Relation among spin operators and magnons

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We present an exact expression of spin operators in terms of magnons and quasielectrons in ferromagnetism. Its relation to the Holstein-Primakoff expression is discussed. The result shows that the spin rotation of electrons is induced by a transformation of magnons and quasielectrons which is different from rotation.

I. INTRODUCTION

Holstein and Primakoff[1] in 1940 introduced a method to diagonalize the Hamiltonian in the exchange interaction model of a ferromagnet, by introducing second-quantized creation and annihilation operators a_l^* and a_l for magnons. In terms of such operators, the spin angular momentum operators are given by[1]

$$S_l^{(+)} = S_l^{(1)} + iS_l^{(2)} = (2S)^{1/2}(1 - a_l^* a_l/2S)^{1/2} a_l , \tag{1.1a}$$

$$S_l^{(-)} = S_l^{(1)} - iS_l^{(2)} = (2S)^{1/2} a_l^*(1 - a_l^* a_l/2S)^{1/2} , \tag{1.1b}$$

$$S_l^{(3)} = S - a_l^* a_l . \tag{1.1c}$$

Here S is the eigenvalue of the third component of total spin and the operators a_l^* and $a_{l'}$ satisfy the usual commutation relations for bosons:

$$[a_l, a_{l'}^*] \equiv a_l a_{l'}^* - a_{l'}^* a_l = \delta_{ll'} , \tag{1.2a}$$

$$[a_l, a_l'] = [a_l^*, a_{l'}^*] = 0 . \tag{1.2b}$$

The spin operators $S_l^{(+)}$, $S_l^{(-)}$, and $S^{(3)}$ satisfy the algebraic relations of $SU(2)$, generating the spin-group (rotation-group) transformations:

$$\begin{aligned} &[S_l^{(+)}, S_{l'}^{(-)}] = 2S_l^{(3)}\delta_{ll'} , \\ &[S_l^{(3)}, S_{l'}^{(\pm)}] = \pm S_l^{\pm}\delta_{ll'} . \end{aligned} \tag{1.3}$$

A crucial role in the Holstein-Primakoff method is played by the approximation used in writing the Hamiltonian in terms of a_l^* and a_l: Essentially they neglected all terms which were not bilinear in a_l^* and a_l. In this way they were able to construct a linear formalism suitable for practical calculations. In this approximation, the factor $(1 - a_l^* a_l/2S)^{1/2}$ in Eqs. (1.1a) and (1.1b) is put equal to 1; therefore the operators $S^{(\pm)}$ are replaced by $s^{(\pm)}$, which are defined by

$$\begin{aligned} &s_l^{(+)} = (2S)^{1/2} a_l , \\ &s_l^{(-)} = (2S)^{1/2} a_l^* , \\ &s_l^{(3)} = S_l^{(3)} = S - a_l^* a_l . \end{aligned} \tag{1.4}$$

These operators satisfy the following commutation relations:

$$\begin{aligned} &[s_l^{(3)}, s_{l'}^{(+)}] = s_l^{(+)}\delta_{ll'} , \\ &[s_l^{(3)}, s_{l'}^{(-)}] = - s_l^{(-)}\delta_{ll'} , \\ &[s_l^{(+)}, s_{l'}^{(-)}] = 2S\delta_{ll'} = (\text{const}) I \delta_{ll'} , \end{aligned} \tag{1.5}$$

where I is the unit matrix.

Let us now introduce

$$S^{(\alpha)} = \sum_l S_l^{(\alpha)} , \tag{1.6}$$

$$s^{(\alpha)} = \sum_l s_l^{(\alpha)} , \tag{1.7}$$

with $\alpha = +, -, 3$ and $s^{\pm} = s^{(1)} \pm is^{(2)}$.

Since $S_l^{(\alpha)}$ and $s_l^{(\alpha)}$ are the operators associated with the position x_l, Eqs. (1.6) and (1.7) mean

$$S^{(\alpha)} = \sum_l S^{(\alpha)}(x_l) , \tag{1.8}$$

$$s^{(\alpha)} = \sum_l s^{(\alpha)}(x_l) . \tag{1.9}$$

Note that $S^{(\alpha)}$ are the spin-rotation generators.

The operators $s_l^{(1)}$ and $s_l^{(2)}$, respectively, generate the "field transformations"

$$\begin{aligned} &a_l \to a_l + i(S/2)^{1/2}\theta_1 , \\ &a_l \to a_l - i(S/2)^{1/2}\theta_2 \end{aligned} \tag{1.10}$$

for all l, while $s^{(3)}$ induces rotation around the third axis:

$$a_l \to e^{i\theta_3} a_l . \tag{1.11}$$

Here θ_1, θ_2, and θ_3 are continuous parameters of the transformations.

The main purpose of this paper is to prove, without any approximation, that when we express the spin-density operators $S^{(\alpha)}(x_l)$ in terms of quasiparticles and sum them over all the space points, then the result is not $S^{(\alpha)}$ but $s^{(\alpha)}$ with the boson Heisenberg operator a_l replaced by the free-field operator of the quasiboson (i.e., the magnon). We may express this situation by saying that $S^{(\alpha)}$ becomes $s^{(\alpha)}$ in the quasiparticle picture. To elaborate this statement, let us first note some characteristics of the quasiparticle picture. The theory of ferromagnetism begins with a Hamiltonian, say H, which is expressed in terms of the Heisenberg operators for the electrons [say $\psi_H(x)$]. The spin operators $S^{(\alpha)}$ generate the rotations of ψ_H [i.e., $\psi_H(x) \to \psi_H'(x)$] and the Hamiltonian is invariant under these rotations (spin rotational symmetry). On the other hand, the ferromagnetic phenomena are usually described in terms of quasiparticles (i.e., quasielectrons and magnons). Let $B(x)$ and $\phi(x)$ denote the magnon field and quasielectron field, respectively. To show the existence of magnons and quasielectrons, starting with the Hamiltonian, is a lengthy computation. Results of such a computation lead to an expression of ψ_H in terms of ϕ and B. Such an expression can be obtained only through calculation of *matrix elements* of ψ_H:

$$\langle i|\psi_H(x)|j\rangle = \langle i|F(x,\phi,B)|j\rangle . \tag{1.12}$$

Here $|i\rangle$ and $|j\rangle$ are quasiparticle states and F is a certain combination of ϕ and B.

Since quasiparticle fields satisfy certain free-field equations (i.e., linear homogeneous differential equations), their Hamiltonian is a free Hamiltonian H_0. Then, we obtain from Eq. (1.12) the relation

$$\begin{aligned}\langle i|[\psi_H(x),H]|j\rangle &= i\hbar\langle i|\frac{d}{dt}\psi_H(x)|j\rangle \\ &= i\hbar\langle i|\frac{d}{dt}F(x,\phi,B)|j\rangle \\ &= \langle i|[F(x,\phi,B),H_0]|j\rangle ,\end{aligned}$$

i.e.,

$$\langle i|[\psi_H(x),H]|j\rangle = \langle i|[F(x,\phi,B),H_0]|j\rangle .$$

In a similar way, we can obtain

$$\langle i|[[\psi_H(x),H],H]|j\rangle = \langle i|[[F(x,\phi,B),H_0],H_0]|j\rangle ,$$

and so on. In this sense, the free Hamiltonian H_0 acts as the time translation operator just as H does. As a matter of fact, all the matrix elements of H agree with those of H_0, i.e.,

$$\langle i|H|j\rangle = \langle i|H_0|j\rangle$$

when we adjust the c number of H_0 such that $\langle 0|H|0\rangle = \langle 0|H_0|0\rangle$. This does not mean that the quasiparticles do not interact. In fact, construction of the S matrix in Sec. III will show that they do interact among themselves despite the fact that they satisfy the free-field equations.[2]

In the quasiparticle picture one tries to express all the local operators in terms of quasiparticles. For example, we will present an expression of the spin-density operators $S^{(\alpha)}(x,\phi,B)$ in terms of the quasielectron ϕ and magnon B. In general, when we consider an operator A which is obtained by integrating a local density $A(x)$ over the space, we first express $A(x)$ in terms of quasiparticles [say $A(x,\phi,B)$] and then integrate it over the space. The result is the operator A in the quasiparticle picture. The path-integral method[3] presents a simple and general way of writing $A(x,\phi,B)$ [e.g., $F(x,\phi,B)$ in Eq. (1.12), $S^{(\alpha)}(x,\phi,B)$, etc.]. The path-integral method is used extensively in this paper.

In Sec. II, we present a simple derivation of magnons. In Sec. III, we prove the following: When the quasiparticles (ϕ, B) perform the $E(2)$-group transformations $(\phi\to\phi'; B\to B')$ then the transformations of Heisenberg operators, $S^{(\alpha)}(x,\phi,B)\to S^{(\alpha)}(x,\phi',B')$, are spin rotations. Furthermore, it will be shown that the $E(2)$ transformations of (ϕ, B) are the only ones which induce the spin rotations on $S^{(\alpha)}(x,\phi,B)$.

We therefore state that the rotational symmetry associated with electrons becomes hidden behind the ferromagnetic behavior in which the magnon exhibits the $E(2)$-group symmetry. We may say that, owing to spin-spin interactions, the rotational symmetry is dynamically rearranged[4,5] into the $E(2)$ symmetry and that the observable loss of rotational symmetry in a ferromagnet is a result of such a rearrangement of symmetries.

One reason why the observable symmetry group can be different from the original symmetry group is based on the fact that any macroscopic observation on an infinite system is a collection of local observations. For example, the spin rotation of the system is induced by creating spin rotation at every point. Therefore, there always exists a possibility that in each local observation, one misses an infinitesimal effect of the order of magnitude of $1/V$ (with the volume $V\to\infty$). This missing effect can be accumulated as a finite amount when it is integrated over the whole system. This is the origin of the difference between $\vec{S}$ and $\vec{s}$. Such a locally infinitesimal effect is called the infrared effect. In Sec. IV, it will be proved that the difference of $\vec{S}$ and $\vec{s}$ is indeed an infrared effect.

Let us now introduce

$$S_f^{(\alpha)} = \sum_l f^{(\alpha)}(x_l) S^{(\alpha)}(x_l) , \tag{1.13}$$

$$s_f^{(\alpha)} = \sum_l f^{(\alpha)}(x_l) s^{(\alpha)}(x_l) , \tag{1.14}$$

where $\alpha = +, -, 3$ and $f^{(\alpha)}(x_l)$ are square-integrable functions which are equal to 1 in a large domain (range R). Contributions to $S_f^{(\alpha)}$ and $s_f^{(\alpha)}$ come only from the domain to which the functions $f^{(\alpha)}(\vec{x})$ are confined. In Sec. IV, we make an explicit calculation of the differences $S_f^{(\alpha)} - s_f^{(\alpha)}$. We will find that the matrix elements of the differences $S_f^{(\alpha)} - s_f^{(\alpha)}$ when $\alpha = \pm$, are of the order of magnitude of $1/V$, while $S_f^{(3)}$ and $s_f^{(3)}$ are exactly equal, due to the fact that the rotational symmetry around the third axis is preserved in observation. Since we must regard $1/V$ as zero, we obtain

$$\langle i | S_f^{(\pm)} | j \rangle = \langle i | s_f^{(\pm)} | j \rangle \, . \tag{1.15}$$

However, in the limit $f \to 1$ (i.e., $R \to \infty$) the integration of the effect of the order $1/V$ becomes finite, thus creating the difference between $S^{(\pm)}$ and $s^{(\pm)}$.

For the third components, we have

$$\langle i | S^{(3)} | j \rangle = \langle i | s^{(3)} | j \rangle \tag{1.16}$$

because there is no infrared effect for these quantities.

There is a mathematical reason for using $S_f^{(\pm)}$ and $s_f^{(\pm)}$ instead of $S^{(\pm)}$ and $s^{(\pm)}$. It has been known that the transformations in Eq. (1.10) with a_l replaced by $B(x_l)$ cannot be induced by any unitary operator (i.e., there are no generators). However, they can be regarded as limits of unitary transformations, as follows:

$$\lim_{f \to 1} \exp(-i\theta_i s_f^{(i)}) B(x_l) \exp(i\theta_i s_f^{(i)}) = B(x_l) \pm i(S/2)^{1/2}\theta_i \, , \tag{1.17}$$

where + (−) for $i = 1$ (2). Our study of rearrangement of symmetries in Sec. II, which relies on the path-integral method and does not require the use of generators, also reveals that the transformations in Eq. (1.10) should be regarded as the limit for $f^{(i)} \to 1$ of the following transformation:

$$B(x_l) \to B(x_l) \pm i(S/2)^{1/2}\theta_i f^{(i)}(x_l) \, . \tag{1.18}$$

Furthermore, we will find in Sec. III that $f^{(i)}(x)$ in Eq. (1.18) should satisfy the magnon equation for B. Thus the transformations in Eq. (1.18) are those under which the magnon equations are invariant: The original spin rotational invariance is replaced by the $E(2)$ transformation invariance in the quasiparticle formulation.

Indeed, it will be shown in Sec. III, that when the functions $f^{(i)}(x)$ satisfy the magnon equation, then the operators $s_f^{(\alpha)}$ are independent of time, and the transformations in Eq. (1.18) induce the magnon condensations.

Although our conclusion is that $S^{(\alpha)}$ becomes $s^{(\alpha)}$ in the quasiparticle picture, this does not justify the Holstein-Primakoff approximation. In the Holstein-Primakoff paper, the operator a_l acts as the Heisenberg operator and not as an operator for the quasiboson. When a_l is expressed in terms of the quasiboson, i.e., magnon B, the expression is a very complicated one. In fact, a_l is an infinite power series in normal products of B and $B^\dagger$. Thus we are not considering the linear approximation, which has been clarified by Dyson.[6] To establish the quasiparticle picture, we need to consider infinite-power expansions of Heisenberg operators in terms of creation-annihilation operators of quasiparticles. Through this process the quasiboson (magnon) emerges as a bound state of electrons.[7]

We find in this paper that, when the spin operators $\vec{S}$ are expressed in terms of B and not in terms of Heisenberg operators, they take the form shown in Eq. (1.4), with a_l replaced by $B(x_l)$. We used the notation B for magnon instead of a_l in order to avoid a possible confusion between quasiparticles and Heisenberg operators.

The explicit expression of the spin operators in terms of the Heisenberg operators depends on a specific model. For example, the expressions (1.1a)–(1.1c) in terms of a boson Heisenberg operator a_l presents a boson model. In this paper we assume that the spin operator is made up of electron Heisenberg fields (electron model). In this model, the quasiboson is a bound state made of original electrons. However, this assumption is really not required, because our arguments for magnons in the following sections do not use any specific assumption for the structure of spin operators. Our only requirement is the invariance of the Lagrangian under the spin rotation. Furthermore, our arguments apply to both the localized spin case and the continuous spin distribution. Therefore, our results are true for the Heisenberg model and the itinerant-electron model of the ferromagnet.

Summarizing, we conclude that the three-parameter rotation symmetry is not simply reduced to the cylindrical rotation symmetry of one parameter, but is replaced by the $E(2)$ symmetry of three parameters. In observation, we can detect the $E(2)$ symmetry, for example, through the well-known low-energy theorem, which states that reactions among magnons vanish in the low-momentum limit. We shall show in Sec. V that this theorem is based on the $E(2)$ symmetry.

In this paper, all considerations are restricted to the $T = 0\,^\circ$K case.

II. EXISTENCE OF MAGNONS

The ferromagnetic systems are characterized by the Lagrangian, which is made of the electron field $\psi(x)$:

$$\psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \, .$$

The Lagrangian is invariant under the spin rotation

$$\psi(x) \to e^{i\theta_i \lambda_i} \psi(x) \, , \quad i = 1, 2, 3. \tag{2.1}$$

Here θ_i are real parameters and λ_i are the spin matrices

$$\lambda_i = \tfrac{1}{2}\sigma_i \;, \tag{2.2}$$

where σ_i are the Pauli matrices.

The path-integral formalism begins with the generating functional

$$W[J,j,n] = \frac{1}{N}\int [d\psi][d\psi^\dagger]\exp\Big(i\int d^4x\, \{\mathcal{L}[\psi(x)] + J^\dagger(x)\psi(x) + \psi^\dagger(x)J(x) + j^\dagger(x)S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) + S_\psi^{(3)}(x)n(x) - i\epsilon S_\psi^{(3)}(x)\}\Big) \tag{2.3a}$$

where

$$N = \int [d\psi][d\psi^\dagger]\exp\Big(i\int d^4x\,\{\mathcal{L}[\psi(x)] - i\epsilon S_\psi^{(3)}(x)\}\Big). \tag{2.3b}$$

Here $S_\psi^{(\alpha)}(x)$ ($\alpha = +, -, 3$) is the spin density which is made of $\psi(x)$. In the following we do not need the explicit form of $S_\psi^{(\alpha)}(x)$ in terms of $\psi(x)$. The electron fields ψ, $\psi^\dagger$ and their sources $J^\dagger$, J anticommute, while the sources of spin fields j, $j^\dagger$ are c numbers. The ϵ term in Eqs. (2.3a) and (2.3b) is a symmetry-breaking term; we perform the limit $\epsilon \to 0$ at the end of the computations. The role played by this term will be explained later.

Let $F[\psi]$ denote a linear combination of products of ψ and $\psi^\dagger$. We will use the following notation:

$$\langle F[\psi]\rangle_{\epsilon,j,J,n} \equiv \frac{1}{N}\int [d\psi][d\psi^\dagger]F[\psi]\exp\Big(i\int d^4x\{\mathcal{L}[\psi(x)] + J^\dagger(x)\psi(x) + \psi^\dagger(x)J(x) + j^+(x)S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) + S_\psi^{(3)}(x)n(x) - i\epsilon S_\psi^{(3)}(x)\}\Big),$$

$$\langle F[\psi]\rangle_{\epsilon,j,n} \equiv \langle F[\psi]\rangle_{\epsilon,j,J,n} \quad \text{with } J\to 0 \;, \tag{2.4}$$

$$\langle F[\psi]\rangle_{\epsilon,J,n} \equiv \langle F[\psi]\rangle_{\epsilon,j,J,n} \quad \text{with } j\to 0 \;,$$

$$\langle F[\psi]\rangle_{\epsilon} \equiv \langle F[\psi]\rangle_{\epsilon,j,J,n} \quad \text{with } j\to 0,\; J\to 0,\; n\to 0 \;,$$

$$\langle F[\psi]\rangle \equiv \langle F[\psi]\rangle_{\epsilon,j,J,n} \quad \text{with all } j, J, n, \epsilon \to \text{zero.}$$

Generally, the parameters that are missing in the suffixes are taken to be zero.

An important fact in the path-integral formalism is that the quantity $\langle F[\psi]\rangle$ agrees with the ground-state expectation value of $T(F[\psi_H(x)])$, in which all the products of Heisenberg operators ψ_H and $\psi_H^\dagger$ are chronological products

$$\langle F[\psi]\rangle = \langle 0|\, T(F[\psi_H(x)])\,|0\rangle \;.$$

Since each functional derivative $\delta/\delta J(x)$ [$\delta/\delta J^\dagger(x)$] acting on $W[J,j,n]$ brings down the factor $i\psi^\dagger(x)$ [$i\psi(x)$], the ground-state expectation value of any chronological product of $\psi^\dagger$ and ψ (i.e., the Green's function) can be obtained from $W[J,j,n]$ by repeated operations of $\delta/\delta J$ and $\delta/\delta J^\dagger$, followed by limits J, j, and n going to zero. The presence of sources j, $j^\dagger$, and n for the spin fields $S_\psi^{(\alpha)}(x)$ are not required for the calculation of the Green's functions. However, use of j, $j^\dagger$, and n makes it possible to study the behavior of spin in ferromagnets without specifying the explicit dependence of $S_\psi^{(\alpha)}(x)$ on $\psi(x)$.

In writing down (2.3a) and (2.3b) we have in our minds both the cases of localized spins and itinerant electrons. In the case of localized spin,

$$\int d^4x\{\mathcal{L}[\psi(x)] + J^\dagger(x)\psi(x) + \psi^\dagger(x)J(x) + j^\dagger(x)S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) + S_\psi^{(3)}(x)n(x) - i\epsilon S_\psi^{(3)}(x)\}$$

should be read as

$$\sum_l \int dt\,\{\mathcal{L}[\psi(x_l)] + J^\dagger(x_l)\psi(x_l) + \psi^\dagger(x_l)J(x_l) + j^\dagger(x_l)S_\psi^{(-)}(x_l) + S_\psi^{(+)}(x_l)j(x_l) + S_\psi^{(3)}(x_l)n(x_l) - i\epsilon S_\psi^{(3)}(x_l)\} \;. \tag{2.5}$$

In general, for the case of localized spin the following replacement is understood:

$$\int d^4x\, F(x) \to \sum_l \int dt\, F(x_l) \tag{2.6}$$

for any quantity $F(x)$.

Let us now put $J=0$ and $n=0$ and perform the change of variables (2.1) in the numerator of Eq. (2.3a). When θ_i are infinitesimal the change of variables (2.1) should induce rotation of the spin fields:

$$S_\psi^{(i)}(x) \to S_\psi^{(i)}(x) - \theta_j \epsilon_{ijk} S_\psi^{(k)}(x) \;. \tag{2.7}$$

Here ϵ_{ijk} is the completely antisymmetric tensor: $\epsilon_{ijk} = (-1)^p$, where p is the number of permutations of 1, 2, and 3. Since a change of variables does not influence the integration, we have

$$\frac{\partial W}{\partial \theta_l} = 0 \;. \tag{2.8}$$

This gives

$$\int d^4x\,\langle\{\epsilon_{1lk}[j(x) + j^\dagger(x)] + i\epsilon_{2lk}[j(x) - j^\dagger(x)] - i\epsilon\epsilon_{3lk}\}\, S_\psi^k(x)\rangle_{\epsilon,j} = 0 \;. \tag{2.9}$$

Operating $\delta/\delta j(y)$ on this and then putting $j=0$, we obtain

$$(\epsilon_{1lk} + i\epsilon_{2lk})\langle S_\psi^k(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x \times \langle S_\psi^{(k)}(x)S_\psi^{(+)}(y)\rangle_\epsilon \;. \tag{2.10}$$

In a similar way, use of $\delta/\delta j^\dagger(y)$ leads to

$$(\epsilon_{1lk} - i\epsilon_{2lk})\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x\,\langle S_\psi^{(k)}(x)S_\psi^{(-)}(y)\rangle_\epsilon \;. \tag{2.11}$$

Equations (2.10) and (2.11) give

$$\epsilon_{1lk}\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x\langle S_\psi^{(k)}(x)S_\psi^{(1)}(y)\rangle_\epsilon\,, \qquad (2.12)$$
$$\epsilon_{2lk}\langle S_\psi^{(k)}(y)\rangle_\epsilon = -\epsilon\epsilon_{3lk}\int d^4x\langle S_\psi^{(k)}(x)S_\psi^{(2)}(y)\rangle_\epsilon\,,$$

where

$$S_\psi^{(\pm)}(x) = S_\psi^{(1)}(x) \pm iS_\psi^{(2)}(x)\,.$$

When $l=3$, we get from Eqs. (2.12)

$$\langle S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(y)\rangle_\epsilon = 0\,. \qquad (2.13)$$

With $l=1$ and $l=2$ we have

$$\epsilon\int d^4x\langle S_\psi^{(2)}(x)S_\psi^{(1)}(y)\rangle_\epsilon = 0\,, \qquad (2.14)$$

$$\langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x\langle S_\psi^{(1)}(x)S_\psi^{(1)}(y)\rangle_\epsilon\,, \qquad (2.15)$$

$$\langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x\langle S_\psi^{(2)}(x)S_\psi^{(2)}(y)\rangle_\epsilon\,. \qquad (2.16)$$

Let us now write

$$\langle S_\psi^{(i)}(x)S_\psi^{(i)}(y)\rangle_\epsilon = i\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\rho_i(p)\times\left(\frac{1}{p_0-\omega_p+i\epsilon a_i}-\frac{1}{p_0+\omega_p-i\epsilon a_i}\right)+(\text{continuum contribution}),\quad i=1,2,3\,. \qquad (2.17)$$

Here we used the notation

$$p(x-y) = -\vec{p}\cdot(\vec{x}-\vec{y}) + ip_0(t_x-t_y)\,.$$

In Eq. (2.17) ω_p is the energy of a quasiparticle which is a bound state of electrons. We will prove that $\rho_i(p)\neq 0$, which proves the existence of such a bound state. The continuum contribution comes from those states which contain more than one quasiparticle. As is well known, the pole singularities in the Feynman Green's functions are defined by putting $\omega_p - i\eta$ for ω_p with an infinitesimal η. In (2.17) we introduced $a_i = \eta/\epsilon$, so that ω_p in the denominator carries $i\epsilon a_i$. The spectral functions $\rho_i(p)$ cannot be negative because $S_\psi^{(i)}$ are Hermitian.

When we consider the case of localized spins, Eq. (2.17) should be replaced by

$$\langle S_\psi^{(i)}(x_l)S_\psi^{(i)}(x_k)\rangle_\epsilon = iv\int\frac{dp_0}{2\pi}\int\frac{d^3p}{(2\pi)^3}e^{-ip(x_l-x_k)}\rho_i(p)\times\left(\frac{1}{p_0-\omega_p+i\epsilon a_i}-\frac{1}{p_0+\omega_p-i\epsilon a_i}\right)+(\text{continuum contribution})\,, \qquad (2.18)$$

in which v is the volume of the unit lattice and the integration of $\vec{p}$ is confined to the domain

$$-\pi/d < p_i < \pi/d\,,$$

where d is the lattice length. Equations (2.15) and (2.16) should be read as

$$\langle S_\psi^{(3)}(x_k)\rangle_\epsilon = \epsilon\sum_l\int dt\langle S_\psi^{(i)}(x_l)S_\psi^{(i)}(x_k)\rangle_\epsilon\,. \qquad (2.19)$$

Also, operating $[\delta/\delta j^\dagger(z)][\delta/\delta j(y)]$ and $[\delta/\delta j(z)][\delta/\delta j^\dagger(y)]$ on (2.9), putting then $j=0$ and subtracting, we obtain

$$\langle S_\psi^{(1)}(x)S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(x)S_\psi^{(2)}(y)\rangle_\epsilon\,,$$

which gives

$$\rho_1(p) = \rho_2(p)\,, \qquad (2.20)$$

$$a_1 = a_2\,. \qquad (2.21)$$

The magnetization is given by $g\mu_B\langle S_\psi^{(3)}(x)\rangle$, where μ_B is the Bohr magneton. We shall use the notation

$$M(\epsilon) \equiv \langle S_\psi^{(3)}(x)\rangle_\epsilon \qquad (2.22)$$

together with

$$M = \lim_{\epsilon\to 0} M(\epsilon)\,. \qquad (2.23)$$

Equations (2.15) and (2.16) then say that there should be a bound state of gapless energy:

$$\omega_p = 0 \quad \text{at } p=0\,. \qquad (2.24)$$

Indeed Eqs. (2.15) and (2.16) give

$$M(\epsilon) = i\epsilon\Delta_i(\epsilon,0),\quad i=1,2 \qquad (2.25)$$

which can lead to nonvanishing M with $\epsilon\to 0$ only when $\omega_p = 0$ at $p=0$.[8] We further have

$$M = 2\rho/a\,. \qquad (2.26)$$

In Eq. (2.25) we used

$$\Delta_i(\epsilon,p) = \rho_i(p)\left(\frac{1}{p_0-\omega_p+i\epsilon a_i}-\frac{1}{p_0+\omega_p-i\epsilon a_i}\right), \qquad (2.27)$$

and in Eq. (2.26) we put $\rho_1(0)=\rho_2(0)=\rho$ and $a_1 = a_2 = a$.

In the case of localized spins, derivation of Eq. (2.25) requires the formula

$$\frac{v}{(2\pi)^3}\sum_l e^{-ipx_l} = \delta^{(3)}(p)\,. \qquad (2.28)$$

Summarizing, we have shown that the relation (2.22) along with nonzero M requires the existence of gapless bosons, i.e., the magnons. The role of the ϵ term in Eq. (2.3) is now evident, because without this term we cannot realize the case $M \neq 0$.[5,9]

Let us now calculate ρ. Since M is the local spin density in the third direction, the total spin in this direction is NM, where N is the number of lattice points. Then the ground-state expectation value of $\vec{S}^2$ is given by

$$\langle 0|\vec{S}^2|0\rangle = NM(NM+1)\ . \tag{2.29}$$

Assuming $t_k < t_l$ in Eq. (2.18), with $i=1, 2$, and then performing the limit $t_k \to t_l$ (same results can be obtained by assuming $t_l < t_k$), we find, using Eq. (2.28), that

$$\langle 0|S^{(i)}S^{(i)}|0\rangle = \rho N \text{ for } i=1,2.$$

Thus

$$\langle 0|\vec{S}^2|0\rangle = 2\rho N + (NM)^2\ . \tag{2.30}$$

Comparing this with (2.29) we get

$$\rho = \tfrac{1}{2}M\ . \tag{2.31}$$

This gives

$$M/2\rho^{1/2} = (M/2)^{1/2}\ . \tag{2.32}$$

Equation (2.31) also shows that $a=1$ [cf. (2.26)].

III. DYNAMICAL REARRANGEMENT OF SYMMETRY

Let us introduce the field for the magnons:

$$B(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B_{\vec{k}} e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \tag{3.1}$$

and

$$B^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}} B^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\ . \tag{3.2}$$

The commutation relations are

$$[B(x), B^\dagger(y)]_{t_x = t_y} = \delta(\vec{x} - \vec{y})\ ,$$
$$[B(x), B(y)] = [B^\dagger(x), B^\dagger(y)] = 0\ . \tag{3.3}$$

Equations (3.1) and (3.2) satisfy the equations

$$K(\vec{\partial})B^\dagger(x) = 0\ ,$$
$$B(x)K(-\overleftarrow{\partial}) = 0\ , \tag{3.4}$$

with

$$K(\vec{\partial}) = -\left(i\frac{\vec{\partial}}{\partial t} + \omega\right)\ . \tag{3.5}$$

Here arrows on the derivatives indicate the directions in which the derivatives operate.

We write also the free-field equations for the quasielectron $\phi(x)$ as

$$\Lambda(\vec{\partial})\phi(x) = 0\ ,$$
$$\phi^\dagger(x)\Lambda(-\overleftarrow{\partial}) = 0\ . \tag{3.6}$$

Let us denote the S matrix and the spin-density operators for the Heisenberg electron field by $\mathcal{S}$ and $S^{(i)}(x)$, respectively. The next step is to express these quantities in terms of quasielectron ϕ and magnon B as $\mathcal{S}(\phi, \phi^\dagger, B, B^\dagger)$ and $S^{(i)}(x, \phi, \phi^\dagger, B, B^\dagger)$.[10] Use of the functional formalism together with the Lehmann-Symanzik-Zimmerman (LSZ) formula[11] gives[5,12]

$$\mathcal{S}(\phi, \phi^\dagger, B, B^\dagger) = \langle : \exp[-iA(\phi, \phi^\dagger, B, B^\dagger)] : \rangle \tag{3.7}$$

and

$$\mathcal{S}S^{(i)}(x, \phi, \phi^\dagger, B, B^\dagger) = \langle S^{(i)}_\psi(x) : \exp[-iA(\phi, \phi^\dagger, B, B^\dagger)] : \rangle\ , \tag{3.8}$$

where

$$A(\phi, \phi^\dagger, B, B^\dagger) = \int d^4x [\rho^{-1/2}B(x)K(\vec{\partial})S^{(-)}_\psi(x) + \rho^{-1/2}S^{(+)}_\psi(x)K(-\overleftarrow{\partial})B^\dagger(x) + Z^{-1/2}\phi^\dagger(x)\Lambda(\vec{\partial})\psi(x) + Z^{-1/2}\psi^\dagger(x)\Lambda(-\overleftarrow{\partial})\phi(x)] \tag{3.9}$$

and $i=1, 2, 3$. Here Z is the wave-function renormalization of the electron. Equation (3.8) together with (3.7) gives us the expressions of the spin-density operator in terms of quasiparticles. In (3.7) and (3.8), the symbol $:\cdots:$ means the normal products in which all the creation operators of quasiparticles are to the left of the annihilation operators. The symbol $\langle\ \rangle$ was introduced in (2.4).

Our task in this section is to study the following question: How do $\phi, \phi^\dagger, B, B^\dagger$ in (3.7) and (3.8) transform so that spin rotation of $S^{(i)}(\phi, \phi^\dagger, B, B^\dagger)$ is induced? Let us write the transformed fields $\phi_\theta(x)$, $\phi^\dagger_\theta(x)$, $B_\theta(x)$, $B^\dagger_\theta(x)$ as

$$\phi_\theta(x) = \Phi(x, \theta_i, \phi, \phi^\dagger, B, B^\dagger)\ ,$$
$$B_\theta(x) = \tilde{B}(x, \theta_i, \phi, \phi^\dagger, B, B^\dagger)\ , \tag{3.10}$$

etc., $i=1, 2, 3$, and require that they satisfy the equations for quasiparticles (3.6) and (3.4),

$$\Lambda(\vec{\partial})\phi_\theta(x) = 0\ ,$$
$$K(\vec{\partial})B^\dagger_\theta(x) = 0\ ,$$
$$\phi^\dagger_\theta(x)\Lambda(-\overleftarrow{\partial}) = 0\ ,$$
$$B_\theta(x)K(-\overleftarrow{\partial}) = 0\ , \tag{3.11}$$

and that

$$\frac{\partial}{\partial\theta_l}\mathcal{S}(\phi_\theta, \phi^\dagger_\theta, B_\theta, B^\dagger_\theta) = 0\ , \tag{3.12}$$

$$\frac{\partial}{\partial\theta_l}S^{(i)}(x, \phi_\theta, \phi^\dagger_\theta, B_\theta, B^\dagger_\theta) = -\epsilon_{ilk}S^{(k)}(x, \phi_\theta, \phi^\dagger_\theta, B_\theta, B^\dagger_\theta)\ . \tag{3.13}$$

When (3.7)–(3.9) are considered, the conditions (3.12) and (3.13) can be rewritten, respectively, as

$$\left\langle : -i\left(\frac{\partial}{\partial\theta_l}A_\theta\right) e^{-iA_\theta} : \right\rangle = 0\ , \quad l=1, 2, 3 \tag{3.14}$$

and

$$\left\langle S^{(i)}_\psi(y) : i\left(\frac{\partial}{\partial\theta_l}A_\theta\right) e^{-iA_\theta} : \right\rangle = \epsilon_{ilk}\langle S^{(k)}_\psi(y) : e^{-iA_\theta} : \rangle\ , \quad l=1, 2, 3 \tag{3.15}$$

where we use the notation $A_\theta \equiv A(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ and

$$\frac{\partial}{\partial\theta_l}A_\theta = \int d^4x \left[\rho^{-1/2}\left(\frac{\partial B_\theta(x)}{\partial\theta_l}\right)K(\vec{\partial})S_\psi^{(-)}(x) + \rho^{-1/2}S_\psi^{(+)}(x)K(-\overleftarrow{\partial})\left(\frac{\partial B_\theta^\dagger(x)}{\partial\theta_l}\right) + Z^{-1/2}\left(\frac{\partial\phi_\theta^\dagger(x)}{\partial\theta_l}\right)\Lambda(\vec{\partial})\psi(x) + Z^{-1/2}\psi^\dagger(x)\Lambda(-\overleftarrow{\partial})\left(\frac{\partial\phi_\theta(x)}{\partial\theta_l}\right)\right]. \tag{3.16}$$

Our problem is to solve Eqs. (3.14) and (3.15) for B_θ, $B_\theta^\dagger$, ϕ_θ, and $\phi_\theta^\dagger$. To do this, we note that when the choices

$$J(x) = -\Lambda(-\overleftarrow{\partial})\phi_\theta(x)Z^{-1/2}, \quad j(x) = -K(-\overleftarrow{\partial})B_\theta^\dagger(x)\rho^{-1/2}, \quad n(x) = 0 \tag{3.17}$$

are made, then $W[J, j, n]$ becomes the transformed S matrix $\mathcal{S}(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$. Now we need several formulas which are derived in the Appendix. Equations (A1), (A2), and (A3), together with (3.17), give, for $l = 3$,

$$\langle :\tilde{A}_\theta e^{-iA_\theta}: \rangle = 0, \tag{3.18}$$

$$\langle S_\psi^{(+)}(y) : e^{-iA_\theta} : \rangle = \langle S_\psi^{(+)}(y) : -i\tilde{A}_\theta e^{-iA_\theta} : \rangle, \tag{3.19}$$

and

$$\langle S_\psi^{(-)}(y) : e^{-iA_\theta} : \rangle = \langle S_\psi^{(-)}(y) : i\tilde{A}_\theta e^{-iA_\theta} : \rangle, \tag{3.20}$$

respectively. In (3.18)–(3.20) we used the notation

$$\tilde{A}_\theta \equiv \int d^4x [Z^{-1/2}\phi_\theta^\dagger(x)\Lambda(\vec{\partial})\lambda_3\psi(x) - Z^{-1/2}\psi^\dagger(x)\lambda_3\Lambda(-\overleftarrow{\partial})\phi_\theta(x) + \rho^{-1/2}B_\theta K(\vec{\partial})S_\psi^{(-)}(x) - \rho^{-1/2}S_\psi^{(+)}(x)K(-\overleftarrow{\partial})B_\theta^\dagger(x)]. \tag{3.21}$$

We shall now introduce the following formulas:

$$\lim_{\epsilon\to 0} \epsilon \int d^3x \langle S_\psi^{(1)}(x)Q\rangle_\epsilon = -i\int d^3x [\langle K(\vec{\partial})S_\psi^{(-)}(x)Q\rangle + \langle S_\psi^{(+)}(x)K(-\overleftarrow{\partial})Q\rangle], \tag{3.22}$$

$$\lim_{\epsilon\to 0} \epsilon \int d^4x \langle S_\psi^{(2)}(x)Q\rangle_\epsilon = \int d^4x [\langle K(\vec{\partial})S_\psi^{(-)}(x)Q\rangle - \langle S_\psi^{(+)}(x)K(-\overleftarrow{\partial})Q\rangle]. \tag{3.23}$$

Also these relations will be derived in the Appendix.

Using (3.22) and (3.23), we can write (A5) and (A11)–(A14) as

$$\int d^4x \langle [K(\vec{\partial})S_\psi^{(-)}(x) \pm S_\psi^{(+)}(x)K(-\overleftarrow{\partial})] : e^{-iA_\theta} : \rangle = 0, \tag{3.24}$$

$$\langle S_\psi^{(3)}(y) : e^{-iA_\theta} : \rangle = -i\int d^4x \langle S_\psi^{(1)}(y)[K(\vec{\partial})S_\psi^{(-)}(x) + S_\psi^{(+)}(x)K(-\overleftarrow{\partial})] : e^{-iA_\theta} : \rangle \tag{3.25}$$

$$= \int d^4x \langle S_\psi^{(2)}(y)[K(\vec{\partial})S_\psi^{(-)}(x) - S_\psi^{(+)}(x)K(-\overleftarrow{\partial})] : e^{-iA_\theta} : \rangle, \tag{3.26}$$

$$\langle S_\psi^{(2)}(y) : e^{-iA_\theta} : \rangle = -\int d^4x \langle S_\psi^{(3)}(y)[K(\vec{\partial})S_\psi^{(-)}(x) - S_\psi^{(+)}(x)K(-\overleftarrow{\partial})] : e^{-iA_\theta} : \rangle, \tag{3.27}$$

$$\langle S_\psi^{(1)}(y) : e^{-iA_\theta} : \rangle = i\int d^4x \langle S_\psi^{(3)}(y)[K(\vec{\partial})S_\psi^{(-)}(x) + S_\psi^{(+)}(x)K(-\overleftarrow{\partial})] : e^{-iA_\theta} : \rangle, \tag{3.28}$$

respectively.

Our task now is to solve the two equations, (3.14) and (3.15), by using the formulas (3.18)–(3.20) and (3.24)–(3.28).

When (3.24)–(3.28) are used, Eqs. (3.14) and (3.15) with $l = 1$ give

$$\frac{\partial}{\partial\theta_1}B_\theta(x) = i(M/2)^{1/2}, \quad \frac{\partial}{\partial\theta_1}B_\theta^\dagger(x) = -i(M/2)^{1/2}, \quad \frac{\partial}{\partial\theta_1}\phi_\theta(x) = 0, \quad \frac{\partial}{\partial\theta_1}\phi_\theta^\dagger(x) = 0, \tag{3.29}$$

while the choice $l = 2$ leads to

$$\frac{\partial}{\partial\theta_2}B_\theta(x) = -(M/2)^{1/2}, \quad \frac{\partial}{\partial\theta_2}B_\theta^\dagger(x) = -(M/2)^{1/2}, \quad \frac{\partial}{\partial\theta_2}\phi_\theta(x) = 0, \quad \frac{\partial}{\partial\theta_2}\phi_\theta^\dagger(x) = 0. \tag{3.30}$$

Here the relation $2\rho = M$ [cf. (2.31)] was used.

When $l=3$, (3.14) and (3.15) together with (3.18)–(3.20) give

$$\begin{aligned}
&\frac{\partial}{\partial\theta_3} B_\theta(x) = -iB_\theta(x),\\
&\frac{\partial}{\partial\theta_3} B_\theta^\dagger(x) = iB_\theta^\dagger(x),\\
&\frac{\partial}{\partial\theta_3} \phi_\theta(x) = i\lambda_3\phi_\theta(x),\\
&\frac{\partial}{\partial\theta_3} \phi_\theta^\dagger(x) = -i\phi_\theta^\dagger(x)\lambda_3 .
\end{aligned} \tag{3.31}$$

Using the relations

$$\phi_\theta(x) = \phi(x),$$
$$B_\theta(x) = B(x), \text{ etc.},$$

for $\theta = 0$, we derive from (3.29), (3.30), and (3.31), respectively,

$$\begin{aligned}
&B_\theta(x) = B(x) + i\theta_1(M/2)^{1/2},\\
&B_\theta^\dagger(x) = B^\dagger(x) - i\theta_1(M/2)^{1/2},\\
&\phi_\theta(x) = \phi(x),\\
&\phi_\theta^\dagger(x) = \phi^\dagger(x)
\end{aligned} \tag{3.32}$$

for $\theta_2 = \theta_3 = 0$,

$$\begin{aligned}
&B_\theta(x) = B(x) - \theta_2(M/2)^{1/2},\\
&B_\theta^\dagger(x) = B_\theta^\dagger(x) - \theta_2(M/2)^{1/2},\\
&\phi_\theta(x) = \phi(x),\\
&\phi_\theta^\dagger(x) = \phi^\dagger(x)
\end{aligned} \tag{3.33}$$

for $\theta_1 = \theta_3 = 0$, and

$$\begin{aligned}
&B_\theta(x) = e^{-i\theta_3} B(x),\\
&B_\theta^\dagger(x) = e^{i\theta_3} B^\dagger(x),\\
&\phi_\theta(x) = e^{i\theta_3\lambda_3}\phi(x),\\
&\phi_\theta^\dagger(x) = \phi^\dagger(x) e^{-i\theta_3\lambda_3}
\end{aligned} \tag{3.34}$$

for $\theta_1 = \theta_2 = 0$, respectively. Thus we conclude that the spin rotation for the electron Heisenberg field is induced by the transformations (3.32)–(3.34) of the quasiparticle fields; note that the transformations (3.32)–(3.34) exactly agree with the $E(2)$ transformation given in (1.10) and (1.11).

We now note that c-number translations in the transformations for B in (3.32) and (3.33) must be understood, respectively, as the limit for $f(x) \to 1$ of the transformations[13]

$$B(x) \to B_\theta(x) = \lim_{f\to 1}[B(x) + if(x)\theta_1(M/2)^{1/2}] \tag{3.35a}$$

and

$$B(x) \to B_\theta(x) = \lim_{f\to 1}[B(x) - f(x)\theta_2(M/2)^{1/2}]. \tag{3.35b}$$

Here $f(x)$ stands for any square-integrable function which satisfies the magnon equation. Such a function is necessary, because without it the quantities on the left-hand sides of (3.7) and (3.8) are not well defined. For example, when (3.32) is used, $A(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ contains the term $\theta_1(M/2\rho) \times K(\partial)S_\phi^{(-)}(x)$ which contributes to Feynman diagrams in $\mathcal{S}(\phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ and $S^{(i)}(x, \phi_\theta, \phi_\theta^\dagger, B_\theta, B_\theta^\dagger)$ by energyless and momentumless external lines, and therefore these Feynman diagrams can contain a power of zero-energy singularities. To avoid such an infrared catastrophe, we replace θ_i by $\theta_i f(x)$ $(i=1,2)$ in $B_\theta(x)$ in (3.17); then the limit $f(x)\to 1$ should be taken at the end of computations. Since $B_\theta(x)$ must satisfy the magnon equation in order to have the equation in (3.21) satisfied, it is necessary that $f(x)$ satisfy the magnon equation. Note that the magnon equations are invariant under the transformations in (3.35a) and (3.35b) even before the limit $f(x)\to 1$ is taken, exhibiting the $E(2)$ invariance of the theory.

The generators of the transformations (3.32)–(3.34) [with θ_1 and θ_2 replaced by $\theta_1 f(x)$ and $\theta_2 f(x)$, respectively] are

$$s_f^{(1)} = (M/2)^{1/2}\int d^3x[B(x)f(x) + B^\dagger(x)f^*(x)], \tag{3.36a}$$

$$s_f^{(2)} = -i(M/2)^{1/2}\int d^3x[B(x)f(x) - B^\dagger(x)f^*(x)], \tag{3.36b}$$

$$s^{(3)} = \int d^3x[\phi^\dagger(x)\lambda_3\phi(x) - B^\dagger(x)B(x)]. \tag{3.36c}$$

As was pointed out in the Introduction, the presence of the function $f(x)$ is essential for $s_f^{(1)}$ and $s_f^{(2)}$ to be well defined.[4,12] Note that (3.36a)–(3.36c) agree exactly with the expression (1.4) when we recall $s^{(\pm)} = s^{(1)} \pm is^{(2)}$ and (2.32). We also note that the generators (3.36a)–(3.36c) are time independent since $f(x)$ satisfies the magnon equation.

The generators (3.36a)–(3.36c) satisfy the following commutation relations:

$$\begin{aligned}
&[s_f^{(1)}, s_f^{(2)}] = iM\int d^3x\,|f(x)|^2 = (\text{const})I,\\
&[s^{(3)}, s_f^{(1)}] = is_f^{(2)},\\
&[s^{(3)}, s_f^{(2)}] = -is_f^{(1)},
\end{aligned} \tag{3.37}$$

or in terms of $s_f^{(\pm)}$:

$$\begin{aligned}
&[s_f^{(+)}, s_f^{(-)}] = 2M\int d^3x\,|f(x)|^2 = (\text{const})\,I,\\
&[s^{(3)}, s_f^{(\pm)}] = \pm s_f^{(\pm)}.
\end{aligned} \tag{3.38}$$

This shows that the algebra of the generators for the transformations of the quasiparticle fields is the same as in (1.5).[14]

IV. ORIGIN OF THE CHANGE OF SYMMETRY GROUP

In this section we will study what really causes the change of the original spin-rotation algebra

(1.3) into the algebra (3.38) [(1.5)].

Let us decompose the magnon field $B(x)$ into the sum of two parts as

$$B(x)=B_t(x)+B_\eta(x), \tag{4.1}$$

where B_η contains only momenta smaller than η while momenta in B_t are larger than η. Here η is infinitesimal. There are many ways of constructing such B_η; for example,

$$B_\eta(x)=\tfrac{1}{2}\eta\int_{-\infty}^{+\infty} dt\, e^{-\eta|t|}\, B(x) . \tag{4.2}$$

Using (3.1), we can write

$$B_\eta(x)=\frac{\eta}{2(2\pi)^{1/2}}\int d^3k\,\delta_\eta(k) B_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} . \tag{4.3}$$

Here $\delta_\eta(k)$ is a function which approaches $\delta(k)$ in the limit $\eta\to 0$. Therefore $B_\eta(x)$ is of order η and independent of x in the limit $\eta\to 0$.

Using (3.8) we can write $\mathcal{S}S^{(i)}(x,\phi,\phi^\dagger,B,B^\dagger)$, up to the first order in η, as follows:

$$\begin{aligned}\mathcal{S}S^{(i)}(y)&\equiv \mathcal{S}S^{(i)}(y,\phi,\phi^\dagger,B,B^\dagger)\\ &=\langle S_\psi^{(i)}(y):\exp[-iA(\phi,\phi^\dagger,B_t+B_\eta,B_t^\dagger+B_\eta^\dagger)]:\rangle\\ &=\mathcal{S}s_t^{(i)}(y)-i\rho^{-1/2}B_\eta\int d^4x\,\langle S_\psi^{(i)}(y)\\ &\quad\times K(\vec{\partial})S_\psi^{(-)}(x):e^{-iA_t}:\rangle\\ &\quad-i\rho^{-1/2}B_\eta^\dagger\int d^4x\,\langle S_\psi^{(i)}(y)S_\psi^{(+)}(x)K(-\overleftarrow{\partial}):e^{-iA_t}:\rangle,\end{aligned} \tag{4.4}$$

where $s_t^{(i)}(y)$ is obtained from $S^{(i)}(y)$ by ignoring the infrared fields, B_η and $B_\eta^\dagger$:

$$\mathcal{S}s_t^{(i)}(y)\equiv\langle S_\psi^{(i)}(y):e^{-iA_t}:\rangle . \tag{4.5}$$

The symbol A_t is A with the infrared field disregarded:

$$A_t\equiv A(\phi,\phi^\dagger,B_t,B_t^\dagger) . \tag{4.6}$$

Let us first consider (4.4) with $i=1$. Since (A8b) and (A9b) together with (3.14) and (3.15) give

$$\int d^4x\,\langle S_\psi^{(1)}(y)[K(\vec{\partial})S_\psi^{(-)}(x)-S_\psi^{(+)}(x)K(-\overleftarrow{\partial})]:e^{-iA}:\rangle=0$$

and

$$\int d^4x\,\langle S_\psi^{(2)}(y)[K(\vec{\partial})S_\psi^{(-)}(x)+S_\psi^{(+)}(x)K(-\overleftarrow{\partial})]:e^{-iA}:\rangle=0,$$

(4.4) with $i=1$ gives

$$\begin{aligned}\mathcal{S}S^{(1)}(y)&=\mathcal{S}s_t^{(1)}(y)-\tfrac{1}{2}i\rho^{-1/2}(B_\eta+B_\eta^\dagger)\int d^4x\,\langle S_\psi^{(1)}(y)\\ &\quad\times[K(\vec{\partial})S_\psi^{(-)}(x)+S_\psi^{(+)}(x)K(-\overleftarrow{\partial})]:e^{-iA_t}:\rangle,\end{aligned}$$

which together with (3.25) and the relation $\rho=M/2$ [cf. (2.31)] leads to

$$S^{(1)}(y)=s_t^{(1)}(y)+(1/2M)^{1/2}(B_\eta+B_\eta^\dagger)s_t^{(3)}(y) . \tag{4.7}$$

In a similar way (4.4) with $i=2$ and $i=3$ gives

$$S^{(2)}(y)=s_t^{(2)}(y)-i(1/2M)^{1/2}(B_\eta-B_\eta^\dagger)s_t^{(3)}(y) \tag{4.8}$$

and

$$\begin{aligned}S^{(3)}(y)&=s_t^{(3)}(y)+(1/2M)^{1/2}[i(B_\eta-B_\eta^\dagger)s_t^{(2)}(y)\\ &\quad-(B_\eta+B_\eta^\dagger)s_t^{(1)}(y)] ,\end{aligned} \tag{4.9}$$

respectively.

Note that, in the limit $\eta\to 0$, the matrix elements[15] of $S^{(i)}(y)$ are equal to those of $s_t^{(i)}(y)$:

$$\langle i|S^{(i)}(y)|j\rangle=\langle i|s_t^{(i)}(y)|j\rangle . \tag{4.10}$$

In particular, this for $i=3$ gives

$$\langle 0|s_t^{(3)}(y)|0\rangle=\langle 0|S^{(3)}(y)|0\rangle=M . \tag{4.11}$$

We can therefore write

$$s_t^{(3)}(y)=M+:s_t^{(3)}(y): . \tag{4.12}$$

Equations (4.7)–(4.9) show that, when we express the spin-density operators $S^{(i)}(y)$ in terms of quasiparticles and then ignore the infrared operators B_η and $B_\eta^\dagger$, we obtain $s_t^{(i)}(y)$. Therefore, the space integration of $s_t^{(i)}(y)$ must be the generators in (3.36a)–(3.36c), which $B(x)$ is replaced by $B_t(x)$. Therefore, we can write (4.7)–(4.9) as

$$S_f^{(1)}=s_t^{(1)}+(1/2M)^{1/2}(B_\eta+B_\eta^\dagger)s_t^{(3)} , \tag{4.13}$$

$$S_f^{(2)}=s_t^{(2)}-i(1/2M)^{1/2}(B_\eta-B_\eta^\dagger)s_t^{(3)} , \tag{4.14}$$

$$S_f^{(3)}=s_t^{(3)}+(1/2M)^{1/2}[i(B_\eta-B_\eta^\dagger)s_t^{(2)}-(B_\eta+B_\eta^\dagger)s_t^{(1)}] . \tag{4.15}$$

Here

$$s_t^{(1)}=(M/2)^{1/2}\int d^3x[B_t(x)f(x)+B_t^\dagger(x)f^*(x)],$$

$$s_t^{(2)}=-i(M/2)^{1/2}\int d^3x[B_t(x)f(x)-B_t^\dagger(x)f^*(x)],$$

$$\begin{aligned}s_t^{(3)}&=\int d^3x:\phi^\dagger(x)\lambda_3\phi(x)f(x):\\ &\quad+\int d^3x[M-B_t^\dagger(x)B_t(x)]f(x),\end{aligned}$$

according to (3.36a)–(3.36c). Here $f(x)$ is the square-integrable function which appeared in (3.36a)–(3.36c) and which is extremely close to 1. The spin-rotation generators $S^{(i)}$ should be given by $S_f^{(i)}$ in the limit $f\to 1$.

Use of (4.12) gives

$$\begin{aligned}&s_t^{(1)}+(1/2M)^{1/2}(B_\eta+B_\eta^\dagger)s_t^{(3)}\\ &\quad=s_f^{(1)}+(1/2M)^{1/2}(B_\eta+B_\eta^\dagger):s_t^{(3)}: ,\ \text{etc.},\end{aligned}$$

because $B=B_t+B_\eta$. Here $s_f^{(1)}$ is given by (3.36a). In this way we can rewrite (4.13) and (4.14) as

$$S_f^{(1)}=s_f^{(1)}+(1/2M)^{1/2}(B_\eta+B_\eta^\dagger):s_t^{(3)}: , \tag{4.16}$$

$$S_f^{(2)}=s_f^{(2)}-i(1/2M)^{1/2}(B_\eta-B_\eta^\dagger):s_t^{(3)}: , \tag{4.17}$$

where $s_f^{(1)}$ and $s_f^{(2)}$ are given in (3.36a) and (3.36b).

Our task now is to show that the spin operators $S_f^{(i)}$ in (4.13)–(4.15) indeed satisfy the commutators for the rotation group (1.3) when the limit

$f \to 1$ is taken. We first note the following commutation relations:

$$
\begin{aligned}
&[s_f^{(1)}, s_f^{(2)}] = iM\int d^3x\, |f(x)|^2 ,\\
&[:s_t^{(3)}:, s_t^{(1)}] = [s_t^{(3)}, s_f^{(1)}] = is_t^{(2)} ,\\
&[:s_t^{(3)}:, s_t^{(2)}] = [s_t^{(3)}, s_f^{(2)}] = -is_t^{(1)} ,\\
&[s_f^{(1)}, B_\eta(x)] = -(M/2)^{1/2} f_\eta^*(x) ,\\
&[s_f^{(1)}, B_\eta^\dagger(x)] = (M/2)^{1/2} f_\eta(x) ,\\
&[s_f^{(2)}, B_\eta(x)] = -i(M/2)^{1/2} f_\eta^*(x) ,\\
&[s_f^{(2)}, B_\eta^\dagger(x)] = -i(M/2)^{1/2} f_\eta(x) ,\\
&[s^{(3)}, B_\eta(x)] = B_\eta(x) ,\\
&[s^{(3)}, B_\eta^\dagger(x)] = -B_\eta^\dagger(x) .
\end{aligned} \tag{4.18}
$$

Here $f_\eta(x)$ is the infrared part of the square-integrable function $f(x)$:

$$f(x) = f_t(x) + f_\eta(x) .$$

In other words, $f_\eta(x)$ contains only momenta smaller than η and therefore has a domain of range $1/\eta$. Thus $f_\eta(x)$ vanishes in the limit $\eta \to 0$ because $f(x)$ is square integrable. Let us note that, since we consider commutators of two generators (i.e., two successive rotations) we need two infrared cutoffs η and $\bar\eta$: two limits $\eta \to 0$ and $\bar\eta \to 0$ are to be performed successively. Let us assume that the limit $\bar\eta \to 0$ is to be performed before the limit $\eta \to 0$ and therefore assume that $\eta \gg \bar\eta$ (we find a same result when we exchange the order of two limits). To take care of the locally infinitesimal effect, the space integration must extend to infinity. We must, therefore, take the limit $f \to 1$ before η and $\bar\eta$ tend to zero in order to recognize the differences between $S_f^{(i)}$ and $s_f^{(i)}$. Using (4.16)–(4.18) we find

$$
\begin{aligned}
[S_f^{(1)}, S_f^{(2)}] = iM\int d^3x\, |f(x)|^2 + i\tfrac{1}{2}[f_\eta^*(x) + f_\eta(x) \\
+ f_{\bar\eta}^*(x) + f_{\bar\eta}(x)] :s_t^{(3)}: - (1/2M)^{1/2} \\
\times (B_{\bar\eta} - B_{\bar\eta}^\dagger) s_t^{(2)} - i(1/2M)^{1/2}(B_\eta + B_\eta^\dagger) s_t^{(1)} .
\end{aligned} \tag{4.19}
$$

Since $\bar\eta \ll \eta$ implies that $|f_{\bar\eta}| \ll |f_\eta|$ for the square-integrable function f, we ignore $f_{\bar\eta}$ and $f_{\bar\eta}^*$ in the second term in the right-hand side of (4.19). Taking the limit $f \to 1$, (4.19) leads to

$$[S^{(1)}, S^{(2)}] = iS^{(3)} , \tag{4.20}$$

where (4.12) and (4.9) are taken into consideration.

We also obtain

$$[S^{(3)}, S^{(1)}] = iS^{(2)}, \quad [S^{(3)}, S^{(2)}] = -iS^{(1)} . \tag{4.21}$$

Thus we find that

$$\lim_{\eta\to 0}\lim_{\bar\eta\to 0}\lim_{f\to 1}[S_f^{(i)}, S_f^{(j)}] = i\epsilon_{ijk} S^{(k)} , \tag{4.22}$$

while

$$
\begin{aligned}
&\lim_{f\to 1}\lim_{\bar\eta\to 0}[S_f^{(1)}, S_f^{(2)}] = iM\lim_{f\to 1}\int d^3x\, |f(x)|^2 = (\text{const})\, I ,\\
&\lim_{f\to 1}\lim_{\bar\eta\to 0}[S_f^{(3)}, S_f^{(1)}] = \lim_{f\to 1}\lim_{\bar\eta\to 0} iS^{(2)} ,\\
&\lim_{f\to 1}\lim_{\bar\eta\to 0}[S_f^{(3)}, S_f^{(2)}] = -\lim_{f\to 1}\lim_{\bar\eta= 0} iS^{(1)} ,
\end{aligned} \tag{4.23}
$$

where $\tilde\eta \equiv$ (minimum of η and $\bar\eta$).

Equation (4.22), in which the limit $f \to 1$ is performed before the limit $\bar\eta \to 0$ and $\eta \to 0$, corresponds to the rotational-group symmetry, while (4.23) correspond to the $E(2)$-group symmetry. We have thus proved that the differences between $S^{(i)}$ and $s^{(i)}$ are due to the infrared effects: $\lim f \to 1$ and $\lim \tilde\eta \to 0$ are not commutable. The infrared term, although locally infinitesimal, gives, however, a finite global contribution to the commutators of the generators $S^{(i)}$ of the electron transformation. Its locally infinitesimal nature makes it, instead, commutable with any local operator and thus it does not contribute to the commutators of the generators for the quasiparticles, which are directly related to the (local) observations.

V. CONCLUSION

We derived, without any approximation, the commutation relations between the generators of the transformations for quasiparticles in a ferromagnetic system. The algebra found is the one used by Holstein and Primakoff.[1] We analyzed the reason why the algebra for quasiparticle transformation differs from that for electron transformation. To summarize, we first proved the existence of a boson (i.e., magnon) without energy gap as a consequence of the nonzero vacuum expectation value of $S^{(3)}(x)$. Next, we expressed the Heisenberg operator $S^{(i)}(x)$ for spin density in terms of the quasiparticle fields (quasielectron ϕ and magnon B) [cf. Eq. (3.8)] which satisfy the linear homogeneous (i.e., free field) Eqs. (3.6) and (3.4). In this way we were able to reconstruct the transformations (3.32)–(3.34) for the quasiparticles which induce the transformation (2.1) for the electron. We found that the nature of such quasiparticle transformations is different from rotation; we called this phenomena the dynamical rearrangement of symmetry. Such a difference is caused by the effects of infinitesimal momenta (infrared effects) which are locally infinitesimal but contribute to global objects such as the total spin operators, thus creating the difference between $S^{(i)}$ and $s^{(i)}$ ($i = 1, 2$). Let us note that the $E(2)$ symmetry (3.38) is related with observable results, since quasiparticles are related to observable energy levels. The fact that the magnons are associated with the $E(2)$ symmetry can be expressed by saying that the magnons form an irreducible representation of the $E(2)$ symmetry group. As we have seen in Sec.

III, the S-matrix $\mathcal{S}$ is not unity; i.e., the quasiparticles do interact with each other though such interactions do not contribute to any energy shift and the Hamiltonian of quasiparticles is the free one.

Although we assumed the electron model, all the arguments in connection with the magnons are true in any model for the spin, because we did not assume any specific form of spin-density operator $S^{(i)}(x)$. In case of the electron model, quasielectron ϕ appears in addition to the magnons. It is remarkable that under the $E(2)$ transformations in (3.32) and (3.33), ϕ does not change at all; the magnon is the only agent for the transformation generated by $S^{(1)}$ and $S^{(2)}$.

It is also easy to show that the $E(2)$ transformations of quasiparticles in (3.32)–(3.34) induce the spin transformation on the Heisenberg operator of the electron ψ_H, i.e., $\psi_H(x)\to e^{i\theta_i\lambda_i}\psi_H(x)$. To do this we use the path-integral formalism to express $\psi_H(x)$ in terms of ϕ and B:

$$\mathcal{S}\psi_H(x)=\langle\psi(x):\exp[-iA(\phi,\phi^\dagger,B,B^\dagger)]:\rangle \ .$$

We then perform $E(2)$ transformations (3.32)–(3.34); this results in the spin rotation of ψ_H. The proof for this follows the steps of the argument in Sec. III, where we proved that the $E(2)$ transformation of quasiparticles induces the spin rotation of $S^{(i)}(x)$.

Dyson concluded[6] that interaction between low-frequency magnons is very small. We note that this is a manifestation of the $E(2)$ symmetry.

Indeed we proved [cf. (3.12)] that the S matrix is invariant under the transformation $B\to B+\text{const}$ [cf. (3.32)–(3.34)]; this implies that the magnon operator B always appears with its derivatives in the S matrix, and thus the magnon interaction disappears in the zero-momentum limit. In this connection we note that the study of interactions among quasiparticles requires the study of relations among vertices with many external lines. This can be done by the path-integral technique, too.[5,9,12]

Although our conclusion looks similar to the one obtained by Holstein and Primakoff,[1] its implication is quite different, as discussed in the Introduction.

Finally, we recall that our arguments are completely general; no assumption is made on the Lagrangian except its invariance under the spin-rotation transformation (2.1); furthermore, our study covers the cases of localized spin (such as the Heisenberg model) and of continuous spin distribution (such as the itinerant electron).

Although the path-integral method presents various results which are independent of specific model and of any approximation, it rarely helps us in computing model-dependent quantities. However, it gives us a framework of rigorous statements with which no practical calculations are allowed to contradict. It is therefore advisable to combine use of the path-integral method with practical model calculations.

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APPENDIX

Here we derive several formulas which are used in the text. We first recall $W[J,j,n]$, which was introduced in (2.3a). We make the change of variables (2.1) in the numerator of (2.3a). Then, $\partial W[J,j,n]/\partial\theta_l=0$ gives

$$\langle I\rangle_{\epsilon,J,j,n}=-\epsilon\,\epsilon_{3lk}\int d^4x\,\langle S_\psi^{(k)}(x)\rangle_{\epsilon,J,j,n}\ , \tag{A1}$$

where

$$I\equiv\int d^4x\,[J^\dagger(x)\lambda_l\psi(x)-\psi^\dagger(x)\lambda_lJ(x)+i(\epsilon_{1lk}-i\epsilon_{2lk})\times j^\dagger(x)S_\psi^{(k)}(x)+i(\epsilon_{1lk}+i\epsilon_{2lk})S_\psi^{(k)}(x)j(x)+i\epsilon_{3lk}S_\psi^{(k)}(x)n(x)]\ .$$

Equation (A1) reduces to (2.9) when $J=0$ and $n=0$. Operating $\delta/\delta j(y)$ on (A1) we obtain

$$(\epsilon_{1lk}+i\epsilon_{2lk})\langle S_\psi^{(k)}(y)\rangle_{\epsilon,J,j,n}=-\langle S_\psi^{(+)}(y)I\rangle_{\epsilon,J,j,n}-\epsilon\epsilon_{3lk}\int d^4x\,\langle S_\psi^{(+)}(y)S_\psi^{(k)}(x)\rangle_{\epsilon,J,j,n}\ . \tag{A2}$$

The operation of $\delta/\delta j^\dagger(y)$ gives

$$(\epsilon_{1lk}-i\epsilon_{2lk})\langle S_\psi^{(k)}(y)\rangle_{\epsilon,J,j,n}=-\langle S_\psi^{(-)}(y)I\rangle_{\epsilon,J,j,n}-\epsilon\epsilon_{3lk}\int d^4x\,\langle S_\psi^{(-)}(y)S_\psi^{(k)}(x)\rangle_{\epsilon,J,j,n}\ . \tag{A3}$$

On the other hand, when $\delta/\delta n(y)$ operates on (A1) we obtain

$$\epsilon_{3lk}\langle S_\psi^{(k)}(y)\rangle_{\epsilon,J,j,n}=-\langle S_\psi^{(3)}(y)I\rangle_{\epsilon,J,j,n}-\epsilon\epsilon_{3lk}\int d^4x\langle S_\psi^{(3)}(y)S_\psi^{(k)}(x)\rangle_{\epsilon,J,j,n}\ . \tag{A4}$$

When (3.17) is used, Eq. (A1) gives

$$\epsilon\int d^4x\ \langle S_\psi^{(k)}(x):e^{-iA_\theta}:\rangle_\epsilon=0 \text{ for } k=1,2 \tag{A5}$$

when $l=1$ or 2; in deriving (A5) we used the relations

$$\int d^4x\,\langle\phi_\theta^\dagger(x)\Lambda(\bar{\partial})\lambda_i\psi(x)Q\rangle=0\ ,\qquad \int d^4x\,\langle B_\theta(x)K(\bar{\partial})S_\psi^{(3)}(x)Q\rangle=0\ , \tag{A6}$$

$i=1,2$, with arbitrary operator Q. The second relation in (A6) is obtained when we recall the fact that $\langle S_\psi^{(3)}(x)Q\rangle$ does not contain the magnon Green's function associated with the point x. The first relation is derived by using the fact that λ_i is an off-diagonal matrix when $i=1$ or 2. Equation (A2), together with (3.17), gives, for $l=1$ and 2,

$$\langle S_\psi^{(3)}(x) : e^{-iA_\theta} : \rangle = \tfrac{1}{2}\lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(+)}(y)S_\psi^{(-)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,, \tag{A7a}$$

$$\epsilon \int d^4x \langle S_\psi^{(+)}(y)S_\psi^{(+)}(x) : e^{-iA_\theta} : \rangle_\epsilon = 0 \,. \tag{A7b}$$

Equation (A3) with $l=1$ and 2 gives

$$\langle S_\psi^{(3)}(y) : e^{-iA_\theta} : \rangle = \tfrac{1}{2}\lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(-)}(y)S^{(+)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,, \tag{A8a}$$

$$\epsilon \int d^4x \langle S_\psi^{(-)}(y)S_\psi^{(-)}(x) : e^{-iA_\theta} : \rangle_\epsilon = 0 \,. \tag{A8b}$$

It is more useful for our purpose to write (A7a) by using (A8b):

$$\langle S_\psi^{(3)}(y) : e^{-iA_\theta} : \rangle = \lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(1)}(y)S_\psi^{(-)}(x) : e^{-iA_\theta} : \rangle_\epsilon \tag{A9a}$$

$$= i\lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(2)}(y)S_\psi^{(-)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,. \tag{A9b}$$

In a similar way (A8a) together with (A7b) leads to

$$\langle S_\psi^{(3)}(y) : e^{-iA_\theta} : \rangle$$

$$= \lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(1)}(y)S_\psi^{(+)}(x) : e^{-iA_\theta} : \rangle_\epsilon \tag{A10a}$$

$$= -i\lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(2)}(y)S_\psi^{(+)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,. \tag{A10b}$$

We can therefore write

$$\langle S_\psi^{(3)}(y) : e^{-iA_\theta} : \rangle$$

$$= \lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(1)}(y)S_\psi^{(1)}(x) : e^{-iA_\theta} : \rangle_\epsilon \tag{A11}$$

$$= \lim_{\epsilon\to0} \epsilon \int d^4x \langle S_\psi^{(2)}(y)S_\psi^{(2)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,. \tag{A12}$$

When (3.17) is used, (A4) together with (A6) gives

$$\langle S_\psi^{(2)}(y) : e^{-iA_\theta} : \rangle_\epsilon$$

$$= -\epsilon \int d^4x \langle S_\psi^{(3)}(y)S_\psi^{(2)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,, \tag{A13}$$

$$\langle S_\psi^{(1)}(y) : e^{-iA_\theta} : \rangle_\epsilon = -\epsilon \int d^4x \langle S_\psi^{(3)}(y)S_\psi^{(1)}(x) : e^{-iA_\theta} : \rangle_\epsilon \,. \tag{A14}$$

Now we derive the relations (3.22) and (3.23). When Q denotes any operator, a study of the Feynman diagrams leads us to the formula

$$\int d^4x \langle S^{(i)}(x)Q\rangle = \int d^4x \int d^4z\, G(x-z)q(z) \,, \qquad i=1,2$$

with certain quantity $q(z)$. Here G is the magnon Green's function defined by [cf. (2.17)]

$$G(x) = i\int \frac{d^4p}{(2\pi)^4} e^{-ipx}\Delta(\epsilon,p) \,.$$

We thus have

$$-i\int d^4x K(\vec{\partial})\langle S^{(i)}(x)Q\rangle$$

$$= \rho \int d^4x\, q(x)$$

$$= \tfrac{1}{2}a \lim_{\epsilon\to0} \epsilon \int d^4x \int d^4z\, G(x-z)q(z)$$

$$= \tfrac{1}{2}a \lim_{\epsilon\to0} \epsilon \int d^4x \langle S^{(i)}(x)Q\rangle_\epsilon \,, \quad i=1,2$$

which leads to the formula

$$\lim_{\epsilon\to0} \epsilon \int d^4x \langle S^{(i)}(x)Q\rangle_\epsilon$$

$$= -i\frac{M}{\rho}\int d^4x K(\vec{\partial})\langle S^{(i)}(x)Q\rangle \,, \quad i=1,2 \,.$$

In a similar way we can derive

$$\lim_{\epsilon\to0} \epsilon \int d^4x \langle S^{(i)}(x)Q\rangle_\epsilon$$

$$= -i\frac{M}{\rho}\int d^4x \langle S^{(i)}(x)K(-\overleftarrow{\partial})Q\rangle \,, \quad i=1,2 \,.$$

These relations give

$$\lim_{\epsilon\to0} \int d^4x \langle S^{(1)}(x)Q\rangle_\epsilon$$

$$= -i\frac{M}{2\rho}\int d^4x \left[\langle K(\vec{\partial})S^{(-)}(x)Q\rangle + \langle S^{(+)}(x)K(-\overleftarrow{\partial})Q\rangle\right]$$

and

$$\lim_{\epsilon\to0} \epsilon \int d^4x \langle S^{(2)}(x)Q\rangle_\epsilon$$

$$= \frac{M}{2\rho}\int d^4x \left[\langle K(\vec{\partial})S^{(-)}(x)Q\rangle - \langle S^{(+)}(x)K(-\overleftarrow{\partial})Q\rangle\right] \,.$$

Using the relation $2\rho = M$ [cf. (2.31)], we find the relations (3.22) and (3.23).

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[1]T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).

[2]For a more detailed description of the quasiparticle picture, see L. Leplae, F. Mancini, and H. Umezawa, Phys. Rep. 10, No. 4 (1974); J. Rest. V. Srinivasan, and H. Umezawa, Phys. Rev. D 3, 1890 (1971). A derivation of a ferromagnetic Hamiltonian bilinear in the quasiparticle operators for the Heisenberg exchange model is given in A. K. Benson, Phys. Rev. B 7, 4158 (1973).

[3]We do not mean the path-integral method in quantum mechanics but the one in quantum field theory. This method also is called the functional-integral method. See R. P. Feynman, Rev. Mod. Phys. 20, 367 (1947); Phys. Rev. 74, 1430 (1948); J. Schwinger, Proc. Natl. Acad. Sci. USA 37, 452 (1851); H. Umezawa and A. Visconti, Nuovo Cimento 1, 1079 (1955); K. Symanzik, Z. Naturforsch. 9, 809 (1954); D. Luriè, *Particles and Fields* (Interscience, New York, 1968).

[4]H. Umezawa, Nuovo Cimento 40, 450 (1965); L. Leplae and H. Umezawa, Nuovo Cimento 44, 410 (1966); L. Leplae, R. N. Sen, and H. Umezawa, Prog. Theor. Phys. Suppl., Extra No. 637 (1965).

[5]The rearrangement of symmetry is a subject which has been well studied in a recent development of quantum field theory. In particular, it has recently been proved that, when the Hamiltonian has the isospin invariance, while the ground state does not have the isospin invariance, the isospin symmetry (which is a rotational symmetry associated with the isospace) is replaced by the $E(2)$ symmetry in observable phenomena. The consideration in the present paper is an application of such a study to the case of spin symmetry. See H. Matsumoto, H. Umezawa, G. Vitiello, and J. K. Wyly, Phys. Rev. D 9, 2806 (1974).

[6]F. J. Dyson, Phys. Rev. 102, 1217 (1956). The representation considered by Holstein and Primakoff is a nonlinear representation of the rotation group. In the present work we show that if we express the spin-density operators in terms of quasiparticles, this naturally leads us to the linear representation of the $E(2)$ group.

[7]In practical computations, this magnon is a bound state of electrons and it is better to treat it by the Bethe-Salpeter equations instead of an explicit counting of spin flips of electrons. See, for example, the first article quoted in Ref. 2 in the case of a superconductor.

[8]In the case $\langle S_\psi^{(3)}(x)\rangle \neq 0$ [i.e., nonvanishing M, cf. (2.22)], we say that "the symmetry is spontaneously broken." The particle without energy gap is also called the Goldstone boson. See J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[9]H. Matsumoto, N. J. Papastamatiou, and H. Umezawa, Phys. Lett. B 46, 73 (1973); Nucl. Phys. B 68, 236 (1974).

[10]The expansion of the Heisenberg field operator in terms of quasiparticles is called the dynamical mapping. Quasiparticle fields correspond to the free fields in the terminology of quantum field theory. In connection with dynamical mapping, see Ref. 4.

[11]In the case of the localized spins, as was pointed out before, the space integration is replaced by the summation over the lattice points. However, the derivation of the LSZ formula can be reproduced without any change. This can be done because of the relation (2.28) and because there is time integration covering the entire time domain $(-\infty, +\infty)$.

[12]H. Umezawa, in Notes for the School on Renormalization and Invariance in Quantum Field Theory, Capri 1973 (unpublished); H. Matsumoto, N. J. Papastamatiou, and H. Umezawa, University of Wisconsin-Milwaukee report, 1974 (unpublished).

[13]Such transformations are called boson transformations. As shown in Refs. 5 and 12, and in the first article quoted in Ref. 2, they play an important role in any theory which presents spontaneous breakdown of symmetry.

[14]For an analysis from the viewpoint of algebraic realizations of spin-wave theory see Y. Chow and C. J. Liu, J. Math. Phys. 12, 2144 (1971); Int. J. Quantum Chem. Symp., No. 7, 521 (1973).

[15]Recall that the quasiparticles in the states $|i\rangle$ and $|j\rangle$ must be in wave-packet states: Their wave functions approach zero at infinity. Thus no infrared effects influence these matrix elements.