Band nonparabolicities, broadening, and internal field distributions: The spectroscopy of Franz-Keldysh oscillations

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The complex dielectric function of a semiconductor or insulator in a uniform electric field is expressed asymptotically as the sum of a zero-field, a third-derivative, and an oscillatory (or exponential) term by appropriately deforming a contour of integration in the complex plane. The latter term describes separately the field-induced change in the dielectric function arising from Franz-Keldysh oscillations (or exponential edges). The present expansion extends the region of validity of the previously developed low-field approximation to a wide range of energies and applied fields. The stationary-phase formalism by which this term is evaluated allows the asymptotic behavior of the Franz-Keldysh oscillations to be calculated by a series of algebraic steps, a simple procedure that permits the influence of various perturbations on the Franz-Keldysh oscillations to be studied generally. We find by application of the theory that the period of the oscillations is relatively unaffected by band nonparabolicities, broadening, or field-distribuion effects, showing that the value of the interband mass at the critical point determined from them is highly accurate. By contrast, the envelope of the oscillations is shown to depend strongly upon these interactions. Since Franz-Keldysh oscillations appear to be relatively unaffected by exciton effects and can be measured for several different critical points of the same crystal, measurement of their field dependence can provide a stringent means for evaluating current theories of internal-field distributions and represent an attractive alternative to exponential-absorption-edge studies for systematic investigations of these effects.

I. INTRODUCTION

The representation of the complex dielectric function ϵ of a semiconductor or insulator in a uniform electric field as a low-field perturbation expansion[1,2] has proved to be particularly useful in solid-state spectroscopy. The low-field expression for $\Delta\epsilon$, the field-induced change in ϵ, is related to the third derivative of the unperturbed dielectric function and is sufficiently simple to allow certain critical-point characteristics to be obtained directly from low-field electroreflectance line shapes.[1-4] At high fields, $\Delta\epsilon$ evolves from a third-derivative response into a more complicated form characterized by exponential or oscillatory behavior at energies well below or well above, respectively, the critical-point energy.[5,6]

It is well known that the period of the high-field or Franz-Keldysh oscillations can be analyzed to obtain the interband reduced masses of the associated critical points.[7-9] Using as relevant examples the theoretical investigations[10-15] of the role of electric field effects in determining the exponentially dependent absorption edge commonly observed in semiconducting or insulating materials,[16] it can be anticipated that studies of the systematics of the energy and field dependence of the envelope of these oscillations should also be fruitful in determining how internal-field distributions influence dielectric properties, and in determining in detail the nature of these internal-field distributions. Since recent experimental results have indicated that Franz-Keldysh oscillations can be measured conveniently for a number of critical points,[17-20] this opens up the possibility of being able to perform these investigations for a range of conditions within the same crystal, instead of being restricted to analyzing data obtained only below the fundamental absorption edge.

An investigation of the effects of various influences on Franz-Keldysh oscillations appears not to have been performed previously, a situation due in part to the complicated closed-form mathematical representations[6] of simple parabolic line shapes, a limiting case of the general one-electron theory.[21] The objectives of this paper are to obtain a simple mathematical expression for these oscillations, and to investigate the effects on their envelope and phase of various interactions treated as perturbations to the basic one-electron electric field theory of a simple parabolic band. The asymptotic expression for the Franz-Keldysh oscillations is derived in Sec. II. This expression and the previously obtained third-derivative term[1,2] together provide a complete asymptotic representation of the exact one-electron expression for $\Delta\epsilon$, which is valid for a wide range of energy for both low and high fields. The high-field term is evaluated for simple parabolic bands of one, two, and three dimensions, and the results summarized in Table I. In Sec. III, the effects on the envelope and phase of deviations from the uniform-field, simple parabolic model, in the form of band nonparabolicities, lifetime broadening, and various approaches to the theory of the effects of internal-field distributions, are treated as perturbations in order to investigate systematics. The results of Sec. III are summarized in Table II, which shows how the interactions modify functionally the energy dependence of the envelope and phase. Applications and limitations are discussed in Sec. IV, with particular emphasis on the evaluation of current theories of internal-field distributions, and the relationship of the closed-form asymptotic representations of the one-electron theory given here to numerically calculated line shapes incorporating Coulomb interaction effects.[22,23]

II. THEORY

A. General expression

The complete expression for the dielectric function of a semiconductor or insulator in a uniform electric field in the one-electron, one-band approximation is given by[21]

$$\epsilon(E_+, \vec{\mathcal{E}}) = 1 + \sum_{c,v} [\epsilon_{cv}(E_+, \vec{\mathcal{E}}) + \epsilon^*_{cv}(-E_+, \vec{\mathcal{E}})], \quad (1)$$

where for a single conduction-valence band pair c, v

$$\epsilon_{cv}(E_+, \vec{\mathcal{E}}) = \frac{ie^2\hbar^2}{\pi^2 m^2 E_+^2} \int_{BZ} d^3k \int_0^\infty ds\, [\hat{e}\cdot \vec{p}_{vc}(\vec{k} - e\vec{\mathcal{E}}s/2)] \times [\hat{e}\cdot\vec{p}_{cv}(\vec{k} + e\vec{\mathcal{E}}s/2)] \times \exp\left(i \int_{-s/2}^{s/2} ds' [E_+ - E_{cv}(\vec{k} - e\vec{\mathcal{E}}s')]\right), \quad (2)$$

where

$$E_+ = E + i\Gamma, \quad (3a)$$

$$E_{cv}(\vec{k}) = E_c(\vec{k}) - E_v(\vec{k}), \quad (3b)$$

and E is the energy, $\vec{k}$ is the wave vector integrated over the Brillouin zone, s is an integration variable with dimensions of inverse energy, $\hat{e}$ is the unit polarization vector, $\vec{p}_{cv}(\vec{k})$ is the momentum matrix element between valence (v)- and conduction (c)-band states of energy $E_v(\vec{k})$ and $E_c(\vec{k})$, respectively, and Γ is the phenomenological broadening parameter. To reduce Eq. (2) to a simpler starting expression without sacrificing any of the essential details, we shall assume (in accordance with experiment) that the dominant observable contributions to the field-induced change in ϵ come from regions near critical points, for which the momentum matrix element is locally $\vec{k}$ independent and the energy bands are locally parabolic (the latter restriction will be lifted partially in Sec. III). Then

$$E_{cv}(\vec{k} - e\vec{\mathcal{E}}s) \cong E_{cv}(\vec{k}) - e\vec{\mathcal{E}}s\cdot\nabla E_{cv}(\vec{k}) + 4s^2(\hbar\Omega)^3, \quad (4)$$

where $\hbar\Omega$ is the characteristic energy that measures the average energy gained from the uniform field by the optically excited carriers,[4] and is given by

$$(\hbar\Omega)^3 = e^2\mathcal{E}^2\hbar^2/8\mu_\parallel, \quad (5)$$

where $\mu_\parallel$ is the interband reduced mass in the field direction. With these simplifications, Eq. (2) becomes

$$\epsilon_{cv}(E_+, \vec{\mathcal{E}}) \cong \frac{e^2 C_0^2 \hbar^2}{\pi^2 m^2 E_+^2} \int_{loc} d^3k\, L_{cv}(E_+, \vec{\mathcal{E}}, \vec{k}), \quad (6)$$

where

$$L_{cv}(E_+, \vec{\mathcal{E}}, \vec{k}) = i\int_0^\infty ds \exp\{is[E_+ - E_{cv}(\vec{k})] - i\tfrac{1}{3}s^3(\hbar\Omega)^3\} \quad (7)$$

is the function giving the line shape of the contribution to ϵ of a single valence-conduction transition at wave vector $\vec{k}$. From Eq. (7),

$$L_{cv}(E_+, 0, \vec{k}) = \frac{-i}{E - E_{cv}(\vec{k}) + i\Gamma} \quad (8)$$

is the usual Lorentzian line shape describing an optical absorption process.

B. Complete asymptotic representation of the line-shape function

The line-shape function can be expressed in closed form as[6]

$$L_{cv}(E_+, \vec{\mathcal{E}}, \vec{k}) = (\pi/\hbar\Omega)[\mathrm{Gi}(-u_+) + i\,\mathrm{sgn}(\Omega)\,\mathrm{Ai}(-u_+)], \quad (9a)$$

where

$$u_+ = [E_+ - E_{cv}(\vec{k})]/\hbar\Omega \quad (9b)$$

and $\mathrm{Ai}(z)$ and $\mathrm{Gi}(z)$ are Airy functions,[24] but this form is not practical since Airy functions of complex argument must be computed numerically. At low fields, greater insight has been obtained by asymptotically expanding Eq. (7) in this limit by substituting the approximation

$$\exp[-i\tfrac{1}{3}s^3(\hbar\Omega)^3] \cong 1 - i\tfrac{1}{3}s^3(\hbar\Omega)^3. \quad (10)$$

This procedure yields separately the zero-field (Lorentzian) and low-field (third-derivative) components of the line shape,[2] but it is not adequate at high fields since it does not contain a description of the Franz-Keldysh oscillations.

We now show that an additional term describing the Franz-Keldysh oscillation component of the line shape can be obtained by suitably deforming the contour of integration in Eq. (7) in the complex s plane. By the method of stationary phase,[25] the dominant contributions to Eq. (7), other than possible contributions from end points of the path of integration, arise from points in the complex plane where the exponential term is invariant with respect to the integration variable, i.e., for those points s_j about which a power-series expansion of the exponent has no term linear in $s-s_j$. Once these points have been located, the integral can be evaluated asymptotically by deforming the integration path so as to pass through the accessible points s_i of s_j in a contour that provides maximum convergence, i.e., along paths for which the nonvanishing quadratic term, $c(s-s_i)^2$ of the expansion about s_i, is real and negative.

The integrand in Eq. (7) has two stationary-phase points,

$$s_1=+\frac{1}{\hbar\Omega}\left(\frac{E_+-E_{cv}(\vec{k})}{\hbar\Omega}\right)^{1/2}, \tag{11a}$$

$$s_2=-s_1, \tag{11b}$$

where the branch is chosen so that $\lim_{x\to\infty}(x+iy)^{1/2}=+x^{1/2}$. Suppose first that $(E-E_g)/\hbar\Omega\gg 0$, which is the condition for Franz-Keldysh oscillations to occur, and $\hbar\Omega>0$, which is the more important case experimentally. The location of the stationary-phase points s_1 and s_2 in the complex s plane for this situation is indicated schematically in Fig. 1. It is straightforward to calculate the contours for which the integrand converges most rapidly near $s=0$, s_1, and s_2. If we assume $\Gamma=0$ for simplicity, then these are given by paths whose azimuth angles are $\pi/2$, $\pi/4\pm\pi/2$, and $-\pi/4\pm\pi/2$, respectively. The only accessible stationary-phase point is s_1 because the contour must terminate at $s=+\infty$. The resulting contour is indicated schematically in Fig. 1, and we find

$$L_{cv}(E_+,\vec{\mathcal{E}},\vec{k})\sim-\int_0^\infty ds\exp\{-s[E_+-E_{cv}(\vec{k})]-\tfrac{1}{3}s^3(\hbar\Omega)^3\}+\frac{1}{\hbar\Omega}\exp\left(i\tfrac{2}{3}u_+^{3/2}+i\frac{\pi}{4}\right)\times\int_{-\infty}^{\infty}dy\exp(-y^2u_+^{1/2}-y^3e^{-i\pi/4}), \tag{12}$$

where $y=s\hbar\Omega$ and where u_+ is defined in Eq. (9b). The first and second integrals represent the contributions from paths (2) and (3), respectively, of Fig. 1. No contribution is obtained from path (4) since the integrand vanishes in the limit of large $|s|$ in this sector. The zero-field and third-derivative terms can be obtained from the first integral in the usual manner by means of Eq. (10). The term describing the Franz-Keldysh oscillations can be obtained in a closed-form approximation by substituting

$$\exp(-y^3e^{-i\pi/4})\cong 1 \tag{13}$$

in the second integral.

In the case $\hbar\Omega<0$, the same considerations apply, except the convergence and nonconvergence sectors in Fig. 1 reverse roles and the new contour is obtained by reflection about the real axis. For either sign of $\hbar\Omega$ but for $[E-E_{cv}(\vec{k})]/\hbar\Omega\gg 0$, we find

$$L_{cv}(E_+,\vec{\mathcal{E}},\vec{k})\sim L_{cv}(E_+,0,\vec{k})+\Delta L, \tag{14}$$

where $L_{cv}(E_+,0,\vec{k})$ is given by Eq. (8) and

$$\Delta L=\Delta L_3+\Delta L_{osc}=\frac{2(\hbar\Omega)^3}{[E-E_{cv}(\vec{k})+i\Gamma]^4}+\frac{\sqrt{\pi}}{\hbar\Omega}\left(\frac{\hbar\Omega}{E-E_{cv}(\vec{k})+i\Gamma}\right)^{1/4}\times\exp\left[\pm i\frac{\pi}{4}\pm i\frac{2}{3}\left(\frac{E-E_{cv}(\vec{k})+i\Gamma}{\hbar\Omega}\right)^{3/2}\right], \tag{15}$$

where the upper (lower) sign in the exponent refers to $\hbar\Omega>0$ ($\hbar\Omega<0$). In Eq. (15), the field-induced change in the line shape is expressed directly as a sum of third-derivative and oscillatory terms.

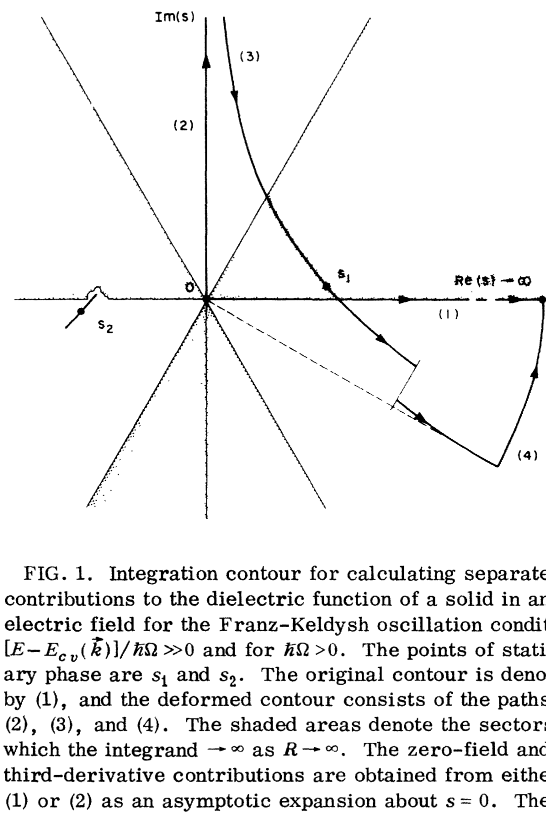

FIG. 1. Integration contour for calculating separate contributions to the dielectric function of a solid in an electric field for the Franz-Keldysh oscillation condition $[E-E_{cv}(\vec{k})]/\hbar\Omega\gg 0$ and for $\hbar\Omega>0$. The points of stationary phase are s_1 and s_2. The original contour is denoted by (1), and the deformed contour consists of the paths (2), (3), and (4). The shaded areas denote the sectors in which the integrand $\to\infty$ as $R\to\infty$. The zero-field and third-derivative contributions are obtained from either (1) or (2) as an asymptotic expansion about $s=0$. The oscillatory contribution is obtained from (3) as an asymptotic expansion about $s=s_1$.

If $[E - E_{cv}(\vec{k})]/\hbar\Omega \ll 0$, the high-field limit is characterized by exponential behavior instead of Franz-Keldysh oscillations. The stationary-phase points s_1 and s_2 now are located as shown in Fig. 2, where the contour is given for $\hbar\Omega > 0$. Following the same procedure as before, we find, for $[E - E_{cv}(k)]/\hbar\Omega \ll 0$,

$$\Delta L \sim \frac{2(\hbar\Omega)^3}{[E - E_{cv}(\vec{k}) + i\Gamma]^4} + \frac{i\sqrt{\pi}}{\hbar\Omega}\left(\frac{\hbar\Omega}{E_{cv}(\vec{k}) - E - i\Gamma}\right)^{1/4} \times \exp\left[-\frac{2}{3}\left(\frac{E_{cv}(\vec{k}) - E - i\Gamma}{\hbar\Omega}\right)^{3/2}\right]. \qquad (16)$$

We mention that these results can be obtained from the asymptotic expansions of $\mathrm{Ai}(z)$ and $\mathrm{Gi}(z)$ for large z,[24] if the real component of the oscillatory term in Eq. (15) is calculated by means of a Kramers-Kronig transformation. However, the present treatment illustrates graphically the regions of contribution to the field-dependent complex dielectric function, showing that the asymptotic expansion should be expected to be a very good approximation if the two contributing regions centered at 0 and s_1 (or s_2) are reasonably well separated. To put this on a quantitative basis, we note that the integrands in Eq. (12) drop to about 1% of their values at $|y_1 u_+| \cong 4$ and $|y_2^2 u_+^{1/2}| \cong 4$, where y_1 and y_2 are the effective radii in units of $|\hbar\Omega|$ of the respective contributing regions in Fig. 1. If we require $y_1 + y_2 < |u_+|$, so the regions do not overlap, then it is straightforward to show that

$$|E - E_{cv}(\vec{k}) + i\Gamma| \gtrsim 3.5\,|\hbar\Omega|, \qquad (17)$$

which we take to be the working definition of the range of validity of the asymptotic expansions (15) and (16). This is essentially equivalent to the inequality $\Gamma \gtrsim 3|\hbar\Omega|$ which was obtained previously[1,2] for the range of validity of the low-field limit. However, with *both* terms present in Eqs. (15) and (16), Eq. (17) gives the validity condition for the field-induced change in the line-shape function for arbitrary field strengths.

Several points concerning the above derivation are worth mentioning. It is evident that the appearance of subsidiary oscillations for $\Gamma < 3|\hbar\Omega|$ is due to the oscillatory term in Eq. (15), and not due to higher-order terms in the nonlinear susceptibility, as was stated in Ref. 2. Higher-order nonlinear susceptibility terms are analytic and come from perturbation theory, i.e., from higher-order terms in the expansion of the integral centered at $s = 0$. These can be seen by inspection to be relatively unimportant for large energy. The oscillatory term cannot be obtained from any finite perturbation expansion. The results indicate further that it is essential to calculate both the real and the imaginary parts of ϵ in order to obtain the zero-field and third-derivative contributions to the line shape. Calculation of ϵ_2 alone by the standard Franz-Keldysh approach yields the oscillatory term[6,13] but cannot yield the third-derivative contribution, which arises entirely because the contour is terminated at $s = 0$. Next, the oscillatory and exponentially decaying terms in Eqs. (15) and (16), respectively, are different functions; they do not evolve into each other as E passes through the value E_g. This is an example of the Stokes phenomenon which occurs in asymptotic expansions.[26] Thus, for computational purposes, the proper forms must be chosen prior to a line-shape calculation. Finally, some modifications of the form of the oscillatory and exponential contributions to Eqs. (15) and (16) should be expected to occur if Coulomb interaction effects are taken into account. The derivation of a closed-form asymptotic solution for the line shape with both Coulomb and electric field is an unsolved problem, but numerical calculations have shown that the $(E_g - E)^{-3/2}$ dependence of the logarithm of the field-induced absorption in the energy range below the fundamental absorption edge is modified to a linear $(E_g - E)$ form[12] if the Coulomb interaction is included, although the period of the Franz-Keldysh

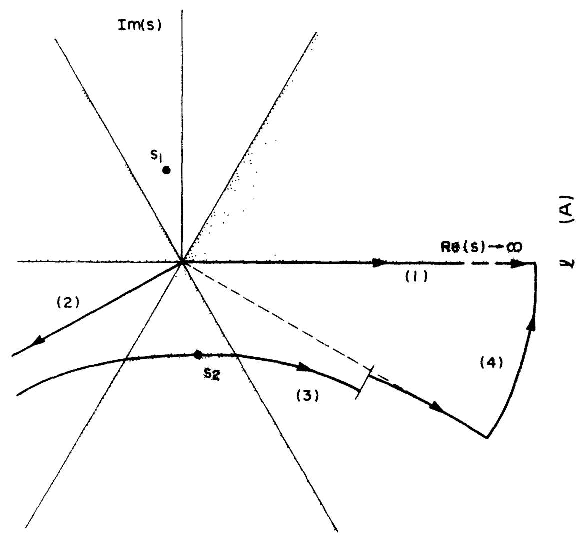

FIG. 2. Integration contour for calculating separate contributions to the dielectric function of a solid in an electric field for the exponential-edge condition $[E - E_{cv}(k)]/\hbar\Omega \ll 0$ and for $\hbar\Omega > 0$. The original contour is denoted by (1), and the deformed contour consists of the paths (2), (3), and (4). The zero-field and third-derivative contributions are obtained from either (1) or (2) as an asymptotic expansion about $s = 0$. The exponential contribution is obtained from (3) as an asymptotic expansion about $s = s_2$.

oscillations in the range of validity of Eq. (16) appears to be relatively unaffected.[22,23] Since Coulomb effects are probably less significant for higher-energy states, we conclude that line-shape analysis of Franz-Keldysh oscillations will be qualitatively similar for both cases. This is discussed more fully in Sec. IV.

To summarize the results of this section, we have separated the line-shape contribution of a single transition to the complex dielectric function, given in closed form by Eq. (9a), into zero-field, third-derivative, and oscillatory (exponential) parts by means of an asymptotic expansion valid for $|[E-E_{cv}(\vec{k})+i\Gamma]/\hbar\Omega|\gtrsim 3.5$. For the oscillatory case, where $[E-E_{cv}(\vec{k})]/\hbar\Omega>0$, the field-induced change in the line shape is given by Eq. (15). For the exponential case, where $[E-E_{cv}(\vec{k})]/\hbar\Omega<0$, the field-induced change in the line shape is given by Eq. (16). The complex dielectric function can be calculated from these results by means of Eqs. (1), (6), (8), and (14). The field-induced change, $\Delta\epsilon$, in the complex dielectric function is given by Eqs. (15) or (16) and

$$\Delta\epsilon=\frac{e^2C_0^2\hbar^2}{\pi^2m^2E_+^2}\int_{\text{loc}}d^3k\,\Delta L\,. \tag{18}$$

Finally, the field-induced relative reflectance change $\Delta R/R$, measured in an electroreflectance experiment, is given by[4,27]

$$\frac{\Delta R}{R}=\text{Re}\left(\frac{2n_a\Delta\epsilon}{n(\epsilon-\epsilon_a)}\right), \tag{19}$$

where $n_a^2=\epsilon_a$ and $n^2=\epsilon$ are the complex refractive index and dielectric function of the ambient and substrate, respectively.

C. Wave-vector integration

The results of Sec. II B must be integrated over $\vec{k}$ to obtain line shapes for direct comparison with experiment. Since our principal interest here lies in the effect of various perturbations on the oscillatory part of the line shape, we shall consider only this term in Eq. (15) and for simplicity, assume isotropic M_0 critical points described by a positive scalar reduced mass μ. By Eq. (18),

$$\Delta\epsilon=\frac{e^2C_0^2\hbar^2}{\pi^2m^2E_+}\int d^3k\,\frac{\sqrt{\pi}}{\hbar\Omega}\left(v_+-\frac{\hbar^2k^2}{2\mu}\right)^{-1/4}\times\exp\left[i\frac{\pi}{4}+i\tfrac{2}{3}\left(v_+-\frac{\hbar^2k^2}{2\mu}\right)^{3/2}\right], \tag{20}$$

where $\vec{k}$ is the radial wave vector in a local coordinate system with origin at the critical point, and

$$v_+=(E-E_g+i\Gamma)/\hbar\Omega \tag{21}$$

is the asymptotic expansion parameter. Applying the method of stationary phase to Eq. (20), the dominant contribution is seen to come from the region about $\vec{k}=0$; hence to lowest order in k^2

$$\Delta\epsilon_{\text{osc}}=\left(\frac{e^2C_0^2\hbar^2}{\pi^2m^2E_+^2}\right)\frac{\sqrt{\pi}}{\hbar\Omega}v_+^{-1/4}\exp\left(i\frac{\pi}{4}+i\tfrac{2}{3}v_+^{3/2}\right)\times\int d^3k\exp\left(-\frac{i\hbar k^2}{2\mu\Omega v_+}\right). \tag{22}$$

By defining appropriate cutoff values K_y and K_z for one- and two-dimensional cases, Eq. (22) can be integrated in closed form for all three dimensions. The results are summarized in Table I, together with the third-derivative contribution calculated previously.[1,2]

Each oscillatory term in Table I includes the approximation introduced in the integration over $\vec{k}$, as well as the original approximation introduced by using the asymptotic representation for the line shape in Eq. (20). Thus, a comparison of the complete asymptotic representation of $\Delta\epsilon$ for a given dimension, obtained by summing the appropriate two terms from Table I, and that calculated exactly from Eqs. (6) and (7), will provide a stringent test of the accuracy of the asymptotic approach. We shall make this comparison in terms of the previously defined electro-optic functions[6] F, G, which give the exact one-electron line shape for $\Delta\epsilon$ for an isotropic three-dimensional M_0 critical point. These functions are defined by[6]

$$\Delta\epsilon_{cv}(E_+,\vec{\mathcal{E}})=(2e^2C_0^2\hbar^2/m^2E^2)D^3(\hbar\Theta)^{1/2}\times[G(z)+iF(z)], \tag{23}$$

where $D=(2\mu/\hbar^2)^{1/2}$ is the density-of-states prefactor, $C_0^2=|\hat{e}\cdot\vec{p}_{cv}|^2$ is the square of the momentum matrix element,

$$\hbar\Theta=2^{2/3}\hbar\Omega\,, \tag{24}$$

and

$$z=(E-E_g+i\Gamma)/\hbar\Theta \tag{25}$$

($\hbar\Theta$ is the natural energy unit for the three-dimensional case). The line-shape function $G+iF$ has the closed-form representation[6,28]

$$G(z)+iF(z)=2\pi i[e^{i\pi/3}\text{Ai}'(z)\text{Ai}'(w)+w\,\text{Ai}(z)\text{Ai}(w)]-iz^{1/2}, \tag{26}$$

where

$$w=ze^{i2\pi/3}. \tag{27}$$

By scaling the results of Table I to the energy unit $\hbar\Theta$, we find that $G(z)+iF(z)$ should have the asymptotic form

$$G(z)+iF(z)\sim\frac{i}{32}z^{-5/2}-\frac{i}{4z}\exp(i\tfrac{4}{3}z^{3/2}). \tag{28}$$

This can be shown to be identical to that calculated

TABLE I. Complete asymptotic representation of the field-induced change in the dielectric function in the one-electron approximation for isotropic M_0 critical points of one, two, and three dimensions, for $E > E_g$. Here, $D = (2\mu/\hbar^2)^{1/2}$ is the density-of-states factor, K_y and K_z are cutoff lengths in remaining Brillouin-zone directions, $C_0^2 = |\hat{e} \cdot \vec{p}_{cv}|^2$ is the square of the momentum matrix element, and $v_+ = (E - E_g + i\Gamma)/\hbar\Omega$ is the asymptotic-expansion parameter.

$$\Delta\epsilon = \frac{e^2 C_0^2 \hbar^2}{\pi^2 m^2 E_+^2} \int d^3k\, \Delta L, \text{ where } \Delta L = \Delta L_3 + \Delta L_{\text{osc}}, \text{ and}$$

Dimension	$\int d^3k\, \Delta L_3 =$	$\int d^3k\, \Delta L_{\text{osc}} =$
1	$-i\frac{5}{8}\pi D K_y K_z (\hbar\Omega)^{-1/2} v_+^{-7/2}$	$\pi D K_y K_z (\hbar\Omega)^{-1/2} v_+^{-1/2} \exp(i\frac{2}{3}v_+^{3/2})$
2	$-\frac{2}{3}\pi D^2 K_z v_+^{-3}$	$\pi^{3/2} D^2 K_z v_+^{-3/4} \exp(-i\pi/4) \exp(i\frac{2}{3}v_+^{3/2})$
3	$i\frac{1}{4}\pi^2 D^3 (\hbar\Omega)^{1/2} v_+^{-5/2}$	$-i\pi^2 D^3 (\hbar\Omega)^{1/2} v_+^{-1} \exp(i\frac{2}{3}v_+^{3/2})$

from the standard asymptotic expansions for the functions Ai(z) and Bi(z).[24]

The line shapes obtained from Eqs. (26) and (28) are compared in Fig. 3 for two values of the broadening parameter, $\Gamma = 0$ and $\Gamma = \hbar\Theta/2$. It is evident that Eq. (28) provides a good representation of the exact line shape for $(E - E_g) \gtrsim 2\hbar\Theta \cong 3\hbar\Omega$, in accordance with Eq. (17). This corresponds roughly to the second subsidiary or Franz-Keldysh half-oscillation. We conclude that the results given in Table I can be used to describe mathematically the Franz-Keldysh oscillations for all presently practical purposes.

Inspection of the oscillatory term in Table I shows that the dominant line-shape determining factor is the exponential $\exp(i\frac{2}{3}v_+^{3/2})$. For the three-dimensional case, the phase and amplitude of this quantity may be obtained to within a phase factor θ from experimental electroreflectance data, $\Delta R/R$, by suppressing the baseline offset arising from the third-derivative term and forming the product $E^2(E - E_g)(\Delta R/R)$. By Table I and Eq. (19), we find

$$E^2(E - E_g)(\Delta R/R)$$

$$\sim \text{Re}\left[\left(1 + i\,\frac{\Gamma}{E - E_g}\right)\exp(i\theta + i\tfrac{2}{3}v_+^{3/2})\right]$$

$$\cong \text{Re}[\exp(i\theta + i\tfrac{2}{3}v_+^{3/2})] \qquad (29)$$

if $\Gamma \ll E - E_g$. The critical-point information is contained in the term v_+, given by Eq. (21).

We note that the dominant asymptotic behavior of the Franz-Keldysh oscillations, given by Eq. (29), has been reduced in Secs. II A and II B to a form which can be obtained *entirely by algebraic steps*. First, the point of stationary phase, s_1, is calculated from Eq. (2) by finding that point in the accessible contour about which a power-series expansion of the exponent has no term linear in $s - s_1$. Second, the dominant line-shape behavior is given by evaluating the exponent at $s = s_1$ and setting $\vec{k} = 0$. This represents a powerful method by which the effects of various perturbations on the Franz-Keldysh oscillations can be studied systematically.

III. APPLICATION

In this section, we apply the theory developed in Sec. II to investigate the dependence of the amplitude and phase of the Franz-Keldysh oscillations on

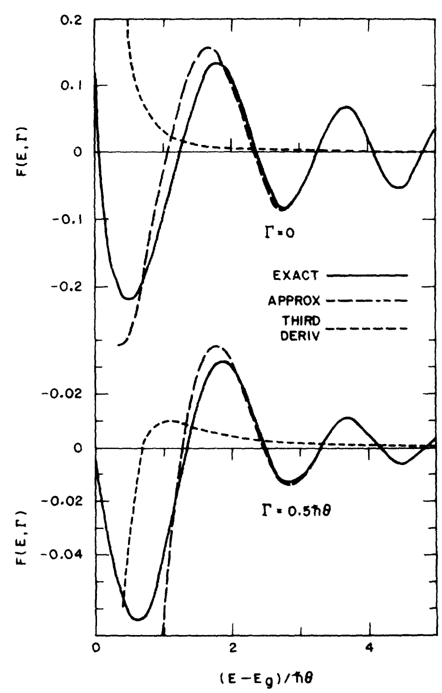

FIG. 3. Comparison of the exact (solid line) and asymptotic (broken line) line-shape functions, given by Eqs. (26) and (28), respectively, for a three-dimensional isotropic M_0 critical point. Energy units $\hbar\Theta = 2^{2/3}\hbar\Omega$ appropriate to three dimensions have been used. The third-derivative contribution (dashed line), given by the first term in Eq. (28), is shown explicitly.

various interactions. These will be treated as small perturbations on the basic theory of the isotropic simple parabolic three-dimensional critical point in order to examine general systematics. The results are summarized in Table II.

A. Broadening-parameter effects

From Eq. (29) we have in this case

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left(\frac{(E-E_g)^{1/2}\Gamma}{(\hbar\Omega)^{3/2}}\right) \times \cos\left[\theta+\frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}\right]. \tag{30}$$

If there exists ν observable tangent points between the oscillation spectrum and its envelope, at energies E_ν and with amplitudes $|\Delta R/R|_\nu$, then by Eq. (30)

$$\nu\pi=\theta+\tfrac{2}{3}[(E_\nu-E_g)/\hbar\Omega]^{3/2}. \tag{31}$$

Therefore, a plot of $(E_\nu-E_g)^{3/2}$ vs ν should yield a straight line with slope $(2/3\pi)(\hbar\Omega)^{-3/2}$, and a plot of $\ln[E_\nu^2(E_\nu-E_g)|\Delta R/R|_\nu]$ vs $(E_\nu-E_g)^{1/2}$ should yield a straight line with slope $-\Gamma(\hbar\Omega)^{-3/2}$ in this simple case.[20] If $\mathcal{E}$ is known, then the mass μ can be determined from $\hbar\Omega$.[7-9] If $\mathcal{E}$ is not known but data are available for a number of critical points, then accurate relative values of μ for different critical points may be obtained.[18]

The above analysis method is known to work well for effective-mass determination, but it has been shown in at least one case[20] to fail badly for broadening-parameter determination. The extreme sensitivity of the amplitude of oscillatory functions to destructive interference suggests here that other factors are significant.

B. Energy-band nonparabolicity effects

For nonparabolic energy bands, the expansion given in Eq. (4) does not terminate at the quadratic term. Assuming an isotropic but weakly nonparabolic interband energy variation, we write in local coordinates with origin at the critical point

$$E_{cv}(\vec{k})=E_g+(\hbar^2/2\mu)\vec{k}^2-C(\vec{k}^2)^2, \tag{32}$$

where C is a coefficient describing the strength of the nonparabolicity. If $C>0$, the energy bands flatten and the interband mass increases with increasing $\vec{k}^2$, which is the usual situation. The exponent in Eq. (2) therefore becomes

$$F(s)=is(E-E_g-\hbar^2k^2/2\mu+Ck^4+i\Gamma)-i\tfrac{1}{3}s^3 \times[(\hbar\Omega)^3-\tfrac{1}{2}Ck^2e^2\mathcal{E}^2-Ce^2(\vec{\mathcal{E}}\cdot\vec{k})^2] +i\tfrac{1}{5}s^5(Ce^4\mathcal{E}^4/16). \tag{33}$$

The presence of a fifth-order term in s substantially modifies the contours in Figs. 1 and 2 for large s, but if C is small, it is expected that the dominant contribution to the oscillatory term still arises from the stationary-phase point s_1 of Eq. (11a), which is shifted slightly from the parabolic-band value due to the presence of the nonparabolicity terms in Eq. (33). Therefore, the algebraic procedure described in Sec. II for determining the stationary-phase point may be used directly, and we find, to first order in C,

$$s_1(\vec{k}=0)=\left(\frac{E_+-E_g}{(\hbar\Omega)^3}\right)^{1/2}\left(1+\frac{Ce^4\mathcal{E}^4}{(\hbar\Omega)^6}(E_+-E_g)\right), \tag{34}$$

whence

TABLE II. Dependence of envelope amplitude and oscillation phase of Franz-Keldysh oscillations for various interactions and models in the one-electron approximation. The equations listed provide the analytic expressions of the oscillatory contribution to the line shape of $\Delta\epsilon$.

Interaction	Line-shape expression	Characteristic parameters	ln(envelope)	Phase
Simple parabolic	Table I	$\hbar\Omega$	~ 0 (power-law decrease)	$\sim(E-E_g)^{3/2}\mu^{1/2}/\mathcal{E}$
Lifetime broadening	Eq. (30)	$\hbar\Omega$, Γ	$\sim-(E-E_g)^{1/2}\Gamma\mu^{1/2}/\mathcal{E}$	$\sim(E-E_g)^{3/2}\mu^{1/2}/\mathcal{E}$
Nonparabolic band	Eq. (35)	$\hbar\Omega$, Γ, C	$\sim-C\mu^{5/2}\Gamma(E-E_g)^{3/2}/\mathcal{E}$	$\sim(E-E_g)^{3/2}\mu^{1/2}/\mathcal{E}$
Field averaging:				(correction only)
Gaussian internal	Eqs. (45), (48)	$\Delta\mathcal{E}$	$\sim-(E-E_g)^3\mu(\Delta\mathcal{E})^2/\mathcal{E}^4$	$\sim\Gamma(E-E_g)^2\mu(\Delta\mathcal{E})^2/\mathcal{E}^4$
Configuration avg.	Eq. (56)	φ_2	$\sim-(E-E_g)^3\mu\varphi_2/\mathcal{E}^4$	$\sim\Gamma(E-E_g)^2\mu\varphi_2/\mathcal{E}^4$
Nonuniform applied	Eq. (51)	β	$\sim-(E-E_g)^3\beta^2\mu/\mathcal{E}^2$	$\sim\Gamma(E-E_g)^2\mu\beta^2/\mathcal{E}^2$
Momentum correlation	Eqs. (52), (55)	σ^2	$\sim-(E-E_g)\sigma^2\mu/\mathcal{E}^2$	$\sim\Gamma\sigma^2\mu/\mathcal{E}^2$

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left\{ i\theta + i\frac{2}{3}\left(\frac{E_+ - E_g}{\hbar\Omega}\right)^{3/2}\left[1+\frac{6C\mu^2(E_+ - E_g)}{5\hbar^4}\right]\right\} \tag{35}$$

$$\cong \exp\left\{-\frac{(E-E_g)^{1/2}\Gamma}{(\hbar\Omega)^{3/2}} - \frac{4C\mu^2\Gamma}{5\hbar^4}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}\right\}\cos\left\{\theta+\frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}\left[1+\frac{6C\mu^2(E-E_g)}{5\hbar^4}\right]\right\}. \tag{36}$$

Thus, for $C>0$ the oscillation period slowly increases with E as compared to the simple parabolic case, which is expected because the interband reduced mass is increasing with energy. Also, a higher-order damping term appears which is proportional to $\mu^{5/2}$, $(E-E_g)^{3/2}$, Γ, and $\mathcal{E}^{-1}$.

It is instructive to compare Eq. (35) with the corresponding expression obtained by simply substituting the local effective mass at energy $E-E_g$ into $\hbar\Omega$, given by Eq. (5). We have by Eq. (32)

$$\frac{1}{\mu} = \frac{1}{\hbar^2}\frac{\partial^2}{\partial k^2}E_{cv}(k) \cong \frac{1}{\mu} - \frac{24\mu(E-E_g)C}{\hbar^2}, \tag{37}$$

to first order in C. Substituting Eq. (37) into Eq. (5) and expanding to first order in C, Eq. (29) becomes

$$(E-E_g)E^2(\Delta R/R) \sim \exp\left\{ i\theta + i\frac{2}{3}\left(\frac{E_+ - E_g}{\hbar\Omega}\right)^{3/2}\left[1+\frac{12C\mu^2(E_+ - E_g)}{\hbar^4}\right]\right\}. \tag{38}$$

The correction term in the above (incorrect) expression is 10 times larger than that of the correct expression given by Eq. (35). Thus, the strongly oscillatory behavior of the integrand in Eq. (20) acts to suppress nonparabolicity effects to a high degree. This remarkable result explains why the energy dependence of the period of the Franz-Keldysh oscillations has been observed[18] to follow so well the simple parabolic form, even at energies well above E_g where the variation of $E_{cv}(k)$ is nonparabolic.

C. Field-distribution effects

1. Internal-field averaging (Redfield model)

In the simplest model[10] of internal-field effects, the uniform-field dielectric function is averaged over a distribution of internal (locally uniform) fields according to a probability distribution function $\mathcal{P}(\vec{\mathcal{E}})$:

$$\langle \epsilon_{cv}(E,\Gamma,\vec{\mathcal{E}})\rangle = \int d\vec{\mathcal{E}}'\mathcal{P}(\vec{\mathcal{E}}'+\vec{\mathcal{E}})\epsilon_{cv}(E,\Gamma,\vec{\mathcal{E}}'), \tag{39}$$

where $\vec{\mathcal{E}}$ is the externally applied field and

$$\int d\vec{\mathcal{E}}'\mathcal{P}(\vec{\mathcal{E}}') = 1. \tag{40}$$

We suppose that the external field can be made much larger than the width, $\Delta\mathcal{E}$, of the probability distribution in order to determine to lowest order the effect of internal fields on the Franz-Keldysh oscillations. Then

$$\langle L\rangle = i\int d\vec{\mathcal{E}}'\mathcal{P}(\vec{\mathcal{E}}'+\vec{\mathcal{E}})\int_0^\infty ds\,\exp\{is[E-E_{cv}(\vec{k})+i\Gamma] - i\tfrac{1}{3}s^3(\hbar\Omega')^3\}$$

$$\cong i\int_0^\infty ds\,\exp[isE_+ - i\tfrac{1}{3}s^3(\hbar\Omega)^3][1-i\tfrac{1}{3}as^3(\hbar\Omega)^3 - bs^6(\hbar\Omega)^6], \tag{41}$$

where the quantities

$$a = 2\int d\vec{\mathcal{E}}'\mathcal{P}(\vec{\mathcal{E}}')\frac{\vec{\mathcal{E}}'\cdot\vec{\mathcal{E}}}{\mathcal{E}^2} + \int d\mathcal{E}'\mathcal{P}(\vec{\mathcal{E}}')\left(\frac{\mathcal{E}'^2}{\mathcal{E}^2}\right), \tag{42a}$$

$$b = \tfrac{2}{9}\int d\vec{\mathcal{E}}'\mathcal{P}(\mathcal{E}')\left(\frac{\vec{\mathcal{E}}\cdot\vec{\mathcal{E}}'}{\mathcal{E}^2}\right)^2 \tag{42b}$$

describe the effect of field averaging for any relatively narrow distribution $\mathcal{P}(\vec{\mathcal{E}})$. To obtain Eq. (41), we have retained only the lowest-order terms in $\Delta\mathcal{E}$, using the fact that $\int d\mathcal{E}'\mathcal{P}(\mathcal{E}')\mathcal{E}'^n \sim (\Delta\mathcal{E})^n$. Evaluating s_1 to the same order leads to

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left[i\frac{2}{3}\left(\frac{E_+ - E_g}{\hbar\Omega}\right)^{3/2}(1-\tfrac{1}{2}a) - b(E_+ - E_g)\right] \tag{43a}$$

$$\cong \exp\left[-\frac{(E-E_g)^{1/2}}{(\hbar\Omega)^{3/2}} - b\left(\frac{E-E_g}{\hbar\Omega}\right)^3\right] \times \cos\left[\theta + \frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}(1-\tfrac{1}{2}a) - \frac{3b\Gamma(E-E_g)^2}{(\hbar\Omega)^3}\right], \tag{43b}$$

where $\Gamma/(E-E_g)$ is also assumed to be small. Therefore, field averaging introduces a new damping term proportional to $(E-E_g)^3$, as well as modifying the energy dependence of the phase.

If the internal-field distribution is Gaussian with width $\Delta\mathcal{E}$, then[15]

$$\mathcal{P}(\vec{\mathcal{E}}) = (\Delta\mathcal{E})^{-3}\pi^{-3/2}\exp[-\vec{\mathcal{E}}^2/(\Delta\mathcal{E})^2], \quad (44)$$

and the integration over the field distribution can be given in closed form. The resulting expression is

$$\langle L\rangle = i\int_0^\infty ds\, g(s,\Delta\mathcal{E})^{-3/2} \times \exp\{is[E - E_{cv}(k) + i\Gamma] - i\tfrac{1}{3}s^3(\hbar\Omega)^3 g(s,\Delta\mathcal{E})^{-1}\}, \quad (45)$$

where

$$g(s,\Delta\mathcal{E}) = 1 + i\tfrac{1}{3}s^3(\hbar\Omega_{\Delta\mathcal{E}})^3, \quad (46a)$$

$$(\hbar\Omega_{\Delta\mathcal{E}})^3 = e^2(\Delta\mathcal{E})^2\hbar^2/8\mu. \quad (46b)$$

For $\Delta\mathcal{E} \ll \mathcal{E}$, the asymptotic oscillation behavior can be obtained either by taking the appropriate limits of Eq. (45) or by using the results

$$a = \tfrac{3}{2}(\Delta\mathcal{E}/\mathcal{E})^2, \quad (47a)$$

$$b = \tfrac{1}{9}(\Delta\mathcal{E}/\mathcal{E})^2 \quad (47b)$$

in Eqs. (43). We find in either case

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left[-\frac{(E-E_g)^{1/2}\Gamma}{(\hbar\Omega)^{3/2}} - \frac{1}{9}\left(\frac{E-E_g}{\hbar\Omega}\right)^3\left(\frac{\Delta\mathcal{E}}{\mathcal{E}}\right)^2\right]\cos\left\{\theta + \frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}\left[1 - \frac{3}{4}\left(\frac{\Delta\mathcal{E}}{\mathcal{E}}\right)^2\right] - \frac{1}{3}\left(\frac{\Gamma}{\hbar\Omega}\right)\left(\frac{E-E_g}{(\hbar\Omega)}\right)^2\left(\frac{\Delta\mathcal{E}}{\mathcal{E}}\right)^2\right\}. \quad (48)$$

Since $(\hbar\Omega)^3 \sim \mathcal{E}^2$, the additional envelope term converges exponentially as $-\text{const}(E-E_g)^3\mu(\Delta\mathcal{E})^2/\mathcal{E}^4$, which strongly suppresses the Franz-Keldysh oscillations at high energy $[\propto(E-E_g)^3]$ and/or low applied fields ($\propto\mathcal{E}^{-4}$). However, the effect is small when the external-field strength appreciably exceeds the width of the internal distribution, and the damping is less severe for critical points of small interband mass.

If the internal-field distribution is a result of a nonuniform applied field over the reflecting surface, which may be caused, for example, by resistive voltage drops in contacts, then the distribution width is expected to be proportional to the applied field. We consider explicitly a small linear variation described by the distribution function

$$\mathcal{P}(\vec{\mathcal{E}}') = \begin{cases} (\beta\mathcal{E})^{-1}, & |\mathcal{E}' - \mathcal{E}| \le \beta\mathcal{E}/2 \\ 0 & \text{elsewhere}, \end{cases} \quad (49)$$

where $\vec{\mathcal{E}}'$ is parallel to $\vec{\mathcal{E}}$, and $\beta \ll 1$ is the width factor. From Eqs. (42) and (43)

$$a = \beta^2/12, \quad (50a)$$

$$b = \beta^2/54, \quad (50b)$$

so

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left[-\frac{(E-E_g)^{1/2}\Gamma}{(\hbar\Omega)^{3/2}} - \frac{\beta^2}{54}\left(\frac{E-E_g}{\hbar\Omega}\right)^3\right]\cos\left[\theta + \frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2}\left(1 - \frac{\beta^2}{24}\right) - \frac{\beta^2\Gamma(E-E_g)^2}{18(\hbar\Omega)^3}\right]. \quad (51)$$

The additional envelope term remains exponentially dependent as $-(E-E_g)^3\mu$, but now depends only as $-\mathcal{E}^{-2}$ on the applied field. This is consistent with the case of a distribution of internal fields, since the same power dependence is obtained if it is assumed that $\Delta\mathcal{E} \sim \mathcal{E}$.

2. Momentum correlation (Lukes-Somaratna) model

The Feynman path-integral approach[29] has been used by Lukes and Somaratna[13] to obtain a line-shape function for a heavily doped semiconductor in a model where the internal-field distribution is assumed to appear as a momentum correlation term, $\sigma^2(0)$. Their result, expressed as Eq. (3.11) of Ref. 13, can be put in the form

$$L(E,\Gamma,\mathcal{E}) = i\int_0^\infty ds\,\exp\{is[E - E_{cv}(\vec{k}) + i\Gamma] - i\tfrac{1}{3}s^3(\hbar\Omega)^3\} \times \exp[-2s^2\sigma^2(0)], \quad (52)$$

where

$$\sigma^2(0) = \rho\int V^2(\vec{R})\,d\vec{R} \quad (53)$$

and $V(\vec{R})$ is the spherically symmetric potential of a single impurity of a randomly distributed set of density ρ. If $\sigma^2(0)$ is small, so that the momentum correlation term can be treated to first order, the stationary-phase point is given approximately by

$$s_1 = E_+^{1/2}/(\hbar\Omega)^{3/2} + i\sigma^2/(\hbar\Omega)^3 , \tag{54}$$

whence

$$E^2(E-E_g)(\Delta R/R) \sim \exp\left(-\frac{(E-E_g)^{1/2}\Gamma}{(\hbar\Omega)^{3/2}} - \frac{(E-E_g)\sigma^2(0)}{(\hbar\Omega)^3}\right) \times \cos\left[\theta + \frac{2}{3}\left(\frac{E-E_g}{\hbar\Omega}\right)^{3/2} - \frac{\Gamma\sigma^2(0)}{(\hbar\Omega)^3}\right] . \tag{55}$$

The additional envelope term depends linearly on energy $[\propto(E-E_g)]$ and depends on the applied field $(\propto\mathcal{E}^{-2})$ through the characteristic energy $(\hbar\Omega)^3$, exactly as in the case of field-proportional inhomogeneity.

3. Configurational averaging (Bonch-Bruevich-Esser) model

A third method of treating internal-field distributions differs in the method of performing the average[30] over internal fields,[11,15] although the locally uniform approach of the Redfield model[10] is used to formulate the problem. Upon performing the integration over $\vec{q}$ explicitly in Eq. (16) of Ref. 15 one obtains

$$\langle L\rangle = i\int_0^\infty ds\, g(s,\varphi_2)^{-3/2} \exp\{is[E - E_{cv}(\vec{k}) + i\Gamma] - i\tfrac{1}{3}s^3(\hbar\Omega)^3 g(s,\varphi_2)^{-1}\} , \tag{56}$$

where

$$g(s,\varphi_2) = 1 + i\tfrac{1}{3}s^3(\hbar\Omega_{\varphi_2})^2 , \tag{57a}$$

$$(\hbar\Omega_{\varphi_2})^3 = e^2\varphi_2\hbar^2/8\mu , \tag{57b}$$

$$\varphi_2 = \tfrac{2}{3}\rho\int d\vec{r}\,[\nabla V(\vec{r})]^2 , \tag{57c}$$

where $V(\vec{r})$ is the impurity potential associated with a single impurity of a set of density ρ. The form of Eq. (57) is identical to that of Eq. (45),[15] consequently in this model the Franz-Keldysh oscillations have the form given in Eq. (48).

IV. DISCUSSION

In this paper we have obtained a complete asymptotic expansion describing field-induced changes in the dielectric function, and have used the oscillatory term to calculate the functional dependence of the envelope amplitude and oscillation phase of Franz-Keldysh oscillations for a number of interactions and for several model calculations of the effects of internal-field distributions. The results of this investigation are summarized in Table II.

Previous investigations of the applicability of the various models[10-15] have concentrated entirely on the field-induced exponential absorption edge measured in electroabsorption. While similar to the oscillatory part of the waveform in mathematical formulation (as shown explicitly in Sec. II), the inability to compare the results to spectra obtained for a number of critical points of different μ and Γ within the same crystal represents a severe limitation. Moreover, the exponential absorption edge is influenced by exciton effects.[12] It is evident that the oscillatory parts of the waveform offer much richer possibilities in testing models and, ultimately, in using these models to determine actual internal-field distributions.

One of the results of the present paper shows that the period of the Franz-Keldysh oscillations is remarkably insensitive to various perturbation effects, including band nonparabolicities, broadening parameters, and internal-field distributions. This suggests that the interband masses obtained from Franz-Keldysh oscillations should be highly accurate and also explains why the oscillations are observed to follow accurately the simple parabolic form, even over large energy ranges[18] where the bands are nonparabolic.

Although Franz-Keldysh oscillation data are limited at present to a very few systems, inspection of systematics enables some qualitative conclusions to be obtained concerning the applicability of various models, particularly if internal fields are involved. We have noted already the inadequacy of the lifetime-broadening mechanism to describe the decay of the envelope of the oscillations for GaAs.[19] Similar statements also can be made for higher interband transitions in Ge,[18] and quite possibly also for the fundamental direct edge in Ge.[31] The most stringent test for the various field-distribution models, by Table II, is found in the dependence of the envelope amplitude on energy and field. Published data on Ge (Ref. 18) show the existence of well-defined oscillations with relatively slow decay rates at relatively low field strength (39 kV cm^{-1}) for the E_0 transition, and relatively fast decay rates at higher field strength (115 kV cm^{-1}) for the higher-energy $E_0'+\Delta_0'+\Delta_0$ transition. Since the principal difference between the two examples above concerns the broadening parameter (with a small contribution from the mass), and since the assumption of the same distribution of internal fields appears to be a reasonable one, this result argues for a relatively weak dependence of envelope decay on *both* $E-E_g$ and

$\mathcal{E}$. Thus from Table II the momentum correlation (Lukes-Somaratna) model[13] would appear to be favored over the remaining averaging models, if internal-field effects are indeed responsible for this result.

The theory outlined here is a one-electron approximation which does not take into account electron-hole scattering by means of the Coulomb interaction. This effect is known from numerical calculations to modify the asymptotic $(E_g - E)^{3/2}$ dependence of the logarithm of the absorption coefficient into a linear $(E_g - E)$ response,[12] and by analogy, it may be expected that the envelope and phase behavior of the oscillations should also be modified if such interactions are included. We have chosen here to restrict our calculations to the one-electron approximation for four reasons: First, at values of $E - E_g \gg E_{ex}$, where E_{ex} is the exciton binding energy, the Coulomb interaction should cause only low-angle scattering, which can be represented by a convolution formalism of the type used in Sec. II.[32] Second, at the values of $\Gamma > E_{ex}$ typically encountered in higher-interband spectra, Coulomb effects tend to be buried in the zero-field background. Third, no closed-form expression equivalent to Eq. (2) exists, although series *expansions* are available[33] and Coulomb interactions have been considered in random-potential situations.[11,15] Finally, final-state interaction effects do not seem to exert a significant effect on the phase of the oscillations, as can be seen by inspection of the numerical calculations of the energy dependence of the period of these oscillations, shown in Fig. 12 of Ref. 22. The next step is clearly to check the applicability of the one-electron approximation and validity of these assumptions in the region of the Franz-Keldysh oscillations. It appears that a simple asymptotic solution for high fields, including Coulomb interactions, may be obtainable by combining the results of Sec. II with WKB waveforms for combined Coulomb-uniform-field potentials.[23]

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[1]D. E. Aspnes and J. E. Rowe, Solid State Commun. 8, 1145 (1970).

[2]D. E. Aspnes and J. E. Rowe, Phys. Rev. B 5, 4022 (1972).

[3]J. E. Rowe and D. E. Aspnes, Phys. Rev. Lett. 25, 162 (1970).

[4]D. E. Aspnes, Surf. Sci. 37, 418 (1973).

[5]K. Tharmalingam, Phys. Rev. 130, 2204 (1963).

[6]D. E. Aspnes, Phys. Rev. 147, 554 (1966); 153, 972 (1967).

[7]Y. Hamakawa, F. A. Germano, and P. Handler, Phys. Rev. 167, 703 (1968); Y. Hamakawa, P. Handler, and F. A. Germano, Phys. Rev. 167, 709 (1968).

[8]P. Handler, S. Jasperson, and S. Koeppen, Phys. Rev. Lett. 23, 1387 (1969); in *Electronic Density of States*, edited by L. H. Bennett, Natl. Bur. Stand. (U.S.) Spec. Publ. 323 (U.S. GPO, Washington, D.C., 1971), p. 417.

[9]J. E. Fischer and N. Bottka, Phys. Rev. Lett. 24, 1292 (1970).

[10]D. Redfield, Phys. Rev. 130, 916 (1963); 140, A2056 (1965).

[11]V. L. Bonch-Bruevich, Opt. Commun. 1, 78 (1969); Phys. Status Solidi 42, 35 (1970).

[12]J. D. Dow and D. Redfield, Phys. Rev. B 1, 3358 (1970); 5, 594 (1972); J. D. Dow, D. L. Smith, and F. L. Lederman, Phys. Rev. B 8, 4612 (1973).

[13]T. Lukes and K. T. S. Somaratna, J. Phys. C 3, 2044 (1970).

[14]B. I. Shklovskii and A. L. Efros, Zh. Eksp. Teor. Fiz. 59, 1343 (1970) [Sov. Phys.—JETP 32, 733 (1971)].

[15]B. Esser, Phys. Status Solidi B 51, 735 (1972).

[16]F. Urbach, Phys. Rev. 92, 1324 (1953); F. Moser and F. Urbach, Phys. Rev. 102, 1519 (1956).

[17]S. F. Pond and P. Handler, Phys. Rev. B 6, 2248 (1972); 8, 2869 (1973); S. F. Pond, Surf. Sci. 37, 596 (1973).

[18]D. E. Aspnes, Phys. Rev. Lett. 31, 230 (1973).

[19]D. E. Aspnes and A. A. Studna, Surf. Sci. 37, 631 (1973).

[20]D. E. Aspnes and A. A. Studna, Phys. Rev. B 7, 4605 (1973).

[21]D. E. Aspnes, P. Handler, and D. F. Blossey, Phys. Rev. 166, 921 (1968).

[22]F. C. Weinstein, J. D. Dow, and B. Y. Lao, Phys. Rev. B 4, 3502 (1971).

[23]D. F. Blossey, Phys. Rev. B 2, 3976 (1970); 3, 1382 (1971).

[24]H. A. Antosiewicz, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U.S. Department of Commerce, Natl. Bur. Stand., Washington, D.C., 1964), Appl. Math. Ser. Vol. 55, pp. 446 ff.

[25]G. N. Watson, *Theory of Bessel Functions* (Cambridge U.P., Cambridge, England, 1962), pp. 229 ff.

[26]Reference 25, pp. 201 ff.

[27]D. E. Aspnes, J. Opt. Soc. Am. 63, 1380 (1973).

[28]In Ref. 6, the functions F and G were defined in terms of the linear combination $F + iG$. We have modified these expressions by the transformation $G + iF = i(F + iG)^*$ to obtain the more physical representation given by Eqs. (25)–(27).

[29]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[30]V. L. Bonch-Bruevich, *The Electronic Theory of Heavily Doped Semiconductors* (Elsevier, New York, 1966), Chap. 8.

[31]A. M. Joullie, S. Gaillard, and C. Alibert, Solid State Commun. 13, 665 (1973).

[32]H. D. Rees, Solid State Commun. 5, 365 (1967); J. Phys. Chem. Solids 29, 143 (1968).

[33]E. Yang, Phys. Rev. B 4, 2046 (1971).

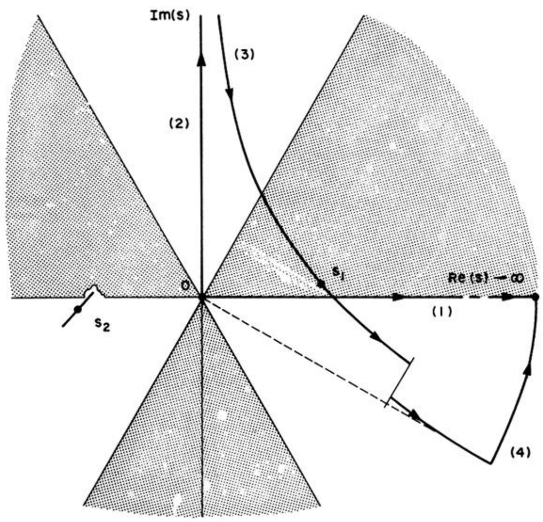

FIG. 1. Integration contour for calculating separate contributions to the dielectric function of a solid in an electric field for the Franz-Keldysh oscillation condition $[E-E_{cv}(\vec{k})]/\hbar\Omega \gg 0$ and for $\hbar\Omega > 0$. The points of stationary phase are s_1 and s_2. The original contour is denoted by (1), and the deformed contour consists of the paths (2), (3), and (4). The shaded areas denote the sectors in which the integrand $\rightarrow \infty$ as $R \rightarrow \infty$. The zero-field and third-derivative contributions are obtained from either (1) or (2) as an asymptotic expansion about $s = 0$. The oscillatory contribution is obtained from (3) as an asymptotic expansion about $s = s_1$.

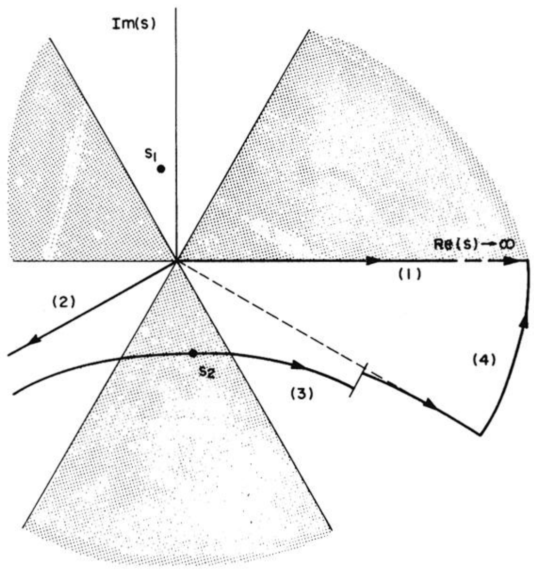

FIG. 2. Integration contour for calculating separate contributions to the dielectric function of a solid in an electric field for the exponential-edge condition $[E-E_{cv}(k)]/\hbar\Omega \ll 0$ and for $\hbar\Omega > 0$. The original contour is denoted by (1), and the deformed contour consists of the paths (2), (3), and (4). The zero-field and third-derivative contributions are obtained from either (1) or (2) as an asymptotic expansion about $s=0$. The exponential contribution is obtained from (3) as an asymptotic expansion about $s=s_2$.