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Nonlinear renormalization groups*

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Most renormalization groups studied heretofore are linear, in that the block-spin variables are linearly related to the old spin variables. These renormalization groups all require an adjustable parameter that must be properly fixed in order that the renormalization-group transformation have a useful fixed point. Niemeijer and van Leeuwen, however, have investigated the critical behavior of the two-dimensional Ising model with a nonlinear renormalization group, and find a fixed point without introducing such a parameter. We introduce here another nonlinear renormalization group and study it in detail for the Gaussian model. We find that (i) within certain limits, no such parameter need be adjusted in order to reach a fixed point; (ii) an eigenvalue of the linearized transformation with no physical significance depends on the nonlinearity of the transformation; and (iii) a physically significant eigenvalue is unchanged (to the order examined).

I. INTRODUCTION

A successful renormalization-group approach[1] to the determination of the singular behavior of a physical system near its critical point requires the construction of a renormalization-group transformation appropriate to the problem, and the location of a fixed point of the transformation related to the original Hamiltonian of the physical system. We shall discuss here an element common to the structures of many renormalization groups, a parameter in the transformation whose value must be properly chosen in order to find an interesting fixed point for the transformation.

It will be convenient to speak in terms of a specific physical system, a system of interacting spins σ on a lattice which undergoes a ferromagnetic transition at some critical temperature. To determine the critical behavior of such a system with the renormalization group, one first constructs a transformation $\mathcal{T}$ that converts the old Hamiltonian of the system, $\mathcal{H}[\sigma]$ (assumed to include the factor $-1/k_BT$), into a new Hamiltonian $\mathcal{H}'[S]$,

$$\mathcal{H}'[S] = \mathcal{T}\{\mathcal{H}[\sigma]\}, \tag{1.1}$$

where $\mathcal{H}'[S]$ represents the effective interactions between blocks of old spins now treated as single new block spins S. This transformation has two important properties. First, it must preserve the partition function of the system; that is, the partition function calculated from $\mathcal{H}'[S]$ must equal the partition function for $\mathcal{H}[\sigma]$. Second, it must have a nontrivial fixed point, a local Hamiltonian $\mathcal{H}^*$ (without long-range interactions), satisfying the equation

$$\mathcal{H}^* = \mathcal{T}\{\mathcal{H}^*\}, \tag{1.2}$$

with the same critical behavior as the original Hamiltonian, and to which the original critical Hamiltonian is carried by repeated applications of the transformation $\mathcal{T}$.

One example of such a transformation is discussed in Ref. 1. The old Hamiltonian is expressed as a function of the spin variables $\sigma_{\vec{q}}$ in momentum space (Fourier transforms of $\sigma_{\vec{r}}$). The transformation consists in carrying out the configuration sums for the partition function only over the spin variables with $|\vec{q}| > \frac{1}{2}$, leaving unintegrated the longer wavelength components. The new block-spin variables $S_{\vec{q}}$ are defined to be the unintegrated old variables multiplied by a constant factor ζ^{-1}.

This transformation is of the type that we shall refer to as *linear*, because the transformation relates the block-spin variables to the old spin variables linearly. More importantly, the spin-spin correlation functions for $\mathcal{H}[\sigma]$ and $\mathcal{H}'[S]$ are also linearly related. At the critical point of the system, the behavior of both correlation functions is characterized by the critical exponent η, and if the transformation has a nontrivial fixed point, then this relation fixes the adjustable parameter ζ in terms of η (another example is given later in this paper). Thus, a fixed point cannot be found unless ζ is appropriately fixed.

Another example of a renormalization-group transformation appears in the work of Niemeijer and van Leeuwen.[2] They define a transformation for Hamiltonian functions of Ising spins ($\sigma = \pm 1$) on a triangular lattice. The lattice is divided up into blocks containing three old spins each, and a block spin is defined to be the sign of the sum of the old spins in a block. This transformation is of the type that we shall refer to as *nonlinear*, because the block spins are not linearly related to the old spins.[3]

Linear and nonlinear renormalization-group transformations differ considerably in their behaviors. For instance, although the linear transformation requires an adjustable parameter in order to function properly, Niemeijer and van Leeuwen succeeded in finding a fixed point without introducing such a parameter. In order to elucidate some of these differences, we have examined a nonlinear renormalization-group transformation whose qualitative differences from the linear transformation can be determined from perturbation theory.

First, we introduce a linear transformation related to the transformation in Ref. 1. The transformation has an adjustable parameter b that must be fixed at a particular value b^* in order that the transformation have a fixed point. The transformation then has a *line* of fixed points (as did the transformation in Ref. 1), parametrized by the over-all normalization of the spins in the Hamiltonian, a parameter that has no physical significance.

Next, a nonlinear transformation is introduced with a second adjustable parameter c in addition to the parameter b, defined so that the transformation is linear for $c=0$. When the behavior of the new transformation is examined to first order in c, two situations can be distinguished:

(i) For $c<0$, the parameter b may take any value in region I of the parameter space indicated in Fig. 1, and a fixed point will be reached without trouble. The transformation now has a single fixed point for given b and c rather than a line of fixed points.

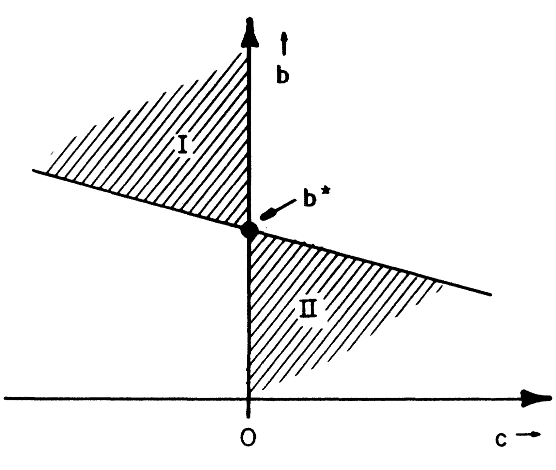

FIG. 1. Parameter space for the nonlinear renormalization group transformation. For $c=0$, the transformation is linear, and for $b=b^*$ it has a line of fixed points. In region I, $c<0$, the transformation has a single fixed point that is reached without trouble, starting from the original critical Hamiltonian. In region II, $c>0$, the fixed point can be reached only by appropriately adjusting an unphysical parameter in the original Hamiltonian.

(ii) For $c>0$, the parameter b may take any value in region II of Fig. 1, and the transformation will have a fixed point, but it can in this case only be reached by fixing a second, unphysical parameter in the original Hamiltonian, the over-all normalization of the spins.[4] Alternatively, one may leave the original Hamiltonian alone, and adjust the parameter b until a fixed point is found, as for the linear transformation. Again, once b and c are chosen, the transformation will have a single fixed point rather than a line of fixed points.

These conclusions follow from an examination of the stability of the fixed points. If $\mathcal{H}^*$ is a fixed point of the transformation $\mathcal{T}$, and a small perturbation $\epsilon\mathcal{O}[\sigma]$ is added to $\mathcal{H}^*[\sigma]$, then under the repeated application of the transformation, denoted symbolically by $\mathcal{T}^k$, we expect to observe behavior like[1]

$$\mathcal{T}^k\{\mathcal{H}^*+\epsilon\mathcal{O}\}=\mathcal{H}^*+\epsilon(\Lambda_1^k\mathcal{R}_1^*+\Lambda_2^k\mathcal{R}_2^*+\cdots)+O(\epsilon^2), \tag{1.3}$$

where the numbers $\Lambda_1\geq\Lambda_2\geq\cdots$ are eigenvalues of the transformation, and $\mathcal{R}_1^*[\sigma]$, $\mathcal{R}_2^*[\sigma]$, etc., are eigenoperators. The terminology arises from the behavior of the fixed point when perturbed in the direction of an eigenoperator $\mathcal{R}^*$:

$$\mathcal{T}\{\mathcal{H}^*+\epsilon\mathcal{R}^*\}=\mathcal{H}^*+\Lambda\epsilon\mathcal{R}^* \tag{1.4}$$

(to order ϵ).

Ordinarily, one expects to find an eigenvalue greater than 1 associated with changing the temperature of the original system. The eigenvalue may be used to determine some of the critical exponents of the system.[1] The linear transformation also has an eigenoperator with eigenvalue equal to 1 (a marginal operator) associated with changes in the over-all normalization of the spins in the original Hamiltonian. It originates from perturbations in the direction along the line of fixed points of the transformation. When the transformation becomes nonlinear, this unphysical eigenvalue is increased or decreased depending on the sign of c, although the physically significant eigenvalue is unaffected (to first order in c). This example confirms a result of the general analysis by Wegner,[5] who concludes that there are two classes of eigenoperators, one physically significant, the other with eigenvalues that depend on the choice of renormalization group. The eigenvalue's change is responsible for the qualitative differences in the behavior of the nonlinear transformations.

The nonlinear renormalization group transformation offers some noteworthy advantages over the linear transformation in its application to systems like the Ising model.[6] The ability to locate a useful fixed point without necessarily having to adjust a parameter in the transformation (as one must in the linear case) is an obvious one. However, as we have seen above, one is not guaranteed success in this respect, and may in fact have more trouble here than with the linear transformation. The freedom to adjust the parameters in the transformation in order to minimize the amount of interaction between distant spins in the fixed-point Hamiltonian is another advantage, since this can improve the accuracy of approximate calculations confined to a finite portion of the lattice, as was the calculation of Niemeijer and van Leeuwen.[2] On the other hand, the linear renormalization group transformation is easy to manipulate and its behavior readily predicted. Which are the decisive factors for a renormalization-group calculation will probably have to be determined case by case.

In the following sections, we show in detail how the above results were obtained. In Sec. II, the linear continuous-spin renormalization-group transformation is presented, and the line of fixed points with mean-field behavior is located. In Sec. III, the nonlinear transformation is introduced and its fixed points examined to first order in c. Further discussion of the results appears in Sec. IV. Some details of the calculation are exhibited in the Appendix.

II. LINEAR RENORMALIZATION-GROUP TRANSFORMATION

We shall study renormalization-group transformations that act on the space of Hamiltonians used in Ref. 1, assumed to have the form

$$\mathcal{H}[\sigma]=\text{const}-\tfrac{1}{2}\int_{\vec{q}} u_2(\vec{q})\sigma_{\vec{q}}\sigma_{-\vec{q}}-\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\int_{\vec{q}_4} u_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\sigma_{\vec{q}_1}\sigma_{\vec{q}_2}\sigma_{\vec{q}_3}\sigma_{\vec{q}_4}\,\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)-\cdots, \quad (2.1)$$

with $\sigma_{\vec{q}}=\sigma^*_{-\vec{q}}$ taking all complex values, where $\vec{q}$ is a momentum vector in d dimensions, and with

$$\int_{\vec{q}}\equiv\int_{|\vec{q}|\le 1} đ^d q, \quad (2.2a)$$

$$đ^d q\equiv(2\pi)^{-d}d^d q, \quad (2.2b)$$

$$\delta(\vec{q})\equiv(2\pi)^d\,\delta^{(d)}(\vec{q}). \quad (2.2c)$$

As explained in Ref. 1, the requirement that $\mathcal{H}[\sigma]$ contain only local interactions between spins in coordinate space imposes some constraints on the analyticity of the functions $u_2(\vec{q})$, $u_4(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)$, etc. In particular, the function $u_2(\vec{q})$ must be an analytic function of the components of $\vec{q}$ at $\vec{q}=0$.

The transformation that we shall examine first is parametrized by a and b:

$$e^{\mathcal{H}'[S]}=T_{a,b}[S]\,e^{\mathcal{H}[\sigma]}, \quad (2.3)$$

$$T_{a,b}[S]\exp\{\mathcal{H}[\sigma]\}\equiv\int_\sigma \exp\left(-\tfrac{1}{2}a\int_{\vec{q}}|S_{\vec{q}}-b\sigma_{\vec{q}/2}|^2\right)\times\exp\{\mathcal{H}[\sigma]\}, \quad (2.4)$$

where $\int_\sigma$ denotes integration over the real and imaginary parts of all spin variables $\sigma_{\vec{q}}$.[7] A few points should be noted about this transformation. First, the parameter a is to some extent superfluous, since results obtained for one value of a may be related to results for another value of a (and a different Hamiltonian) by an appropriate scale change in the spins σ and S. We shall nevertheless keep it. Second, the transformation in Eq. (2.4) is a simple extension of the renormalization-group transformation of Ref. 1, and equivalent to it when a is taken to infinity, aside from normalization factors, which will be consistently ignored throughout this paper, since they do not affect any of the results. Half of the old spin variables $\sigma_{\vec{q}}$ (those with $|\vec{q}|>\frac{1}{2}$) are integrated out, and the other half are associated with new spin variables $S_{\vec{q}}$, with $0\le|\vec{q}|\le 1$. Finally, the transformation has the desired property of preserving the partition function of the system; that is, if one calculates the partition function corresponding to the new Hamiltonian $\mathcal{H}'$, by integrating over all of the spin variables $S_{\vec{q}}$ on the left-hand side of Eq. (2.3), one finds

$$\int_S \exp\{\mathcal{H}'[S]\}=\text{const}\times\int_\sigma\exp\{\mathcal{H}[\sigma]\},$$

where the constant comes from performing the integrals over the transformation kernel on the right-hand side of Eq. (2.4) with respect to $S_{\vec{q}}$. The value of the constant is independent of $\mathcal{H}[\sigma]$, and has no effect on the calculation of thermodynamic averages.

In order to find a fixed point of the transformation, we shall iterate it. The effect of repeated application of the transformation to an arbitrary Hamiltonian is easy to discover, since it only involves integrals of Gaussian functions. It is represented by an effective transformation of the form

$$[T_{a,b}]^k\exp\{\mathcal{H}[\sigma]\}\equiv T^k_{a,b}\exp\{\mathcal{H}[\sigma]\} \quad (2.5)$$

$$=\int_\sigma\exp\left(-\tfrac{1}{2}a_k\int_{\vec{q}}|S_{\vec{q}}-b_k\sigma_{\vec{q}/L_k}|^2\right)\times\exp\{\mathcal{H}[\sigma]\}, \quad (2.6)$$

with

$$a_k = a(1 - 2^d b^2)/[1 - (2^d b^2)^k], \tag{2.7a}$$

$$b_k = b^k, \tag{2.7b}$$

$$L_k = 2^k. \tag{2.7c}$$

The quantity L_k is just the size of one of the blocks now represented by a spin S.

The transformation of a Hamiltonian quadratic in the spin variables,

$$\mathcal{H}[\sigma] = -\tfrac{1}{2}\int_{\vec{q}} \rho(\vec{q})\sigma_{\vec{q}}\sigma_{-\vec{q}}, \tag{2.8}$$

may be carried out exactly, since the problem separates into a product of Gaussian integrals. After k transformations, the Hamiltonian (2.8) becomes

$$\exp\{\mathcal{H}^{(k)}[S]\} \equiv T^k_{a,b}\exp\{\mathcal{H}[\sigma]\} = \exp\left(-\frac{1}{2}\int_{\vec{q}} \frac{L_k^2\rho(\vec{q}/L_k)S_{\vec{q}}S_{-\vec{q}}}{b_k^2 L_k^{d+2} + L_k^2\rho(\vec{q}/L_k)/a_k}\right). \tag{2.9}$$

The transformation should have an interesting fixed point, the Gaussian fixed point,[1] for

$$\rho(\vec{q}) = zq^2 + wq^4 + \cdots; \tag{2.10}$$

and, in fact, with the choice

$$b = b^* \equiv 2^{-(d+2)/2}, \tag{2.11}$$

we are indeed led to a nontrivial fixed point in the limit $k \to \infty$. If b should be chosen larger than b^*, the limit of $\mathcal{H}^{(k)}$ would vanish and its spin-spin correlation functions diverge; if b should be chosen smaller than b^*, the limit of $\mathcal{H}^{(k)}$ would be a Hamiltonian describing a completely noninteracting system of spins, and the correlation functions trivial constants. In neither case would the fixed-point Hamiltonian be of any use for obtaining critical exponents; for example, the large-distance behavior of the spin-spin correlation function for the original critical Hamiltonian is deduced from the similar behavior of the correlation function for the fixed-point Hamiltonian. With $b = b^*$, the nontrivial fixed point is

$$\mathcal{H}^*[\sigma] = -\tfrac{1}{2}\int_{\vec{q}} \rho^*(q)\sigma_{\vec{q}}\sigma_{-\vec{q}} \tag{2.12}$$

with

$$\rho^*(q) = zq^2/[1 + 4zq^2/(3a)]. \tag{2.13}$$

The fixed point is parametrized by the coefficient z of q^2 in $\rho(\vec{q})$, and so we have, in fact, a *line* of fixed points. The factor z in the original Hamiltonian determines to which fixed point the Hamiltonian is carried by the renormalization-group transformation. The fixed points do not depend on the value of w in Eq. (2.10); w is an irrelevant variable.

That a fixed point can be found for only one choice for b occurs for exactly the reason discussed in the Introduction. The transformation (2.4) implies a linear relation between the new and old spins. This may be made explicit by comparing the spin-spin correlation functions of the original and transformed Hamiltonians. The transformation (2.4) relates them, and this relation fixes b when the transformation has a fixed point. To see this, consider the correlation function for the Hamiltonian $\mathcal{H}'$,

$$\Gamma'(\vec{q})\delta(\vec{q}+\vec{q}\,') = (Z')^{-1}\int_S S_{\vec{q}}S_{\vec{q}\,'}\exp\{\mathcal{H}'[S]\}, \tag{2.14}$$

where Z' is the partition function for the Hamiltonian $\mathcal{H}'$,

$$Z' = \int_S \exp\{\mathcal{H}'[S]\}. \tag{2.15}$$

Use the definition of $\mathcal{H}'[S]$ in Eq. (2.4) to obtain

$$\Gamma'(\vec{q})\delta(\vec{q}+\vec{q}\,') = Z^{-1}\int_S S_{\vec{q}}S_{\vec{q}\,'} \times \int_\sigma \exp\left(-\tfrac{1}{2}a\int_{\vec{q}\,''}|S_{\vec{q}\,''} - b\sigma_{\vec{q}\,''/2}|^2 + \mathcal{H}[\sigma]\right).$$

Interchange the order of integration, and change the integration variables $S_{\vec{q}}$ to $S'_{\vec{q}} = S_{\vec{q}} - b\sigma_{\vec{q}/2}$. The integral over $S'_{\vec{q}}$ may then be performed, and gives

$$\Gamma'(\vec{q})\delta(\vec{q}+\vec{q}\,') = Z^{-1}b^2\int_\sigma \sigma_{\vec{q}/2}\sigma_{\vec{q}\,'/2}\exp\{\mathcal{H}[\sigma]\} + a^{-1}\delta(\vec{q}+\vec{q}\,'). \tag{2.16}$$

Using the definition of $\Gamma(\vec{q})$, we have the relation we were seeking:

$$\Gamma'(\vec{q}) = 2^d b^2\Gamma(\vec{q}/2) + a^{-1}. \tag{2.17}$$

At the critical point, if the correlation function falls off with a power of the distance between the spins, as $r^{-(d-2+\eta)}$, then $\Gamma(\vec{q})$ will have a singularity at $q = 0$, $\Gamma(\vec{q}) \sim q^{-2+\eta}$. Equation (2.17) requires that $\Gamma'(\vec{q})$ likewise have such a behavior, and will require such behavior of the spin-spin correlation functions for each succeeding Hamiltonian generated by the transformation. When a fixed-point Hamiltonian is reached, its spin-spin correlation function must have such a singularity. But if Eq. (2.17) is used at the fixed point, where $\Gamma' = \Gamma$, then it fixes the value of b:

$$b^2 = 2^{-(d+2-\eta)}. \tag{2.18}$$

The critical exponent η vanishes for the mean-field theory, in agreement with our choice for b^* in Eq. (2.11).

Equation (2.17) provides an example of a more general characterization of a linear renormalization-group transformation: A linear renormalization-group transformation, because of its form, relates the spin-spin correlation function of the transformed Hamiltonian to that of the original Hamiltonian.

We next investigate the stability of a fixed point $\mathcal{H}^*$ in the manner discussed in the Introduction. As in Eq. (1.3), for small perturbations of the fixed point, we expect to see behavior like

$$T^k_{a,b^*} e^{\mathcal{H}^*+\epsilon\mathfrak{O}} = \exp[\mathcal{H}^* + \epsilon(\Lambda_1^k \mathfrak{R}_1^* + \Lambda_2^k \mathfrak{R}_2^* + \cdots)], \tag{2.19}$$

to first order in ϵ. For perturbations quadratic in the spin, it is simplest to use the results of Eq. (2.9), where the effect of the iterated transformation is determined for any initial Hamiltonian quadratic in the spin. For $\rho = \rho^* + \delta\rho$, assuming $\delta\rho$ to have the form

$$\delta\rho(q) = \epsilon(q^2)^m, \tag{2.20}$$

m any non-negative integer, one finds

$$\mathcal{H}^{(k)} \to \mathcal{H}^* - 4^{(1-m)k}\epsilon\mathfrak{R}_m^{(2)*}, \tag{2.21}$$

with

$$\mathfrak{R}_m^{(2)*} = \tfrac{1}{2}\int_{\vec{q}} [\rho^*(q)/z]^2 (q^2)^{m-2} \sigma_{\vec{q}}\sigma_{-\vec{q}}. \tag{2.22}$$

In Eq. (2.21) we have omitted terms that fall off faster with k than the one given.

As expected, there is one relevant eigenoperator $\mathfrak{R}_0^{(2)*}$ of the transformation, with eigenvalue $\Lambda_0 = 4$; when the temperature of the original system is moved slightly above the critical temperature, the Hamiltonian develops a term that is increased in size with each iteration. There is also a marginal eigenoperator $\mathfrak{R}_1^{(2)*}$ with eigenvalue $\Lambda_1 = 1$. The marginal eigenoperator originates from perturbations in a direction along the line of fixed points of the transformation (that is, for small changes in z). Since any point on the line is fixed under the transformation, a perturbation from one point to another is left unchanged by the transformation, and therefore has eigenvalue 1. Movement in the direction of the marginal operator accompanies a uniform shift in the normalization of the spin variables of the original Hamiltonian, which may also be parametrized by z. Since this is merely a change in the integration variables σ when computing the partition function, it has no physical significance.

III. NONLINEAR RENORMALIZATION-GROUP TRANSFORMATION

We have shown, in the derivation of Eq. (2.17), that the form of the transformation in Eq. (2.4) implies a relation between the spin-spin correlation functions of the original Hamiltonian $\mathcal{H}$ and the transformed Hamiltonian $\mathcal{H}'$, and that this relation fixes the parameter b. We shall now alter the transformation so that such a relation between spin-spin correlation functions is no longer a consequence of the form of the transformation. This may be accomplished by introducing into the transformation kernel of Eq. (2.4) a term cubic in the old spin variables:

$$T_{a,b,c}[S]\exp\{\mathcal{H}[\sigma]\} = \int_\sigma \exp\left[-\tfrac{1}{2}a\int_{\vec{q}}\left|S_{\vec{q}} - b\sigma_{\vec{q}/2} - c\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\sigma_{\vec{q}_1/2}\sigma_{\vec{q}_2/2}\sigma_{\vec{q}_3/2}\delta\left(\vec{q} - \sum_{i=1}^{3}\vec{q}_i\right)\right|^2\right] \times \exp\{\mathcal{H}[\sigma]\}. \tag{3.1}$$

If one now returns to the derivation of the relation between correlation functions given for Eq. (2.17), and proceeds analogously with the new transformation $T_{a,b,c}$, one finds that the terms in the transformation proportional to c lead to four- and six-spin correlation functions in the right-hand side of Eq. (2.17), in addition to the spin-spin correlation function. The equation no longer provides a condition on the transformation parameters.

One would now like to know the fixed points of the new transformation, and the eigenvalues associated with perturbations about the fixed point, but we are unable to carry out analytically the integrals required in Eq. (3.1). We therefore resort to perturbation in the parameter c. It will prove convenient to include a perturbation in the parameter b as well. We write

$$T_{a,b^*+\delta b,\delta c} = T_{a,b^*} + \delta T, \tag{3.2}$$

with, to first order in δb and δc, from Eq. (3.1),

$$\delta T = \int_\sigma\left[a\delta b\int_{\vec{q}}(S_{\vec{q}} - b^*\sigma_{\vec{q}/2})\sigma_{-\vec{q}/2} + a\delta c\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}(S_{\vec{q}} - b^*\sigma_{\vec{q}/2})\sigma_{\vec{q}_1/2}\sigma_{\vec{q}_2/2}\sigma_{\vec{q}_3/2}\,\delta\left(\sum\vec{q}\right)\right] \times \exp\left(-\tfrac{1}{2}a\int_{\vec{q}\,'}|S_{\vec{q}\,'} - b^*\sigma_{\vec{q}\,'/2}|^2\right), \tag{3.3}$$

where we have used the abbreviation $\sum \vec{q} = \vec{q} + \vec{q}_1 + \vec{q}_2 + \vec{q}_3$, and where the integration $\int_\sigma$ is to be performed only after multiplication by the functional on which δT operates.

Just as in Sec. II, to find the position of the new fixed point, we iterate the transformation. The iterated form of the transformation, to first order in δb and δc, is

$$T^k_{a,\,b^*+\delta b,\,\delta c} \approx T^k_{a,\,b^*} + \sum_{l=1}^{k} T^{k-l}_{a,\,b^*}\,\delta T\, T^{l-1}_{a,\,b^*}\,. \tag{3.4}$$

Let us represent by $\mathcal{H}^*$ a fixed point of the old transformation $T_{a,\,b^*}$, and by $\mathcal{H}^* + \delta\mathcal{H}^*$ (if it exists) the fixed point of the altered transformation $T_{a,\,b^*+\delta b,\,\delta c}$. Since $\mathcal{H}^*$ represents a system at a critical point, we require

$$T^k_{a,\,b^*+\delta b,\,\delta c}\, e^{\mathcal{H}^*} \to e^{\mathcal{H}^*+\delta\mathcal{H}^*}, \tag{3.5}$$

in the limit $k \to \infty$, and expanding both sides of Eq. (3.5) to first order in δb and δc, we have

$$\sum_{l=1}^{k} T^{k-l}_{a,\,b^*}\,\delta T\, T^{l-1}_{a,\,b^*} \exp\{\mathcal{H}^*\} - \delta\mathcal{H}^* \exp\{\mathcal{H}^*\}. \tag{3.6}$$

We shall study the effect of the new transformation on the Gaussian fixed points, given in Eqs. (2.12) and (2.13). The calculation of the left-hand side of Eq. (3.6) is in this case straightforward but long, and a description of it is contained in the Appendix. The calculation is carried out there with the fixed-point Hamiltonian $\mathcal{H}^*$ on the left-hand side of Eq. (3.6) replaced by an arbitrary Hamiltonian $\mathcal{H}$ quadratic in the spin, like the one in Eq. (2.8). Of the terms obtained in the Appendix, we shall use here only those that, in general, diverge in the limit $k \to \infty$, when $\rho(\vec{q})$ is set equal to the fixed-point value $\rho^*(q)$ given in Eq. (2.13). Those terms are, for $d > 2$,

$$\sum_{l=1}^{k} T^{k-l}_{a,\,b^*}\,\delta T\, T^{l-1}_{a,\,b^*} \exp\{\mathcal{H}\} = \left[k\,\delta b/b^* + 3\frac{\delta c}{b^{*3}} \sum_{l=1}^{k} \int_{1/L_{k-l}}^{1} \left(\frac{1}{L_l^2 \rho(\vec{q}\,'/L_l)} + \frac{1}{4a_{l-1}} \right) d^d q' \right] \times \int_{\vec{q}} S_{\vec{q}} S_{-\vec{q}} \frac{L_k^2 \rho(\vec{q}/L_k)}{[1 + L_k^2 \rho(\vec{q}/L_k)/a_k]^2} \exp\{\mathcal{H}^{(k)}[S]\}$$
$$+ \text{ other terms finite in the limit } k \to \infty. \tag{3.7}$$

The limits on the integral over $\vec{q}\,'$ are the limits on the magnitude $q' = |\vec{q}\,'|$. For $\rho = \rho^*$ in Eq. (3.7), the terms enclosed in large square brackets diverge in the limit $k \to \infty$ unless

$$\frac{\delta b}{b^*} = -K_d \frac{\delta c}{a b^{*3}} \left(\frac{3}{d-2}\, \frac{a}{z} + \frac{1}{d} \right), \tag{3.8}$$

where K_d is proportional to the solid angle for a d-dimensional space,

$$K_d = (2\pi)^{-d}\, 2\pi^{d/2}/\Gamma(d/2). \tag{3.9}$$

Thus, for given values of δb and δc, we are able to locate a fixed point of the transformation near the old line of fixed points if we can choose $z > 0$ satisfying Eq. (3.8). But there is no longer a line of fixed points for $\delta c \neq 0$. The situation is illustrated graphically in Fig. 2, where the Hamiltonian space, though infinite in dimensions, is represented by only two axes, labeled by the coefficients Z and W in the expansion of $u_2(\vec{q})$ in expression (3.1) for a given Hamiltonian,

$$u_2(\vec{q}) = \text{const} + Zq^2 + Wq^4 + \cdots.$$

The old line of fixed points parametrized by z is indicated by the line in Fig. 2. By choosing z to satisfy Eq. (3.8), we select one of the fixed points of the old transformation, labeled A in the figure, near the new fixed point, labeled B in the figure. The position of the new fixed point with respect to the old one is found by our perturbation procedure.

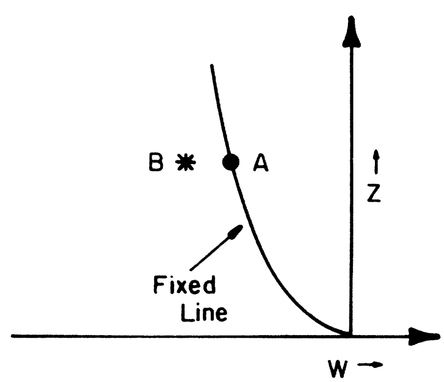

FIG. 2. Projection of the Hamiltonian space defined in Eq. (3.1) onto the W, Z axes in the expansion $u_2(\vec{q}) = Zq^2 + Wq^4 + \cdots$, for the quadratic-spin component of a given Hamiltonian. The curve represents the line of fixed points of the linear transformation of Sec. II. Point B is the projection of the fixed point of the nonlinear transformation, and point A the nearby point on the fixed line singled out by Eq. (3.8), for given δb and δc.

Note, however, that the new fixed point is no longer a simple quadratic function of the spin variables. To this order in δc, for example, $\delta\mathcal{H}^*$ contains quartic polynomials in the spin (see Appendix).

The requirement that Eq. (3.8) have a solution $z>0$ restricts the values of δb and δc for which the transformation $T_{a,\,b^*+\delta b,\delta c}$ can have a fixed point. To this order in perturbation theory, one finds that the transformation will only have fixed points for

$$\delta b < -K_d\,\delta c/(dab^*),\quad \delta c>0;$$
$$\delta b > -K_d\,\delta c/(dab^*),\quad \delta c<0. \tag{3.10}$$

These restrictions were illustrated in Fig. 1.

Given values of δb and δc for which a fixed point exists, we now wish to investigate its stability. We shall adapt the procedure used in Sec. II to perturbation in δb and δc. If $\mathcal{H}^*+\delta\mathcal{H}^*$ is a fixed point of the transformation $T_{a,\,b^*+\delta b,\delta c}$, and we add to the fixed point a perturbation $\epsilon\mathcal{O}$, we expect, as in Eq. (2.19), to see

$$T^k_{a,\,b^*+\delta b,\delta c}\,e^{\mathcal{H}^*+\delta\mathcal{H}^*+\epsilon\mathcal{O}} = \exp\{\mathcal{H}^*+\delta\mathcal{H}^*+\epsilon[(\Lambda_1+\delta\Lambda_1)^k(\mathcal{R}_1^*+\delta\mathcal{R}_1^*)+\cdots]\}$$
$$\approx[1+\delta\mathcal{H}^*+\epsilon(\Lambda_1^k+k\Lambda_1^{k-1}\delta\Lambda_1)\mathcal{R}_1^*+\epsilon\Lambda_1^k\,\delta\mathcal{R}_1^*]\exp\{\mathcal{H}^*\}. \tag{3.11}$$

By comparing the results of the perturbation calculation of the left-hand side of Eq. (3.11) with the right-hand side, we can identify the change $\delta\Lambda$ in an eigenvalue Λ.

Since we are only interested here in the largest eigenvalues of the transformation, we may make use of the results contained in Eq. (3.7). Choose

$$\rho(\vec{q})=\rho^*(q)+\epsilon r(q), \tag{3.12}$$

with the intention of using $r(q)=1$ or $r(q)=q^2$, and let the parameter z in $\rho^*(q)$ satisfy Eq. (3.8). For $\epsilon=0$, the iterated transformation then carries the Hamiltonian $\mathcal{H}^*$ to the fixed point $\mathcal{H}^*+\delta\mathcal{H}^*$. For $\epsilon\neq 0$, the perturbation dominates the approach to the fixed point, and one can easily identify the terms corresponding to the largest eigenvalues. Upon substituting the choice (3.12) for $\rho(\vec{q})$ in Eq. (3.7), we find

$$T^k_{a,\,b^*+\delta b,\delta c}\exp\{\mathcal{H}\}\approx\exp\{\mathcal{H}^*\}\Big(1+\delta\mathcal{H}^*-\tfrac{1}{2}\int_{\vec{q}}S_{\vec{q}}S_{-\vec{q}}[\rho^*(q)/(zq^2)]^2L_k^2\,\epsilon r(q/L_k)$$
$$-\Big[\int_{\vec{q}}S_{\vec{q}}S_{-\vec{q}}\rho^{*2}(q)(zq^2)^{-1}\Big]\sum_{l=1}^{k}\frac{3\delta c}{b^{*3}}\int_{1/L_{k-l}}^{1}\frac{\epsilon r(q'/L_l)}{L_l^2\rho^{*2}(q'/L_l)}\,d^dq'\Big), \tag{3.13}$$

where we have neglected terms that fall off faster with k or are uninteresting for our purposes. The terms on the right-hand side of Eq. (3.7) that were omitted there are again of no interest here.

The choice $r(q)=1$ in Eq. (3.13) gives

$$\Big[1+\delta\mathcal{H}^*-4^k\,\epsilon\mathcal{R}_0^{(2)*}-6\delta c(b^*)^{-3}\Big(\sum_{l=1}^{k}L_l^{-2}\int_{1/L_{k-l}}^{1}z[\rho^*(q'/L_l)]^{-2}d^dq'\Big)\epsilon\mathcal{R}_1^{(2)*}\Big]\exp\{\mathcal{H}^*\}, \tag{3.14}$$

and there are no terms behaving like $k4^{k-1}\mathcal{R}_0^{(2)*}$ that would indicate a change in the eigenvalue 4. Thus, the eigenvalue associated with changing the temperature of the physical system is unaffected (to order δc) by the change in the transformation.

There is, however, an eigenvalue change for the choice $r(q)=q^2$. We find, to first order in δc,

$$T^k_{a,\,b^*+\delta b,\delta c}\exp\{\mathcal{H}\}\approx\exp\{\mathcal{H}^*\}\Big[1+\delta\mathcal{H}^*-\Big(1+\frac{6K_d}{d-2}\frac{\delta c}{b^{*3}z}\Big)^k\epsilon(\mathcal{R}_1^{(2)*}+\delta\mathcal{R}_1^{(2)*})\Big]. \tag{3.15}$$

The terms $\delta\mathcal{H}^*$ and $\delta\mathcal{R}_1^{(2)*}$ in Eq. (3.15) have not actually been calculated, but only verified to be finite. The eigenvalue corresponding to the previously marginal operator $\mathcal{R}_1^{(2)*}$ is now

$$\Lambda_1+\delta\Lambda_1=1+\frac{6K_d}{d-2}\frac{\delta c}{b^{*3}z}, \tag{3.16}$$

provided δb satisfies Eq. (3.8).

IV. CONCLUSION

We have seen that an eigenvalue, the one associated with perturbations of the normalization of the spin variables in the original Hamiltonian, is changed when the renormalization-group transformation is rendered nonlinear. Equation (3.16) indicates the direction of this change, and explains the behavior ascribed in the Introduction to critical Hamiltonians under the repeated application of a nonlinear renormalization-group transformation. If δc is negative, and the transformation parameters are in region I of Fig. 1, then this eigenvalue is smaller than 1, and, regardless of the normalization of the spin variables, every critical Hamiltonian is carried by the iterated transformation to the same fixed point, instead of to one among a line of fixed points. In particular, every Hamiltonian on the line of fixed points of the old linear transformation is carried to this new fixed point. The transformation parameters are much less severely constrained than in the linear case. They may be freely varied independently of each other, except near the boundaries of region I, and a fixed point will always be found. If, on the other hand, δc is positive, and the transformation parameters lie in region II of Fig. 1, then the eigenvalue is larger than 1. Only if the normalization of the spins in the original Hamiltonian is properly fixed can a fixed point be reached. Alternatively, as was pointed out in the Introduction, one may adjust the parameter b in the transformation until a fixed point is found, just as one had to adjust b to this end in the linear transformation. However, in contrast with the linear transformation, if the normalization of the spin variables in the original Hamiltonian is changed, the nonlinear transformation ($c>0$) will no longer reach a useful fixed point, unless b is also appropriately changed.

The nonlinear transformation would appear to be most useful when its parameters lie in region I of Fig. 1, since it is then possible to locate a fixed point without having to adjust the parameters of the transformation with each iteration, as is necessary with a linear transformation. However, nonlinear renormalization-group transformations are rather awkward to use in the continuous-spin case, and are more likely to be profitably employed in studies of systems with discrete variables like the Ising model. The calculation presented here serves as a model for what may be looked for in such applications.

APPENDIX: DETAILS OF CALCULATION

In order to determine the effect of the right-hand side of Eq. (3.4) on a Hamiltonian quadratic in the spin variables, we must evaluate the quantity

$$T^{k-l}_{a,b^*}\,\delta T\,T^{l-1}_{a,b^*}\exp\left[-\tfrac{1}{2}\int_{\vec{q}}\rho(\vec{q})\sigma_{\vec{q}}\sigma_{-\vec{q}}\right], \tag{A1}$$

where T^{k}_{a,b^*} is defined in Eqs. (2.5)–(2.7), δT is given in Eq. (3.3), and b^* is given in Eq. (2.11). The calculation of (A1) is facilitated by introducing appropriate generating functionals. Denote by $\delta\mathcal{T}$ the polynomial in new and old spins occurring in Eq. (3.3), defined so that

$$\delta T=\int_\sigma \delta\mathcal{T}[S_{\vec{q}}-b^*\sigma_{\vec{q}/2},\sigma_{\vec{q}\,'}]\exp\left(-\tfrac{1}{2}a\int_{\vec{q}\,''}|S_{\vec{q}\,''}-b^*\sigma_{\vec{q}\,''/2}|^2\right). \tag{A2}$$

Define the generating functional $\mathcal{F}[J,j]$ to be

$$\begin{aligned}\mathcal{F}[J,j]=\int_{S^{(l)}}\int_{S^{(l-1)}}\exp\Big(&-\tfrac{1}{2}a'\int_{\vec{q}}|S^{(k)}_{\vec{q}}-b'S^{(l)}_{\vec{q}/L'}|^2\\&+\int_{\vec{q}}J_{-\vec{q}}(S^{(l)}_{\vec{q}}-b^*S^{(l-1)}_{\vec{q}/2})-\tfrac{1}{2}a\int_{\vec{q}}|S^{(l)}_{\vec{q}}-b^*S^{(l-1)}_{\vec{q}/2}|^2\\&+\int_0^{1/2}j_{-\vec{q}}S^{(l-1)}_{\vec{q}}d^dq-\tfrac{1}{2}a_{l-1}\int_{\vec{q}}|S^{(l-1)}_{\vec{q}}-b_{l-1}\sigma_{\vec{q}/L_{l-1}}|^2\Big),\end{aligned} \tag{A3}$$

with

$$a'\equiv a_{k-l},$$

$$b'\equiv b_{k-l}=(b^*)^{k-l},$$

$$L'\equiv L_{k-l},$$

where a_k, b_k, L_k are defined in Eqs. (2.7), and where $S^{(k)}$ is the block-spin variable introduced by the kth transformation. In terms of this functional, the transformation (A1) can be written

$$\int_\sigma \delta\mathcal{T}[\delta/\delta J_{-\vec{q}},\delta/\delta j_{-\vec{q}/2}]\,\mathcal{F}[J,j]\big|_{J,j=0}\exp\{\mathcal{H}[\sigma]\}. \tag{A4}$$

Upon evaluating the integrals in Eq. (A3) and simplifying terms by using the identity

$$a_{m+n}=a_ma_n/(a_m+L_n^db_n^2a_n), \tag{A5}$$

one obtains

$$\ln\mathfrak{F}[J,j]=-\tfrac{1}{2}a_k\int_{\vec{q}}\mathcal{S}_{\vec{q}}\mathcal{S}_{-\vec{q}}+b_{l-1}\int_0^{1/2}j_{-\vec{q}}\sigma_{\vec{q}/L_{l-1}}+(L'^{d}b^*b'a_k/a_{l-1})\int_0^{1/L'}j_{-\vec{q}/2}\mathcal{S}_{L'\vec{q}}+L'^{d}b'(a_k/a)\int_0^{1/L'}J_{-\vec{q}}\mathcal{S}_{L'\vec{q}}$$
$$+\tfrac{1}{2}(1/a_{l-1})\int_{1/(2L')}^{1/2}j_{-\vec{q}}j_{\vec{q}}+\tfrac{1}{2}a_k/(a_{l-1}a_{k-l+1})\int_0^{1/(2L')}j_{-\vec{q}}j_{\vec{q}}-b^*b'^2L'^{d}a_k/(aa_{l-1})\int_0^{1/L'}j_{-\vec{q}/2}J_{\vec{q}}+\tfrac{1}{2}(1/a)$$
$$\times\int_{1/L'}^{1}J_{-\vec{q}}J_{\vec{q}}+\tfrac{1}{2}(1/a)(1-L'^{d}b'^2a_k/a)\int_0^{1/L'}J_{-\vec{q}}J_{\vec{q}},\qquad\text{(A6)}$$

with

$$\mathcal{S}_{\vec{q}}\equiv S_{\vec{q}}-b_k\sigma_{\vec{q}/L_k},\qquad\text{(A7a)}$$
$$S_{\vec{q}}\equiv S^{(k)}_{\vec{q}}\qquad\text{(A7b)}$$

and where we have omitted all of the phase-space factors $đ^dq$ from all of the integrals. After using the results of Eq. (A6) in Eq. (A4), there still remains to be performed in the calculation of (A1) the integral over the original spins $\sigma_{\vec{q}}$. The generating functional

$$\mathcal{K}[j]=\int_\sigma\exp\left(-\tfrac{1}{2}a_k\int_{\vec{q}}|S_{\vec{q}}-b_k\sigma_{\vec{q}/L_k}|^2-\tfrac{1}{2}\int_{\vec{q}}\rho(\vec{q})\sigma_{\vec{q}}\sigma_{-\vec{q}}+\int_{\vec{q}}j_{-\vec{q}}\sigma_{\vec{q}}\right)\qquad\text{(A8)}$$

is useful for that purpose. One finds

$$\ln\mathcal{K}[j]=-\tfrac{1}{2}\int_{\vec{q}}R_k(\vec{q})S_{\vec{q}}S_{-\vec{q}}+\int_0^{1/L_k}\frac{b_k^{-1}j_{-\vec{q}}S_{L_k\vec{q}}}{1+L_k^2\rho(\vec{q}/L_k)/a_k}+\tfrac{1}{2}\int_0^{1/L_k}\frac{j_{-\vec{q}}j_{\vec{q}}}{L_k^db_k^2a_k+\rho(\vec{q})}+\tfrac{1}{2}\int_{1/L_k}^{1}\frac{j_{-\vec{q}}j_{\vec{q}}}{\rho(\vec{q})},\qquad\text{(A9)}$$

with

$$R_k(\vec{q})\equiv L_k^2\rho(\vec{q}/L_k)/[1+L_k^2\rho(\vec{q}/L_k)/a_k].\qquad\text{(A10)}$$

As an example of its use, let us calculate the contribution to (A1) from the part of δT proportional to δb. First one finds, from Eqs. (A4) and (A6),

$$a\delta b\int_{\vec{q}}\frac{\delta}{\delta J_{-\vec{q}}}\frac{\delta}{\delta j_{\vec{q}/2}}\mathfrak{F}\bigg|_{J,j=0}=\delta bb'a_k\int_{\vec{q}}\mathcal{S}_{\vec{q}}[2^db^*L'^{d}b'(a_k/a_{l-1})\mathcal{S}_{\vec{q}}+b_{l-1}\sigma_{\vec{q}/L_k}]\exp\left(-\tfrac{1}{2}a_k\int_{\vec{q}'}\mathcal{S}_{\vec{q}'}\mathcal{S}_{-\vec{q}'}\right).\qquad\text{(A11)}$$

The spin-independent constants have been left out in Eq. (A11), since they only contribute to the normalization factor for the transformation. One next calculates the effect of these terms on the quadratic-spin Hamiltonian in (A1), using the generating functional from Eq. (A9), and finds, after some algebraic simplification, that the contribution to (A1) proportional to δb is

$$\delta b(b^*)^{-1}\int_{\vec{q}}R_k(\vec{q})A_{k,l}(\vec{q})S_{\vec{q}}S_{-\vec{q}}\exp\{\mathcal{H}^{(k)}[S]\},\qquad\text{(A12)}$$

with

$$A_{k,l}(\vec{q})\equiv D_k(\vec{q})+2^db^{*2}L'^{-2}R_k(\vec{q})/a_{l-1},\qquad\text{(A13)}$$
$$D_k(\vec{q})\equiv[1+L_k^2\rho(\vec{q}/L_k)/a_k]^{-1},\qquad\text{(A14)}$$

and where $\mathcal{H}^{(k)}$ is defined in Eq. (2.9).

A similar, lengthier calculation yields the contributions to (A1) from the term in δT proportional to δc:

$$(b^*)^{-3}\delta c\left(3\left\{\int_{1/L'}^{1}[L_l^2\rho(\vec{q}'/L_l)]^{-1}đ^dq'\right\}\int_{\vec{q}}R_k(\vec{q})A_{k,l}(\vec{q})S_{\vec{q}}S_{-\vec{q}}+L'^{(2-d)}\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}R_k(\vec{q})A_{k,l}(\vec{q}_1)A_{k,l}(\vec{q}_2)\right.$$
$$\times A_{k,l}(\vec{q}_3)S_{\vec{q}}S_{\vec{q}_1}S_{\vec{q}_2}S_{\vec{q}_3}\delta\left(\sum\vec{q}\right)+3L'^{(2-d)}\frac{a_k}{a_{k-l+1}}\left[\int_{\vec{q}'}D_k(\vec{q}')\right]\int_{\vec{q}}\left[\left(\frac{2}{a_{k-l+1}}-\frac{1}{a_k}\right)R_k(\vec{q})-D_k(\vec{q})\right]$$
$$\times A_{k,l}(\vec{q})S_{\vec{q}}S_{-\vec{q}}-\tfrac{3}{4}(a_k/a_{l-1})L'^{-d}\left(\int_{\vec{q}'}1\right)\int_{\vec{q}}[A_{k,l}(\vec{q})]^2S_{\vec{q}}S_{-\vec{q}}+\tfrac{3}{4}(1/a_{l-1})$$
$$\left.\times\left[(a_k/a_{k-l+1})\int_0^{1/L'}đ^dq'+\int_{1/L'}^{1}đ^dq'\right]\int_{\vec{q}}R_k(\vec{q})A_{k,l}(\vec{q})S_{\vec{q}}S_{-\vec{q}}\right)\exp\{\mathcal{H}^{(k)}[S]\}.\qquad\text{(A15)}$$

Since a_k and the functions $R_k(\vec{q})$, $D_k(\vec{q})$, and $A_{k,l}(\vec{q})$ have finite limits for $k\to\infty$ and $\rho=\rho^*$, the only terms in (A1) that can diverge, when the summation over l in Eq. (3.6) is carried out, are the contributions proportional to δb in (A12) and the first and last terms proportional to δc in (A15):

$$\left\{\frac{\delta b}{b^*}+\frac{\delta c}{b^{*3}}\left[3\int_{1/L'}^{1}\frac{1}{L_l^2\rho(\vec{q}\,'/L_l)}\,đ^d q'+\frac{\frac{3}{4}}{a_{l-1}}\int_{1/L'}^{1}đ^d q'\right]\right\}\left[\int_{\vec{q}}R_k(\vec{q})A_{k,l}(\vec{q})S_{\vec{q}}S_{-\vec{q}}\right]\exp\{\mathcal{H}^{(k)}[S]\}. \tag{A16}$$

The other terms are finite when summed over l, for $d>2$.

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[1] A thorough discussion of the renormalization group is contained in K. G. Wilson and J. Kogut, Phys. Rep. (to be published).

[2] Th. Niemeijer and J. M. J. van Leeuwen, Phys. Rev. Lett. 31, 1411 (1973); Physica 71, 17 (1974).

[3] Note that the word "nonlinear" as we are using it is *not* intended to describe the action of the transformation on the Hamiltonian space, which is always nonlinear in the sense of $\mathcal{T}\{\alpha\mathcal{H}_1+\beta\mathcal{H}_2\}\neq\alpha\mathcal{T}\{\mathcal{H}_1\}+\beta\mathcal{T}\{\mathcal{H}_2\}$.

[4] This fixed point cannot be associated with a system at a tricritical point, because of the unphysical nature of the extra parameter.

[5] F. J. Wegner, Institüt für Festkörperforschung, report of work prior to publication.

[6] See also some general remarks about the possibilities for nonlinear renormalization groups by G. Jona-Lasinio, lecture delivered at Nobel Symposium 24, 1973.

[7] One may write $\int_\sigma=\prod_{\vec{q}}\int_{-\infty}^{+\infty}d(\mathrm{Re}\,\sigma_{\vec{q}})\int_{-\infty}^{+\infty}d(\mathrm{Im}\,\sigma_{\vec{q}})$, where the product over the possible values of $\vec{q}$ is constrained so that the same integration variable does not appear twice, because of the relation $\sigma_{\vec{q}}=\sigma^*_{-\vec{q}}$.