

Griffiths singularities in the randomly dilute one-dimensional Ising model*

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(Received 31 July 1974)

Exact solutions for the thermodynamic functions of the randomly dilute $s = 1/2$ nearest-neighbor Ising chain in a magnetic field are examined. Both site and bond impurities are treated. Behavior is nonanalytic at $T = h = 0$. The divergences of the pure-chain thermodynamics are replaced at nonzero dilution by essential singularities of the Griffiths type at which all functions are finite and infinitely differentiable. The simplicity of the solution allows the origin and form of the Griffiths singularities to be traced in detail.

I. INTRODUCTION

The phenomenology of critical behavior in pure materials is by now reasonably well understood.¹ For appropriate values of the interaction parameters there is typically a unique critical temperature T_c at which the correlation length ξ becomes infinite and the thermodynamic functions become nonanalytic with power-law singularities described by critical exponents. The phenomenology of critical behavior in amorphous or random systems² has not yet been established with similar clarity; however, what evidence there is suggests that it is more complicated.

McCoy and Wu³ have studied a two-dimensional layered Ising model with random interlayer coupling. They find a specific heat which is finite, infinitely differentiable, but nonanalytic and a boundary susceptibility which is divergent not only at the T_c defined by the specific-heat singularity but also in a *finite* temperature interval above and below. Griffiths and Lebowitz⁴ in a rigorous study of the randomly dilute Ising model show that the thermodynamic functions are analytic for nonvanishing magnetic field ($h \neq 0$) at arbitrary concentration p of magnetic sites; however, Griffiths⁵ goes on to show that for nonzero dilution $0 < p < 1$ the magnetization is actually nonanalytic at $h = 0$ as a function of h for a range of temperatures above the temperature $T_c(p)$ at which spontaneous magnetization first appears,⁶ even when p is below the critical concentration p_c , so $T_c(p) = 0$ and the correlation length as conventionally defined is certainly finite. These papers demonstrate that for some random systems (at least) (a) there is no longer a unique T_c but rather a range of temperatures over which nonanalyticity occurs,⁷ (b) there may be nonanalytic behavior for finite ξ , and (c) singularities other than power laws may typically occur.

These nonstandard (i. e., for pure materials) features may or may not be experimentally relevant⁸; however, implications for other theoretical

work cannot be ignored. Rushbrooke *et al.*⁹ and Rapaport¹⁰ have derived and analyzed high-temperature series for randomly dilute Ising and Heisenberg models. Their work appears to favor a continuous concentration dependence of the critical exponents in agreement with a recent proposal of Suzuki¹¹ but in disagreement with earlier speculations by Watson¹² and Domb.¹³ However, the series analysis rests *crucially* on the assumption that the critical behavior of the random system is no more complicated than that of the pure system, so (as the authors, themselves, emphasize) any conclusions based on the series evidence must be treated with caution. Only when the *structure* of the singularities is established more firmly can the series be expected to yield reliable parameters. Finally, initial attempts to apply renormalization-group methods to random systems¹⁴ have yet to exhibit the Griffiths singularities.⁵ Until they do, it will seem possible that some essential ingredient may be missing.¹⁵

The literature contains exact solutions for a variety of random models: Ising models in one dimension¹⁶ ($d = 1$) and two dimensions³ ($d = 2$) and XY models¹⁷ in $d = 1$. None of these solutions have been shown to exhibit the Griffiths singularities.¹⁸ In this paper we examine exact results for the thermodynamic functions of two randomly dilute $d = 1$ Ising models in arbitrary external magnetic field. Because the system is one dimensional, forcing $T_c = 0$, the spreading of the nonanalyticity over a finite temperature interval [feature (a) above] is lost; however, these models *do* exhibit finite ξ and Griffiths singularities [features (b) and (c)] in a form which is simple enough so the origin of the behavior may be traced quite explicitly. We hope that the insight gained may be of some value in providing understanding of physically more interesting systems. In our simple models the Griffiths singularities arise from the fact that for $0 < p < 1$ there is a *finite* probability for the existence of intact chains of n sites, with n arbitrarily large but not infinite.¹⁹ The relevant piece

of the finite-chain free energy varies with large n as e^{-n} and is, thus, totally irrelevant for the pure ($p=1$) system in the thermodynamic limit.

The solution of the $s=\frac{1}{2}$ nearest-neighbor one-dimensional Ising chain without impurities is well known.²⁰ The point $T_c = h_c = 0$ may be said to be a critical point in the sense that the correlation length diverges there ($\xi \rightarrow \infty$) and nonanalyticities occur in the thermodynamic functions. Indeed, both Kadanoff²¹ and Nelson and Fisher²² have shown how to extract the critical behavior via renormalization-group recursion relations. The natural variables are h and $e^{-J/k_B T}$, but one may interpret the behavior in terms of the conventional exponents with²² $\gamma = \nu = \delta = \infty$, $\beta = 0$, and $\eta = 1$. The one-dimensional problem with site dilution was first studied by Katsura and Tsujiyama,^{16b} who worked out the energy, free energy, and susceptibility at $h=0$. It turns out, as we shall show below, that the e^{-n} part of the finite-chain thermodynamic functions vanishes at $h=0$, so the Griffiths singularities are absent in these results. More recently Matsubara *et al.*^{16c} extended this work to $h \neq 0$ and treated the generalization to several kinds of magnetic ions.^{16d,e}

Section II sketches a derivation of the exact thermodynamic functions of the $d=1$, $s=\frac{1}{2}$ randomly dilute nearest-neighbor Ising model at arbitrary concentration, field, and temperature. Results for bond dilution are new but those for site dilution parallel Matsubara *et al.*^{16c} As noted in Ref. 16c, the method is trivially applicable to any $d=1$ randomly dilute system for which the finite-chain thermodynamics is known.^{17b} Section III discusses the origin and form of the Griffiths singularities.

$$P\{\nu(n)\} = \begin{cases} (1-p)^{N_p} p^{N-N_p} N \frac{(N_p-1)!}{(N_p-N_c)!} \prod_{n=1}^{\infty} \frac{1}{\nu(n)!} & \text{(site)} \\ (1-b)^{N_c} b^{N-N_c} N(N_c-1)! \prod_{n=1}^{\infty} \frac{1}{\nu(n)!} & \text{(bond)}. \end{cases} \quad (4)$$

Here $N_c = \sum_{n=1}^{\infty} \nu(n)$ = total number of finite, open-ended chains, which is just the total number of missing bonds in case (ii), and $N - N_p = \sum_{n=1}^{\infty} n\nu(n)$, so N_p is just the total number of inert sites [$N_p \equiv 0$ in case (ii)]. The second factor in (4) is the number of specific configurations corresponding to $\{\nu(n)\}$ and the first factor is the probability of each specific configuration. To get the (average) free energy per lattice site in the thermodynamic limit, one takes

$$\begin{aligned} w(K, h) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\{\nu(n)\}} P\{\nu(n)\} W\{\nu(n)\} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{\infty} \langle \nu(n) \rangle W_n(K, h). \end{aligned} \quad (5)$$

II. EXACT THERMODYNAMICS OF THE RANDOMLY DILUTE ISING CHAIN

The Hamiltonian of the pure $s=\frac{1}{2}$ nearest-neighbor Ising chain is

$$\begin{aligned} -\beta J C_N &= K \sum_{i=1}^{N-1} \mu_i \mu_{i+1} + K \mu_N \mu_1 \\ &+ h \sum_{i=1}^N \mu_i, \quad \mu_i = \pm 1, \end{aligned} \quad (1)$$

where $\beta^{-1} = k_B T$, $K = \beta J$ ($J \equiv$ exchange coupling), $h = \beta H$ ($H =$ magnetic field), and we have taken circular boundary conditions for convenience. We shall consider two types of dilution: (i) "site" dilution, in which each magnetic ion either remains (with probability p) or is removed (with probability $1-p$) at random, and (ii) "bond" dilution, in which each exchange bond is randomly set to K (with probability b) or 0 (with probability $1-b$).

Each specific configuration of sites and bonds remaining after dilution consists of a set of *independent* finite chains with a correspondingly additive free energy,

$$W\{\nu(n)\} = \sum_{n=1}^{\infty} \nu(n) W_n(K, h), \quad (2)$$

where $\nu(n)$ is the number of finite (open-ended) chains of n sites and W_n is the corresponding reduced free energy,

$$W_n(K, h) = \ln \text{Tr} \exp \left(K \sum_{i=1}^{n-1} \mu_i \mu_{i+1} + h \sum_{i=1}^n \mu_i \right). \quad (3)$$

The over-all probability of a configuration $\{\nu(n)\}$ is just²³

An elementary maximization gives

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle \nu(n) \rangle = \begin{cases} (1-p)^2 p^n & \text{(site)} \\ (1-b)^2 b^{n-1} & \text{(bond)} \end{cases}, \quad (6)$$

so the free energy per site is

$$w(K, h) = \begin{cases} (1-p)^2 \sum_{n=1}^{\infty} p^n W_n(K, h) & \text{(site)} \\ (1-b)^2 \sum_{n=1}^{\infty} b^{n-1} W_n(K, h) & \text{(bond)}. \end{cases} \quad (7)$$

The forms (6) are understandable in that each chain of n intact sites has two ends and $n-1$ intact bonds. Equation (7) has appeared previously²⁴ and is quite general, requiring a knowledge of only

the finite-chain free energies W_n .

The finite-chain energies (3) are easy to calculate via transfer-matrix methods. If $\tau = e^{-K}$, the larger and smaller eigenvalues of the transfer matrix may be written

$$\lambda_{\pm} = \tau^{-1} (\cosh h \pm \Delta^{1/2}) \quad (8)$$

with

$$\Delta = \sinh^2 h + \tau^4$$

and

$$\lambda_{\pm} \geq \lambda_{\mp} \geq 0. \quad (9)$$

Then,

$$\begin{aligned} W_n(K, h) &= \ln(a_{\pm} \lambda_{\pm}^{n-1} + a_{\mp} \lambda_{\mp}^{n-1}) \\ &= n \ln \lambda_{\pm} + \ln(a_{\pm}/\lambda_{\pm}) \\ &\quad + \ln \left[1 + \frac{a_{\mp}}{a_{\pm}} \left(\frac{\lambda_{\mp}}{\lambda_{\pm}} \right)^{n-1} \right], \end{aligned} \quad (10)$$

with

$$a_{\pm} = \cosh h \pm [(\sinh^2 h + \tau^2)/\Delta^{1/2}],$$

so

$$a_{\pm} \geq a_{\mp} \geq 0. \quad (11)$$

The three terms on the right-hand side of (10) represent, respectively, the bulk free energy, the surface (i. e., end-point) free energy, and a finite-size contribution to which we shall return in Sec. III B. The result (7) with (8)–(11) can be checked against high-temperature and high-field expansions.

It is useful to record here certain limiting forms of the thermodynamic functions which will be used in the discussion of Sec. III. At $p, b = 0$, one finds

$$w(K, h) = \begin{cases} 0 & \text{(site)} \\ \ln(2 \cosh h) & \text{(bond)} \end{cases} \quad (12)$$

At $p, b = 1$ the pure undiluted Ising chain is regained,^{20,23}

$$w(K, h) = \ln \lambda_{\pm} = K + \ln[\cosh h + (\sinh^2 h + e^{-4K})^{1/2}]. \quad (13)$$

For $h = 0$, $a_{\pm} = 0$ and known results are found^{16b}:

$$w(K, 0) = \begin{cases} p \ln 2 + p^2 \ln \cosh K & \text{(site)} \\ \ln 2 + b \ln \cosh K & \text{(bond)}, \end{cases} \quad (14)$$

and the zero-field (reduced) susceptibility,

$$\begin{aligned} \chi_0(K) &= \left. \frac{\partial^2 w(K, h)}{\partial h^2} \right|_{h=0} \\ &= \begin{cases} \frac{p(1+p \tanh K)}{1-p \tanh K} & \text{(site)} \\ \frac{1+b \tanh K}{1-b \tanh K} & \text{(bond)}. \end{cases} \end{aligned} \quad (15)$$

Finally, for $T = 0$ ($J > 0$, ferromagnetic),

$$\begin{aligned} w(K, h) - p^2 K &\xrightarrow{K \rightarrow \infty} (1-p)^2 \\ &\quad \times \sum_{n=1}^{\infty} p^n \ln(2 \cosh nh) \quad \text{(site)} \end{aligned}$$

$$\begin{aligned} w(K, h) - bK &\xrightarrow{K \rightarrow \infty} (1-b)^2 \\ &\quad \times \sum_{n=1}^{\infty} b^{n-1} \ln(2 \cosh nh) \quad \text{(bond)}, \end{aligned} \quad (16)$$

with the corresponding $T = 0$ magnetization and susceptibility,²⁵

$$M(h) = \begin{cases} (1-p)^2 \sum_{n=1}^{\infty} n p^n \tanh nh & \text{(site)} \\ (1-b)^2 \sum_{n=1}^{\infty} n b^{n-1} \tanh nh & \text{(bond)}, \end{cases} \quad (17)$$

$$\chi(h) = \begin{cases} (1-p)^2 \sum_{n=1}^{\infty} \frac{n^2 p^n}{(\cosh nh)^2} & \text{(site)} \\ (1-b)^2 \sum_{n=1}^{\infty} \frac{n^2 b^{n-1}}{(\cosh nh)^2} & \text{(bond)}. \end{cases} \quad (18)$$

III. GRIFFITHS SINGULARITIES AND OTHER DISCUSSION

A. Singularities of the free energy

The analytic properties of the free energy $w(K, h) = w(K, -h)$ defined by (7)–(11) may now be discussed. For simplicity we choose $J/k_B = +1$ (ferromagnetic) and consider the range of variables $T = K^{-1} \geq 0$, $h \geq 0$. $w(T, h)$ is analytic for real values of T and h provided $T > 0$. The pole which is always present at $T = 0$ [see, e. g., (13) and (16)] is trivial and can be removed by a redefinition of the ground-state energy. In addition there is always an essential (but trivial) singularity at $T = 0$, since the natural temperature variable of the problem $\tau = e^{-K} = e^{-1/T}$ depends on T in a singular manner. The quantities $w(K, h) - p^2 K$ (site) or $w(K, h) - bK$ (bond) regarded as functions of τ and h are analytic except at the critical point $\tau = \tau_c = 0$ ($T = 0$), $h = h_c = 0$.

When $p, b = 1$ we see from (13) that the singular behavior arises entirely from the branch cut belonging to $\Delta^{1/2}$,

$$w(K, h) - K \xrightarrow{h, \tau \text{ small}} (h^2 + \tau^4)^{1/2}, \quad (19)$$

from which it follows that the zero-field susceptibility diverges as τ^{-2} when $\tau \rightarrow 0$ [see also (15)]. The branch cuts persist at nonzero dilution $0 < p, b < 1$, since both λ_{\pm} and a_{\pm} depend on $\Delta^{1/2}$ [see, e. g., the first and second terms on the right-hand side of (10)]; however, in addition there is now singular structure—the Griffiths singularity—which arises from an entirely different source: The finite-chain $W_{n+1}(K, h)$ has singularities whenever

$$a_> \lambda_>^n + a_< \lambda_<^n = 0. \tag{20}$$

Recalling $0 < a_</a_> \leq 1$ (for $h \neq 0$), we see that (20) is satisfied at the n (complex) roots,

$$z = \lambda_</\lambda_> = |a_</a_>|^{1/n} (-1)^{1/n}. \tag{21}$$

None of the roots (21) lies inside the unit circle and none lies in the physical domain (the positive real axis, $0 \leq z \leq 1$); however, as $n \rightarrow \infty$, the roots become dense on the unit circle, so $z = 1$ is a limit point of the roots and, thus, an essential singularity of the sum (7) for $w(K, h)$. Away from the critical point $\Delta > 0$, $\lambda_</\lambda_> < 1$, and the singularities (21) occur finitely away from the physical values of the variables τ and h ; however, at $T = h = 0$ one has $\lambda_</\lambda_> = 1$, so the Griffiths singularity is present as the critical point is approached along any path not parallel to²⁶ $h = 0$, $T \rightarrow 0$ [since $a_<(K, h = 0) = 0$].

All this structure is particularly transparent at $T = 0$, where $\lambda_</\lambda_> = a_</a_> = e^{-2h}$, so the singularities of W_n may be read off from (21),

$$z = e^{-2h} = (-1)^{1/n}$$

or

$$h = i\pi(2m + 1)/n, \quad m = 0, \pm 1, \pm 2, \dots, \tag{22}$$

as is also evident directly from (16)–(18). Poles of $M(h)$, for example, occur at (22). For each finite n the real h axis is free of singularities; however, for $0 < p, b < 1$ all values of n appear in the sum and the Griffiths singularities occur, since the real point $h = 0$ becomes a limit point of poles.

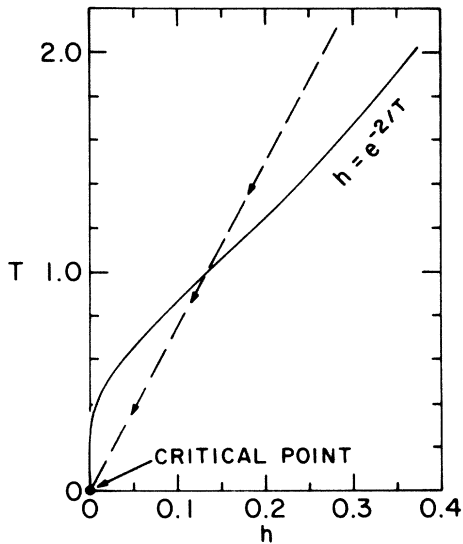


FIG. 1. Plot of $h = \tau^2 = e^{-2/T}$. The Griffiths singularity is absent when the critical point is approached along the T axis. Approach along any other straight path (such as the dashed line) yields $\tau^2/h \rightarrow 0$, and the $T = 0$ forms (16)–(18) apply near enough the origin.

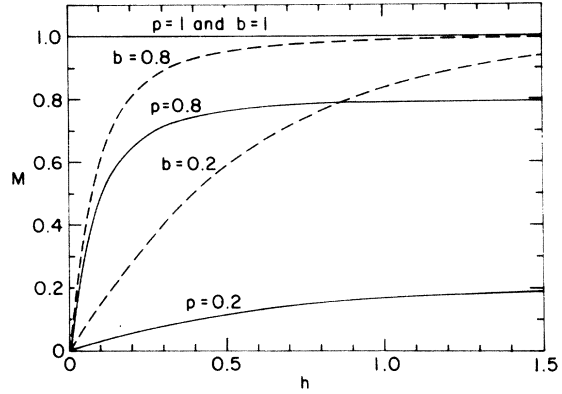


FIG. 2. $M(h)$ at $T = 0$ for a variety of site (p) and bond (b) dilutions. $M(h)$ is essentially singular (Griffiths singularity) at $h = 0$ for $0 < p, b < 1$. The asymptotic behavior is $M(h) \xrightarrow{h \rightarrow \infty} p(1)$ for site (bond) dilution.

B. Origin of Griffiths singularities

Griffiths's⁵ general argument relies on the theorem of Yang and Lee.²⁷ The connection in terms of our calculation is not hard to find. Equation (20) is just the condition that the finite-chain partition function should vanish. Yang and Lee show that this takes place only for (complex) values of h such that $|e^h| = 1$. These zeros close on the real axis as $T \rightarrow 0$, $n \rightarrow \infty$, and, indeed, at $T = 0$ we see from (8) that $\lambda_</\lambda_> = e^{-2h}$.

Note that, when the finite-chain free energies

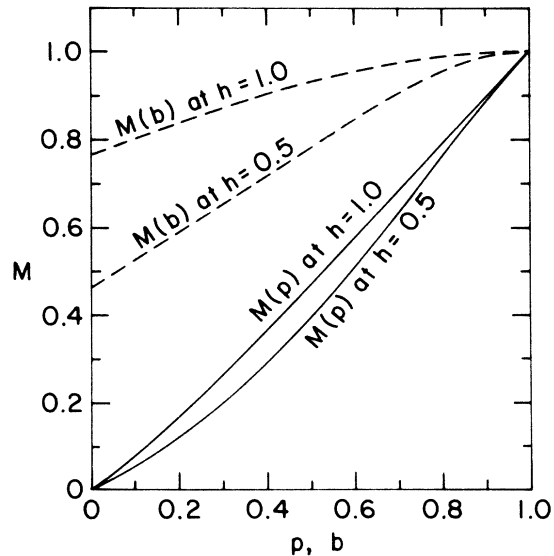


FIG. 3. Concentration dependence of the magnetization at $T = 0$. $M(p = 0) = 0$, since at $p = 0$ there are no magnetic ions. Behavior is linear at low concentrations. Near $p = 1$ ($b = 1$) the dependence is $M(p) \sim p [1 - M(b) \propto (1 - b)^2]$.

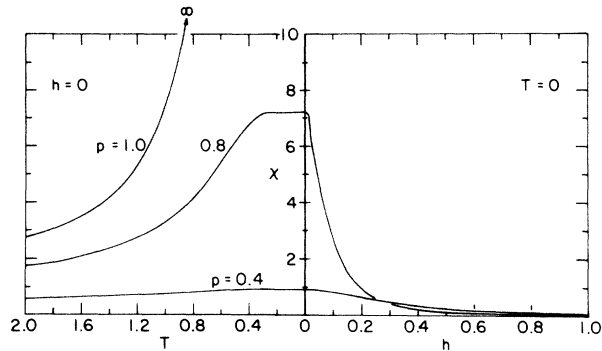


FIG. 4. Reduced susceptibility $\chi = \partial M / \partial h$ near $T = h = 0$ for the ferromagnetic Ising chain with site dilution. $\chi(h)$ has the Griffiths singularity at $h = 0$ for $0 < p < 1$.

are divided as in (10) into bulk, surface, and finite-size contributions, it is only the last of these, varying with size as e^{-n} for large n , which contains the Griffiths singularity. The strong suggestion is that any general treatment of the random dilution problem (e.g., for $d > 1$) must be detailed enough to include subtle finite-cluster effects (not just bulk and surface), if it is to succeed in reproducing the Griffiths singularities.

C. Nonanalyticities at finite ξ

In contrast to all ordinary critical behavior¹ the nonanalyticity (Griffiths singularity) which takes place for nonzero dilution at $T = h = 0$ occurs at a finite value of the correlation length.²⁸ It is easy to show that the $h = 0$ correlations behave as

$$\Gamma(n) = \langle \mu_1 \mu_{n+1} \rangle = A e^{-n/\xi} \quad (23)$$

with²⁹

$$A = \begin{cases} p & \text{and } \xi^{-1} = \begin{cases} |\ln p| + |\ln \tanh K| & \text{(site)} \\ |\ln b| + |\ln \tanh K| & \text{(bond),} \end{cases} \\ 1 & \end{cases} \quad (24)$$

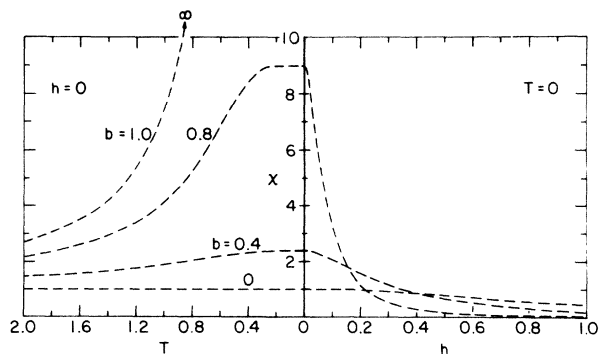


FIG. 5. Reduced susceptibility $\chi = \partial M / \partial h$ near $T = h = 0$ for the ferromagnetic Ising chain with bond dilution. $\chi(h)$ has the Griffiths singularity at $h = 0$ for $0 < b < 1$.

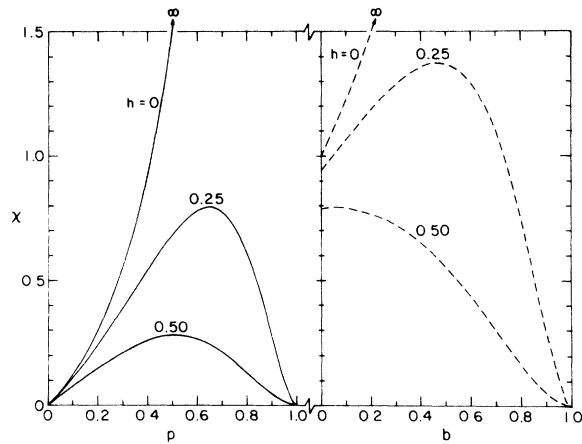


FIG. 6. Concentration dependence of the reduced susceptibility at $T = 0$ for various magnetic fields. $\chi(p) = 0$ at $p = 0$ (since the system becomes magnetically inert), while $\chi(b = 0)$ shows the behavior of a set of independent magnetic ions. For $h > 0$, χ vanishes quadratically at $p, b = 1$, since the $M(h)$ curve of Fig. 2 is flat.

so the correlation length is only divergent at $p, b = 1$. The origin of the finiteness of ξ for $p, b < 1$ is just the finite size of the typical cluster: The probability that a chain of ξ units remains intact after dilution is $p^\xi = e^{-1}$. The Griffiths singularities depend only on the fact that there is a finite probability (however small) for clusters larger than any given size.

D. Examples

The Griffiths singularities are likely to be a very subtle effect from an experimental point of view.⁸ It is easy to show for $0 < p, b < 1$ that, despite the Griffiths singularity, the free energy $w(K, h)$ (regarded as a function of λ_c/λ_s) is finite and has finite derivatives at $z = \lambda_c/\lambda_s = 1$. Correspondingly, one may verify at $T = 0$ directly from (17) and (18) that for $p, b < 1$ both $M(h)$ and $\chi(h)$ are finite and have finite derivatives of all orders at $h = 0$.³⁰

We illustrate in Figs. 2–7 the behavior of the magnetization and susceptibility as functions of h (at $T = 0$) and T (at $h = 0$) for a variety of dilutions [see (15), (17), and (18)]. It is not difficult^{16c} to plot behavior along other paths using the more general expressions (7) and (10). Note, however, that these limiting forms are quite typical near the critical point: Critical behavior is dominated by the ratio of h to τ^2 in Δ . Figure 1 shows a plot in h, T variables of the line $h = \tau^2$. Any straight-line approach to $T = h = 0$ except $h = 0, T \rightarrow 0$ has $\tau^2/h \rightarrow 0$, so the $T = 0$ forms (17) and (18) apply near enough critically.

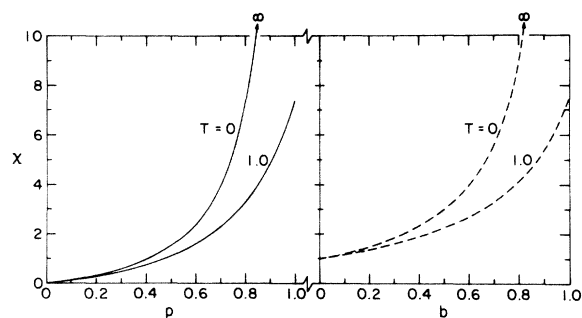


FIG. 7. Concentration dependence of the reduced susceptibility at $h=0$ for various temperatures.

Figures 2 and 3 show $T=0$ magnetization curves. The magnetization $M(h)$ of the undiluted chain is discontinuous at $h=0$. For $p, b < 1$, $M(h=0) = 0$ but the apparently smooth behavior near $h=0$ belies the essential (Griffiths) singularity which is present for $p, b > 0$. The concentration dependence of Fig. 3 is, on the other hand, analytic. Note that for fixed field and dilution the effect of site dilution (which renders *two* bonds inactive) is always more pronounced than that of bond dilution.

Figures 4 and 5 show the temperature and field

dependence of the (reduced) susceptibility near the critical point. For $p, b = 1$ the susceptibility diverges at the critical point. At any nonzero dilution the critical susceptibility is finite and has a horizontal tangent in both T and h . $\chi(h)$ has the Griffiths singularity at $h=0$. $\chi(T)$ varies linearly with τ near $T=0$ and is essentially singular only because of the nonanalytic dependence $\tau = e^{-1/T}$ (not a Griffiths singularity). For the undiluted system $\chi(h) = 0$ for all $h \neq 0$. Note that $\int_0^\infty dh \chi(h) = 1$ at $T=0$ for all concentrations.

Finally, Figs. 6 and 7 show the concentration dependence of the reduced susceptibility at $T=0$ and $h=0$, respectively. Except for the susceptibility divergence of the pure system all behavior is analytic.

ACKNOWLEDGMENTS

The author would like to thank Dr. J. L. Gammel for an interesting correspondence, Dr. R. D. Bourgin for kind help and explication, and the Physics Department of the University of Washington for their gracious hospitality during the course of this work. He is grateful to Dr. S. Katsura and Dr. F. Matsubara for supplying inadvertently omitted references to their recent work.

*Research supported in part by the National Science Foundation under Grant Nos. NSF-GP-40395 and NSF-GH-33167.

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⁶There is the strong suggestion that there is nonanalytic behavior in the entire interval $T_c(p) \leq T \leq T_c(1) \equiv T_c(\text{bulk})$. Such behavior was previously found in the nonrigorous work of M. A. Mikulinski, Sov. Phys. -JETP **26**, 637 (1968).

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⁸There is certainly impurity rounding in real systems; however, practical observations of *nonanalyticities* in

functions which are infinitely differentiable is problematic; see Sec. III D.

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¹⁵The models treated in Refs. 3, 14, 16(a), and 17(a) all involve a small random fluctuation of the exchange coupling about its average value. References 3, 16(a), and 17(a) expand in the width of this distribution. The dilution treated in this paper and in Ref. 5 is more dramatic: Certain sites or couplings are entirely absent. It seems at least possible that the Griffiths singularities actually require this more dramatic randomness or (more weakly) that they fail to show up to lowest order in the width.

¹⁶(a) C. Fan and B. M. McCoy, Phys. Rev. **182**, 614 (1969); (b) S. Katsura and B. Tsujiyama, in *Critical Phenomena*, edited by M. S. Green and J. V. Sengers (Natl. Bur. of Std. Washington, D. C., 1966), p. 219; (c) F. Matsubara, K. Yoshimura, and S. Katsura, Can. J. Phys. **51**, 1053 (1973); (d) F. Matsubara, Prog. Theor. Phys. **51**, 378 (1974); **51**, 1694 (1974); and S.

- Katsura and F. Matsubara, *Can. J. Phys.* **52**, 120 (1974); (e) F. Matsubara, *Prog. Theor. Phys.* (to be published).
- ¹⁷(a) E. R. Smith, *J. Phys. C* **3**, 1419 (1970); (b) F. Matsubara and S. Katsura, *Prog. Theor. Phys.* **49**, 367 (1973).
- ¹⁸The work of Refs. 3 and 16(b) lacks a magnetic field. References 16(a) and 17(a) are restricted to narrow distributions of exchange couplings, as discussed in Ref. 15. The solutions of Refs. 16(c), 16(d), 16(e), and 17(b) contain Griffiths-type singularities, but these singularities remain implicit.
- ¹⁹This feature is clearly exhibited in Griffiths's original proof, Ref. 5.
- ²⁰K. Huang, *Statistical Mechanics* (Wiley, New York, 1965), p. 346.
- ²¹L. P. Kadanoff, Conference on the Renormalization Group in Critical Phenomena and Quantum Field Theory, Temple University, Philadelphia, 1973 (unpublished).
- ²²D. R. Nelson and M. E. Fisher, *AIP Conf. Proc.* **18**, 888 (1974) and unpublished work.
- ²³The unique *undiluted* configuration has probability p^N or b^N and free energy $W_{\text{ring}} = \ln \text{Tr} e^{-\beta \mathcal{H}^N}$. It contains no open-ended chains of the type of (3) and must, strictly speaking, be treated as a special case in (4). Its contribution vanishes, however, as $N \rightarrow \infty$ for all $p, b < 1$, so we neglect the special case in what follows. The final thermodynamic formulas for nonzero dilution approach those for $p, b = 1$ continuously, so it is not necessary to treat the pure case separately.
- ²⁴Reference 16(c), Eqs. (2.1) and (2.2). Also, more generally, Ref. 5, Eq. (1).
- ²⁵As $T \rightarrow 0$ with h nonzero, $K/h \rightarrow \infty$, and each n chain acts like a single ion with magnetic moment n in interaction with the field h . The expressions (16)–(18) are just appropriately weighted averages of contributions from such "giant" moments. Provided only that $T = 0$, similar expressions hold for arbitrary d (Ref. 7) if only an extra multiplicity $g(n)$ is introduced into the summands to reflect the fact that for $d > 1$ there are many topologically distinct connected clusters of n sites [or $(n-1)$ bonds].
- ²⁶The absence of the Griffiths singularity from the T (or τ) dependence at $h = 0$ is connected formally to the fact that the transformation which diagonalizes the transfer matrix is T independent at $h = 0$. Thus, one can write immediately, $w(T) = \int dJ P(J) \ln [2 \cosh(J/k_B T)]$ for an Ising chain with nearest-neighbor bond strengths distributed randomly with $P(J)$, $\int dJ P(J) = 1$. More generally, as pointed out by Griffiths (Ref. 5), the existence of singularities in T at $T = T_c$ (bulk), $h = 0$, might involve some analog in the temperature variable of the Yang-Lee theorem (Ref. 27).
- ²⁷C. N. Yang and T. D. Lee, *Phys. Rev.* **87**, 404 (1952); **87**, 410 (1952).
- ²⁸There is a parallel here with the singular behavior which occurs (at least in the droplet model) at an ordinary *first*-order phase boundary. M. E. Fisher, *Physics* **3**, 255 (1967); J. S. Langer, *Ann. Phys.* **41**, 108 (1967).
- ²⁹We note in passing that (23) and (18) are consistent via the susceptibility sum rule $\chi = \Gamma(0) + 2 \sum_{n=1}^{\infty} \Gamma(n)$.
- ³⁰Despite the existence of all derivatives, the essential singularity makes itself felt in that the formal Taylor expansion $M(h) = \sum_{n=1}^{\infty} a_n h^n$ has zero radius of convergence.