# General dispersion law in a ferromagnetic cubic magnetoelastic conductor

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A general dispersion law has been derived for a cubic, ferromagnetic, elastic, and conductive medium in which the magnetoelastic coupling and the magnetic anisotropy energy parameters can be large. Also, the direction of the external field is taken to be arbitrary and it is not assumed to be collinear with the internal field and the magnetization. Maxwell's equations and the equation of motion for the magnetization and the elastic equations of motion have been combined in a consistent manner without the assumption of collinearity of the fields to yield a general dispersion law which is of seventh order in the square of the propagation constant $k$, and contains three acoustic and four magnetic branches. All of the seven values of $k^2$ belonging to a fixed frequency and bias magnetic field have been calculated numerically by computer. The calculations are applicable to the rare-earth–transition-metal alloy systems which have large magnetic anisotropy and magnetostriction and, thus, may be useful for high-frequency magnetostrictive transducers. Also, the surface impedance is calculated for some simple field configurations.

## I. INTRODUCTION

In this paper we are concerned with magnetoelastic interactions in a cubic ferromagnetic conductor. The phenomena of coupling spin waves to lattice vibrations in magnetic *isotropic* insulators have been studied in a number of papers.[1-4] We have generalized the dispersion law so that it is applicable to magnetic conductors with large magnetocrystalline anisotropy and magnetoacoustic coupling. In a magnetoelastic insulator it is found[3,4] that the dispersion relation which relates frequency $f$ and propagation constant $k$, contains five branches (Fig. 1). Three branches correspond to the two degenerate transverse modes (TE) and to the one longitudinal mode (LE) of the lattice motion. The other two are magnetic branches and correspond to a bulk spin-wave branch and a surface spin-wave branch.

For a pure magnetic conductor (with *no* coupling to the lattice) it is found[5-7] that there are four branches, one of which is nonmagnetic and requires different rf field excitation. Whereas the three magnetic branches require that the rf magnetic field be perpendicular to the static applied field, the nonmagnetic branch requires the rf field to be along the static field. The dispersion curves for the four branches depend on the direction of the applied bias field $\vec{H}_a$. If $\vec{H}_a$ is in the plane of the plate, there is one nonmagnetic branch, referred to as the pure electromagnetic or skin-depth mode, and three magnetic branches (Fig. 2). Although two of the branches are common to both an insulator and metal, one of the branches, which is referred to as the exchange-conductivity branch[5] occurs only in metals. Physically, the exchange-conductivity branch arises from the fact that in a metal the internal rf fields attenuate with depth, and thereby induce an extra exchange field torque on the magnetization. For the case that $\vec{H}_a$ is at an oblique angle,[7] but in a plane which is perpendicular[6] to the plane of the plate, the pure electromagnetic mode is admixed with the other three magnetic branches so that it is no longer a pure skin-depth mode.

Although it is expected that for the general case of a metallic magnetoelastic medium there ought to be seven branches (three acoustic and four magnetic branches), it is not clear which of the magnetic branches couples to the elastic branches for a given direction of $\vec{H}_a$. Since the coupling depends[4] on the direction of $\vec{H}_a$, we have obtained a dispersion relation for arbitrary directions of $\vec{H}_a$.

In order to calculate the general dispersion relation, the equilibrium position of the magnetization is required. The omission in previous calculations of the effect of magnetic anisotropy but the inclusion of anisotropic magnetoelastic interaction and elastic self-energy[8] is somewhat inconsistent. The implicit assumption previously made, that $\vec{H}_a$ and $\vec{H}_0$ (the effective internal static field) are collinear, is not valid in magnetically anisotropic media. The effect of this assumption is to predict an erroneous frequency for "crossing." Because we have included the effect of magnetic anisotropy of arbitrary strength with no assumption of collinearity between $\vec{H}_a$ and $\vec{H}_0$, our calculation should be applicable to systems where the magnetic anisotropy and magnetoelastic interaction energies are unusually[9,10] large in metals.

In a magnetic metal the pure electromagnetic branch and the exchange-conducting branch coalesce as $k \to 0$ (Fig. 3). It is possible that repulsions occur between the above two branches and the acoustic branches as $k\to 0$ for the general case, and, in fact, one might force such interactions to occur by the correct choice of the conductivity, the exchange constant, and the acoustic velocities. We have adopted a more realistic approach to investigate the region of $k\to 0$ for a recently studied material, $Tb_{0.15}Ho_{0.85}Fe_2$.[9]

The method of calculation is basically the same as that introduced in Ref. 11. In Ref. 11 we used the variation principle to determine the rf effective magnetic fields from the free energy. We take small or virtual displacements of the magnetization away from its equilibrium position. However, in this case an rf effective magnetoelastic field must also be calculated by taking small lattice displacements from the equilibrium position of the lattice displacement. Whereas in Ref. 11 only an anisotropic magnetic metal is considered, in the present paper the elastic self-energy, the magnetoelastic interaction, and interaction energy terms which reflect change in volume as the sample is strained, are also included as part of the free energy of the system.

## II. THEORETICAL FORMULATION

In this section a functional relationship between frequency and the propagation constant $k$ is derived for a given magnetic bias field $\vec{H}_a$. The propagation constant vector $\vec{k}$ is assumed to be normal to the metal plate. One surface of the plate is at $y=0$ and the other surface at $y=-\infty$. Magnetostatic and elastic wave propagations in the plane of the plate are *not* considered. The procedure for obtaining the dispersion law is as follows:

(i) Magnetic, elastic, and magnetoelastic energy terms are included in the free-energy expression. (ii) The equations of motion for the magnetization $\vec{M}$ and the elastic displacement $\vec{u}$ are obtained routinely from the free-energy expression. There are *three* unknown field variables ($\vec{M}$, $\vec{u}$, and $\vec{H}$) and two sets of equations. One set of equations relates $\vec{M}$ to $\vec{H}$ and $\vec{u}$, and the other set $\vec{u}$ to $\vec{M}$. Maxwell's equations form the third set of equations which relates $\vec{M}$ to $\vec{H}$. Thus, the required number of equations is obtained which gives rise to nontrivial solutions for $\vec{M}$, $\vec{H}$, and $\vec{u}$.

The total free energy is the sum of elastic ($F_E$), magnetic ($F_M$), magnetoelastic ($F_{ME}$), and induced magnetoelastic energy terms ($F_I$):

$$F=F_E+F_M+F_{ME}+F_I\ . \tag{1}$$

The elastic energy is given as

$$F_E=\frac{1}{2}\left(C_{11}\sum_{n=1}^{3}e_{nn}^2+\sum_{n\neq m}^{3,3}(C_{44}e_{nm}^2+2C_{12}e_{nn}e_{mm})\right). \tag{2}$$

$C_{11}$, $C_{44}$, and $C_{12}$ are the elastic constants appropriate for a material with cubic symmetry and

$$e_{nn}=\frac{\partial u_n}{\partial x_n},\quad e_{nm}=\frac{\partial u_n}{\partial x_m}+\frac{\partial u_m}{\partial x_n},$$

where $\vec{u}_n$ is the lattice displacement along the $n$ direction and $x_n$ is the coordinate axis. The magnetic free energy is given as

$$F_M=-\vec{H}_a\cdot\vec{M}+2\pi M_0^2\alpha_2^2+K_1\sum_{n\neq m}^{3,3}(\alpha_n\alpha_m)^2+K_2(\alpha_1\alpha_2\alpha_3)^2. \tag{3}$$

In Eq. (3), the $\alpha$'s are the directional cosines of $\vec{M}$ with respect to the cubic $\langle 100\rangle$ axes so that $\alpha_1=\sin\theta\cos\varphi$, $\alpha_2=\sin\theta\sin\varphi$, and $\alpha_3=\cos\theta$ (see Fig. 4). $K_1$ and $K_2$ are the first- and second-order anisotropy constants. The first term on the right-hand side represents the magnetizing energy, the second the demagnetizing energy, and the third and fourth the magnetocrystalline anisotropy energies.

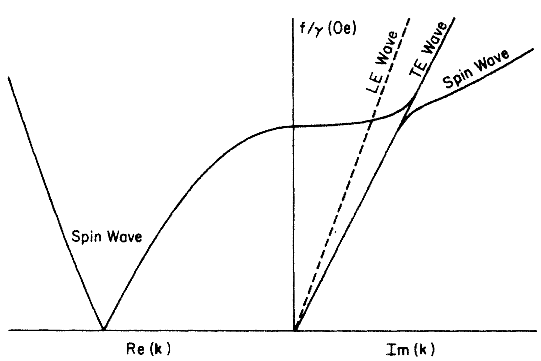


FIG. 1. Sketch of the dispersion curves for a ferromagnetic insulating magnetoelastic medium.

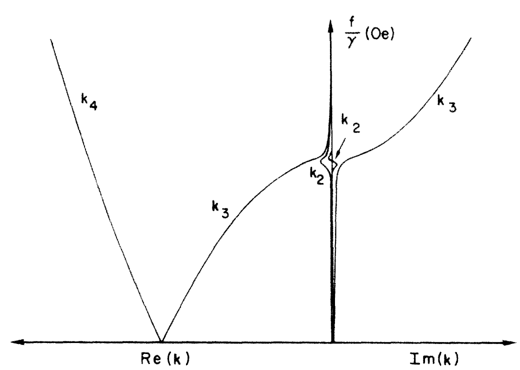


FIG. 2. Sketch of the four branches of the propagation constant as a function of $f/\gamma$ for a ferromagnetic conductor.

The form of the second term in Eq. (3), the demagnetizing energy, is dictated by our having taken the 2 (or $y$) axis to be a $\langle 100\rangle$ axis normal to the plane of the plate as shown in Fig. 4. Further, we assume that, in the static situation, the magnetization is saturated, and that the sample is infinite in the $x$ and $z$ directions, so that the demagnetizing factor is $4\pi$.

Considering[8] only the lowest-order terms dependent on the orientation of $\vec{M}$, we may take from symmetry considerations the following form for the magnetoelastic interaction:

$$F_{\rm ME} = B_1 \sum_{n=1}^{3} e_{nn}\alpha_n^2 + B_2 \sum_{n\neq m}^{3,3} e_{nm}\alpha_n\alpha_m . \tag{3a}$$

For propagation of elastic waves in the $y$ direction this becomes

$$F_{\rm ME} = B_1\alpha_2^2 e_{22} + B_2(\alpha_1\alpha_2 e_{12} + \alpha_2\alpha_3 e_{23}) . \tag{3b}$$

$B_1$ and $B_2$ are the magnetoelastic interaction parameters.

In a magnetoelastic medium the volume of the sample changes as it is magnetized. This implies that the density of magnetic moments must also change. To first order, the change in magnetization is approximately

$$\Delta\vec{M} \cong -\vec{M}\,\vec{\nabla}\cdot\vec{u} . \tag{4}$$

The physical implications of Eq. (4) are as follows: the total number of spins or of magnetic moments in a sample does not change, but the number of magnetic moments per $cm^3$ ($\vec{M}$) can vary as the sample is in tension or compression. When the sample is in tension ($\vec{\nabla}\cdot\vec{u} > 0$), the volume is increased. Therefore, the total magnetic moment per $cm^3$ decreases. The converse is true when the sample is compressed ($\vec{\nabla}\cdot\vec{u} < 0$). Let us now estimate how the free energy is modified (when a sample is under stress) using Eq. (4). For example, the change in demagnetizing energy ($\Delta F_D$) is given as

$$\Delta F_D \cong \int 4\pi\Delta\vec{M}\cdot d\vec{M}$$

which gives

$$\Delta F_D \cong -2\pi M_0^2\alpha_2^2\vec{\nabla}\cdot\vec{u} .$$

For waves propagating in the $y$ direction $\Delta F_D$ becomes

$$F_I = \Delta F_D \simeq -2\pi M_0^2\alpha_2^2 e_{22} . \tag{5}$$

The above term is of the same form as that of the first term in $F_{\rm ME}$ [Eq. (3b)]. Effectively $F_I$ gives rise to a magnetoelastic interaction. For convenience $B_1$ is redefined as $B_1' = B_1 - 2\pi M_0^2$. Henceforth, the prime on $B_1$ will be dropped. The same conclusion has been reached using a different calculating approach.[3,4] Usually, $F_I < F_{\rm ME}$. For example taking $2\pi M_0 \simeq 4400$ G, the induced magnetoelastic interaction strength is $\sim 3\times 10^6$ erg/cm$^3$ (compared to $B_2 \sim 10^8$ erg/cm$^3$). At microwave frequencies changes in magnetic anisotropy and Zeeman energies due to changes in magnetization are smaller in magnitude, and therefore these effects have been neglected in the free energy.

Before we derive the equations of motion for $\vec{M}$ and $\vec{u}$ we must determine (i) the amount of strain imposed at equilibrium on the sample; (ii) the direction in which the sample is magnetized by a given $\vec{H}_a$.

### A. Static equilibrium conditions

When the sample is in the presence of a strong bias magnetic field, the sample is magnetized in

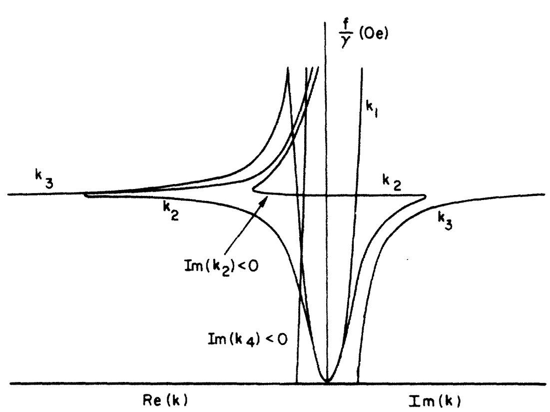


FIG. 3. Same as in Fig. 2, except the region near $k \simeq 0$ is expanded.

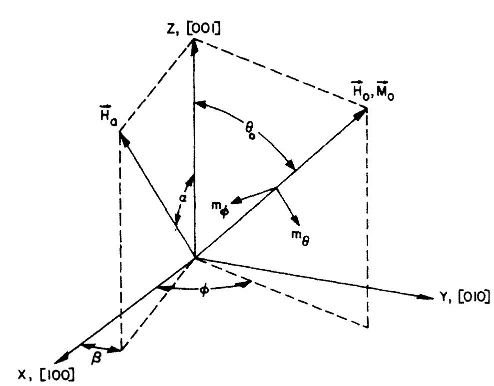


FIG. 4. Geometrical configuration of the various fields with respect to the cubic axes.

some direction and, consequently, the sample is strained. The equilibrium configuration of the crystal may be found by minimizing $F$ with respect to $e_{nn}$ and $e_{nm}$. The solutions at equilibrium are readily found to be[8]

$$e_{nn}^{(0)} = \frac{B_1[C_{12} - \alpha_n^2(C_{11}+C_{12})]}{(C_{11}-C_{12})(C_{11}+2C_{12})} , \tag{6}$$

$$e_{nm}^{(0)} = - B_2 \alpha_n \alpha_m / C_{44} . \tag{7}$$

Substituting $e_{nn}^{(0)}$ and $e_{nm}^{(0)}$ into $F$ gives rise to an energy term of the same symmetry as the first-order anisotropy energy. Thus, it is found that although the sample is strained when magnetized, the symmetry of the magnetic anisotropy is still cubic. This amounts to changing $K_1$ into $K_1 + \Delta K$ where[8] $\Delta K = B_1^2/(C_{11}-C_{12}) - B_2^2/2C_{44}$. Higher-order terms[12] in the magnetoelastic interaction terms are required to modify $K_2$ as in the case of $K_1$. In summary, one needs to redefine the free energy after the sample is magnetized. In this case only $K_1$ is redefined, and for our particular case $\Delta K < K_1$.

Since the equations of motion are written in terms of the internal field $\vec{H}_0$, the direction of $\vec{H}_0$ is required for all directions of $\vec{H}_a$. For a given direction of $\vec{H}_a$ the direction of the static magnetization $\vec{M}_0$, or the internal field $\vec{H}_0$, is uniquely determined from $\vec{H}_0 = -(\vec{\nabla}_\alpha F)/M_0$ where $M_0 \cong |\vec{M}|$. The gradient operator $\vec{\nabla}_\alpha$ is defined with respect to the direction cosines measured from the crystal axes. Thus, we have

$$\vec{H}_0 = \vec{H}_a - 4\pi M_0 \alpha_2 \vec{a}_y - 2K_1[\alpha_1(\alpha_2^2+\alpha_3^2)\vec{a}_x + \alpha_2(\alpha_1^2+\alpha_3^2)\vec{a}_y + \alpha_3(\alpha_1^2+\alpha_2^2)\vec{a}_z]/M_0 - 2K_2(\alpha_1\alpha_2^2\alpha_3^2\vec{a}_x + \alpha_2\alpha_1^2\alpha_3^2\vec{a}_y + \alpha_3\alpha_1^2\alpha_2^2\vec{a}_z)/M_0 . \tag{8}$$

$\vec{H}_a$ is the applied bias field; the second term is the demagnetizing field. The last two terms are the magnetic anisotropy fields. The magnetoelastic static fields due to straining of the sample are included through the redefinition of $K_1$. From Eq. (8) it is readily shown that the equilibrium condition is

$$H_a \sin(\alpha - \theta_0) = 2\pi M_0 \sin 2\theta_0 + (K_1/2M_0)\sin 4\theta_0 \tag{9}$$

for $\vec{H}_a$ in the (100) plane. For $\vec{H}_a$ in the plane of the plate the first term in the right-hand side is omitted. In Eq. (9), $\alpha$ and $\theta_0$ are the angles between $\vec{H}_a$ and $\vec{M}_0$ and the [001] axis, respectively. The equilibrium conditions can also be obtained by setting $\partial F/\partial\theta = \partial F/\partial\varphi = 0$ and by solving for both $\theta$ and $\varphi$.

For $\vec{H}_a$ in the (010) plane (plate plane), $\vec{H}_0$ is parallel to $\vec{H}_a$ *only* for $\alpha = 0$ ([001] axis) and $\alpha = \pi/4$ ([101] axis). For $\vec{H}_a$ in the (100) plane, $\vec{H}_0$ and $\vec{H}_a$ are parallel to each other only for $\alpha = 0$ ([001] axis) and $\alpha = \pi/2$ ([010] axis).

### B. Dynamic ferromagnetic equations of motion

In this section the equations of motion for the rf magnetization and the lattice are derived. The classical equation of motion for $\vec{M}$ is

$$\frac{1}{\gamma}\frac{d\vec{M}}{dt} = \vec{M} \times \left(\vec{H}_{\text{eff}} - \frac{\lambda}{\gamma|\vec{M}|^2}(\vec{M}\times\vec{H}_{\text{eff}})\right), \tag{10}$$

where $\vec{H}_{\text{eff}}$ is an effective field which includes contributions from all *possible* static and rf effective field sources. In Eq. (10) the Landau-Lifshitz magnetic damping mechanism characterized by an isotropic damping parameter $\lambda$ is assumed.

$$\vec{H}_{\text{eff}} = \vec{H}_0 + \vec{h} + \vec{h}_{\text{ex}} + \vec{h}_A + \vec{h}_{\text{ME}} ,$$

where $\vec{H}_0$ is the effective static internal field and it is assumed to be spatially uniform in the plate.

The Maxwellian rf magnetic field $\vec{h}$ is related[11] to $\vec{m}$ via Maxwell's equations by

$$Qh_x + m_x = 0 , \quad h_y + 4\pi m_y = 0 ,$$

$$Qh_z + m_z = 0 , \quad Q = (1 + \tfrac{1}{2} j\delta_0^2 k^2)/4\pi ,$$

and

$$\delta_0^2 = c^2/2\pi\sigma\omega ,$$

where $c$ is the velocity of light and $\sigma$ the conductivity. For an *isotropic* exchange interaction between magnetic moments in a cubic material, the rf exchange field is given as

$$\vec{h}_{\text{ex}} = (2A\nabla^2\vec{m})/M_0^2$$

where $A$ is the exchange stiffness constant. The gradient operator $\vec{\nabla}$ is defined with respect to the rectangular coordinates. The above form of $\vec{h}_{\text{ex}}$ is valid in the long-wavelength approximation.[8] The rf magnetic anisotropy field $\vec{h}_A$ can be derived from

$$\vec{h}_A = -(1/M_0)\delta(\vec{\nabla}_\alpha F_A)$$

where $F_A$ is the magnetic anisotropy energy contribution to $F_M$. The analytical expression for $\vec{h}_A$ is found in Ref. 11 and is readily obtained by taking virtual displacement of $\vec{M}$ about its equilibrium position $\vec{M}_0$. In the same manner, the rf magnetoelastic field $\vec{h}_{\text{ME}}$ is obtained from

$$\vec{h}_{\text{ME}} = -(1/M_0)\delta(\vec{\nabla}_\alpha F_{\text{ME}}) .$$

To first order in the lattice displacement, $\vec{h}_{\text{ME}}$ is found to be

$$\vec{h}_{\text{ME}} = -\{B_2\alpha_2 e_{12}\vec{a}_x + [2B_1\alpha_2 e_{22} + B_2(\alpha_1 e_{12} + \alpha_2 e_{23})]\vec{a}_y + B_2\alpha_2 e_{23}\vec{a}_z\}/M_0 .$$

The above expressions for the rf fields $\vec{h}_A$ and $\vec{h}_{ME}$ apply only to small rf field perturbations or small deviations of $\vec{M}$ and $\vec{u}$ away from $\vec{M}_0$ and $e_{nm}^{(0)}$, respectively. However, Eq. (10) is a nonlinear differential equation and it is linearized by collecting linear terms in $\vec{m}$ and $\vec{u}$. We write $\vec{M}$ and $\vec{H}$ as

$$\vec{M}=\vec{M}_0+\vec{m},\quad \vec{H}=\vec{H}_0+\vec{h}.$$

For small perturbations, $|\vec{m}| \ll |\vec{M}_0|$ and $|\vec{h}| \ll |\vec{H}_0|$ so that quadratic terms of the form $m_\alpha m_\beta$, $m_\alpha h_\beta$, $m_\alpha u_\beta$, etc. can be omitted. Thus, after linearizing Eq. (10) and assuming an exponential variation of the components $m_n$, $u_n$ of the form $e^{j\omega t-ky}$, we have

$$\frac{1}{\gamma}\frac{d\vec{m}}{dt}=(\underline{A}+I\Omega)\vec{m}-k\underline{A}_1\vec{u}. \tag{11}$$

Here, $\underline{A}$ is a $(3\times 3)$ square matrix which was derived in detail in Ref. 11, $I$ is the unit $(3\times 3)$ matrix, $\Omega=j\omega/\gamma$, and $\underline{A}_1$ is a square matrix which contains elements proportional to $B_1$ and $B_2$. Definitions of the matrix elements of $\underline{A}_1$ and $\underline{A}$ are given in the Appendix. It should be pointed out that Maxwell's equations have already been included through the $\vec{h}$ term in Eq. (10), and, therefore, $\underline{A}$ contains elements which are proportional to the conductivity.[11] The Hamiltonian for the lattice motion is given as

$$\mathcal{H}=\left(\sum_{n=1}^{3}\frac{\pi_n^2+C_{11}^2e_{22}}{2}+\frac{C_{44}(e_{12}^2+e_{23}^2)}{2}\right)\Big/2\rho$$

$$+B_1e_{22}\alpha_2^2+B_2(\alpha_1\alpha_2e_{12}+\alpha_2\alpha_3e_{23}), \tag{12}$$

where $\rho$ is the crystal density, and $\pi_n$ is the crystal momentum along the $n$ direction. The equation of motion for the lattice is obtained from the time rate of change of the lattice momentum

$$\rho\ddot{u}_n=\dot{\pi}_n=-\frac{\partial\mathcal{H}}{\partial u_n}+\frac{\partial}{\partial y}\left(\frac{\partial\mathcal{H}}{\partial e_{n2}}\right),\quad n=1,2,3. \tag{13}$$

Again by collecting terms to first order in $\vec{u}$ and $\vec{m}$, Eq. (13) leads to

$$\rho\ddot{\vec{u}}=k^2\underline{A}_2\vec{u}-k\underline{A}_3\vec{m}, \tag{14}$$

where

$$\underline{A}_2=\begin{bmatrix} C_{44} & 0 & 0\\ 0 & C_{11} & 0\\ 0 & 0 & C_{44}\end{bmatrix}$$

and

$$\underline{A}_3=\begin{bmatrix} B_2\alpha_2 & B_2\alpha_1 & 0\\ 0 & 2B_1\alpha_2 & 0\\ 0 & B_2\alpha_2 & B_2\alpha_2\end{bmatrix}\Big/M_0.$$

For $\vec{H}_a$ in the plate plane ($\alpha_2=0$), there is no coupling between the magnetic spin system and the longitudinal wave $u_y$. However, one of the transverse waves ($\vec{u}\parallel\vec{M}_0$) is coupled to $m_y$. Thus, we would expect that for this configuration, the spin wave and the longitudinal acoustic branches intersect each other without any repulsion, but repulsion will occur between the spin wave and one of the transverse acoustic branches. For $\vec{H}_a$ perpendicular to the plate ($\alpha_2=1$) there is coupling between the spin system and the transverse wave but not with the longitudinal wave, since $m_y=0$. Closer examination of $\underline{A}_3$ reveals that the longitudinal wave couples to the spin system for $H_a$ at oblique angles to the plate.

Equation (14) contains *no* provision for elastic damping. Elastic damping can be introduced phenomenologically by making $\underline{A}_2$ complex or, equivalently, making $C_{11}$ and $C_{44}$ complex. The full set of coupled elastic and magnetic equations of motion for the system are displayed by combining Eqs. (11) and (14):

$$\underline{DX}=\underline{BX},$$

where $\underline{D}$ is a $(6\times 6)$ diagonal matrix

$$\underline{D}=\begin{bmatrix} \Omega & 0 & 0 & 0 & 0 & 0\\ 0 & \Omega & 0 & 0 & 0 & 0\\ 0 & 0 & \Omega & 0 & 0 & 0\\ 0 & 0 & 0 & -\rho\omega^2 & 0 & 0\\ 0 & 0 & 0 & 0 & -\rho\omega^2 & 0\\ 0 & 0 & 0 & 0 & 0 & -\rho\omega^2\end{bmatrix}$$

$\underline{X}$ is a six-dimensional vector whose components are $m_x$, $m_y$, $m_z$, $u_x$, $u_y$, $u_z$, and $\underline{B}$ is a $(6\times 6)$ matrix which can be written in terms of $\underline{A}$, $\underline{A}_1$, $\underline{A}_2$, and $\underline{A}_3$, namely

$$\underline{B}=\left[\begin{array}{c|c} \underline{A}+I\Omega & -k\underline{A}_1\\ \hline -k\underline{A}_3 & k^2\underline{A}_2\end{array}\right].$$

By our choice of the Landau-Lifshitz form of the equation of motion, Eq. (10), the magnitude of $\vec{M}$ is necessarily conserved. In the linear small-field approximation this requires that $\vec{M}_0\cdot\vec{m}=0$. As a result $m_x$, $m_y$, and $m_z$ are linearly dependent. We now introduce a transformation to a five-component vector $X'$ whose components are linearly *independent*. Obviously $u_x$, $u_y$, and $u_z$ are independent of each other. The six-component vector is connected to the five-component vector by transformation which reflects $m_x$, $m_y$, $m_z$ into the two components of $\vec{m}$ ($m_\theta$ and $m_\phi$) which are orthogonal to $\vec{M}_0$ (see Fig. 4). Thus write

$$X'=SX \tag{15a}$$

or

$$X = S^{-1}X' . \tag{15b}$$

The matrix representation of this linear transformation is given as

$$S = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} T' & 0 \\ 0 & I \end{bmatrix}, \tag{16}$$

where

$$T = \begin{bmatrix} 0 & 0 & -1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} \Big/ (\alpha_1^2 + \alpha_2^2)^{1/2}$$

and

$$T' = \begin{bmatrix} \alpha_1\alpha_3 & -\alpha_2 \\ \alpha_2\alpha_3 & \alpha_1 \\ -(\alpha_1^2+\alpha_2^2) & 0 \end{bmatrix} \Big/ (\alpha_1^2 + \alpha_2^2)^{1/2} .$$

These transformations now allow us to write the equations of motion for the five components of $X'$ in the following manner. Defining $D'$ as $D' = SDS^{-1}$, where $D'$ is a $(5\times 5)$ diagonal matrix, and using Eq. (15a) write

$$D'X' = D'(SX) .$$

Since $D'S = SD$,

$$D'X' = SDX = SBX . \tag{17}$$

Substituting Eq. (15b) into (17) and defining $B' = SBS^{-1}$, we get

$$(D' - B')X' = 0 .$$

The matrix $B'$ is a $(5\times 5)$ matrix whose elements are defined in the Appendix. There exists nontrivial solutions of $X'$ if

$$\det(D' - B') = 0 .$$

The resultant secular equation which results from expanding the determinant relates frequency and propagation constant $k$:

$$\begin{aligned}&(\omega^2 + C_{11}k^2/\rho)(\omega^2 + C_{44}k^2/\rho)^2 V(f,k) - \alpha_2^2(\omega^2 + C_{44}k^2/\rho)^2(2B_1/M_0\rho)^2 M_0\rho k^2\{(1-\alpha_2^2)[b + \eta(\Omega + a)] + R_1\} \\ &\quad - (\omega^2 + C_{11}k^2/\rho)(\omega^2 + C_{44}k^2/\rho)(B_2/M_0\rho)^2 M_0\rho k^2[(1 - 3\alpha_2^2 + 4\alpha_2^4)\eta\Omega + (\eta a + b)(1 - 4\alpha_2^2 + 5\alpha_2^4) + (c + \eta d)\alpha_2^2(1-\alpha_2^2) + R_2] \\ &\quad + \alpha_2^2(1 - 2\alpha_2^2)^2\beta(\omega^2 + C_{11}k^2/\rho)(B_2/M_0\rho)^4(M_0\rho k^2)^2 + \alpha_2^4(1-\alpha_2^2)\beta(\omega^2 + C_{44}k^2/\rho)(2B_1/M_0\rho)^2(B_2/M_0\rho)^2(M_0\rho k^2)^2 = 0 ,\end{aligned} \tag{18}$$

where $V(f,k)$ is the magnetic dispersion relation for a pure magnetic conductor as found in Ref. 11:

$$\begin{aligned}V(f,k) &= (a^2 + b^2 + 2\Omega a)\alpha_2^2 + (1-\alpha_2^2)[ad + bc + \Omega(a+d)] + \Omega^2 + [(c_{11}c_{22} + c_{11}c_{33} + c_{22}c_{33}) - (c_{31}c_{13} + c_{21}c_{12} + c_{23}c_{32})] \\ &\quad - (a_{31}c_{13} + a_{13}c_{31} + a_{32}c_{23} + a_{23}c_{32} + a_{21}c_{12} + a_{12}c_{21}) - [d(1-\alpha_2^2)(c_{11} + c_{33}) + a(1-\alpha_1^2)c_{33} + a(1-\alpha_3^2)c_{11} \\ &\quad + a(2 - \alpha_1^2 - \alpha_3^2)c_{22}] - \Omega(c_{11} + c_{22} + c_{33}) .\end{aligned}$$

A simple method by which the secular determinant can be expanded with the above form is shown in the Appendix. All of the parameters in Eq. (18) are defined in the Appendix and in Ref. 11.

The first term in Eq. (18) represents the uncoupled magnetic and elastic wave dispersion relations. The second term represents a coupling between the spin waves and a LE wave (longitudinal magnetoelastic coupling), and the third term represents a coupling between spin waves and a Te wave (transverse magnetoelastic coupling). The fourth and fifth terms represent couplings between two TE waves and between a TE wave and the LE wave. As suggested by Eq. (18), the coupling strength between the elastic and magnetic waves depends on the product of $B_1$ or $B_2$, the $C_{ij}$'s and an effective magnetic field which includes $H_0$, the damping fields, and conductivity, as well as $2K_1/M_0$ and $2K_2/M_0$, through the $R_1$ and $R_2$ terms (see the Appendix). The two major roles of magnetic crystalline anisotropy are (i) the selection of the crossover frequency between the elastic and magnetic branches [see Eq. (8) and $V(f,k)$] and (ii) the influence on the coupling strength between the waves [see Eq. (18)]. We now take various limits and show that the present form for the dispersion relation reduces to expressions derived by previous authors. In the limit of $B_1 = B_2 = 0$ the secular equation reduces to the product of the uncoupled elastic and magnetic dispersion relations. Further limits on $V(f,k)$ are discussed in Ref. 11. For the elastic dispersion, the $f/\gamma$ versus $k$ relation is trivial.

In the limit of $B_1 \neq 0$, $B_2 \neq 0$, $\sigma = 0$ (insulator) and $K_1 = K_2 = 0$ (isotropic magnetic medium) Eq. (18) reduces to that of Kobayashi *et al.*[4] In the general case that we are considering where $\sigma \neq 0$ and $K_1 \neq 0$ and $K_2 \neq 0$, the secular equation is seventh order in $k^2$, since $V(f,k)$ is fourth[11] order in $k^2$ and the elastic dispersion is third order in $k^2$. Although the solutions for all the roots of $k^2$ for a given bias field and frequency appear to be complicated, the solutions are trivial for frequencies away from crossover regions between spin waves and elastic branches. In this region four of the roots can be approximately obtained from $V(f, k)$ as has been done already,[11] and the other three can be obtained from the uncoupled elastic dispersion relations. Near the crossover regions, computer solutions have been obtained for $k^2$ using a Newton iteration procedure. Convergence is improved considerably by selecting a small increment of frequency, since the guess for the new root (in the iteration procedure) *is* the root for the previously selected frequency. In the following section, plots of $f/\gamma$(Oe) versus $k$ are presented for a recently[9] characterized rare-earth transition-metal alloy.

## III. DISPERSION CURVES

The solution of the $k$ values for a given value of $f/\gamma$(Oe) in Eq. (18) are given in Figs. 5 and 6 for the case of $Tb_{0.15}Ho_{0.85}Fe_2$ and for which the following parameters were used:

$M_0 = 700$ G (measured[9]),

$A = 0.9\times10^{-6}$ erg/cm,

$2K_1/M_0 = 600$ Oe (measured[9]),

$2K_2/M_0 = 3400$ Oe (measured[9]),

$\lambda = 3.75\times10^7$ Hz,

$B_1 = 0$ (measured[9]),

$B_2 = 10^8$ erg/cm$^3$ (measured[9]),

$C_{11} = 2\times10^{12}(1+0.1j)$ dyn/cm$^2$,

$C_{44} = 10^{12}(1+0.1j)$ dyn/cm$^2$,

$\sigma = 0.37\times10^5$ mho/cm,

$g = 2.20$,

$\rho = 9.4$ (measured[9]).

For the purpose of illustrating the calculations we have assumed reasonable values for the rest of the parameters which are not measured. The roots or branches are identified in each graph by the numbers 1–7. Figure 6 serves the purpose of expanding the scale for $k\to0$.

In Fig. 5 the dispersion curves were obtained for $\theta_0 = 0$, 45°, and 90° and $\varphi_0 = 90°$ [$\vec{M}_0$ in the (100) plane] for the purpose of comparison. The magnitude and direction ($\alpha$) of $\vec{H}_a$ was chosen so that the three sets of dispersion curves overlapped each other for $k\to0$. This is achieved by requiring that for all three angles of $\theta_0$ the dispersion curves have the same intercept (Kittel uniform mode of $f_0/\gamma = 8418$ Oe). The required values of $H_a$ and $\alpha$ are calculated by using $V(f_0, k) = 0$ in the limit of $\sigma=\lambda=k=0$ simultaneously with Eq. (9). Table I shows the required biasing conditions. Branches (1), (2), (3), and (4) are the magnetic branches and (5), (6), and (7) are the acoustic branches as identified in the two figures. Of course, in the region of strong coupling the magnetic and acoustic branches are admixed and lose "character."

Branch (1) as shown in Fig. 6 is recognized as the skin depth or the electromagnetic branch which is simply given as

$$k_1 = (1/\delta_0)(1+j).$$

This branch in Fig. 6 is plotted only for $\theta_0 = 0$, since little deviation from the above expression occurs for $\theta_0 = 45°$ and 90°. There is small coupling between $k_1$ and the other branches for $\theta \neq 0$. For $\theta_0 = 0$ this branch is uncoupled from the rest of the branches, since $V(f, k)$ can be written as the product of a cubic equation in $k^2$ and $Q^2$.

Branch (2) is often referred to as the exchange-conductivity branch.[5] This branch is highly dispersive near $f_0/\gamma \simeq 8418$ Oe. The attenuation of the rf magnetic fields varies sharply near $f_0/\gamma$, since $\mathrm{Re}(k_2)$ changes rapidly. The exchange rf field component induced by attenuation of $m$ goes through a rapid change as $f/\gamma$ is varied near $f_0/\gamma$. As a result the ferromagnetic resonance linewidth in metals is broadened by this mechanism. For $f/\gamma \gg f_0/\gamma$, as in the excitation of higher-order standing-spin-wave modes in films, the exchange-conductivity linewidth broadening mechanism[5] is negligible. At low frequencies so that $f/\gamma \ll f_0/\gamma$, this branch assumes similar features to those of the skin-depth mode.

The spin-wave branch (3) couples with one of the transverse acoustic branches. The repulsion or splitting between the two branches increases with increasing $\theta_0$. The coupling interaction is angular dependent as suggested by the third term in Eq. (18). Since $B_1 = 0$, there is no coupling between the spin-wave branch and the longitudinal acoustic branch. For values of $\mathrm{Im}(k)$ below the interaction region the spin-wave branch behaves in the usual manner

$$f/\gamma \propto [\mathrm{Im}(k_3)]^2.$$

However, the spin-wave branch does *not* intersect the $f/\gamma$ axis as $k_3\to0$, since spin waves are assumed to be damped. The other spin-wave branch (4) is referred to as the surface spin-wave branch, since $\mathrm{Re}(k_4)$ is large. For large values of $\mathrm{Re}(k_4)$ the rf fields are confined to the surface. The two spin-wave branches are approximately separated by

$$[\mathrm{Re}(k_4)]^2 - [\mathrm{Re}(k_3)]^2 \simeq 4\pi M_0/(2A/M_0)$$

for $\theta_0 = 0$ and $f/\gamma = 0$, but are degenerate for $\theta_0 = 90°$. Neglecting magnetic anisotropy, conductivity, and magnetic damping, $k_4$ is given by

$$(2A/M_0)k_4^2 = H_0 + 2\pi M_0 + [(2\pi M_0 \cos^2\theta_0)^2 + (f/\gamma)^2]^{1/2}.$$

When magnetic damping is included, $k_4$ is complex and $\mathrm{Re}(k_4) \gg \mathrm{Im}(k_4)$ (Fig. 6). A simple way to include the effect of magnetic damping on $k_4$ is to write[13] the internal field as

$$H_0' \cong H_0\left[1 + j\left(\frac{\lambda/\gamma}{H_0}\right)\left(\frac{f/\gamma}{M_0}\right)\right].$$

The magnetic anisotropy effect on $k_4$ cannot be included simply except for $\vec{M}_0$ along a $\langle 100\rangle$ axis. For this case one needs only to redefine $H_0$ as

$$H_0'' = H_0' + 2K_1/M_0.$$

Thus, two of the four magnetic branches can be approximately obtained from simple analytical expression. This is very helpful in the computer solution of the roots. Finally, it is pointed out that for the two spin-wave branches $\mathrm{Im}(k_3)$ and $\mathrm{Im}(k_4) \to 0$ as $f/\gamma \to 0$. This is in marked contrast to the pure magnetic case[14] where these two branches intersect the $\mathrm{Im}(k)$ axis. Thus, it appears that there is certain amount of repulsion between the acoustic and spin-wave branches near $\mathrm{Im}(k) \simeq 0$.

As indicated in the second term in the secular equation, there is no coupling between spin waves and the longitudinal acoustic branch (5). Thus, the dispersion relation for this branch is simply given as

$$k_5 = j\,2\pi f(\rho/C_{11})^{1/2}.$$

The dispersion relation for one of the noninteracting transverse acoustic branches (6) is given as

$$k_6 \cong j\,2\pi f(\rho/C_{44})^{1/2}.$$

It is pointed out that both $C_{11}$ and $C_{44}$ are complex quantities, and therefore, $k_5$ and $k_6$ can be complex as indicated in Figs. 5 and 6. The acoustic damping is introduced phenomenologically by taking $C_{11}$ and $C_{44}$ as $C_{ii} = c_{ii}' + jc_{ii}''$, where $c_{ii}''$ is chosen for our case to be 10% of $c_{ii}'$. Finally, as expected the other transverse acoustic mode (7) couples strongly with spin waves. For frequencies near the crossover region there is no simple relationship between frequency and $k_7$.

## IV. APPLICATIONS

So far we have been discussing one branch at a time. Let us now examine the simultaneous excitation of a number of branches. Whether one calculates the magnetic resonance condition or acoustic wave propagation characteristics, one needs to calculate the surface impedance $Z$. Let us calculate $Z$ for a special field configuration, as an example. For simplicity let us assume a semi-infinite plate whose surface is at $y = 0$, $B_1 = B_2 = 0$, and $\vec{H}_a$ perpendicular to the plate (we will also consider the case of $B_2 \neq 0$). Further, the rf magnetization is assumed to be pinned at the surface so that $m_x = 0$ and $m_z = 0$ at $y = 0$. The boundary conditions are written as

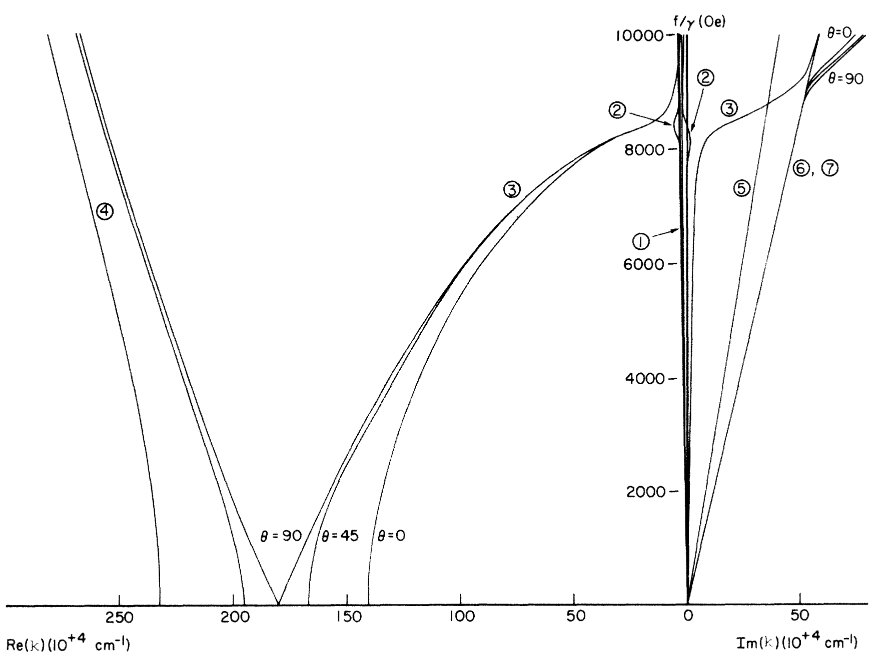


FIG. 5. Quantitative plot of the seven branches of $k$ for a ferromagnetic magnetoelastic conductor.

$$\sum_{n=1}^{4} h_n = h_{0x}\,, \tag{19a}$$

$$\sum_{n=1}^{4} (-1)^p h_n = h_{0z}\,, \tag{19b}$$

$$\sum_{n=1}^{4} k_n h_n = Z_1 h_{0x}\,, \tag{19c}$$

$$\sum_{n=1}^{4} (-1)^p k_n h_n = Z_2 h_{0z}\,, \tag{19d}$$

$$\sum_{n=1}^{4} Q_n h_n = 0\,, \tag{19e}$$

$$\sum_{n=1}^{4} (-1)^p Q_n h_n = 0\,. \tag{19f}$$

Because of the special symmetry of this field geometry, it is well known that the normal modes are circularly polarized. The normal modes are

$$m_\pm = m_x \pm j m_z\,.$$

The problem becomes one of expressing $Z$ in the proper combination of the four $k$ values. There are two ways of selecting the proper combination of $k$ values to obtain the surface impedance for right and left circularly polarized modes. One way is to determine which $k$ values correspond to right and circularly polarized modes as it was done in Ref. 6. The surface impedance can then be expressed in terms of $k$ values with the same corresponding "sense" of polarization.[6]

It is instructive to use the approach of setting up an eigenvalue problem for which the resulting eigenvalues are the two surface impedances, and demonstrate that each surface impedance is, in fact, expressed in terms of the proper combination of $k$ values. Since we are setting up an eigenvalue problem for the surface impedances, the boundary conditions at the surface can be expressed in terms of rf fields which are not necessarily normal mode field solutions. Thus Eqs. (19a)–(19f) are expressed in terms of a convenient set of variables, $m_x$ and $m_z$ or $h_x$ and $h_z$. For this geometry $m_\varphi = -m_x$ and $m_\theta = -m_z$. In the first four equations we require the *total* rf magnetic ($h_x$ and $h_z$) and electric ($e_x$ and $e_z$) fields to be continuous at the surface. The last two equations represent $\vec{m} = 0$. The rf component of $\vec{m}$ normal to the plate ($m_y$) is zero. In Eqs. (19b), (19d), and (19f) we have made use of the fact that $m_z/m_x = +j$ for $k_1$ and $k_4$ and $m_z/m_x = -j$ for $k_2$ and $k_3$. Thus, $k_1$ and $k_4$ correspond to left circularly polarized normal modes of $\vec{m}$, while $k_2$ and $k_3$ correspond to the right circularly polarized waves. The value of $p = 0$ for $n = 1$ and 4 and $p = 1$ for $n = 2$ and 3. We have assumed that the surface impedance may be anisotropic so that it is defined by

$$Z_1 = (4\pi\sigma/c)(e_{0z}/h_{0x})\,,$$

$$Z_2 = (-4\pi\sigma/c)(e_{0x}/h_{0z})\,.$$

$h_n$ are the magnetic field strengths corresponding to the $k_n$ root. There are six unknowns ($h_1, h_2, h_3, h_4, -h_{0x}$, and $-h_{0z}$) and six equations. Nontrivial solutions are obtained if

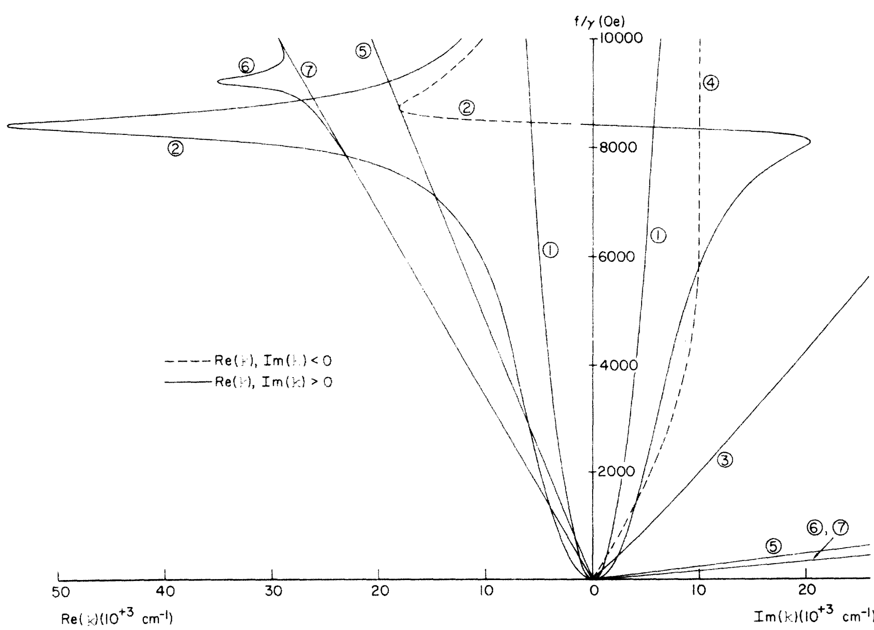


FIG. 6. Same as Fig. 5, except the region near $k \simeq 0$ is expanded.

TABLE I. Required biasing conditions.

| $\phi_0$ (deg) | $\theta_0$ (deg) | $\alpha$ (deg) | $H_a$ (oe) |
|---|---|---|---|
| [010] | [010] | 0 | 4500 |
| [010] | [011] | 67.83 | 11 334 |
| [010] | [001] | 90 | 16 615 |

$$\det\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ Z_1 & 0 & k_1 & k_2 & k_3 & k_4 \\ 0 & Z_2 & k_1 & -k_2 & -k_3 & k_4 \\ 0 & 0 & Q_1 & Q_2 & Q_3 & Q_4 \\ 0 & 0 & Q_1 & -Q_2 & -Q_3 & Q_4 \end{bmatrix} = 0 .$$

Expanding the determinant, one obtains

$$Z_1 Z_2 - \tfrac{1}{2}[(Z_1 + Z_2)(Z^+ + Z^-)] + Z^+ Z^- = 0 , \qquad (20)$$

where

$$Z^+ = (k_2 Q_3 - Q_2 k_3)/(Q_3 - Q_2) , \qquad (21)$$

$$Z^- = (k_4 Q_1 - Q_4 k_1)/(Q_1 - Q_4) . \qquad (22)$$

In seeking the characteristic polarizations of the system, we look for vectors $\vec{h}$ and $\vec{e}$ on the surface, where $\vec{e}$ is a known *linear* function of $\vec{h}$. This implies that $Z_1 = Z_2 = Z$ and leads to an eigenvalue problem for the determination of $Z$, and Eq. (20) becomes quadratic in $Z$. There are two values for $Z$, since the vector space is two dimensional.

The two solutions of $Z$, $Z = Z^+$ and $Z^-$, represent the condition[14] that the incident and reflected waves from the surface have the same polarization. In a magnetic resonance experiment $Z = Z^+$ gives rise to the resonant response of the system while the $Z = Z^-$ solution the "flat" response (nonresonant).

For oblique angles[14] $Z^+$ and $Z^-$ are a function of $k_1$, $k_2$, $k_3$, and $k_4$ instead of only $k_2$ and $k_3$ or $k_1$ and $k_4$. However, reasonable[15] approximate solutions of $Z^+$ at oblique angles can be obtained using only two roots ($k_2$ and $k_3$) instead of all four roots. For in-plane resonance, $\theta_0 = 0$, $k_2$, $k_3$, and $k_4$ are required[5] to calculate $Z$, since $k_1$ is the nonmagnetic branch (the skin-depth branch).

For our case $B_2 \neq 0$ so that $Z^+$ will also be a function of $k_7$ and, therefore, the magnetoelastic parameter. The other two acoustic branches are not coupled to the magnetic system and, therefore, are not included in the expression for $Z^+$. The boundary conditions are the same as that given in Eq. (19) except a new boundary condition on the elastic motion must be introduced. The traction free boundary condition,[4] which implies that the lattice is free to vibrate at the surface, is given as

$$C_{44}\frac{\partial u}{\partial y} + \frac{B_2}{M_0} m = 0 ,$$

where $u$ and $m$ are the circularly polarized amplitudes of the lattice displacement and magnetic rf field or $u = u_x - j u_z$ and $m = m_x - j m$, since $\theta_0 = 90°$. If the lattice is "clamped" at the surface the boundary condition (deformation free) is given simply as

$$u = 0 .$$

After some algebra $Z^+$ is found to be

$$Z^+ = \frac{k_2(v_7 - v_3)/Q_2 + k_3(v_2 - v_7)/Q_3 + k_7(v_3 - v_2)/Q_7}{(v_7 - v_3)/Q_2 + (v_2 - v_7)/Q_3 + (v_3 - v_2)/Q_7} ,$$

where

$$v_n = B_2[1 - C_{44}k_n^2/(C_{44}k_n^2 + \rho\omega^2)]/M_0$$

for traction-free boundary condition. For the deformation-free boundary condition, $v_n$ is defined as

$$v_n = B_2[k_n/(C_{44}k_n^2 + \rho\omega^2)]/M_0 .$$

The same procedure as outlined in Ref. 16 can be used to calculate acoustic power generated in a rod by a magnetic metallic film. The basic idea[16] is to set up an acoustic standing mode in a film with finite thickness. The acoustic impedance of the rod is purposely[16] chosen to be mismatched to the acoustic impedance of the film so that only part of the acoustic power is launched into the rod. Maximum acoustic power transfer is realized when the delicate balance of (i) "tapping" as much of acoustic power out of the film and (ii) still maintaining an acoustic standing wave resonance is achieved. If the rod and film acoustic impedances are matched, an acoustic standing mode cannot be set up. Thus, the thickness is chosen to be half an acoustic wavelength. A material with a high magnetostriction constant is desirable, since the acoustic wave is generated via the magnetic system. For our case the maximum transverse acoustic power is generated for $\theta_0 = 90°$, and only three branches are necessary and they are ($k_2$, $k_3$, and $k_7$), since $k_1$ and $k_4$ are magnetically inactive and $k_5$ and $k_6$ are not coupled to the magnetic systems. However, it may be possible to generate acoustic waves by tuning the frequency of operation to the cross over region. For this type of experiment the substrate, or the rod acoustic impedance, and the thickness is required to be less than the skin depth.

From a practical point of view the rare-earth-transition-metal systems appear promising in view of recent[17] experimental results where an acoustic wave was generated at ~1 GHz.

## ACKNOWLEDGMENT

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## APPENDIX

The matrix elements of $\underline{A}$, $\underline{A}_1$, $\underline{B}'$, and $\underline{D}'$ are defined in this section. The matrix $\underline{A}$ is defined as

$$\underline{A}=\begin{bmatrix} a_{11}+c_{11} & a_{12}+c_{12} & a_{13}+c_{13} \\ a_{21}+c_{21} & a_{22}+c_{22} & a_{23}+c_{23} \\ a_{31}+c_{31} & a_{32}+c_{32} & a_{33}+c_{33} \end{bmatrix}.$$

The elements $a_{ij}$ and $c_{ij}$ have been defined in the appendix of Ref. 11 and in Ref. 18 except the $a$ and $d$ coefficients are defined as

$$a=\frac{\lambda}{\gamma}\left(\frac{H_0-(2A/M_0)k^2}{M_0}+\frac{1}{Q}\right),$$

$$d=\frac{\lambda}{\gamma}\left(\frac{H_0-(2A/M_0)k^2}{M_0}+4\pi\right),$$

in order to include the effect of the exchange field in the magnetic damping terms.

The magnetoelastic matrix $\underline{A}_1$ is given as

$$\underline{A}_1=\begin{bmatrix} B_2[\alpha_1\alpha_3-\eta(1-2\alpha_1^2)\alpha_2] & 2B_1'(\alpha_2\alpha_3+\eta\alpha_1\alpha_2^2) & -B_2[(\alpha_2^2-\alpha_3^2)+2\eta\alpha_1\alpha_2\alpha_3] \\ -B_2[\alpha_2\alpha_3+\eta(1-2\alpha_2^2)\alpha_1] & -2B_1'\eta(1-\alpha_2^2)\alpha_2 & B_2[\alpha_1\alpha_2-\eta(1-2\alpha_2^2)\alpha_3] \\ -B_2[(\alpha_1^2-\alpha_2^2)-2\eta\alpha_1\alpha_2\alpha_3] & -2B_1'(\alpha_1\alpha_2-\eta\alpha_2^2\alpha_3) & -B_2[\alpha_1\alpha_3+\eta(1-2\alpha_3^2)\alpha_2] \end{bmatrix},$$

where $B_1'=B_1-2\pi M_0^2$, $\eta=\lambda/\gamma M_0$, and $\beta=1+\eta^2$.

In obtaining the secular equation, it was required to expand the determinant

$$\det(\underline{D}'-\underline{B}')=0,$$

where

$$\underline{D}'=\begin{bmatrix} \Omega & 0 & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 & 0 \\ 0 & 0 & -\rho\omega^2 & 0 & 0 \\ 0 & 0 & 0 & -\rho\omega^2 & 0 \\ 0 & 0 & 0 & 0 & -\rho\omega^2 \end{bmatrix}$$

and

$$\underline{D}'-\underline{B}'=\frac{1}{l^4}\begin{bmatrix} -\alpha_3(\alpha_1A_{31}+\alpha_2A_{32})+l^2A_{33} & \alpha_2A_{31}-\alpha_1A_{32} & -A_{34} & -A_{35} & -A_{36} \\ \begin{array}{l}-\alpha_2\alpha_3(\alpha_1A_{11}+\alpha_2A_{12}) \\ +\alpha_1\alpha_3(\alpha_1A_{21}+\alpha_2A_{22}) \\ +l^2(\alpha_2A_{13}-\alpha_1A_{23})\end{array} & \begin{array}{l}\alpha_2(\alpha_2A_{11}-\alpha_1A_{12}) \\ +\alpha_1(\alpha_1A_{22}-\alpha_2A_{21})\end{array} & -\alpha_2A_{14}/l+\alpha_1A_{24}/l & -\alpha_2A_{15}/l+\alpha_1A_{25}/l & -\alpha_2A_{16}/l+\alpha_1A_{26}/l \\ -(2B_2l\alpha_1\alpha_2\alpha_3k)/M_0\rho & B_2l(\alpha_2^2-\alpha_1^2)k/M_0\rho & C_{44}k^2/\rho+\omega^2 & 0 & 0 \\ -(2B_1'l\alpha_2^2\alpha_3k)/M_0\rho & -(2B_1'l\alpha_1\alpha_2k)/M_0\rho & 0 & C_{11}k^2/\rho+\omega^2 & 0 \\ B_2l\alpha_2(1-2\alpha_3^2)k/M_0\rho & -B_2l\alpha_1\alpha_3k/M_0\rho & 0 & 0 & C_{44}k^2/\rho+\omega^2 \end{bmatrix}$$

where

$$l=\sin\theta_0,$$

$$A_{ij}=a_{ij}+c_{ij}$$

$$A_{i,j+3}\equiv(A_1)_{ij},\quad i,j=1,2,3.$$

It is noticed that the $5\times 5$ matrix $(\underline{D}'-\underline{B}')$ is of the form

$$(\underline{D}'-\underline{B}')\equiv\begin{bmatrix} W & X \\ Y & Z\end{bmatrix}.$$

It is found that the determinant expansion can be simplified by using the identity

$$\det(\underline{D}'-\underline{B}')=[\det(W-XZ^{-1}Y)]\det Z .$$

The advantage of using the identity is that it reduces the $5\times 5$ determinant expansion to a $2\times 2$ determinant expansion, since $Z$ is a diagonal matrix. Finally, the parameters $R_1$ and $R_2$ in Eq. (18) are defined as

$$R_1=-\{\alpha_1\alpha_2\alpha_3(c_{33}-c_{11})+\alpha_2(\alpha_1^2c_{13}-\alpha_3^2c_{31})+\alpha_1^2\alpha_3c_{21}-\alpha_1^3c_{23}$$
$$+\eta[\alpha_1^2c_{33}+\alpha_2^2\alpha_3^2c_{11}-\alpha_1\alpha_3(c_{31}+\alpha_2^2c_{13})-\alpha_1\alpha_2\alpha_3(\alpha_3c_{21}-\alpha_1c_{23})]\}/(\alpha_1^2+\alpha_2^2)\,,$$

and

$$R_2=\{\alpha_1\alpha_3(1-3\alpha_2^2)(\alpha_2c_{11}-\alpha_1c_{21}+\eta c_{31})+\alpha_2(1-\alpha_1^2-4\alpha_2^2\alpha_3^2)[c_{31}+\eta(\alpha_1c_{21}-\alpha_2c_{11})]$$
$$+(1-\alpha_3^2(1+\alpha_2^2)-4\alpha_1^2\alpha_2^2)(\alpha_1c_{23}-\alpha_2c_{13}-\eta c_{33})+\alpha_1\alpha_2\alpha_3(1-4\alpha_2^2)[\eta(\alpha_2c_{13}-\alpha_1c_{23})-c_{33}]$$
$$+\alpha_2^2\alpha_3(\alpha_2^2c_{12}-\alpha_1\alpha_2c_{22}+\eta\alpha_2c_{32})+\alpha_1\alpha_2^2[\eta(\alpha_2c_{12}-\alpha_1c_{22})-c_{32}]\}/(\alpha_1^2+\alpha_2^2)\,.$$

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