Dispersion relations for third-degree nonlinear systems

F. L. Ridener,Jr. and R. H. Good,Jr.
Physics Department, Pennsylvania State University, University Park, Pennsylvania 16802
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A time-independent casual system is considered, in which the effect depends cubically on the cause. The system's response function depends on three time variables and its transform on three frequency variables. It is shown that there is an analog of the Kramers-Kronig dispersion relations, so that the real part of the transform can be obtained from the imaginary part by integrating over the observed frequency in a certain way, and vice versa. It is shown how the transform function is determined by the effects produced by causes built up from pure frequencies. Also it is shown how the real and imaginary parts of the transform can be separately found by time averaging the product of pure-frequency responses with the cubes of the causes.

I. INTRODUCTION

Consider a time-independent system in which an effect $E(t)$ depends in a third-degree way on a cause $C(t)$ so that

$$E(t)=\int dt'\int dt''\int dt'''\,G(t-t',t-t'',t-t''')\times C(t')C(t'')C(t''').\tag{1}$$

(Unless otherwise specified, integrals range from $-\infty$ to $+\infty$.) Here $C(t)$ and $E(t)$ are real functions so the response function $G(t_1,t_2,t_3)$ also is real. Evidently only symmetric response functions

$$G(t_1,t_2,t_3)=G(t_2,t_1,t_3)=G(t_1,t_3,t_2)\tag{2}$$

need be considered. Suppose further that the system is causal, the effect at a certain time depending only on the cause at earlier times; that is,

$$G(t_1,t_2,t_3)=0 \quad\text{if } t_1<0 \text{ or } t_2<0 \text{ or } t_3<0.\tag{3}$$

We will show that Kramers–Kronig-type dispersion relations exist for such a system and we will develop the physical interpretation of the transform

$$g(\omega_1,\omega_2,\omega_3)=\int dt_1\int dt_2\int dt_3\,e^{i(\omega_1t_1+\omega_2t_2+\omega_3t_3)}\times G(t_1,t_2,t_3)\tag{4}$$

of the response function.

As an elementary example of how this type of response occurs, consider a one-dimensional forced oscillator with first-degree damping, in which the particle vibrates about a center of symmetry. Suppose also that the first nonlinear effect can be treated as a perturbation

$$\ddot{x}+\nu\dot{x}+\omega_n^2x-\alpha x^3=F\ ,\tag{5}$$

where α marks the order of the perturbation. Let the solution of the problem be written as

$$x(t)=x^{(1)}(t)+\alpha x^{(3)}(t)\ .$$

As is well known, the solution of the unperturbed linear problem is

$$x^{(1)}(t)=\int dt'\,G^{(1)}(t-t')F(t')\ ,\tag{6}$$

where the Green's function is causal

$$G^{(1)}(t)=0 \quad\text{if } t<0.\tag{7}$$

The perturbation is here found to be

$$x^{(3)}(t)=\int dt'\int dt''\int dt'''\,G^{(3)}(t-t',t-t'',t-t''')\times F(t')F(t'')F(t''')\ ,\tag{8}$$

where

$$G^{(3)}(t_1,t_2,t_3)=\int d\tau\,G^{(1)}(\tau)G^{(1)}(t_1-\tau)\times G^{(1)}(t_2-\tau)G^{(1)}(t_3-\tau)\ .\tag{9}$$

Here $x^{(3)}$ is the effect, depending in a third-degree way on the cause $F(t)$. The response function $G^{(3)}$ is causal in the sense of Eq. (3), as follows immediately from Eq. (7).

As is discussed in detail below, the function $g(\omega_1,\omega_2,\omega_3)$ in general only needs to be specified inside a certain 60° semi-infinite wedge. The response of the system to pure-frequency input tones is derived in Sec. IV. These responses in principle provide a way to measure $g(\omega_1,\omega_2,\omega_3)$. There are several different types of response, corresponding to different edges, surfaces, and volumes of the wedge. Also it is shown that the real and imaginary parts of $g(\omega_1,\omega_2,\omega_3)$ can be found by making time averages of the effects of pure frequencies times the cubes of the causes.

The dispersion relations apply to the functions

$$\bar{g}_1(\omega_s,\omega_t,\omega_u)=\tfrac{1}{2}\bar{g}(\omega_s,\omega_t,\omega_u)+\tfrac{1}{2}\bar{g}(\omega_s,\omega_u,\omega_t),\tag{10}$$

$$\bar{g}_2(\omega_s,\omega_t,\omega_u)=\tfrac{1}{2}i\bar{g}(\omega_s,\omega_t,\omega_u)-\tfrac{1}{2}i\bar{g}(\omega_s,\omega_u,\omega_t),\tag{11}$$

where $\bar{g}$ is related to g by

$$\bar{g}(\omega_s, \omega_t, \omega_u) = g(\omega_1, \omega_2, \omega_3) \tag{12}$$

and these new frequency variables are defined by

$$\omega_s = \omega_1 + \omega_2 + \omega_3 , \tag{13a}$$

$$\omega_t = \omega_1 - \omega_2 , \tag{13b}$$

$$\omega_u = \omega_2 - \omega_3 . \tag{13c}$$

Except for the parametric dependence on ω_t and ω_u, the equations are identical to the Kramers–Kronig relations

$$\mathrm{Re}\,\bar{g}_i(\omega_s, \omega_t, \omega_u) = \frac{2}{\pi} P\int_0^\infty \frac{d\omega_s' \omega_s' \mathrm{Im}\,\bar{g}_i(\omega_s', \omega_t, \omega_u)}{\omega_s'^2 - \omega_s^2} , \tag{14a}$$

$$\mathrm{Im}\,\bar{g}_i(\omega_s, \omega_t, \omega_u) = -\frac{2}{\pi}\omega_s P\int_0^\infty \frac{d\omega_s' \omega_s' \mathrm{Im}\,\bar{g}_i(\omega_s', \omega_t, \omega_u)}{\omega_s'^2 - \omega_s^2} . \tag{14b}$$

Here $\bar{g}_i$ denotes either $\bar{g}_1$ or $\bar{g}_2$ and $\bar{g}_i = \mathrm{Re}\bar{g}_i + i\,\mathrm{Im}\bar{g}_i$.

Contributions to the understanding of dispersion relations for quadratic nonlinear systems were made by Kogan,[1] Price,[2] and Caspers.[2] Especially Price introduced appropriate frequency variables and found the general Kramers–Kronig-type dispersion relations for the quadratic system. However the third-degree system has not been treated in detail previously.

Before starting on the third-degree problems, we will quote the well-known results for the linear system (Sec. II) and give our discussion of the quadratic system (Sec. III). These problems are much simpler than the third-degree problem but they show the basic ideas and type of reasoning needed in the more complicated case.

II. LINEAR SYSTEM

Here the cause and effect are related by

$$E(t) = \int dt'\, G(t-t')C(t'), \tag{15}$$

where

$$G(t) = 0 \quad \text{if } t < 0. \tag{16}$$

The transform

$$g(\omega) = \int dt\, e^{i\omega t} G(t) , \tag{17}$$

in consequence of the reality of G, satisfies the crossing relation

$$g^*(\omega) = g(-\omega) . \tag{18}$$

Since t actually ranges only over positive values in Eq. (17), $g(\omega)$ is well defined and analytic for complex ω in the upper half plane.

As long as no subtractions are needed, the analyticity and crossing relations imply, by a well-known argument, that $g(\omega)$ satisfies the Kramers-Kronig dispersion relations

$$\mathrm{Re}g(\omega) = \frac{2}{\pi} P\int_0^\infty \frac{d\omega'\, \omega' \mathrm{Im}g(\omega')}{\omega'^2 - \omega^2} , \tag{19a}$$

$$\mathrm{Im}g(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{d\omega'\, \mathrm{Re}g(\omega')}{\omega'^2 - \omega^2} . \tag{19b}$$

In principle, $g(\omega)$ can be determined by observing the response to a pure tone. That is, if

$$C_a(t) = A_a \cos(\omega_a t - \eta_a) , \tag{20}$$

where ω_a is positive, then Eq. (15) implies that

$$E_a(t) = A_a |g| \cos(\omega_a t - \eta_a - \theta) , \tag{21}$$

where $|g|$ and θ are the amplitude and phase of $g(\omega_a)$ according to

$$g(\omega_a) = |g| e^{i\theta} .$$

This response determines g for positive ω and the crossing relation then fixes it for negative ω. Also, in principle, the real and imaginary parts of $g(\omega)$ can be determined by making time averages of products of cause and effects of pure tones:

$$\langle E_a(t) C_a(t)\rangle = \tfrac{1}{2} A_a^2 \mathrm{Re}g(\omega_a) , \tag{22a}$$

$$\left\langle \frac{dE_a(t)}{dt} C_a(t)\right\rangle = \tfrac{1}{2}\omega_a A_a^2 \mathrm{Im}g(\omega_a) . \tag{22b}$$

III. QUADRATIC SYSTEM

In this case we consider a cause and an effect related by

$$E(t) = \int dt' \int dt''\, G(t-t', t-t'')C(t')C(t''), \tag{23}$$

where

$$G(t_1, t_2) = G(t_2, t_1) , \tag{24}$$

$$G(t_1, t_2) = 0 \quad \text{if } t_1 < 0 \text{ or } t_2 < 0. \tag{25}$$

An alternate way of writing this relation, which is close to Kogan's starting point,[1] is

$$E(t) = 2\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\, G(t-t', t-t'')C(t')C(t'') .$$

The transform of the response function is defined by

$$g(\omega_1, \omega_2) = \int dt_1 \int dt_2\, e^{i(\omega_1 t_1 + \omega_2 t_2)} G(t_1, t_2). \tag{26}$$

It has the symmetries

$$g(\omega_1, \omega_2) = g(\omega_2, \omega_1) , \tag{27}$$

$$g^*(\omega_1, \omega_2) = g(-\omega_1, -\omega_2). \tag{28}$$

As emphasized by Price,[2] it is important to introduce new frequency variables

$$\omega_s = \omega_1 + \omega_2 , \tag{29a}$$

$$\omega_t = \omega_1 - \omega_2 \tag{29b}$$

and a new transform function

$$\bar{g}(\omega_s, \omega_t) = g(\omega_1, \omega_2) \ . \tag{30}$$

The symmetries of $\bar{g}$ are

$$\bar{g}(\omega_s, \omega_t) = \bar{g}(\omega_s, -\omega_t) \ , \tag{31a}$$

$$\bar{g}^*(\omega_s, \omega_t) = \bar{g}(-\omega_s, \omega_t). \tag{31b}$$

Consequently only the range

$$\omega_s \geq 0, \ \omega_t \geq 0$$

needs to be considered in specifying $\bar{g}(\omega_s, \omega_t)$. This range is indicated in Fig. 1 as the unshaded quarter plane.

In terms of the original response function, $\bar{g}$ is given by

$$\bar{g}(\omega_s, \omega_t) = \int_0^\infty dt_1 \int_0^\infty dt_2 \, e^{i\omega_s(t_1+t_2)/2} e^{i\omega_t(t_1-t_2)/2} G(t_1, t_2). \tag{32}$$

For fixed ω_t, since $(t_1 + t_2)$ is always positive, this equation serves to define $\bar{g}(\omega_s, \omega_t)$ as an analytic function of a complex variable ω_s in the upper half plane. Also, for fixed ω_t, g has crossing symmetry in ω_s, so it satisfies Kramers–Kronig-type dispersion relations

$$\mathrm{Re}\,\bar{g}(\omega_s, \omega_t) = \frac{2}{\pi} P \int_0^\infty \frac{d\omega_s' \, \omega_s' \, \mathrm{Im}\,\bar{g}(\omega_s', \omega_t)}{\omega_s'^2 - \omega_s^2} \ , \tag{33a}$$

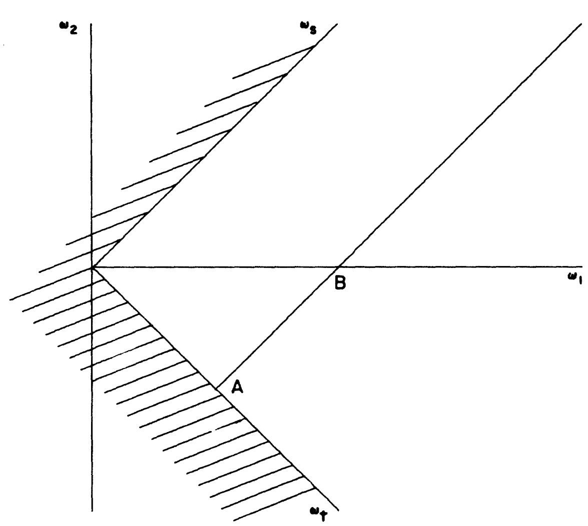

FIG. 1. The fundamental region of the ω_1, ω_2 plane, in which the functions $g(\omega_1, \omega_2)$ and $\bar{g}(\omega_s, \omega_t)$ may be defined, is shown unshaded. The dispersion relations apply to the values of $\bar{g}$ along a line of unit slope, such as the ray from A to B to infinity. The values of g along AB determine difference-frequency responses, at constant sum frequency, and the values along the rest of the ray determine sum-frequency responses, at constant difference frequency, according to Eq. (40).

$$\mathrm{Im}\,\bar{g}(\omega_s, \omega_t) = -\frac{2}{\pi} \omega_s P \int_0^\infty \frac{d\omega_s' \, \mathrm{Re}\,\bar{g}(\omega_s', \omega_t)}{\omega_s'^2 - \omega_s^2} \ . \tag{33b}$$

The values of $g(\omega_1, \omega_2)$ along a line of unit slope in the $\omega_s \geq 0$, $\omega_t \geq 0$ region contribute in the integration, as indicated in Fig. 1.

Consider next the interpretation of $g(\omega_1, \omega_2)$ as the response to pure applied frequencies. For a cause that is a pure tone, say

$$C_a(t) = A_a \cos(\omega_a t - \eta_a) \ , \tag{34}$$

Equations (23) and (26) give the effect

$$E_a(t) = \tilde{E}(\omega_a, \omega_a) + \tilde{E}(\omega_a, -\omega_a) \ , \tag{35}$$

where the first term gives the double-frequency response

$$\tilde{E}(\omega_a, \omega_a) = \tfrac{1}{2} A_a^2 |g(\omega_a, \omega_a)| \cos[2\omega_a t - 2\eta_a - \theta(\omega_a, \omega_a)] \tag{36}$$

and the second term is the constant

$$\tilde{E}(\omega_a, -\omega_a) = \tfrac{1}{2} A_a^2 g(\omega_a, -\omega_a) \ . \tag{37}$$

For a cause that is a superposition of two frequencies

$$C_{a \text{ and } b}(t) = A_a \cos(\omega_a t - \eta_a) + A_b \cos(\omega_b t - \eta_b) \ , \tag{38}$$

where our convention is that $\omega_a > \omega_b > 0$, the effect is

$$E_{a \text{ and } b}(t) = \tilde{E}(\omega_a, \omega_a) + \tilde{E}(\omega_a, -\omega_a) + \tilde{E}(\omega_b, \omega_b) + \tilde{E}(\omega_b, -\omega_b) + \tilde{E}(\omega_a, \omega_b) + \tilde{E}(\omega_a, -\omega_b). \tag{39}$$

The two new terms here are the sum and difference responses,

$$\tilde{E}(\omega_a, \pm\omega_b) = A_a A_b |g(\omega_a, \pm\omega_b)| \times \cos[(\omega_a \pm \omega_b)t - \eta_a \mp \eta_b - \theta(\omega_a, \pm\omega_b)] \ . \tag{40}$$

It is clear that all values of $g(\omega_1, \omega_2)$ can in principle be found this way and that no other types of response occur, even when the cause includes more than two frequencies.

The different types of response correspond to the boundaries and sub-areas of the fundamental region of the ω_1, ω_2 plane. Thus, in Fig. 1, the values of $g(\omega_a, \omega_a)$ for the double-frequency response occur along the upper boundary, the values $g(\omega_a, -\omega_a)$ for the constant occur along the lower boundary, the values $g(\omega_a, \omega_b)$ for the sum-frequency response are in the upper half of the region, and the values $g(\omega_a, -\omega_b)$ for the difference-frequency response are in the lower half.

In the dispersion relations of Eq. (33) the integration extends along a line of unit slope in the ω_2, ω_1 plane at constant $\omega_t = \omega_1 - \omega_2$. In the difference-frequency region this is at constant $\omega_a + \omega_b = \omega_t$ whereas in the sum-frequency region it is at constant $\omega_a - \omega_b = \omega_t$ as indicated in Fig. 2. At first glance one might think these two types of response would be unrelated but it is seen that, when the system is causal, they just give two parts of one smooth function.

The real and imaginary parts of the transform of the response function come in separately when certain time averages are taken. The results are

$$\langle \bar{E}(\omega_a, \omega_a)[C_a(t)]^2\rangle = \tfrac{1}{8} A_a^4 \operatorname{Re} g(\omega_a, \omega_a), \quad (41a)$$

$$\langle \bar{E}(\omega_a, -\omega_a)[C_a(t)]^2\rangle = \tfrac{1}{4} A_a^4 g(\omega_a, -\omega_a), \quad (41b)$$

$$\langle \bar{E}(\omega_a, \pm\omega_b)[C_{a\text{ and }b}(t)]^2\rangle = \tfrac{1}{2} A_a^2 A_b^2 \operatorname{Re} g(\omega_a, \pm\omega_b), \quad (41c)$$

where the angular brackets indicate an average over an indefinitely long time. One notes that $g(\omega_a, -\omega_a)$ itself is real. The same equations apply with $\bar{E}(\omega_1, \omega_2)$ replaced by $d\bar{E}(\omega_1, \omega_2)/dt$ and Re $g(\omega_1, \omega_2)$ replaced by $(\omega_1 + \omega_2)$ Im $g(\omega_1, \omega_2)$.

IV. THIRD DEGREE SYSTEM

Here cause and effect are related by

$$E(t) = \int dt' \int dt'' \int dt''' \, G(t-t', t-t'', t-t''') \times C(t')C(t'')C(t'''), \quad (42)$$

where

$$G(t_1, t_2, t_3) = G(t_2, t_1, t_3) = G(t_1, t_3, t_2), \quad (43)$$

$$G(t_1, t_2, t_3) = 0 \text{ if } t_1 < 0. \quad (44)$$

The transform of the response function,

$$g(\omega_1, \omega_2, \omega_3) = \int dt_1 \int dt_2 \int dt_3 \, G(t_1, t_2, t_3) \times \exp[i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)], \quad (45)$$

has the symmetries

$$g(\omega_1, \omega_2, \omega_3) = g(\omega_2, \omega_1, \omega_3) = g(\omega_1, \omega_3, \omega_2), \quad (46)$$

$$g^*(\omega_1, \omega_2, \omega_3) = g(-\omega_1, -\omega_2, -\omega_3). \quad (47)$$

These symmetries lead to a fundamental region in which g may be defined and to a canonical set of frequency variables. The property $g(\omega_1, \omega_2, \omega_3) = g(\omega_2, \omega_1, \omega_3)$ means that, in the space $(\omega_1, \omega_2, \omega_3)$ the function has reflection symmetry in a plane through the ω_3 axis and midway between the ω_1 and ω_2 axes. There are three such symmetry planes, as indicated in Fig. 3, which intersect in the ray $(\vec{i}+\vec{j}+\vec{k})$ that makes equal angles with all three axes. These planes define six 60° wedges and the symmetry means that the function only needs to be defined inside any one of them. We choose to use the wedge with $\omega_1 > \omega_2$ and $\omega_2 > \omega_3$ as indicated. Furthermore, the crossing symmetry of Eq. (47) implies that only half of the wedge is needed, on one side only of the plane through the origin and perpendicular to $\vec{i}+\vec{j}+\vec{k}$. We choose to use the half

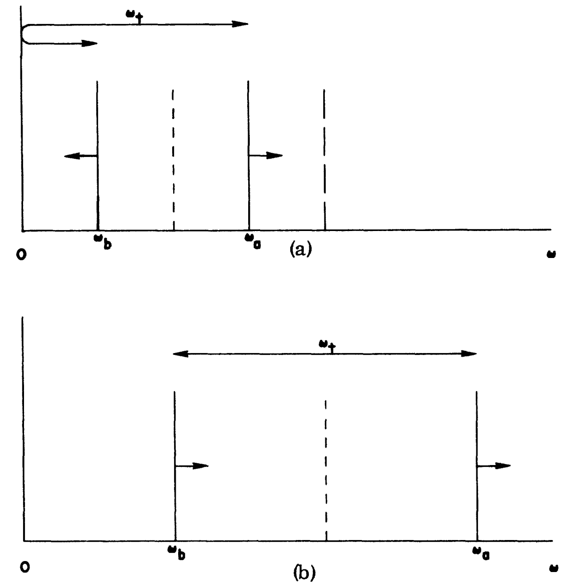

FIG. 2. The applied frequencies ω_a and ω_b needed to determine the values of the transform function $g(\omega_1, \omega_2) = \bar{g}(\omega_s, \omega_t)$ that occur in the dispersion relations, Eqs. (33). The variable ω_s ranges from zero to infinity, ω_t remains constant, as indicated by the line extending from A to B to infinity in Fig. 1. Figure (a) above applies to the AB segment. At $\omega_s = 0$, point A, one has $\omega_a = \omega_b = \frac{1}{2}\omega_t$ and both ω_a and ω_b are at the dotted line. As ω_s increases ω_a and ω_b vary as indicated by the arrows. In this region $\omega_s = \omega_a - \omega_b$ and $\omega_t = \omega_a + \omega_b$, the fixed value. At $\omega_s = \omega_t$, point B, one has $\omega_a = \omega_t$, the dashed line, and $\omega_b = 0$. Figure (b) applies to the rest of the ray, past point B. There $\omega_s = \omega_a + \omega_b$ and $\omega_t = \omega_a - \omega_b$, the fixed value, so ω_a and ω_b increase with ω_s, keeping constant distance between, as indicated by the arrows.

on the same side as the coordinate axes. It is not important which half-infinite wedge you choose to work with. A different choice at this point would lead to different labels at intermediate steps, especially Eqs. (48) below, but not to any different ideas or final results.

This fundamental region suggests what frequency variables to use in the problem:

$$\omega_s = \omega_1 + \omega_2 + \omega_3, \quad (48a)$$

$$\omega_t = \omega_1 - \omega_2, \quad (48b)$$

$$\omega_u = \omega_2 - \omega_3. \quad (48c)$$

The boundaries of the wedge are $\omega_s = 0$, $\omega_t = 0$, and $\omega_u = 0$ and these variables remain positive throughout this fundamental volume, forming a nonorthogonal set of coordinates.

In terms of these variables a new function is defined by

$$\bar{g}(\omega_s, \omega_t, \omega_u) = g(\omega_1, \omega_2, \omega_3) \quad (49)$$

and Eq. (45) leads to

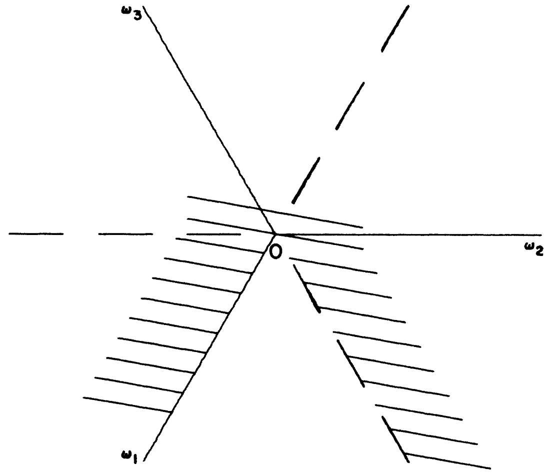

FIG. 3. The fundamental region in which the transform function $g(\omega_1, \omega_2, \omega_3)$ may be defined. The three coordinate axes $\omega_1, \omega_2, \omega_3$ are viewed in the $-\vec{i}-\vec{j}-\vec{k}$ direction. The reflection planes are indicated by the dashed lines. The $\omega_1 > \omega_2$ and $\omega_2 > \omega_3$ wedge is shown unshaded. Only the half of the wedge above the plane through the origin and perpendicular to $\vec{i}+\vec{j}+\vec{k}$ is needed.

$$\bar{g}(\omega_s, \omega_t, \omega_u) = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3\, G(t_1, t_2, t_3) \times \exp\{\tfrac{1}{3} i[\omega_s(t_1+t_2+t_3) + \omega_t(2t_1 - t_2 - t_3) + \omega_u(t_1 + t_2 - 2t_3)]\} \,. \tag{50}$$

For fixed ω_t and ω_u, since $(t_1+t_2+t_3)$ is always positive, this equation serves to define $\bar{g}(\omega_s, \omega_t, \omega_u)$ as an analytic function of a complex variable ω_s in the upper half plane.

Although $\bar{g}$ has the correct analyticity in ω_s, one cannot immediately write dispersion relations completely analogous to those of Kramers and Kronig because $\bar{g}$ lacks the correct crossing symmetry. However, it does have the property

$$\bar{g}^*(\omega_s, \omega_t, \omega_u) = \bar{g}(-\omega_s, \omega_u, \omega_t), \tag{51}$$

as follows from Eqs. (46) and (47). Consequently, one introduces the functions

$$\bar{g}_1(\omega_s, \omega_t, \omega_u) = \tfrac{1}{2}\bar{g}(\omega_s, \omega_t, \omega_u) + \tfrac{1}{2}\bar{g}(\omega_s, \omega_u, \omega_t), \tag{52a}$$

$$\bar{g}_2(\omega_s, \omega_t, \omega_u) = \tfrac{1}{2} i\bar{g}(\omega_s, \omega_t, \omega_u) - \tfrac{1}{2} i\bar{g}(\omega_s, \omega_u, \omega_t). \tag{52b}$$

These each have the property

$$\bar{g}_i^*(\omega_s, \omega_t, \omega_u) = \bar{g}_i(-\omega_s, \omega_t, \omega_u)$$

and are analytic functions of complex ω_s in the upper half plane. Assuming no subtractions, one can write

$$\operatorname{Re}\bar{g}_i(\omega_s, \omega_t, \omega_u) = \frac{2}{\pi} P\int_0^\infty \frac{d\omega_s'\, \omega_s' \operatorname{Im}\bar{g}_i(\omega_s', \omega_t, \omega_u)}{\omega_s'^2 - \omega_s^2}\,, \tag{53a}$$

$$\operatorname{Im}\bar{g}_i(\omega_s, \omega_t, \omega_u) = -\frac{2}{\pi}\omega_s P\int_0^\infty \frac{d\omega_s' \operatorname{Re}\bar{g}_i(\omega_s', \omega_t, \omega_u)}{\omega_s'^2 - \omega_s^2}\,. \tag{53b}$$

Consider next the interpretation of $g(\omega_1, \omega_2, \omega_3)$ as the response to pure applied frequencies. For the pure-tone cause,

$$C_a(t) = A_a \cos(\omega_a t - \eta_a), \tag{54}$$

where $\omega_a > 0$, Eqs. (42) and (45) give the effect

$$E_a(t) = \tilde{E}(\omega_a, \omega_a, \omega_a) + \tilde{E}(\omega_a, \omega_a, -\omega_a), \tag{55}$$

where the first term gives the triple-frequency response

$$\tilde{E}(\omega_a, \omega_a, \omega_a) = \tfrac{1}{4} A_a^3 |g(\omega_a, \omega_a, \omega_a)| \times \cos[3\omega_a t - 3\eta_a - \theta(\omega_a, \omega_a, \omega_a)] \tag{56}$$

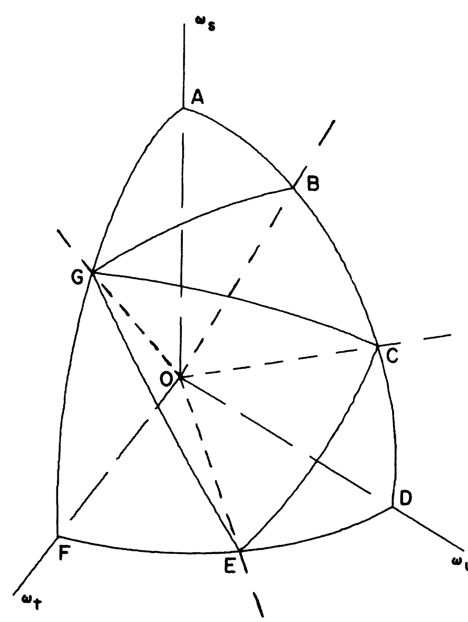

FIG. 4. Sketch of the fundamental wedge-shaped region for the transform function $g(\omega_1, \omega_2, \omega_3) = \bar{g}(\omega_s, \omega_t, \omega_u)$, showing the various subregions. The corner of the wedge is at the origin O and the sharp edge of the wedge is along the ray OA so that angle FOD is 60°. Intersection with a finite sphere is shown for purposes of visualization. However the faces, rays, and volumes extend radially to infinity. The equations of the planes OGB, OGC, OGE, OEC are $\omega_s - \omega_t - 2\omega_u = 0$, $2\omega_s - 2\omega_t - \omega_u = 0$, $\omega_s - \omega_t + \omega_u = 0$, $2\omega_s + \omega_t - \omega_u = 0$, respectively.

TABLE I. Regions that correspond to different types of response to two input frequencies ω_a and ω_b, where $\omega_a > \omega_b > 0$.

Output frequency	Parametric description of the region	Location in Fig. 4
$2\omega_a + \omega_b$	$\omega_s = 2\omega_a + \omega_b$, $\omega_t = 0$, $\omega_u = \omega_a - \omega_b$	*OAB*
$2\omega_a - \omega_b$	$\omega_s = 2\omega_a - \omega_b$, $\omega_t = 0$, $\omega_u = \omega_a + \omega_b$	*OBC*
ω_b	$\omega_s = \omega_b$, $\omega_t = \omega_a - \omega_b$, $\omega_u = \omega_a + \omega_b$	*OCE*
$2\omega_b + \omega_a$	$\omega_s = 2\omega_b + \omega_a$, $\omega_t = \omega_a - \omega_b$, $\omega_u = 0$	*OGA*
$2\omega_b - \omega_a$	$\omega_s = 2\omega_b - \omega_a$, $\omega_t = 0$, $\omega_u = \omega_a + \omega_b$	*OCD*
$\omega_a - 2\omega_b$	$\omega_s = \omega_a - 2\omega_b$, $\omega_t = \omega_a + \omega_b$, $\omega_u = 0$	*OGF*
ω_a	$\omega_s = \omega_a$, $\omega_t = \omega_a - \omega_b$, $\omega_u = 2\omega_b$	*OGC*

and the second term is the response at the original frequency

$$\tilde{E}(\omega_a, \omega_a, -\omega_a) = \tfrac{3}{4} A_a^3 |g(\omega_a, \omega_a, -\omega_a)| \times \cos[\omega_a t - \eta_a - \theta(\omega_a, \omega_a, -\omega_a)] . \quad (57)$$

In general, each type of response involves the transform $g(\omega_1, \omega_2, \omega_3)$ in or on some part of the fundamental volume. For the triple-frequency response the values of g along the line $\omega_1 = \omega_a$, $\omega_2 = \omega_a$, $\omega_3 = \omega_a$ or $\omega_s = 3\omega_a$, $\omega_t = 0$, $\omega_u = 0$ (ω_a considered as parameter) contribute. This is the edge of the wedge, ray *OA* in Fig. 4. For the single-frequency response the values of g along the line $\omega_s = \omega_a$, $\omega_t = 0$, $\omega_u = 2\omega_a$ contribute. This is ray *OC* in Fig. 4.

The cause which includes two frequencies

$$C_{a\,\text{and}\,b}(t) = A_a \cos(\omega_a t - \eta_a) + A_b \cos(\omega_b t - \eta_b) \quad (58)$$

(where $\omega_a > \omega_b > 0$) produces this effect:

$$E_{a\,\text{and}\,b}(t) = \tilde{E}(\omega_a, \omega_a, \omega_b) + \tilde{E}(\omega_a, \omega_a, -\omega_b) + \tilde{E}(\omega_a, -\omega_a, \omega_b) + \tilde{E}(\omega_b, \omega_b, \omega_a) + \tilde{E}(\omega_b, \omega_b, -\omega_a) + \tilde{E}(\omega_b, -\omega_b, \omega_a) + E_a(t) + E_b(t). \quad (59)$$

Here the new types of terms that come in are given by

$$\tilde{E}(\omega_a, \omega_a, \pm\omega_b) = \tfrac{3}{4} A_a^2 A_b |g(\omega_a, \omega_a, \pm\omega_b)| \times \cos[(2\omega_a \pm \omega_b)t - 2\eta_a \mp \eta_b - \theta(\omega_a, \omega_a, \pm\omega_b)] , \quad (60)$$

$$\tilde{E}(\omega_a, -\omega_a, \omega_b) = \tfrac{3}{2} A_a^2 A_b |g(\omega_a, -\omega_a, \omega_b)| \times \cos[\omega_b t - \eta_b - \theta(\omega_a, -\omega_a, \omega_b)], \quad (61)$$

and other equations that are the same except with subscripts a and b interchanged. Each type of output frequency occurs on a plane in or on the fundamental region, given conveniently by using ω_a and ω_b as parameters. Table I gives the equations of the planes and their locations in Fig. 4. In determining these results one must give special consideration to the $\cos[(2\omega_b - \omega_a)t\text{-phase}]$ term; in case $2\omega_b > \omega_a$ this is a frequency $2\omega_b - \omega_a$ response, in case $2\omega_b < \omega_a$ it is a frequency $\omega_a - 2\omega_b$.

Finally, we consider a cause which includes three frequencies

TABLE II. Regions that correspond to different types of response to three input frequencies ω_a, ω_b, and ω_c, where $\omega_a > \omega_b > \omega_c > 0$.

Output frequency	Parametric description of the region	Location in Fig. 4
$\omega_a + \omega_b + \omega_c$	$\omega_s = \omega_a + \omega_b + \omega_c$, $\omega_t = \omega_a - \omega_b$, $\omega_u = \omega_b - \omega_c$	*OAGB*
$\omega_a + \omega_b - \omega_c$	$\omega_s = \omega_a + \omega_b - \omega_c$, $\omega_t = \omega_a - \omega_b$, $\omega_u = \omega_b + \omega_c$	*OGBC*
$\omega_a - \omega_b + \omega_c$	$\omega_s = \omega_a - \omega_b + \omega_c$, $\omega_t = \omega_a - \omega_c$, $\omega_u = \omega_b + \omega_c$	*OGCE*
$\omega_a - \omega_b - \omega_c$	$\omega_s = \omega_a - \omega_b - \omega_c$, $\omega_t = \omega_a + \omega_c$, $\omega_u = \omega_b - \omega_c$	*OGFE*
$\omega_b + \omega_c - \omega_a$	$\omega_s = \omega_b + \omega_c - \omega_a$, $\omega_t = \omega_b - \omega_c$, $\omega_u = \omega_a + \omega_c$	*OEDC*

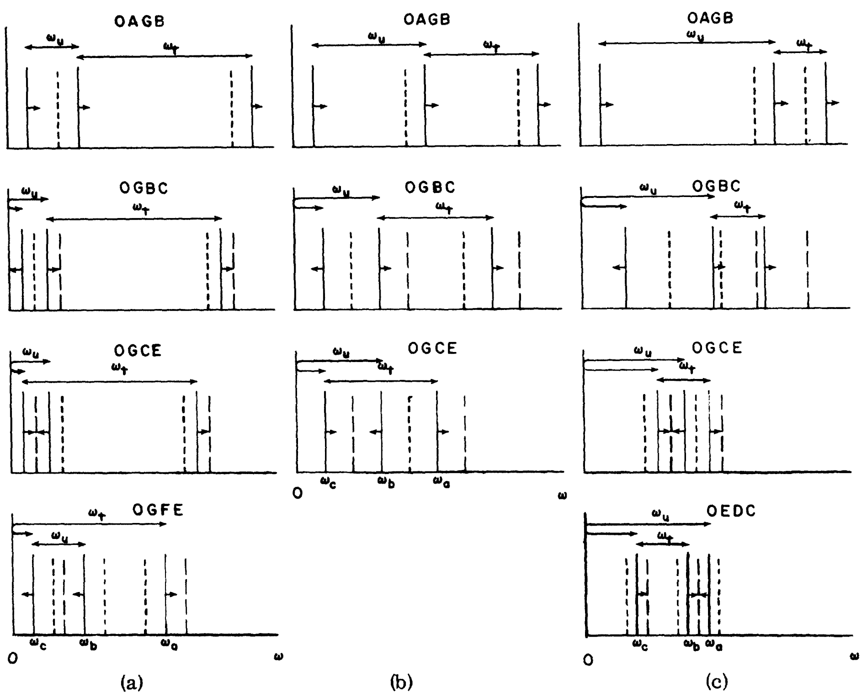

FIG. 5. System for varying the applied frequencies to determine the responses contributing in a dispersion relation. The three applied frequencies ω_a, ω_b, ω_c are indicated by solid lines in each diagram, where $\omega_a > \omega_b > \omega_c > 0$. Column (a) applies when $\omega_t > \omega_u$, column (b) when $\omega_t = \omega_u$, and column (c) when $\omega_t < \omega_u$. There are different diagrams for the different regions shown in Fig. 4 as indicated. The direction in which the frequencies change as ω_s increases, ω_t and ω_u remaining constant, is indicated by arrows. The dotted lines show where the frequencies start at the lower boundaries of the regions, the dashed lines show where the frequencies are at the upper boundaries. The starting values, at $\omega_s = 0$, are (a) $\omega_a = \frac{1}{3}(2\omega_t + \omega_u)$, $\omega_b = \frac{1}{3}(\omega_t + 2\omega_u)$, $\omega_c = \frac{1}{3}(\omega_t - \omega_u)$; (b) $\omega_a = \omega_t$, $\omega_b = \omega_t$, $\omega_c = 0$; (c) $\omega_a = \frac{1}{3}(\omega_t + 2\omega_u)$, $\omega_b = \frac{1}{3}(2\omega_t + \omega_u)$, $\omega_c = \frac{1}{3}(\omega_u - \omega_t)$. The values used for ω_t and ω_u in (a) were used for ω_u and ω_t in (c). Conjugate pairs like this are needed to determine $\bar{g}_1$ and $\bar{g}_2$.

$$C_{a,\,b,\,\text{and}\,c}(t) = A_a \cos(\omega_a t - \eta_a) + A_b \cos(\omega_b t - \eta_b) + A_c \cos(\omega_c t - \eta_c), \tag{62}$$

where $\omega_a > \omega_b > \omega_c > 0$. The effect produced is

$$E_{a,\,b,\,\text{and}\,c}(t) = \tilde{E}(\omega_a, \omega_b, \omega_c) + \tilde{E}(\omega_a, \omega_b, -\omega_c) + \tilde{E}(\omega_a, \omega_c, -\omega_b) + \tilde{E}(\omega_b, \omega_c, -\omega_a) + \cdots , \tag{63}$$

where the dots indicate responses to single-frequency and two-frequency causes as already discussed and where these new types of response are given by

$$\tilde{E}(\omega_a, \omega_b, \pm\omega_c) = \tfrac{3}{2} A_a A_b A_c \left| g(\omega_a, \omega_b, \pm\omega_c)\right| \times \cos[(\omega_a + \omega_b \pm \omega_c)t - \eta_a - \eta_b \mp \eta_c - \theta(\omega_a, \omega_b, \pm\omega_c)] \tag{64}$$

and other equations with the subscripts interchanged. Here each type of output frequency occurs in a certain subvolume of the fundamental region, given conveniently by using ω_a, ω_b, and ω_c as parameters, see Table II. Special consideration must be given to the $\cos[(\omega_b + \omega_c - \omega_a)t\text{-phase}]$ term; it gives frequency $\omega_b + \omega_c - \omega_a$ or $\omega_a - \omega_b - \omega_c$ depending on whether $\omega_b + \omega_c$ is greater than or less than ω_a.

In the dispersion relations of Eqs. (53) the integration extends along a line parallel to the ω_s axis at fixed values of ω_t and ω_u. Three cases must be distinguished, depending on where the integration starts in the $\omega_s = 0$ plane (Fig. 4). If $\omega_t > \omega_u$ so the integration starts in area *OFE* it goes through region *OGFE* and three others. If $\omega_t = \omega_u$ it starts on line *OE* and goes through three regions only. In this case $g = g_1$ and the dispersion relations apply to the function $\bar{g}(\omega_s, \omega_t, \omega_u)$ directly. Finally, if $\omega_t < \omega_u$ the integration starts in area *OED*, and goes through region *OEDC* and the three higher. To observe the values of $g(\omega_1, \omega_2, \omega_3)$ that contribute in a dispersion relation one must vary ω_a, ω_b, ω_c in a special way, different in each subvolume, as illustrated in Fig. 5.

As in the linear and quadratic cases, the real and imaginary parts of g can be found separately if certain types of time averages can be made. The formal results are

$$\langle \tilde{E}(\omega_a, \omega_a, \omega_a)[C_a(t)]^3\rangle = \tfrac{1}{32} A_a^6 \,\mathrm{Re}\, g(\omega_a, \omega_a, \omega_a) , \tag{65a}$$

$$\langle \tilde{E}(\omega_a, \omega_a, -\omega_a)[C_a(t)]^3\rangle = \tfrac{9}{32} A_a^6 \,\mathrm{Re}\, g(\omega_a, \omega_a, -\omega_a), \tag{65b}$$

$$\langle \tilde{E}(\omega_a, \omega_a, \pm\omega_b)[C_{a\,\text{and}\,b}(t)]^3\rangle = \tfrac{9}{32} A_a^4 A_b^2 \,\mathrm{Re}\, g(\omega_a, \omega_a, \pm\omega_b), \tag{65c}$$

$$\langle \tilde{E}(\omega_a, -\omega_a, \omega_b)[C_{a\,\text{and}\,b}(t)]^3\rangle = \tfrac{9}{16} A_a^5 A_b \,\mathrm{Re}\, g(\omega_a, -\omega_a, \omega_b), \tag{65d}$$

$$\langle \tilde{E}(\omega_a, \omega_b, \pm\omega_c)[C_{a,\,b,\,\text{and}\,c}(t)]^3\rangle = \tfrac{9}{8}A_a^2A_b^2A_c^2 \times \mathrm{Re}\, g(\omega_a, \omega_b, \pm\omega_c), \quad (65e)$$

and similar equations with the subscripts interchanged. The same equations apply with $\tilde{E}(\omega_1, \omega_2, \omega_3)$ replaced by $d\tilde{E}(\omega_1, \omega_2, \omega_3)/dt$ and $\mathrm{Re}\, g(\omega_1, \omega_2, \omega_3)$ replaced by $(\omega_1 + \omega_2 + \omega_3)\,\mathrm{Im}\, g(\omega_1, \omega_2, \omega_3)$.

To actually use these time averages to determine $g(\omega_1, \omega_2, \omega_3)$, a possible procedure is to apply the various causes, select the response $\tilde{E}(\omega_1, \omega_2, \omega_3)$ by its frequency, and then make the time average. However, there is a minor difficulty in carrying out this program in certain special cases when different terms in Eqs. (59) or (63)occur at the same output frequency. As a first example, $C_{a\,\text{and}\,b}(t)$ produces effects at frequency ω_b in the single-frequency response $E(\omega_b, \omega_b, -\omega_b)$ and also in the two-frequency response $E(\omega_a, -\omega_a, \omega_b)$ so these responses cannot be separated by their frequencies. This is not a major difficulty because it only involves values of g on the inner surfaces *OCE* and *OGC* of Fig. 4 but does not prevent this type of determination of g elsewhere in the fundamental volume. There is a similar occurrence at ω_a. This first type of difficulty occurs whenever a response at an output frequency which is the same as one of several input frequencies is measured. As a second example, $C_{a,\,b,\,\text{and}\,c}(t)$ produces effects at frequency $2\omega_b + \omega_c$ in the two-frequency response $\tilde{E}(\omega_b, \omega_b, \omega_c)$ and also in the three-frequency response $\tilde{E}(\omega_a, \omega_b, -\omega_c)$ when $\omega_a = \omega_b + 2\omega_c$. There are several occurrences of this second type but again they only affect determination of g on certain internal planes in the fundamental volume.

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