tfrac{1}{2}N)h\nu]\sin\varphi_0 + J_{N-n}(\gamma)J_n(\gamma)I_{J2}[(n-\tfrac{1}{2}N)h\nu]\cos\varphi_0\}\,, \tag{4}$$

where $\gamma = ev_{\mathrm{rf}}/h\nu$, v_{rf} is the rf-induced voltage across the junction, ν is the rf frequency, and J_n is the nth-order Bessel function. The constant φ_0 is determined by the circuit to which the junc-

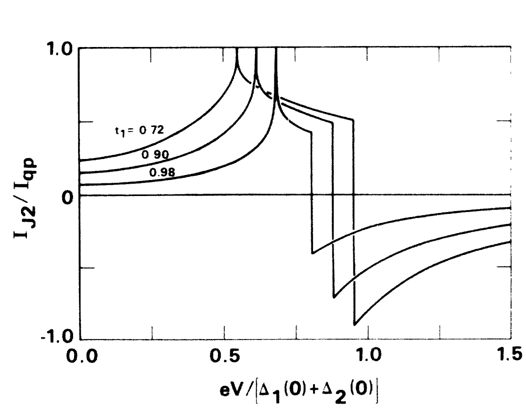

FIG. 9. Ratio I_{J2}/I_{qp} of the amplitude of the cosine term to the quasiparticle term for different superconductors.

tion is attached. Actually φ_0 is indeterminate when a true voltage source is used. It is assumed, however, that one can use a nearly-constant-voltage source which holds constant both the voltage and the phase φ_0 to a sufficiently good approximation.

The first term in Eq. (4) describes the effect of the rf field on the quasiparticle term, the process being referred to as photon-assisted tunneling.[33] We will not discuss it further. The second and third terms give the effect of the rf field on the phase-dependent terms of the tunneling current.

Considering only terms two and three, we notice that the sum is over all positive and negative values of n. Thus for every Bessel function product $J_{N-n}J_n$, there is another equal product J_nJ_{N-n} in the series. Grouping these terms together one obtains the following expressions for odd step number N:

$$I_J^{\rm dc} = \sum_{l=1}^{\infty} J_{(N/2-1/2+l)}(\gamma) J_{(N/2+1/2-l)}(\gamma) \times \{I_{J1}[(l-\tfrac{1}{2})h\nu] + I_{J1}[(-l+\tfrac{1}{2})h\nu]\} \sin\varphi_0 + \{I_{J2}[(l-\tfrac{1}{2})h\nu] + I_{J2}[(-l+\tfrac{1}{2})h\nu]\} \cos\varphi_0 \,.$$

We see that the Bessel-function products are multiplied by sums of I_{J1} and I_{J2} evaluated with opposite arguments. Because I_{J1} is an even function of its argument those two terms add. However, I_{J2} is an odd function of its argument so that those two terms cancel. This discussion is for steps of odd number N, but a similar one can be repeated for even N. We conclude that for a constant-voltage bias, the cosine term has no effect on the rf-induced steps.

In the case of a circuit having finite impedance, there may be an effect due to the cosine term. It also might appear in a mixing experiment or in a high-speed Josephson switching device where the voltage is changing rapidly with time.

VII. COHERENCE EFFECTS AND THE TUNNELING CURRENT

To obtain some feeling for the origin of the cosine term in the theory of superconductivity, one can characterize the theoretical expression for the tunneling current in a way very similar to that used in the original BCS paper[28,34,35] for other experiments on superconductors. In that paper BCS divided the matrix elements one must calculate to predict the results of experiments into two cases which they called I and II. The matrix element for scattering of an electron from a state $\vec{k}$ to a state $\vec{k}'$ is written symbolically $B_{\vec{k},\vec{k}'}$. For simplicity we consider only cases where there is no spin flip. The Hamiltonian for many experiments can thus be written as the sum over all initial states $\vec{k}$, final states $\vec{k}'$, and spins σ:

$$H = \sum_{\vec{k},\vec{k}',\sigma} B_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k}'\sigma} c_{\vec{k}\sigma} + \text{H.c.}$$

For case I there is no change in the matrix element for scattering from $-\vec{k}$ into $-\vec{k}'$; i.e., $B_{\vec{k},\vec{k}'} = B_{-\vec{k},-\vec{k}'}$. This is true for an experiment such as acoustic attenuation where the scattering is from density variations in the lattice. In this case the interaction is related only to the magnitude of the momentum $\vec{k}$. For other experiments such as far-infrared absorption where the interaction energy is given by $\vec{A}\cdot\vec{p}$, where $\vec{A}$ is the vector potential due to the far-infrared radiation, the Hamiltonian obviously changes sign when the momentum changes sign. Here $B_{\vec{k},\vec{k}'} = -B_{-\vec{k},-\vec{k}'}$, and this situation is called case II.

BCS showed that for these experiments the normal-state properties of the metal enter only in an average way so that if experimentally determined quantities such as acoustic attenuation per unit length α_S, or far-infrared conductivity σ_1, are normalized to the normal-state results, the remaining variation can be fully characterized by coherence factors determined only by the properties of the BCS wave function and the distinction between cases I and II.

In fact, these theoretical predictions have been verified experimentally. In the local limit they can be written as follows[34,35]:

Acoustic attenuation (case I):

$$\alpha_S/\alpha_N = \text{Re}[N(\omega) + P(\omega)] \,, \tag{5a}$$

Far-infrared absorption (case II):

$$\sigma_1/\sigma_N = \text{Re}[N(\omega) - P(\omega)] \,. \tag{5b}$$

The quantity ω is the appropriate acoustic or far-infrared frequency. The functions $N(\omega)$ and $P(\omega)$ are given below. As can be seen, the only difference between the two results is in the sign of the function $P(\omega)$. This difference can be traced directly to the sign change, or lack of it, in the matrix element $B_{\vec{k},\vec{k}'}$ under inversion of $\vec{k}$ and $\vec{k}'$.

In the case of tunneling we note, simply by comparison with the above functional dependence, that the tunneling current, normalized to the normal-state current, can be written in the following manner:

$$I/I_N = \text{Re}[N(\omega) - e^{i\varphi}P(\omega)] \,. \tag{6}$$

We have replaced the voltage by ω. The similarity to Eqs. (5) is immediately apparent[36,37] and it is now clear that $\mathrm{Re}\,N(\omega) = (1/\omega)I_{qp}$ and $P(\omega) = (i/\omega) \times (I_{J1} + iI_{J2})$. For zero phase difference ($\varphi = 0$) the normalized tunneling current has the same form as the real part of the normalized far-infrared conductivity σ_1/σ_N. More generally, it appears that in a tunneling experiment there is a continuous oscillation with frequency $2eV/h$ between cases I and II.

We can partially see how this similarity arises from the interaction energy for tunneling:

$$H = \sum_{\vec{k},\vec{q},\sigma} T_{\vec{k},\vec{q}}\, c^{\dagger}_{\vec{k}} c_{\vec{q}} + \text{H.c.}$$

This has the same form as cases I and II. Furthermore, in the absence of magnetic impurities, we have $T_{\vec{k},\vec{q}} = T_{-\vec{k},-\vec{q}}$. Since this is like case I rather than case II, the minus sign in Eq. (6) is not immediately explained. Although the explanation lies in details of the calculations, Tinkham[38] has pointed out to the author that a simple understanding of the minus sign is possible. One imagines forming a tunnel junction from a single superconductor by gradually increasing the thickness of an insulating layer within the superconductor. The conductivity of Eq. (5b) describes the response to an electromagnetic field for zero insulator thickness. Since there is no thickness at which the coherence effects can abruptly switch type, a Josephson device must exhibit essentially the same type observed in a far-infrared experiment on a single superconductor.

Another difference, the presence of the phase, remains. A detailed theoretical examination of the theory would be necessary to understand exactly how the phase enters the expression for the tunneling current. We simply note that the transfer of a pair from one side of the tunneling barrier to the other multiples the wave function by $e^{i\varphi}$.[39] This does not happen in the case of the other kinds of experiments because there the initial and final states are all in the same superconductor. For tunneling the initial and final states are separated by the tunneling barrier permitting the phase to appear.

Thus it seems that a slight generalization of the BCS concept of case I and case II matrix elements can include tunneling. From this approach one can see that the terms ReN and ReP correspond to the lossy processes in tunneling just as they do in acoustic attenuation and far-infrared absorption. This correlates with Josephson's statement[8,9] that the cosine term corresponds to a phase dependence of the quasiparticle current. His discussion is in terms of conductivities rather than currents. We can write the quasiparticle and cosine terms in the following way:

$$I_{\text{loss}} = I_{qp}\{1 + (|I_{J2}|/I_{qp})[\mathrm{sgn}(I_{J2})\cos\varphi]\} \ .$$

Thus the net quasiparticle current is I_{qp} modulated by the phase-dependent term. The magnitude of the modulation function is always less than or equal to unity because $I_{qp} \leq |I_{J2}|$. We can think of the cosine term as a phase modulation of the quasiparticle current. Of course, when one uses a constant-voltage bias to measure the quasiparticle current the result agrees with I_{qp}. On the other hand, one obtains I_{qp} for real situations where the bias is only approximately constant voltage. This is true for three possible reasons: First, a constant voltage bias is a good approximation for tunnel junctions whose inherent capacitance tends to hold the voltage constant. Thus the cosine term averages to zero. Second, experimenters sometimes apply a magnetic field to "quench" the Josephson effect. The field produces a linear change in the phase along the length of the junction. Thus, the Josephson current has alternating signs along the junction. The total dc Josephson current is the sum over the length of the junction and is thus reduced. A similar thing happens to the ac Josephson currents, reducing their amplitude also. The latter applies to the cosine term as well as the sine term. The result is that one measures the function I_{qp}. Finally, one must not dismiss the possibility that detailed analysis may show that the cosine term has no effect on dc current-voltage characteristics regardless of the circuit which biases the junction.

VIII. SUMMARY AND APPLICABILITY OF PRESENT TUNNELING THEORY

We have examined the predictions of the BCS theory applied to tunneling between different superconductors. Numerical results have been graphically displayed, revealing at least one measurable result: the step in the sine term at the difference in the energy gaps. It has been shown that the cosine term has no effect on the amplitude of the rf-induced steps in the case of a voltage bias. It has further been shown that tunneling can be described in terms of the coherence effects important in other experiments on superconductors.

While the Werthamer theory is the most general that is available for tunnel junctions, it does have important limitations. It has only been shown to assume a tractible form in the special case of zero-impedance dc and rf sources (voltage sources), although Werthamer does state a more general result. Additional parameters such as possible shunt conductance, capacitance, and inductance are also not included. It is important to note in this context that the widely used Eq. (1) is not Werthamer's most general result and is rigorously valid only for dc voltages.

Undoubtedly the general problem is exceedingly complicated. On the other hand, an understanding of combined circuit and high-frequency ($\nu \gtrsim 2\Delta/h$) effects is becoming increasingly important in view of recent experiments in which two signals from CO_2 lasers were mixed using a point-contact junction.[40] One needs to know, for example, how to simplify the general Werthamer result for finite impedance sources which give rise to voltages having nonsinusoidal time dependence, and whether additional terms, other than a simple capacitive one, must be added.

Theories of other devices[41,42] such as microbridges, proximity effect bridges, and point contacts do not seem to include high-frequency effects as does the Werthamer theory. Because of the widespread use of these other devices there is a significant need for a detailed understanding of them. For example, the simple way in which the coherence effects produce the sine and cosine terms in tunnel junctions possibly suggests that one can show on general grounds that theories of the other devices will have a cosine term which is the Kramers-Kronig transform of the sine term. The high-frequency theory of these devices may be based on a different physical mechanism and awaits further development.

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