Critical temperatures of classical n-vector models on hypercubic lattices

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The critical temperature $T_c(n,d)$ of a classical n-component spin model on a d-dimensional hypercubic lattice is expanded in powers of $1/d$ to fifth order. In the spherical-model limit, $n \rightarrow \infty$, the expansion is derived exactly to all orders and shown rigorously to be only asymptotic, although $T_c(\infty,d)$ is analytic in d for $2 < d < \infty$.

I. INTRODUCTION

Among those nonuniversal parameters describing a critical point, the critical temperature T_c is perhaps the most significant. Some time ago Fisher and Gaunt[1] showed that the critical temperature of an Ising model with nearest-neighbor interactions on a d-dimensional hypercubic lattice, of coordination number $q=\sigma+1=2d$, could be expanded in inverse powers of d. Fisher and Gaunt carried the expansion to order $1/d^5$. The expansion appeared to be of asymptotic character although this could not be established definitely. As a matter of fact, truncation of the expansion at the smallest term gave reasonable estimates even for $d=3$ or $d=2$.

In this paper we extend the work of Fisher and Gaunt to general classical spin models with n-component spins[2] $\vec{s}=[s^\alpha]$ ($\alpha=1,2,\ldots n$) of fixed length $|\vec{s}|=n^{1/2}$.[3] We calculate the $1/d$ expansion for general n, obtaining

$$\frac{T_c(n,d)}{T_c^0}=1-q^{-1}-q^{-2}\left(1+\frac{n}{n+2}\right)-q^{-3}\left(3+\frac{4n}{n+2}\right)-q^{-4}\left(16+\frac{(21n+32)n}{(n+2)^2}-\frac{2n^2}{(n+2)(n+4)}\right)-q^{-5}\left(102+\frac{(129n^2+422n+340)n}{(n+2)^3}-\frac{16n^2}{(n+2)(n+4)}\right)+O(q^{-6}), \quad (1.1)$$

where

$$T_c^0=qJ/k_B=2dJ/k_B \quad (1.2)$$

is the mean-field critical temperature. For $n=1$ this reduces to the Ising-model result of Fisher and Gaunt.[1] For $n=0$ it is equivalent to the result obtained by Fisher and Gaunt for μ/q, where $\mu(d)=\lim_{l\to\infty}(c_l)^{1/l}$ is the self-avoiding-walk limit [$c_l(d)$ being the number of self-avoiding walks of l steps from the origin of the hypercubic lattice[1]]. This is precisely in accord with expectations[4,5] that the self-avoiding-walk problem should be described by the limit $n\to 0$. Our result can be regarded as the first explicit demonstration of this identity for a nonuniversal critical parameter, although it is implied by the proportionality of the self-avoiding walks c_l to the high-temperature series-expansion coefficients $a_l(n,d)$ of the susceptibility when $n\to 0$, as demonstrated recently by Bowers and McKerrell.[5]

In the result (1.1) we may also take the limit $n\to\infty$. As shown by Stanley[6] this limit should correspond to the spherical model.[6,7] For this reason we have made a separate analytical investigation of the critical temperature of the d-dimensional spherical model. This first demonstrates the existence of a $1/d$ expansion. Second, it confirms the expansion (1.1) for $n\to\infty$. However, the analysis also shows how to calculate the general expansion coefficient of $T_c(\infty,d)$ and hence establishes rigorously that the expansion is, in fact, only asymptotic. (The magnitudes of the coefficients grow factorially fast.) On the other hand we also prove that $T_c(\infty,d)$ extends to an analytic function of d in the whole interval $2<d<\infty$. This result is inconsistent with a recent speculation of Baker[8] to the effect that $T_c(1,d)$ might have an essential singularity at $d=4$. It is remarkable that the critical temperature $T_c(\infty,d)$ varies analytically through $d=4$ even though the critical exponents are nonanalytic at $d=4$ [e.g., $\gamma(\infty,d)=2/(d-2)$ for $2<d\le 4$ but $\gamma\equiv 1$ for $d\ge 4$].[7]

The layout of this paper is as follows: In Sec. II we discuss the critical temperature of the spherical model analytically. In Sec. III the graphical methods used to expand $T_c(n,d)$ for general n are expounded. The results are examined numerically and discussed in Sec. IV. An Appendix explains some of the computational details.

II. SPHERICAL MODEL

The critical point of a spherical model with nearest-neighbor interactions of strength J on a d-dimensional hypercubic lattice of coordination number $q=2d$ may be written[7,9]

$$K_c(d)=J/k_B T_c(\infty,d)=W_d(0), \quad (2.1)$$

for $d>2$, where the generalized Watson functions

$$W_d(z)=\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{d\theta_1\,d\theta_2\cdots d\theta_d}{z+\sum_{j=1}^d(1-\cos\theta_j)} \quad (2.2)$$

can be written[7,9]

$$W_d(z)=\tfrac{1}{2}\int_0^\infty e^{-zs/2}[e^{-s}I_0(s)]^d\,ds\ , \qquad (2.3)$$

in which $I_0(s)$ is the Bessel function of zero order and pure-imaginary argument. This last expression provides an appropriate analytic continuation in d (see Sec. IV) and yields

$$K_c(d)=\tfrac{1}{2}\int_0^\infty e^{-dg(s)}\,ds\ , \qquad (2.4)$$

where

$$g(s)=s-\ln I_0(s)\ . \qquad (2.5)$$

From the integral representation[10]

$$e^{-s}I_0(s)=\pi^{-1}\int_0^\pi e^{-s(1+\cos\theta)}\,d\theta\ , \qquad (2.6)$$

we see that $g(s)$ increases monotonically from $g(0)=0$. Furthermore, as $s\to\infty$ we have[10]

$$g(s)=\tfrac{1}{2}\ln s+\tfrac{1}{2}\ln 2\pi+o(1)\ , \qquad (2.7)$$

and thus one can find bounds of the form

$$g(s)\le \ln(s+c^2)^{1/2}+B=y(s) \qquad (2.8)$$

and

$$e^{-g(s)}\le A/(s+c^2)^{1/2}=Ce^{-y(s)}\ , \qquad (2.9)$$

where A, B, C, and c are constants.

Now by differentiating (2.4) with respect to d we obtain

$$\begin{aligned}Q_m(d)&=(-1)^m\left(\frac{\partial}{\partial d}\right)^m K_c(d)\\ &=\tfrac{1}{2}\int_0^\infty [g(s)]^m e^{-dg(s)}\,ds\\ &\le C^d e^{-2B}\int_{y_0}^\infty y^m e^{-(d-2)y}\,dy\ ,\end{aligned} \qquad (2.10)$$

with $y_0=\ln|c|+B\ge 0$. Extending the lower limit of integration to zero, yields

$$Q_m(d)/m!\le C^d e^{-2B}(d-2)^{-m-1}\ . \qquad (2.11)$$

It follows from this bound that $K_c(d)$ has an expansion about any dimensionality d_0 (with $2<d_0<\infty$) with a radius of convergence of at least (d_0-2). Since $K_c(d)\to\infty$ as $d\to 2+$ the radius of convergence cannot exceed (d_0-2). Hence we conclude that $K_c(d)$ is analytic in d for $2<d<\infty$. Evidently $K_c(d)$ is also a monotonic function of d in the same interval.

To obtain an expansion for large d we set $z=g(s)$, with inverse $s=f(z)$, and obtain

$$\begin{aligned}K_c(d)&=\frac{1}{2}\int_0^\infty e^{-dz}\frac{df}{dz}\,dz\\ &=\frac{1}{2}d\int_0^\infty f(z)e^{-dz}\,dz\ ,\end{aligned} \qquad (2.12)$$

where we have integrated by parts. Now, from the expansion

$$f(z)=\sum_{l=1}^\infty f_l z^l\ , \qquad (2.13)$$

we obtain by termwise integration the desired $1/d$ expansion

$$K_c(d)=\frac{1}{2}\sum_{m=1}^\infty f_m m!\ /\ d^m=\sum_{m=1}^\infty K_m\ /\ q^m\ . \qquad (2.14)$$

Reversion of the series for $g(s)$ derived from (2.5) yields explicitly

$$\begin{aligned}2dK_c(d)=T_c^0/T_c(\infty,d)&=1+q^{-1}+3q^{-2}+12q^{-3}+60q^{-4}\\ &+355q^{-5}+2380q^{-6}+17430q^{-7}+134190q^{-8}\\ &+1027656q^{-9}+6922146q^{-10}+21248073q^{-11}\\ &-601744143q^{-12}-20802115620q^{-13}-\cdots\ ,\end{aligned} \qquad (2.15)$$

which, indeed, checks the result (1.1) in the limit $n\to\infty$. The terms become positive again at order q^{-20}.

From (2.14) we see that if the expansion for $K_c(d)$ is to have a nonzero radius of convergence, say $|1/d|=1/d_\infty$, one must have $|f_m m!|\sim d_\infty^m$ or $f_m\sim d_\infty^m/m!$. Thus a necessary condition that the $1/d$ expansion have a circle of convergence is that $f(z)$ is an entire function. To show that $f(z)$ *cannot* be entire we look for zeros of the derivative of the inverse function, namely,[10]

$$\frac{dz}{ds}=\frac{dg}{ds}(s)=1-I_1(s)/I_0(s)\ . \qquad (2.16)$$

Equivalently, since $I_0(s)$ and $I_1(s)$ have no common zeros, we may look for zeros of

$$L(s)=I_0(s)-I_1(s)\ . \qquad (2.17)$$

If $s=s_0$ is such a zero then $f(z)$ has a branch point singularity at $z_0=g(s_0)$.

Now $L(s)$ is an entire function of order (of growth) unity.[11] It follows by Hadamard's theorem[11] that $L(s)$ has a representation of the form

$$L(s)=e^{h(s)}\prod_i(1-s/s_i)e^{s/s_i}\ , \qquad (2.18)$$

where $h(s)=h_0+h_1 s$ is a linear function of s while the product runs over the zeros s_i, if any, of $L(s)$. From the power series expansion for $\ln L(s)$ we find $h(s)=-\tfrac{1}{2}s$. However, $L(s)$ is clearly not equal to $e^{-s/2}$; thus the canonical product over the zeros cannot be empty. Hence $L(s)$ has at least one finite zero, implying a finite singularity in $f(z)$, which cannot then be entire.

To find an upper bound on the radius of convergence ρ of the expansion (2.13) for $f(z)$ we have determined numerically several of the zeros s_i of $L(s)$, which occur in complex-conjugate pairs.

The pair of zeros nearest the origin can also be determined by application of the formula

$$\sum_i s_i^{-m} = -\frac{1}{(m-1)!}\left[\left(\frac{d}{ds}\right)^m \ln L(s)\right]_{s=0}, \qquad (2.19)$$

which follows readily from (2.18) for $m \geq 2$. In Table I the first five pairs of zeros are listed together with the corresponding singular points $z_i = g(s_i)$. Although we have no formal proof, it seems most likely that the closest zeros $s_1^{\pm}$ do, in fact, also determine the nearest singularities $z_1^{\pm}$ in the z plane. These singularities, in turn, determine the radius of convergence of the expansion for $f(z)$. Accepting this, the expansion coefficients for $K_c(d)$ vary as

$$|K_m| \approx m!\,(2/\rho)^m k_m , \qquad (2.20)$$

as $m \to \infty$, where $\rho \simeq 1.62006$ and $|k_m|^{1/m} \to 1$. From this explicit estimate it is again clear that the series for $K_c(d)$ can only be asymptotic.

From the expression (2.12) one concludes that one may reasonably truncate the expansion in powers of $1/d$ at that term, $l = l^*$, for which the maximum of the integrand $z^l e^{-dz}$ lies within the radius of convergence of the $f(z)$ expansion. This yields the prescription

$$l^* \simeq \lfloor \rho d \rfloor - 1 \simeq \lfloor 1.62d \rfloor - 1 . \qquad (2.21)$$

Thus in three dimensions it is reasonable to truncate the series (2.15) after the fourth-order term.

Finally we may note that one may construct an expansion about $d = 2$ for $d - 2 > 0$ by (a) splitting the range of the integral in (2.4) at $s = 1$ and (b) subtracting off the asymptotic piece of the integrand, which varies as $(2\pi s)^{-d/2}$, in the integral for $s > 1$. This leads straightforwardly to

$$T_c(\infty, d)/T_c^0 = p_1(d-2) - p_2(d-2)^2 + O((d-2)^3) , \qquad (2.22)$$

where $p_1 = \frac{1}{2}\pi$ and

$$\begin{aligned} p_2 &= \tfrac{1}{4}\pi\left(1 - \ln 2\pi + \int_0^\infty \left[2\pi e^{-2s} I_0^2(s) - \theta(s-1)/s\right] ds\right) \\ &= \tfrac{1}{4}\pi\left(1 - \ln 2\pi + \lim_{\epsilon \to 0}\left[2(1-\tfrac{1}{2}\epsilon)\,\tilde{K}(1-\tfrac{1}{2}\epsilon) + \mathrm{Ei}(-\epsilon)\right]\right) \\ &= \tfrac{1}{4}\pi\left[1 + \ln(8/\pi) + \gamma_E\right] \\ &= 1.972863\ldots , \end{aligned} \qquad (2.23)$$

where $\gamma_E = 0.577\ldots$ is Euler's constant, $\mathrm{Ei}(x)$ is the exponential integral, $\tilde{K}(k)$ is the complete elliptic integral, and $\theta(x)$ is the unit step function.

TABLE I. Zeros of the function $L(s) = I_0(s) - I_1(s)$ and corresponding singular points $z_i = g(s_i)$ of $s = f(z)$.

i	$\mathrm{Re}(s_i)$	$\mathrm{Im}(s_i)$	$\mathrm{Re}(z_i)$	$\mathrm{Im}(z_i)$
1	1.27960	± 2.98038	1.52405	± 0.54943
2	1.61872	± 6.17515	1.85140	± 0.63872
3	1.81887	± 9.34196	2.04831	± 0.67651
4	1.96146	± 12.49851	2.18954	± 0.69787
5	2.07231	± 15.65010	2.29968	± 0.71176

III. GENERAL n-VECTOR MODEL

We consider now the general n-vector spin model with nearest-neighbor interactions, as described by the Hamiltonian

$$\mathcal{H} = -I \sum_{(ij)} \vec{s}_i \cdot \vec{s}_j - g\mu_B H \sum_i s_i^1 . \qquad (3.1)$$

As mentioned, the spin vectors $\vec{s}_i$ have n components and are of fixed length $|\vec{s}| = n^{1/2}$. The reduced zero-field susceptibility above $T_c(n,d)$ is given by

$$\chi_0 = \chi_T k_B T / g^2 \mu_B^2 = n^{-1} \sum_i \langle \vec{s}_0 \cdot \vec{s}_i \rangle . \qquad (3.2)$$

In order to find the critical temperature we follow Fisher and Gaunt[1] by calculating the high-temperature expansion

$$\chi_0 = 1 + \sum_{l=1}^{\infty} a_l K^l , \qquad K = J/k_B T . \qquad (3.3)$$

If the coefficients $a_l(n,d)$ are all positive, the radius of convergence, and hence the critical temperature, is given by

$$\ln K_c(n,d) = -\limsup_{l \to \infty} l^{-1} \ln a_1(n,d) . \qquad (3.4)$$

We aim to calculate the $a_l(n,d)$ to leading order in d for all l.

A systematic graphical method for calculating the susceptibility coefficients $a_l(n,d)$ has been developed by Stanley and Kaplan.[12] Following Ref. 12c we may write

$$a_l(n,d) = 2 \sum_{G_l^2} (G_l^2; \mathcal{L}_d) A(n; G_l^2) , \qquad (3.5)$$

where the sum runs over multigraphs[13] G_l^2 of l lines or edges (counting multiedges with appropriate multiplicity) which (i) are connected, (ii) have precisely two vertices of odd degree (as indicated by the superscript 2), and (iii) the graphs G_l^{2+} formed by joining the two odd points by an additional (wavy) line are stars; i.e., they have no articulation points (or cut vertices).[13] The symbol $(G; \mathcal{L}_d)$ denotes the weak lattice constant per site[13] of the graph G in the hypercubical lattice $\mathcal{L}_d$ of dimensionality d. The graphical weights $A(n; G_l^2)$ satisfy the recurrence relations

$$A(n; G_l^2) = N(n; G_l^2) - {\sum_{G_k^2}}' M(n; G_{l-k}^0) A(n; G_k^2) , \qquad (3.6)$$

where the sum runs over all proper subgraphs G_k^2 of G_l^2 which fulfill the same conditions as G_l^2. Then G_{l-k}^0 denotes the complementary subgraph of $l - k$

lines (and no odd vertices) such that $G^0_{l-k} \cup G^2_k = G^2_l$. Note that G^0_{l-k} may contain articulation points and, in general, is not even connected. The basic graphical weights N and M are defined by

$$M(n; G^0) = \left\langle \prod_{(ij)} (\vec{s}_i \cdot \vec{s}_j)^{m_{ij}} (m_{ij}!)^{-1} \right\rangle_0 , \qquad (3.7)$$

$$N(n; G^2) = n^{-1} \left\langle \vec{s}_0 \cdot \vec{s}_1 \prod_{(ij)} (\vec{s}_i \cdot \vec{s}_j)^{m_{ij}} (m_{ij}!)^{-1} \right\rangle_0 , \qquad (3.8)$$

where $\langle \cdot \rangle_0$ denotes the noninteracting statistical average (i.e., taken at $K=0$ or $T=\infty$) normalized so that $\langle 1 \rangle_0 = 1$. The vertices of G are labeled i, j, ... and the products run over adjacent pairs of vertices (ij); the two odd vertices of G^2 carry the labels 0 and 1. The multiplicity of the edge (ij) is denoted by m_{ij}.

Following Fisher and Gaunt we now select those graphs of l lines which have lattice constants of highest order in d, the dimensionality. According to this criterion the graphs of largest diameter are the most important. In leading order these are just the l-step chains or self-avoiding walks $C_l = G_l^{[a]}$, see Fig. 1(a), whose lattice constant $c_l(d) = 2(G_l^{[a]}; \mathcal{L}_d)$ on the hypercubical lattices is known[1] to fifth order in $1/\sigma$, where $\sigma = 2d-1$. To leading order one easily sees that $c_l(d) \approx q\sigma^{l-1}$. Accordingly we define the reduced lattice constants

$$\lambda(\tilde{G}^{[t]}) = 2q^{-1}\sigma^{-l+1} \sum_{G \subset \tilde{G}^{[t]}}^{\sim} (G; \mathcal{L}_d) , \qquad (3.9)$$

where the significance of the tilde and corresponding summation will be explained shortly; for chains, both are to be ignored. From the calculations of Fisher and Gaunt [Ref. 1, Eq. (5.15)] we thus obtain

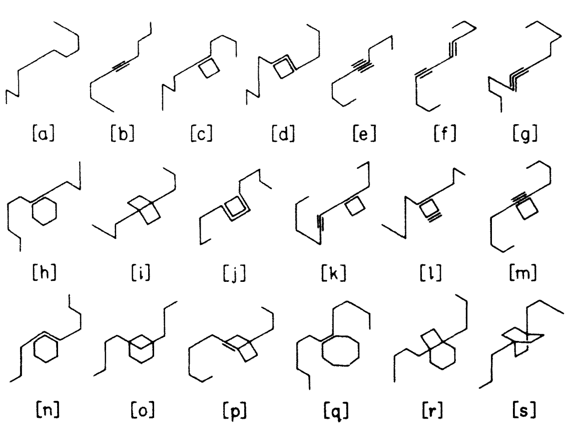

FIG. 1. Generic graphs $\tilde{G}^{[t]}$ needed to calculate the susceptibility expansion coefficients to relative order $1/d^5$. The graphs (k) and (l) also include the cases where the triple bond shares a vertex with the double bond, since, as in the case of (f) and (g), the weights are the same.

$$\lambda(G_l^{[a]}) = 1 - (l-3)\sigma^{-2} - (2l-13)\sigma^{-3} + (\tfrac{1}{2}l^2 - \tfrac{29}{2}l + 107)\sigma^{-4} + (2l^2 - 83l + 895)\sigma^{-5} + O(\sigma^{-6}) . \qquad (3.10)$$

The next most important graphs are $(l-2)$-step chains with one triple bond $G_l^{[b]}$, see Fig. 1(b); the lattice constant is clearly just $c_{l-2}(d)$. Such graphs did not arise in the Ising-model calculations[1] since repeated bonds or multiedges are not needed when $v = \tanh K$ is used as expansion variable. It is shown in the Appendix that the weights of such graphs are independent of the position of the triple bond along the chain of $(l-2)$ links. More generally this is true for any distribution of bonds amongst the free "tails" of a graph. Thus we may class all such graphs together under one generic graph $\tilde{G}_l^{[b]}$ and take the sum in (3.9) over the particular graphs which are members of the same generic class. In the present case this simply means multiplying the single lattice constant by $(l-2)$ to account for the different possible positions of the triple bond. Thus we obtain

$$\lambda(\tilde{G}_l^{[b]}) = (l-2)\sigma^{-2} - (l^2 - 7l + 10)\sigma^{-4} - (2l^2 - 21l + 34)\sigma^{-5} + O(\sigma^{-6}) . \qquad (3.11)$$

The class of graphs of next significance, $\tilde{G}_l^{[c]}$, consists of a square with a double bond and two tails, see Fig. 1(c). These tails must intersect neither themselves, each other, nor the square. Let the lengths of the tails be k and k', with $k, k' \geq 4$. The first step of each tail (starting at the square) can point in $\sigma - 1$ possible directions. In leading order the number of possible tails is hence $(\sigma-1)^2\sigma^{k+k'-2}$, but this overcounts by allowing the following possibilities: (i) each tail forms a square loop with the original square, giving $2(2\sigma-4) \times (\sigma-1)\sigma^{k+k'-4} \approx 4\sigma^{k+k'-2}$; (ii) the two tails together form a square loop with the original square, giving $2(\sigma-2)\sigma^{k+k'-3} \approx 2\sigma^{k+k'-2}$; (iii) one or the other tail contains a square loop giving $\sim (k-3+k'-3)\sigma^{k+k'-2}$. To relative order σ^{-2} these are all the possibilities. Collecting them together gives

$$\sigma^{k+k'}[1 - 2\sigma^{-1} - (k+k'-1)\sigma^{-2} + O(\sigma^{-3})] ,$$

where $k + k' = l - 5$. This is to be multiplied by the number of ways of choosing k and k' with $k, k' \geq 4$, namely, $k + k' - 7 = l - 12$. Similar considerations can be applied when one or the other tail has a length 1, 2, or 3. The final result, which also includes the lattice constant of the original square, is then

$$\lambda(\tilde{G}_l^{[c]}) \approx (l-4)\sigma^{-3} - 3l\sigma^{-4} - (l^2 - 12l)\sigma^{-5} , \qquad (3.12)$$

where terms of order l^0 in the coefficients of σ^{-4} and σ^{-5} have been neglected, since they cannot contribute to the final expression for $K_c(d)$ to order $1/\sigma^5$.

The calculations of the other lattice constants needed to orders $1/\sigma^5$ and l is somewhat simpler since they are all of order $1/\sigma^4$ or higher, so that only $1/\sigma$ corrections are needed. The required graphs are shown in Fig. 1 and the results are summarized in Table II.

Table II also lists the corresponding class weights $A(n; \tilde{G})$; these data must be supplemented by

$$\begin{aligned} A(n; G_l^{[a]}) &= 1 , \\ A(n; G_l^{[b]}) &= -n/(n+2) , \\ A(n; G_l^{[c]}) &= -3n/(n+2) . \end{aligned} \tag{3.13}$$

The calculation of these weights is straightforward if performed with the aid of the Funk-Hencke theorem.[14] The crucial steps are sketched in the Appendix. It can be seen from (3.13), from Table II, and, more generally, from the Appendix that the weight $A(n; \tilde{G})$ for every class of graphs except the self-avoiding chain $G_l^{[a]} = C_l$ is proportional to n and hence vanishes when $n = 0$. This represents an alternative derivation of the Bowers-McKerrell result[5] referred to in the Introduction.

On combining (3.9) and (3.5) in (3.4) we obtain finally

$$qK_c(n, d) = (q/\sigma)\exp\left[-\limsup_{l\to\infty} l^{-1} \ln\left(\sum_{\tilde{G}} \lambda(\tilde{G})A(n, \tilde{G})\right)\right]. \tag{3.14}$$

By considering $1/\sigma$ as a small parameter one may expand the logarithm. In this process all explicit dependence on l cancels and one may formally take the limit $l \to \infty$. This step is only formal since the expansion, as such, is valid only for $l/\sigma^2 \ll 1$.

TABLE II. Reduced lattice constants $\lambda(\tilde{G}^{[t]})$ and corresponding graphical weights $A(n, \tilde{G}^{[t]})$ needed to calculate the susceptibility coefficients $a_l(n,d)$ to relative order σ^{-5}. Graphs labeled $[t]$ are shown in Fig. 1.

$[t]$	$\lambda(\tilde{G}_l^{[t]})$	$A(n;\ \tilde{G}_l^{[t]})$
$[d]$	$l\sigma^{-4} - 3l\sigma^{-5}$	$-(5n+4)n/(n+2)^2$
$[e]$	$l\sigma^{-4}$	$2n^2/(n+2)(n+4)$
$[f]+[g]$	$\frac{1}{2}(l^2 - 9l)\sigma^{-4}$	$n^2/(n+2)^2$
$[h]$	$4l\sigma^{-4} - 23l\sigma^{-5}$	$-3n/(n+2)$
$[i]$	$\frac{1}{2}l\sigma^{-4} - 3\frac{1}{2}l\sigma^{-5}$	$-6n/(n+2)$
$[j]$	$l\sigma^{-5}$	$-(7n^2+12n+8)n/(n+2)^3$
$[k]+[l]$	$(l^2 - 10l)\sigma^{-5}$	$3n^2/(n+2)^2$
$[m]$	$l\sigma^{-5}$	$10n^2/(n+2)(n+4)$
$[n]$	$4l\sigma^{-5}$	$-(5n+4)n/(n+2)^2$
$[o]$	$l\sigma^{-5}$	$-6n/(n+2)$
$[p]$	$2l\sigma^{-5}$	$-2(5n+4)n/(n+2)^2$
$[q]$	[illegible]σ^{-5}	$-3n/(n+2)$
$[r]$	[illegible]$^{-5}$	$-6n/(n+2)$
$[s]$	[illegible]$l\sigma^{-5}$	$-6n/(n+2)$

Thus one must not be surprised if the final expansion is only asymptotic. Lastly we may replace $1/\sigma$ by $q^{-1}(1-q^{-1})^{-1}$ and expand the exponential to obtain

$$\begin{aligned} \frac{2dJ}{k_B T_c} = \frac{T_c^0}{T_c} &= 1 + q^{-1} + q^{-2}\left(2 + \frac{n}{n+2}\right) + q^{-3}\left(6 + \frac{6n}{n+2}\right) \\ &+ q^{-4}\left[27 + \frac{n}{n+2}\left(33 - \frac{12}{n+2} + \frac{8}{n+4}\right)\right] \\ &+ q^{-5}\left[157 + \frac{n}{n+2}\left(198 - \frac{136}{n+2} + \frac{80}{n+4} + \frac{12}{(n+2)^2}\right)\right] \\ &+ O(q^{-6}) . \end{aligned} \tag{3.15}$$

On taking the reciprocal this yields the result already quoted in (1.1).

IV. DISCUSSION

As observed in the Introduction, the results (1.1) and (3.15) reduce to the previous Ising-model expressions[1] for $n=1$ and to those for the self-avoiding-walk or excluded-volume problem for $n=0$.[1,4] In the limit[5] $n\to\infty$ the exact spherical model expansion (2.15) is recaptured to order $1/q^5$. It is also evident, from the fact that n dependence enters only in the $1/q^2$ and higher-order terms, that the critical temperatures for large $q = 2d$ are less sensitive to n than in lower dimensionalities. As $n \to -2$ or -4 (and, presumably, -6, -8, ...) the coefficients of the $1/q$ expansion diverge. It is interesting that these values of n are also just those at which pure Gaussian behavior is expected, at least for continuous-spin systems.[3]

In Fig. 2 the critical temperatures $T_c(n,d)$ predicted by (1.1), with truncation after l_0 terms, are plotted vs $n/(n+2)$ for various values of d (solid and dashed lines, which are almost straight in this representation). The circled dots indicate the exact critical temperatures[9,15] or best series estimates, based on series expansions[1,2,12]; the dotted lines serve to show the trend of these precise data but have no other significance. When the criterion $l_0 = l^*$, following from (2.21), is used, the results improve with decreasing n. Improvement with increasing d is, of course, to be expected.

It is interesting that the $1/d$ series do *not* always yield the best numerical values when truncated at the *smallest* term. For example, the exact spherical model result for $d=4$ is[9] $T_c(\infty, 4)/T_c^0 = 0.80680$. Truncation at fifth order, following (2.21), yields $T_c/T_c^0 \simeq 0.81497$. The best value is obtained by including the sixth-order term, giving $T_c/T_c^0 \simeq 0.80925$, but truncation after the smallest term, which is the eleventh, gives $T_c/T_c^0 \simeq 0.78941$. Evidently the criterion (2.21) is fairly reliable although inclusion of a following term or two may improve the accuracy.

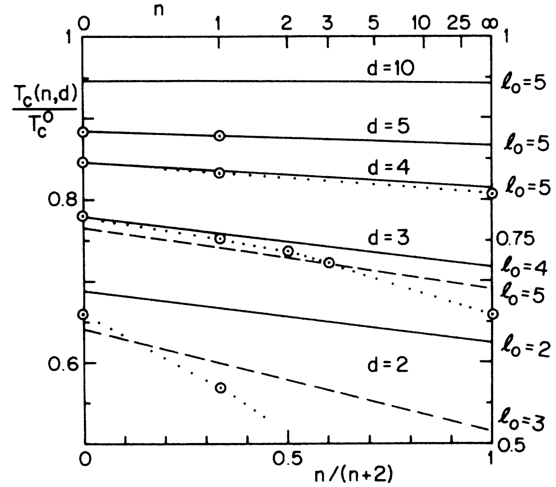

FIG. 2. Variation of critical temperatures $T_c(n,d)$ with n for various dimensionalites d. Solid and dashed lines represent the expansion (1.1) truncated at the l_0 term. Circled dots show the exact (or best-estimate) critical temperatures.

Although we were able to show that $T_c(\infty, d)$ was analytic in d for $2<d<\infty$, we cannot do the same for general n; however, such analyticity seems very plausible for $n>-2$. For this point, recall[1] first that each lattice constant $(G_l, \mathcal{L}_d)$ which enters into the high-temperature expansion can be written as a finite polynomial in the dimensionality d, of degree l or less. Hence the susceptibility expansion coefficients $a_l(n,d)$ are also polynomials in d of degree l; as such they have a unique, natural analytic continuation to all real (and complex) d. This continuation is what we have implicitly used in deriving $T_c(n,d)$ from (3.4). The analytic continuation for the spherical model provided by (2.3) is easily seen (by expanding in powers of $z^{-1} \sim K$) to conform to the same prescription. Thus it is reasonable to expect similar analyticity properties, at least from positive integral n.

Finally we note that although the first five $1/d$ expansion coefficients are positive for $n \geq 0$, this cannot be true for all higher-order terms since the spherical model expansion (2.15) contains terms of both signs and the coefficients remain continuous as $n\to\infty$.

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APPENDIX

In order to calculate the graphical weights $A(n;G)$ we need the averages $M(n;G)$ and $N(n;G)$ defined in (3.7) and (3.8). To this end, note that we only have to evaluate weights of the form

$$P(n;G^0)=\left\langle \prod_{(ij)} (\vec{s}_i\cdot\vec{s}_j)^{m_{ij}}\right\rangle_0 , \tag{A1}$$

where G^0 is a graph of even vertices only. Furthermore, if G^0 is the union of connected graphs $G^0_{(1)}, G^0_{(2)}, \ldots$, then $P(G^0)=P(G^0_{(1)})P(G^0_{(2)})\cdots$. This factorization property also holds for the pieces[13] $G^0_{(i)}$ and $G^0_{(ii)}$ of a graph relative to an articulation point at, say, the vertex 0; by holding the spin vector $\vec{s}_0$ fixed while all others are integrated over, the final result is seen to be independent of $\vec{s}_0$ and equal to the product of the separate graphical weights $P(G^0_{(i)})$ and $P(G^0_{(ii)})$. This is, of course, a direct reflection of the isotropy of the spin coupling.

In addition, every chain of single edges connecting two points can be reduced to a single edge connecting the points by use of the Funk-Hencke theorem[14] in the form

$$\int d\Omega(\vec{s}_i)\,(\vec{s}_i\cdot\vec{s}_j)^m S_k(\vec{s}_i)=\omega_n\lambda_k^m(n)S_k(\vec{s}_j) , \tag{A2}$$

where $d\Omega(\vec{s}_i)$ is the element of area of the unit n-hypersphere of total surface area ω_n, while $S_k(\vec{s})$ is a surface harmonic of degree k. The eigenvalue is given by

$$\omega_n\lambda_k^m(n)=\omega_{n-1}n^m[C_k^{\bar{n}}(1)]^{-1} \times\int_{-1}^{1} x^m C_k^{\bar{n}}(x)\,(1-x^2)^{\bar{n}-1/2}\,dx , \tag{A3}$$

where $\bar{n}=\frac{1}{2}n-1$ and $C_k^{\bar{n}}(x)$ is a Gegenbauer polynomial. After a k-fold integration by parts[14] we obtain the explicit result

$$\lambda_k^{k+2l}(n)=2^k n^{k+2l}\,\frac{\Gamma(\frac{1}{2}n)\Gamma(n-2)\Gamma(\frac{1}{2}n+k-1)\Gamma(\frac{1}{2}n+k-\frac{1}{2})\Gamma(k+2l+1)\Gamma(l+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n-\frac{1}{2})\Gamma(\frac{1}{2}n-1)\Gamma(n+2k-2)\Gamma(\frac{1}{2}n+k+l)\Gamma(2l+1)} , \tag{A4}$$

for $l=0, 1, 2, \ldots$.

The most valuable result is $\lambda_1^1(n)\equiv 1$. Then in integrating over the spin $\vec{s}_i$ in a chain of single bonds the scalar product with the neighbor $\vec{s}_{i+1}$ may be considered as a spherical harmonic of degree unity, $S_1(\vec{s}_i)=\vec{s}_i\cdot\vec{s}_{i+1}$. Thus integration over $\vec{s}_i$ (with $\vec{s}_j=\vec{s}_{i-1}$ being the other neighbor) introduces the coupling $\vec{s}_{i-1}\cdot\vec{s}_{i+1}$ and hence reduces the length of the chain by one. This result is actually just a different form of a lemma obtained by Paul and

Stanley[16] and justifies the grouping of graphs into classes used in Sec. III.

The P weights of the remaining chain-free diagrams may be calculated in a similar way by choosing parts of the integrands as surface harmonics. For the weights shown in Table II, one needs the values $k=0$, 1, and 2 and $l=0$, 1, 2, and 3 in (A4). Finally note that one always has $m=k+2l\geq 1$, so that it follows from (A4) that $\lambda_k^m(n)\to 0$ as $n\to 0$ unless $m=k=1$. This is sufficient to show, in agreement with Bowers and McKerrell,[5] that the only graphs contributing to the susceptibility expansion when $n\to 0$ are the self-avoiding chains $G_l^{[a]}=C_l$.[17]

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[17]Thus with the exception of $P(n, P_l)=n$, where P_l denotes a self-avoiding ring or polygon, all the $p(n, G^0)$ of (A1) are of order n^2 or higher as $n\to 0$. Hence by (3.7) and (3.8) the weights M and N are at least of order n for all graphs except the chain for which $N(n, C_l)=1$. The result then follows from (3.6).