Theory of layered Ising models. II. Spin correlation functions parallel to the layering

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We study the spin correlation $\langle\sigma_{00}\sigma_{0N}\rangle$, which is expressable as a block Toeplitz determinant, of a nth-order layered Ising model, in the direction parallel to the layering. The generating function of this block Toeplitz determinant can be written as a scalar function times a 2×2 matrix whose elements are nth-order trigonometric polynomials. We show that it is possible to calculate asymptotically for large N the block Toeplitz determinant generated by such a matrix function. As an example we compute, for large N, $\langle\sigma_{00}\sigma_{0N}\rangle$ for the second-order layered Ising model whose horizontal bonds are all equal and whose vertical bonds between the jth row and $(j+1)$th row is E_2, if j even and E_2', if j odd.

I. INTRODUCTION

In this paper, we study $\langle\sigma_{M0}\sigma_{MN}\rangle$, the spin-correlation function of two spins on the same row, for the nth-ordered layered Ising model.[1] As defined in Paper I,[1] an nth-ordered layered Ising lattice is a rectangular Ising lattice whose horizontal bonds are all equal to E_1, and whose vertical bonds between the jth row and $(j+1)$th row are $E_2(j)$, such that $E_2(j+n)=E_2(j)$ (see Fig. 1). Since the vertical bonds can take n different values and are different from row to row, we are calculating the spin correlation function in the direction parallel to the layering.

The purpose of our work is to study how the system tends to behave when there are some inhomogeneities present. Moreover, by studying this problem, one can gain insight of how to handle the spin correlation function of a random lattice (whose verticle bonds are random variables).

It is known[2] that the spin correlation $\langle\sigma_{00}\sigma_{0N}\rangle$ is a Toeplitz determinant for the Ising model with $n=1$. A Toeplitz determinant is a determinant of the following form:

$$D_N[c]=\begin{vmatrix} c_0 & c_{-1} & \cdot & \cdot & \cdot & c_{-N+1} \\ c_1 & c_0 & c_{-1} & \cdot & \cdot & \cdot \\ \cdot & c_1 & c_0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_0 & c_{-1} \\ c_{N-1} & \cdot & \cdot & \cdot & c_1 & c_0 \end{vmatrix},$$

$$1\le i,\ j<N, \tag{1.1}$$

where c_l are $2N-1$ constants. If the c_l's in Eq. (1.1) are replaced by $2N-1$ constant matrices a_l, the determinant of the resultant matrix is called a block Toeplitz determinant, and is denoted by $D_N[a]$. The spin-correlation function of two spins on the same row (parallel to the layering) for a layered Ising model is found[3] to be a block Toeplitz determinant whose entries a_l's are 2×2 matrices.

The generating function

$$c(\xi)=\sum_{l=-\infty}^{\infty} c_l\xi^l, \quad |\xi|=1 \tag{1.2a}$$

of a Toeplitz determinant is said to admit a Wiener-Hopf factorization,[4] if $c(\xi)$ can be written as a product

$$c(\xi)=c_+(\xi)c_-(\xi), \quad |\xi|=1, \tag{1.3}$$

where $c_+(\xi)$ [$c_-(\xi)$] is an invertible analytic function inside [outside] the unit circle. When $c(\xi)$ satisfy the above condition, the Toeplitz determinant (1.1) can be evaluated by Szegö's theorem for $N\to\infty$.[5] This gives the famous result of Onsager and Yang[6] that the spontaneous magnetization of the Ising model behaves as $M\sim(1-T/T_c)^{1/8}$ for $T\le T_c$. Furthermore, the asymptotic expansion of the Toeplitz determinant for large N is also known.[7] Therefore, the asymptotic behavior of the spin-correlation function can be evaluated.[7]

Analogously,

$$a(\xi)=\sum_{l=-\infty}^{\infty} a_l\xi^l, \quad |\xi|=1 \tag{1.2b}$$

is called the generating function of the block Toeplitz determinant. However, no theorem, analogous to Szegö's theorem, exists (except for a very special case discussed in Sec. II), and it is not possible to calculate the block Toeplitz determinant in general.

However, we shall show in Sec. III, that the generating function of the correlation function $\langle\sigma_{00}\sigma_{0N}\rangle^2$, for the nth-order layered Ising model, is a scalar function $1/\chi(e^{i\theta})$ times a 2×2 matrix $b(e^{i\theta})$, whose elements are nth-order trigonometric polynomials, i.e.,

$$a(e^{i\theta})=b(e^{i\theta})/\chi(e^{i\theta}), \tag{1.4a}$$

where

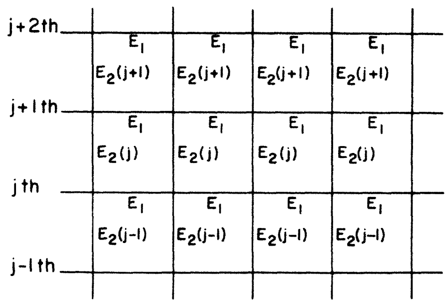

FIG. 1. nth-ordered layered Ising model, whose horizontal bonds are all equal to E_1, and whose vertical bonds, $E_2(j)$, between the jth and $(j+1)$th rows are periodical function of n, i.e., $E_2(j+n)=E_2(j)$.

$$b(e^{i\theta})=\sum_{l=-n}^{n} b_l e^{il\theta}. \tag{1.4b}$$

A block Toeplitz determinant that has this particular form (1.4) has been studied previously by McCoy and Wu,[8] when they studied the correlation of the Ising model in the presence of an imaginary magnetic field $H=\frac{1}{2}i\pi kT$. We shall generalize their method to calculate the spontaneous magnetization and the correlation function $\langle\sigma_{00}\sigma_{0N}\rangle$ for large N.

We include in Sec. II some known theorems on block Toeplitz determinants. In Sec. IV, we shall show how to make use of these theorems to evaluate the spontaneous magnetization for the layered Ising model, and also $\langle\sigma_{00}\sigma_{0N}\rangle$ for large separations at $T<T_c$. Although it is possible, in principle, to calculate these quantities for any n, the algebra becomes increasingly tedious as n increases. We shall therefore restrict ourselves to the simplest nontrivial case of the layered Ising model, when the verticle bonds can take only two different values: $E_2(2K)=E_2$, $E_2(2K+1)=E_2'$. In Sec. V, we shall calculate the spontaneous magnetization for this case. We find that the spontaneous magnetization M, as $T\to T_c$, approaches zero in the form $M\sim A(E_1, E_2, E_2')(1-T/T_c)^{1/8}$, where $A(E_1,E_2,E_2')$ is given by Eq. (5.52). Here the critical temperature is determined by

$$(1-z_{1c})^2/(1+z_{1c})^2=z_{2c}z_{2c}', \tag{1.5}$$

with

$$z_{1c}=\tanh\beta_c E_1, \tag{1.6}$$

$$z_{2c}=\tanh\beta_c E_2,$$

$$z_{2c}'=\tanh\beta_c E_2',$$

and

$$\beta_c=1/kT_c. \tag{1.7}$$

From Eq. (1.5), one sees that we can fix T_c by fixing $z_{2c}z_{2c}'$. We find that $A(E_1, E_2, E_2')$, for fixed T_c, is minimum when $E_1=E_2=E_2'$ (see Fig. 2). In Sec. VI, we calculate the spin-correlation function $\langle\sigma_{00}\sigma_{0N}\rangle$ for large N at $T<T_c$, we find the correlation length $\xi(T)$ is given by

$$1/\xi(T)=|\ln\gamma_1|, \tag{1.8}$$

where

$$\gamma_1=\{(1+z_1^2)(1-z_2z_2')-[(1+z_1^2)^2(1-z_2z_2')^2 - 4z_1^2(1-z_2^2)(1-z_2'^2)]^{1/2}\}/[2z_1(1-z_2)(1-z_2')], \tag{1.9}$$

and

$$\langle\sigma_{00}\sigma_{0N}\rangle\sim M^2\left(1+\frac{(\gamma_1-\gamma_1^{-1})^{-2}\gamma_1^{-2N}}{2\pi N^2}\right) \tag{1.10}$$

for large N and $T<T_c$.

When $T>T_c$, the generating function does not possess a factorization of the form analogous to Eq. (1.3). However, we are still able to find in Sec. VII [Eq. (7.51)] that

$$\langle\sigma_{00}\sigma_{0N}\rangle\sim f(z_2, z_2', z_1)\gamma_1^N/(\pi N)^{1/2}. \tag{1.11}$$

II. KNOWN THEOREMS ON BLOCK TOEPLITZ DETERMINANT

Before stating these theorems, we shall repeat some of the definitions. We define a block Toeplitz determinant as a determinant of the form

$$D_N[a]=|a_{i-j}|, \quad 1\le i, j\le N, \tag{2.1}$$

where a_l are $r\times r$ matrices. The matrix function

$$a(\xi)=\sum_{l=-\infty}^{\infty} a_l\xi^l, \quad |\xi|=1 \tag{2.2}$$

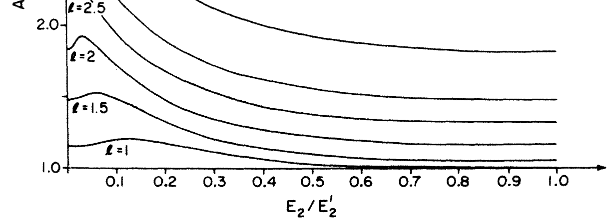

FIG. 2. Plot of the coefficient of the spontaneous magnetization $[A(E_1, E_2, E_2')/A(E_1, E_1, E_1)]$ (Ref. 8) as a function of E_2/E_2' for fixed T_c, and for $\exp(-4\beta_c E_1) = z_{2c}z_{2c}'=(\sqrt{2}-1)^{2l}$.

is called the generating function of the block Toeplitz determinant.

It was shown[9] by Gohberg and Krein that a $r\times r$ matrix $a(e^{i\theta})$, whose elements have convergent Fourier expansions $[a(\xi)\in R_{r\times r}]$, and whose determinant does not vanish for any real θ, admit a *right standard factorization* of the form

$$a(\xi)=a_-(\xi)\begin{pmatrix}\xi^{\nu_1} & 0 & \cdots & 0\\ 0 & \xi^{\nu_2} & \cdots & 0\\ \vdots & & \ddots & \\ 0 & 0 & & \xi^{\nu_r}\end{pmatrix}a_+(\xi), \qquad (2.3)$$

where $a_+(\xi)$ $[a_-(\xi)]$ is a $r\times r$ invertible analytic matrix function for $|\xi|<1$ $[|\xi|>1]$, and ν_i $(i=1, \ldots, r)$ are uniquely determined integers (except for ordering) called the "right exponents."

In the scalar case ($r=1$), this reduces to the known theorems of Wiener and Wiener-Levy.[4] When all the right exponents are identically zero, the matrix $a(\xi)$ is said to admit a *right canonical factorization*, or simply a *right factorization*,

$$a(\xi)=a_-(\xi)a_+(\xi). \qquad (2.4)$$

On the other hand, if any of these integers are not zero, i.e., $\nu_i\neq 0$ for some i, the matrix $a(\xi)$ does not admit a right factorization.

Similarly, $a(\xi)$ is shown[9] to possess a *left standard factorization*

$$a(\xi)=\tilde a_+(\xi)\begin{pmatrix}\xi^{\nu_1'} & 0 & \cdots & 0\\ 0 & \xi^{\nu_2'} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \xi^{\nu_2'}\end{pmatrix}\tilde a_-(\xi), \qquad (2.5)$$

where ν_1' $(i=1, \cdots, r)$ are called the left exponents. When all the left exponents vanish, the matrix $a(\xi)$ is said to admit a *left factorization*

$$a(\xi)=\tilde a_+(\xi)\tilde a_-(\xi). \qquad (2.6)$$

Although every nonsingular matrix that belongs to $R_{r\times r}$ admits *standard factorizations*, there is no method known to determine the left or right exponents, nor the left and right factors $a_\pm(\xi)$, except for some special cases.

Assuming the matrix $a(\xi)$ admits factorizations, we have the following theorems on the block Toeplitz determinant generated by $a(\xi)$.

Theorem I. If $a(\xi)$ admits a right factorization, then

$$\lim_{N\to\infty} D_{N+1}[a]/D_N[a]=\mu =\exp\left\{\frac{1}{2\pi}\int_0^{2\pi} d\theta\,\ln[\det a(e^{i\theta})]\right\}. \qquad (2.7)$$

(Hirschman[10]).

Theorem II. Assume that $a(\xi)$ admits a right factorization and also a left factorization, then the limit

$$E[a]=\lim_{N\to\infty} D_N[a]/\mu^N \qquad (2.8)$$

exists and is nonzero (Widom[11]).

In the case of a scalar ($r=1$), $E[a]$ is given by the formula

$$E[a]=\exp\left(\sum_{n=1}^{\infty} n(k_n)(k_{-n})\right), \qquad (2.9)$$

where

$$k_n=\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-in\theta}\ln a(e^{i\theta}).$$

This is known as Szegö's theorem. When the generating function is a matrix, no analogous theorem exists except for the following very special case.

Theorem III. If $a(\xi)$ is a $r\times r$ matrix that can be written

$$a(\xi)=f(\xi)a_+(\xi)$$

or (2.10)

$$a(\xi)=f(\xi)a_-(\xi),$$

where $a_+(\xi)$ $[a_-(\xi)]$ is a invertible analytic matrix function of ξ for $|\xi|<1$ $[|\xi|>1]$, and $f(\xi)$ is a scalar function which satisfies

(i) $f(\xi)\neq 0$ for $|\xi|=1$,

(ii) $\operatorname{ind} f=\frac{1}{2\pi}\int_0^{2\pi}\arg f(e^{i\theta})\,d\theta=0$;

then

$$E[a]=\exp\left(\frac{1}{r}\sum_{l=1}^{\infty} l(k_l)(k_{-l})\right),$$

where

$$k_l=\frac{1}{2\pi}\int_0^{2\pi} e^{il\theta}\ln[\det a(e^{i\theta})]\,d\theta \qquad (2.11)$$

(Widom[11]).

Theorem IV. Assume the matrix $a(\xi)$ admits a right factorization of (2.4) and a left factorization of (2.5), then the asymptotic behavior for $D_N[a]$ for large N is

$$D_N[a]\sim\mu^N E[a](1-\mathrm{Tr}\Omega), \qquad (2.12a)$$

where Ω is a $r\times r$ matrix given by

$$\Omega=\frac{1}{(2\pi i)^2}\oint_{|\xi|=1} d\xi\,\xi^N M(\xi^{-1}) \times\oint_{|\eta|=1}\eta^{-N}(\xi-\eta)^{-2}L(\eta) \qquad (2.12b)$$

with

$$M(\xi^{-1}) = [a_-(\xi)]^{-1} a_+(\xi) = L^{-1}(\xi) \quad (2.12c)$$

(McCoy and Wu[8]). Formula (2.12) is derived in Ref. 8 for a 2×2 matrix $a(\xi)$. However, an examination of that proof shows that the same formula holds for a $r\times r$ matrix.

Since there is no such formula (2.9) available to calculate $E[a]$ for an arbitrary matrix $a(\xi)$, we find the following identities and theorem most useful in calculating the limit of $D_N[a]$ as $N\to\infty$ for a 2×2 matrix $a(\xi)$ of the type (1.4).

If we multiply all the matrices $a_{ij} = a_{i-j}$ ($j = 0, N$) on the same block row (or block column) by a constant $r\times r$ matrix whose determinant is 1, the block Toeplitz determinant remain unchanged. Hence, we have the identity

$$D_N[Ka] = D_N[aK] = |K|^N D_N[a] = D_N[a]. \quad (2.13)$$

Since the determinant of a matrix is equal to the determinant of its transpose, we find

$$D_N[a] = D_N[a'], \quad (2.14)$$

where $a'(\xi) = a^T(\xi^{-1})$.

When two adjacent block rows (or columns) of matrices in the block Toeplitz determinant are interchanged, the determinant of the resultant matrix is equal to the original matrix multiplied by a factor $(-1)^{r\times r}$. Using this fact, one can show

$$D_N[a] = D_N[\bar{a}], \quad (2.15)$$

where $\bar{a}(\xi) = a(\xi^{-1})$. Combining Eqs. (2.14) and (2.15), we find,

$$D_N[a] = D_N[a^T]. \quad (2.16)$$

Let us now restrict ourself to the 2×2 matrices. It is implicit, in Ref. 8, the following theorem holds.

Theorem V. If the two 2×2 matrices $g(\xi)$ and $h(\xi)$ are related to each other as

$$g(\xi) = h(\xi)\begin{pmatrix} 1-\alpha\xi^{-1} & 0 \\ 0 & 1-\beta\xi^{-1}\end{pmatrix} \quad (2.17)$$

and $h(\xi)$ admits a right factorization

$$h(\xi) = h_-(\xi)\,h_+(\xi), \quad (2.18)$$

then in the limit $N\to\infty$, the ratio of the block Toeplitz determinant generated by $g(\xi)$ and $h(\xi)$ is

$$\lim_{N\to\infty}\frac{D_N[g]}{D_N[h]} = \begin{vmatrix} h_+(\alpha)_{22} & h_+(\beta)_{12} \\ h_+(\alpha)_{21} & h_+(\beta)_{11}\end{vmatrix}\frac{1}{\det h_+(\alpha)\det h_+(\beta)}. \quad (2.19)$$

III. SPIN-SPIN CORRELATIONS

In this section we demonstrate that the (square of) correlation function $\langle\sigma_{M0}\sigma_{MN}\rangle$ of the nth-order layered Ising model may be written as a block Toeplitz determinant whose generating function is of the form (1.4). This correlation function will in general depend on M, but it is surely no loss of generality to take $M=0$ since by cyclically permuting the n energies $E_2(l)$ all other n distinct correlation functions $\langle\sigma_{M0}\sigma_{MN}\rangle$ can be obtained.

It was previously shown[3] that $\langle\sigma_{00}\sigma_{0N}\rangle^2$ can be written as a block Toeplitz determinant

$$\langle\sigma_{00}\sigma_{0N}\rangle^2 = \begin{vmatrix} (a_0)_{11} & (a_0)_{12} & (a_{-1})_{11} & (a_{-1})_{12} & (a_{-N+1})_{11} & (a_{-N+1})_{12} \\ (a_0)_{21} & (a_0)_{22} & (a_{-1})_{21} & (a_{-1})_{22} & \cdots(a_{-N+1})_{21} & (a_{-N+1})_{22} \\ (a_1)_{11} & (a_1)_{12} & (a_0)_{11} & (a_0)_{12} & (a_{-N+2})_{11} & (a_{-N+2})_{12} \\ (a_1)_{21} & (a_1)_{22} & (a_0)_{21} & (a_0)_{22} & (a_{-N+2})_{21} & (a_{-N+2})_{22} \\ & \cdots & & & \cdots & \\ (a_{N-1})_{11} & (a_{N-1})_{12} & (a_{N-2})_{11} & (a_{N-2})_{12} & \cdots(a_0)_{11} & (a_0)_{12} \\ (a_{N-1})_{21} & (a_{N-1})_{22} & (a_{N-2})_{21} & (a_{N-2})_{22} & (a_0)_{21} & (a_0)_{22}\end{vmatrix}, \quad (3.1)$$

where the 2×2 matrix generating function

$$a(e^{i\theta}) = \sum_{l=-\infty}^{\infty} a_l e^{il\theta} \quad (3.2)$$

is

$$a_{11}(e^{i\theta}) = a_{22}(e^{i\theta}) = \frac{i\mathrm{b}(\bar{x}-x)}{\mathrm{b}^2 + (\bar{x}-\mathrm{a})(x-\mathrm{a})} \quad (3.3a)$$

and

$$a_{21}(e^{i\theta}) = -a_{12}^*(e^{i\theta}) = \frac{e^{i\theta}(1+z_1e^{-i\theta})}{1+z_1e^{i\theta}} \times\left[1 + \frac{i\mathrm{b}(x+\bar{x}-2\mathrm{a}) - 2(x-\mathrm{a})(\bar{x}-\mathrm{a})}{\mathrm{b}^2+(\bar{x}-\mathrm{a})(x-\mathrm{a})}\right] \quad (3.3b)$$

Here we use the notation

$$\mathrm{a} = -2z_1\sin\theta\,|1+z_1e^{i\theta}|^{-2}, \quad (3.4)$$

$$\mathrm{b} = (1-z_1^2)\,|1+z_1e^{i\theta}|^{-2}, \quad (3.5)$$

with

$$z_1 = \tanh\beta E_1, \quad (3.6a)$$

$$z_2(j) = \tanh\beta E_2(j), \quad (3.6b)$$

and $\beta = (kT)^{-1}$. Furthermore, x, which is a function of θ and n, satisfies

$$x = (T_{11}x + T_{12})/(T_{21}x + T_{22}) \quad (3.7a)$$

and $\bar{x}$ satisfies

$$\bar{x} = (\bar{T}_{11}x + \bar{T}_{12})/(\bar{T}_{21}x + \bar{T}_{22}), \quad (3.7b)$$

where

$$T = T(n) = A(n-1)A(n-2)\cdots A(1)A(0)\,, \tag{3.8a}$$

$$\overline{T} = \overline{T}(n) = A(0)A(1)\cdots A(n-1)\,, \tag{3.8b}$$

and

$$A(l) = \begin{pmatrix} \mathrm{a}^2+\mathrm{b}^2 & \mathrm{a}z_2^2(l) \\ \mathrm{a} & z_2^2(l) \end{pmatrix}. \tag{3.9}$$

More explicitly we may solve Eq. (3.7) and find

$$x = \frac{T_{11} - T_{22} + [(\mathrm{Tr}\,T)^2 - 4\det T\,]^{1/2}}{2T_{21}} \tag{3.10a}$$

and

$$\overline{x} = \frac{\overline{T}_{11} - \overline{T}_{22} + [(\mathrm{Tr}\,\overline{T})^2 - 4\det \overline{T}\,]^{1/2}}{2\overline{T}_{21}}\,. \tag{3.10b}$$

We now demonstrate that $a(e^{i\theta})$ can be written in the form

$$a(e^{i\theta}) = b(e^{i\theta})/\chi(e^{i\theta})\,, \tag{3.11}$$

where $b(\xi)$ is a 2×2 matrix of the form

$$b(\xi) = \sum_{j=-n}^{n} b_j \xi^j\,, \quad |\xi| = 1\,, \tag{3.12}$$

and $\chi(\xi)$ is a scalar function such that

$$\chi^2(\xi) = \sum_{j=-2n}^{2n} \chi_j \xi^j\,, \quad |\xi| = 1\,. \tag{3.13}$$

Furthermore,

$$\chi(\xi) = \chi(\xi^{-1}) \tag{3.14}$$

and

$$\det b(\xi) = \chi^2(\xi)\,. \tag{3.15}$$

We first note from Eq. (3.8) that

$$\det T = \det \overline{T}\,. \tag{3.16}$$

Moreover, let us write

$$A(l) = AB(l)\,, \tag{3.17}$$

where

$$A = \begin{pmatrix} \mathrm{a}^2+\mathrm{b}^2 & \mathrm{a} \\ \mathrm{a} & 1 \end{pmatrix} \tag{3.18}$$

and

$$B(l) = \begin{pmatrix} 1 & 0 \\ 0 & z_2^2(l) \end{pmatrix}. \tag{3.19}$$

Then from Eq. (3.8),

$$T = AB(n-1)AB(n-2)\cdots AB(0) \tag{3.20}$$

and

$$T = AT^TA^{-1}\,. \tag{3.21}$$

Consequently,

$$\mathrm{Tr}\,T = \mathrm{Tr}\,\overline{T}\,. \tag{3.22}$$

Therefore, if we denote

$$X = (\mathrm{Tr}\,T)^2 - 4\det T\,, \tag{3.23}$$

Eq. (3.10) becomes

$$x = (T_{11} - T_{22} + X^{1/2})(2T_{21}) \tag{3.24a}$$

and

$$\overline{x} = (\overline{T}_{11} - \overline{T}_{22} + X^{1/2})/(2\overline{T}_{21})\,. \tag{3.24b}$$

Consider next

$$x - a = \frac{T_{11} - T_{22} - 2\mathrm{a}T_{21} + X^{1/2}}{2T_{21}} = \frac{Y + X^{1/2}}{2T_{21}}\,, \tag{3.25a}$$

where

$$Y = T_{11} - T_{22} - 2\mathrm{a}T_{21} \tag{3.26}$$

and the corresponding formula for $\overline{x} - \mathrm{a}$. Write out Eq. (3.21) to obtain

$$\overline{T} = \mathrm{b}^{-2}\begin{pmatrix} (\mathrm{a}^2+\mathrm{b}^2)T_{11} + \mathrm{a}T_{12} - \mathrm{a}(\mathrm{a}^2+\mathrm{b}^2)T_{21} - \mathrm{a}^2T_{22} & -\mathrm{a}(\mathrm{a}^2+\mathrm{b}^2)T_{11} - \mathrm{a}^2T_{12} + (\mathrm{a}^2+\mathrm{b}^2)^2T_{21} + \mathrm{a}(\mathrm{a}^2+\mathrm{b}^2)T_{22} \\ \mathrm{a}T_{11} + T_{12} - \mathrm{a}^2T_{21} - \mathrm{a}T_{22} & -\mathrm{a}^2T_{11} - \mathrm{a}T_{12} + \mathrm{a}(\mathrm{a}^2+\mathrm{b}^2)T_{21} + (\mathrm{a}^2+\mathrm{b}^2)T_{22} \end{pmatrix}. \tag{3.27}$$

Then it is easily seen that

$$T_{22} + \mathrm{a}T_{21} = \overline{T}_{22} + \mathrm{a}\overline{T}_{21} \tag{3.28}$$

and therefore

$$Y = \overline{T}_{11} - \overline{T}_{22} - 2\mathrm{a}\overline{T}_{21}\,, \tag{3.29}$$

so that

$$\overline{x} - \mathrm{a} = (Y + X^{1/2})/(2\overline{T}_{21})\,. \tag{3.25b}$$

Furthermore, using the 21 element of Eqs. (3.27) and (3.23) and (3.26) we see that

$$4T_{21}\overline{T}_{21}\mathrm{b}^2 + Y^2 = X\,. \tag{3.30}$$

Then substitute (3.25) into (3.3) and use (3.30) to obtain

$$a_{11}(e^{i\theta}) = a_{22}(e^{i\theta}) = i\mathrm{b}(T_{21} - T_{21})X^{-1/2} \tag{3.31a}$$

and

$$a_{21}(e^{i\theta}) = -a_{12}^*(e^{i\theta})$$

$$= \frac{e^{i\theta}(1 + z_1e^{-i\theta})}{1 + z_1e^{i\theta}}\,\frac{i\mathrm{b}(T_{21} + \overline{T}_{21}) - Y}{X^{1/2}}\,. \tag{3.31b}$$

Thus, defining

$$\chi^2(e^{i\theta}) = |1 + z_1e^{i\theta}|^{4n}X\,, \tag{3.32}$$

$$b_{11}(e^{i\theta}) = b_{22}(e^{i\theta}) = |1+z_1e^{i\theta}|^{2n} i\text{b}(T_{21} - \overline{T}_{21}), \tag{3.33a}$$

and

$$b_{21}(e^{i\theta}) = -b_{12}^*(e^{i\theta}) = |1+z_1e^{i\theta}|^{2(n-1)} \times e^{i\theta}(1+z_1e^{-i\theta})^2[i\text{b}(T_{21}+\overline{T}_{21}) - Y], \tag{3.33b}$$

we see from (3.31) that $a(e^{i\theta})$ is of the form (3.11).

The relation (3.15) immediately follows from

$$\det a(\xi) = 1, \tag{3.34}$$

which is computed directly from Eq. (3.3). Moreover, it is clear from Eqs. (3.4), (3.8), (3.9), (3.23), and (3.32) that $\chi(\xi)$ is of the form (3.13). Furthermore, since $\det T(\xi)$ and $\operatorname{Tr} T(\xi)$ are invariant under $\xi \to \xi^{-1}$, Eq. (3.14) holds.

It remains to verify Eq. (3.12). Consider first Eq. (3.33a) and use (3.5) to write

$$b_{11}(e^{i\theta}) = i(1-z_1^2)|1+z_1e^{i\theta}|^{2(n-1)}(T_{21} - \overline{T}_{21}). \tag{3.35}$$

Now neither the term with T_{21} nor the term with $\overline{T}_{21}$ is of the form (3.12) since $T(n)_{21}$ and $\overline{T}(n)_{21}$ each have a denominator of $|1+z_1e^{i\theta}|^{2n}$. However, we may expand Eqs. (3.20) and (3.21) to obtain

$$T_{21} = \text{a}[A(n-1)\cdots A(0)]_{11} + z_2^2(n-1)[A(n-2)\cdots A(0)]_{21} \tag{3.36a}$$

and

$$\overline{T}_{21} = \text{a}[A(1)\cdots A(n-1)]_{11} + z_2^2(0)[A(1)\cdots A(n-1)]_{21}. \tag{3.36b}$$

Then use the fact that $B(l)$ is diagonal and $B(l)_{11} = 1$ to show that

$$[B(0)A\cdots B(n-2)A]_{11} = [AB(1)\cdots AB(n-1)]_{11} \tag{3.37}$$

and thus

$$[B(0)A\cdots B(n-2)A]_{11}^T = [A(1)\cdots A(n-1)]_{11} = [A(n-2)\cdots A(0)]_{11}. \tag{3.38}$$

Therefore

$$T_{21} - \overline{T}_{21} = z_2^2(n-1)[A(n-2)\cdots A(0)]_{21} - z_2^2(0)[A(1)\cdots A(n-1)]_{21}. \tag{3.39}$$

Each of the two terms here involves only a product of $n-1$ factors $A(l)$. Therefore $b_{11}(e^{i\theta})$ and $b_{22}(e^{i\theta})$ are of the form (3.12). Indeed, the somewhat strong statement also holds

$$b_{11}(e^{i\theta}) = b_{22}(e^{i\theta}) = \sum_{l=-n+1}^{n-1} (b_l)_{11} e^{il\theta}. \tag{3.12a}$$

Finally, consider $b_{21}(e^{i\theta})$ as given by Eq. (3.33b). Expand (3.8) to obtain

$$T_{22} = z_2^2(n-1)[A(n-2)\cdots A(0)]_{22} + \text{a}[A(n-2)\cdots A(0)]_{12} \tag{3.40a}$$

and

$$\overline{T}_{11} = (\text{a}^2+\text{b}^2)[A(1)\cdots A(n-1)]_{11} + z_2^2(0)\text{a}[A(1)\cdots A(n-1)]_{21}, \tag{3.40b}$$

we use

$$[A(n-2)\cdots A(0)]_{12} = z_2^2(0)[A(1)\cdots A(n-1)]_{21} \tag{3.41}$$

together with Eq. (3.36) to show that

$$i\text{b}[\overline{T}_{21}+T_{21}] - Y = (\text{a}+i\text{b})[\overline{T}_{21}+T_{21}] - \overline{T}_{11} + T_{22} = (\text{a}+i\text{b})\{z_2^2(n-1)[A(n-1)\cdots A(0)]_{21} + z_2^2(0)[A(1)\cdots A(n-1)]_{21}\} + (\text{a}+i\text{b})^2[A(1)\cdots A(n-1)]_{11} + z_2^2(n-1)[A(n-2)\cdots A(0)]_{22} \tag{3.42}$$

Therefore, noting that

$$\text{a}+i\text{b} = (1-ze^{-i\theta})(1+z_1e^{-i\theta})^{-1}, \tag{3.43}$$

we find the desired result

$$b_{21}(e^{i\theta}) = |1+z_1e^{i\theta}|^{2(n-1)}\Big((e^{i\theta} - z_1^2e^{-i\theta})\{z_2^2(n-1)[A(n-2)\cdots A(0)]_{21} + z_2^2(0)[A(1)\cdots A(n-1)]_{21}\} + e^{i\theta}(1-z_1e^{-i\theta})^2[A(1)\cdots A(n-1)]_{11} + e^{i\theta}(1+z_1e^{-i\theta})^2 z_2^2(n-1)[A(n-2)\cdots A(0)]_{22}\Big), \tag{3.44}$$

which is term by term of the form (3.12).

IV. FACTORIZATION PROCEDURE

It is clear from Sec. II, that most of the theorems are valid if the generating function admits a factorization, and that we need to have the explicit factors in the factorization to evaluate the asymptotic behavior of the block Toeplitz determinant. We shall first show how to carry out the "factorization" for a 2×2 matrix of the form (1.4). In particular, we shall determine the condition that such a matrix admits a factorization.

From (3.13) and (3.14), we find

$$\chi^2(\xi) = \chi^2(\xi^{-1}) = \sum_{j=-2n}^{2n} \chi_j \xi^j.$$

Therefore $\chi^2(\xi)$ has $4n$ roots; and if $\chi^2(\gamma_i)=0$, $\chi^2(\gamma_i^{-1})=0$. Let α_i $(i=1,\ldots,2n)$ be the $2n$ roots of

$\chi^2(\xi)$ inside or on the unit circle. We can write

$$\chi(\xi)=A\left(\prod_{i=1}^{2n}(1-\alpha_i\xi)(1-\alpha_i\xi^{-1})\right)^{1/2}. \qquad (4.1)$$

Clearly, when $\chi(e^{i\theta})\neq 0$ (none of the $4n$ roots lies on the unit circle), $\chi(\xi)$ always admits a factorization

$$\chi(\xi)=\chi_+(\xi)\chi_-(\xi), \qquad (4.2a)$$

where

$$\chi_+(\xi)=A\prod_{i=1}^{2n}(1-\alpha_i\xi)^{1/2} \qquad (4.2b)$$

and

$$\chi_-(\xi)=\prod_{i=1}^{2n}(1-\alpha_i\xi^{-1})^{1/2}. \qquad (4.2c)$$

Under the condition

$$\chi(\xi)\neq 0 \text{ for } |\xi|=1,$$

the matrix

$$a(\xi)=b(\xi)/\chi(\xi) \qquad (4.3)$$

is factorizable if and only if $b(\xi)$ admits a factorization. We shall prove in the Appendix the following lemma.

Lemma. Let $f(e^{i\theta})$ be a function whose Fourier coefficients f_{-j} vanish for $j>n$, and $f_{-n}\neq 0$. If $f(\xi)$ admits a factorization, $f(\xi)$ has exactly n roots inside the unit circle.

Let us consider the following example. According to this lemma, the function

$$f(\xi)=(1-\alpha\xi)(1-\beta\xi^{-1})$$

admits a factorization, only if $f(\xi)$ has one root outside and one root inside the unit circle. Indeed this agrees with the fact that such a function is factorizable if and only if either $|\alpha|, |\beta|<1$, or $|\alpha|, |\beta|>1$.

Using this lemma, we can prove the following theorem.

Theorem 4.1. Let $F(e^{i\theta})$ be a 2×2 matrix whose Fourier coefficient F_{-j} vanishes for $j>n$, and whose determinant $\Delta(\xi)$ has a nonvanishing $(-2n)$th Fourier coefficient ($\Delta_{-2n}\neq 0$). If $F(\xi)$ admits a left (or a right) factorization, then $\Delta(\xi)$ has exactly $2n$ roots inside the unit circle.

Proof. If $F(\xi)$ admit a left factorization

$$F(\xi)=F_+(\xi)F_-(\xi),$$

then the function $\Delta(\xi)=|F(\xi)|$ must admit a Wiener-Hopf factorization

$$\Delta(\xi)=|F(\xi)|=|F_+(\xi)||F_-(\xi)|.$$

Since the Fourier coefficient $F_{-j}\equiv 0$ for $j>n$, we have

$$\Delta(e^{i\theta})=\sum_{j=-2n}^{\infty}\Delta_{2j}e^{(i\theta)j},$$

with $\Delta_{-2n}\neq 0$. One can conclude from the above lemma that $\Delta(\xi)$ has exactly $2n$ roots inside the unit circle. The same proof holds for a right factorization.

This theorem provides us a necessary condition that a 2×2 matrix of the form (1.4b) admit a factorization. From Eqs. (3.25) and (3.26), we have

$$\Delta(\xi)=\chi^2(\xi)=\Delta(\xi^{-1}).$$

Therefore, this necessary condition is always satisfied for the generating function of the layered Ising model.

Next we shall describe the factorization procedure. (a) Let us consider a 2×2 matrix $F(\xi)$ of the form

$$F(\xi)=\sum_{j=-l}^{\infty}F_j\,\xi^j, \quad |\xi|=1. \qquad (4.4)$$

Let $\det F(\xi)$ have $2l$ roots (α_i, $i=1,\ldots,-2l$) inside the unit circle. We shall show this matrix $F(\xi)$ can be related to a matrix $H(\xi)$ that has the form

$$H(\xi)=\sum_{j=-l+1}^{\infty}H_j\xi^j, \quad |\xi|=1, \qquad (4.5)$$

where $\det H(\xi)$ has $2l-2$ roots inside the unit circle, provided certain condition is satisfied.

Assume among the $2l$ roots of $\det F(\xi)$, there exist a pair α_1 and α_2, such that

$$\frac{F_{11}(\alpha_1)}{F_{21}(\alpha_1)}\neq\frac{F_{11}(\alpha_2)}{F_{21}(\alpha_2)}. \qquad (4.6)$$

Let[12]

$$K_{12}=\frac{F_{11}(\alpha_1)}{F_{21}(\alpha_1)}=\frac{F_{12}(\alpha_1)}{F_{22}(\alpha_1)} \qquad (4.7)$$

[the second equality holds, because α_1 is a root of $\det F(\xi)$] and

$$K_{21}=\frac{F_{21}(\alpha_2)}{F_{11}(\alpha_2)}=\frac{F_{22}(\alpha_2)}{F_{12}(\alpha_2)}. \qquad (4.8)$$

We can always identify the roots in such a way so that K_{12} and K_{21} are finite.

Consider now the matrix

$$G(\xi)=K^{-1}F(\xi) \qquad (4.9a)$$

$$=(1-K_{12}K_{21})^{-1/2}\begin{pmatrix}1 & -K_{12}\\ -K_{21} & 1\end{pmatrix}\times\begin{pmatrix}F_{11}(\xi) & F_{12}(\xi)\\ F_{21}(\xi) & F_{22}(\xi)\end{pmatrix}. \qquad (4.9b)$$

From Eqs. (4.7) and (4.9), we find

$$G(\alpha_1)_{11}=(1-K_{12}K_{21})^{-1/2}[(F_{11}(\alpha_1)-K_{12}F_{21}(\alpha_1)]=0 \qquad (4.10)$$

and

$$G(\alpha_1)_{12} = 0 .$$

Likewise, from Eqs. (4.8) and (4.9), we find

$$G(\alpha_2)_{21} = G(\alpha_2)_{22} = 0 . \tag{4.11}$$

This means that we can write

$$G(\xi)_{11} = (1 - \alpha_1 \xi^{-1}) H(\xi)_{11} ,$$
$$G(\xi)_{12} = (1 - \alpha_1 \xi^{-1}) H(\xi)_{12}$$

and (4.12)

$$G(\xi)_{21} = (1 - \alpha_2 \xi^{-1}) H(\xi)_{21} ,$$
$$G(\xi)_{22} = (1 - \alpha_2 \xi^{-1}) H(\xi)_{22} ,$$

where $H(\xi)$ (γ, $s = 1, 2$) have the following forms:

$$H(\xi)_{\gamma s} = \sum_{j=-l+1}^{\infty} (H_j)_{\gamma s} \xi^j , \quad |\xi| = 1, \quad \gamma, s = 1, 2. \tag{4.13}$$

We can now rewrite Eqs. (4.9) as

$$G(\xi) = K \begin{pmatrix} 1 - \alpha_1 \xi^{-1} & 0 \\ 0 & 1 - \alpha_2 \xi^{-1} \end{pmatrix} H(\xi) , \tag{4.14}$$

where $H(\xi)$ is defined by Eqs. (4.12) and (4.13).

When Eq. (4.6) is satisfied, so that $(1 - K_{12} K_{21}) \neq 0$, the constant matrix K defined by Eq. (4.9) is a well defined matrix whose determinant is unity. According to Eq. (4.14), we also have

$$\det G(\xi) = (1 - \alpha_1 \xi^{-1})(1 - \alpha_2 \xi^{-1}) \det H(\xi) . \tag{4.15}$$

Since $|G(\xi)|$ has $2l$ roots inside the unit circle, the determinant of $H(\xi)$ has $2l - 2$ roots (α_i, $i = 3, \ldots, 2n$) inside the unit circle.

(b) Similarly, we can show that if

$$\frac{F_{11}(\alpha_1)}{F_{12}(\alpha_1)} \neq \frac{F_{11}(\alpha_2)}{F_{12}(\alpha_2)} , \tag{4.16}$$

there exists a matrix $H^*(\xi)$ of the form

$$H^*(\xi) = \sum_{j=-l+1}^{\infty} H_j^* \xi^j , \quad |\xi| = 1 , \tag{4.17}$$

whose determinant has $2l - 2$ roots (α_i, $i = 3, \ldots, 2l$) inside the unit circle, such that

$$F(\xi) = H^*(\xi) \begin{pmatrix} 1 - \alpha_1 \xi^{-1} & 0 \\ 0 & 1 - \alpha_2 \xi^{-1} \end{pmatrix} K^* , \tag{4.18}$$

where

$$K^* = \frac{1}{c} \begin{pmatrix} 1 & \dfrac{-F_{12}(\alpha_2)}{F_{11}(\alpha_2)} \\ \dfrac{-F_{11}(\alpha_1)}{F_{12}(\alpha_1)} & 1 \end{pmatrix} , \tag{4.19a}$$

with

$$c = \left[1 - \left(\frac{F_{11}(\alpha_1)}{F_{12}(\alpha_1)} \right) \left(\frac{F_{12}(\alpha_2)}{F_{11}(\alpha_2)} \right) \right]^{1/2} . \tag{4.19b}$$

(c) Consider now the 2×2 matrix $b(\xi)$ of Eq. (3.12), whose determinant has $2n$ roots ($\alpha_i^{\pm 1}$; $i = 1, \ldots, 2n$; $|\alpha_i| < 1$). When

$$\frac{b_{11}(\alpha_1)}{b_{21}(\alpha_1)} \neq \frac{b_{11}(\alpha_2)}{b_{21}(\alpha_2)} , \tag{4.20}$$

according to (a), there is a matrix $b^{(1)}(\xi)$ of the form

$$b^{(1)}(\xi) = \sum_{j=-n+1}^{n} b_j^{(1)} \xi^j , \tag{4.21}$$

whose determinant $\det b^{(1)}(\xi)$ has $4n - 2$ roots (α_1^{-1}; α_2^{-1}; $\alpha_i^{\pm 1}$; $i = 3, \ldots, 2n$; $\alpha_i < 1$), such that

$$b(\xi) = K^{(1)} \begin{pmatrix} 1 - \alpha_1 \xi^{-1} & 0 \\ 0 & 1 - \alpha_2 \xi^{-1} \end{pmatrix} b^{(1)}(\xi) . \tag{4.22}$$

Again, if

$$\frac{b_{11}^{(1)}(\alpha_3)}{b_{21}^{(1)}(\alpha_3)} \neq \frac{b_{11}^{(1)}(\alpha_4)}{b_{21}^{(1)}(\alpha_4)} , \tag{4.23}$$

there exists a matrix $b^{(2)}(\xi)$ of the form

$$b^{(2)}(\xi) = \sum_{j=n-2}^{\infty} b_j^{(2)} \xi^j , \quad |\xi| = 1 , \tag{4.24}$$

such that

$$b^{(1)}(\xi) = K^{(2)} \begin{pmatrix} 1 - \alpha_3 \xi^{-1} & 0 \\ 0 & 1 - \alpha_4 \xi^{-1} \end{pmatrix} b^{(2)}(\xi) . \tag{4.25}$$

We can repeat the same argument n times. We find, that when n such conditions (4.20), (4.23), etc., are satisfied, there exists a matrix $b^{(n)}(\xi)$ of the form

$$b^{(n)}(\xi) = \sum_{j=0}^{n} b_j^{(n)} \xi^i , \quad |\xi| = 1 \tag{4.26}$$

whose determinant has $2n$ roots (α_i^{-1}, $i = 1, \ldots, 2n$) that lie outside the unit circle, such that

$$b(\xi) = e^{(1)}(\xi) e^{(2)}(\xi) \cdots e^{(n)}(\xi) b^{(n)}(\xi) , \tag{4.27}$$

where

$$e^{(i)}(\xi) = K^{(i)} \begin{pmatrix} 1 - \alpha_{2i-1} \xi^{-1} & 0 \\ 0 & 1 - \alpha_{2i} \xi^{-1} \end{pmatrix} . \tag{4.28}$$

Clearly, $b^{(n)}(\xi)$ given by Eq. (4.21) is analytic inside the unit circle. Since $\det b^{(n)}(\xi) \neq 0$ for $|\xi| < 1$ (all its roots lie outside the unit circle), $b^{(n)}(\xi)$ is invertible for $|\xi| < 1$. On the other hand, the matrices $e^{(i)}(\xi)$ are analytic invertible functions outside the unit circle. Therefore, we have carried out the right factorization for the matrix $b(\xi)$, with

$$b_-(\xi) = e^{(1)}(\xi) e^{(2)}(\xi) \cdots e^{(n)}(\xi) [K^{(1)} K^{(2)} \cdots K^{(n)}]^{-1} \tag{4.29}$$

and

$$b_+(\xi) = K^{(1)} K^{(2)} \cdots K^{(n)} b^{(n)}(\xi) . \tag{4.30}$$

We find it convenient to choose $b_-(\xi)$ such that

$$b_-(\infty) = I . \tag{4.31}$$

(d) We shall consider the case of $n=2$, to show what happens, if the n conditions (here $n=2$), as given by Eqs. (4.20) and (4.23), are not satisfied.

Consider a 2×2 matrix $b(\xi)$ of the form

$$b(\xi) = \sum_{j=-2}^{2} b_j \xi^j , \quad |\xi| = 1 , \tag{4.32a}$$

whose determinant has the property

$$\det b(e^{i\theta}) = \det b(e^{-i\theta}) \neq 0 . \tag{4.32b}$$

Let α_i ($i=1, \ldots, 4$) be the four roots of $\det b(\xi)$, that lie inside the unit circle.

Case 1. Assume

$$\frac{b_{11}(\alpha_1)}{b_{21}(\alpha_1)} = \frac{b_{11}(\alpha_2)}{b_{21}(\alpha_2)} = \frac{b_{11}(\alpha_3)}{b_{21}(\alpha_3)} = \frac{b_{11}(\alpha_4)}{b_{21}(\alpha_4)} . \tag{4.33}$$

This means condition (4.20) is not satisfied. Consider the matrix

$$A(\xi) = R^{-1} b(\xi) = \begin{pmatrix} 1 & -b_{11}(\alpha_1)/b_{21}(\alpha_1) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} b_{11}(\xi) & b_{12}(\xi) \\ b_{21}(\xi) & b_{22}(\xi) \end{pmatrix} . \tag{4.34}$$

We find

$$A_{1l}(\xi) = b_{1l}(\xi) - \frac{b_{11}(\alpha_1)}{b_{21}(\alpha_1)} b_{2l}(\xi), \quad l = 1, 2 \tag{4.35}$$

and also

$$A_{2l}(\xi) = b_{2l}(\xi) , \quad l = 1, 2 . \tag{4.36}$$

By the assumption of Eq. (4.33), we have

$$A_{11}(\alpha_i) = 0 \quad \text{for } i = 1, 4 . \tag{4.37}$$

Since

$$b_{11}(\alpha_i) b_{22}(\alpha_i) - b_{12}(\alpha_i) b_{21}(\alpha_i) = 0 \quad \text{for } i = 1, \ldots, 4$$

we also have, from Eq. (4.33), that

$$A_{12}(\alpha_i) = 0 \quad \text{for } i = 1, \ldots, 4 . \tag{4.38}$$

It is evident from Eqs. (4.35), (4.36), and (4.32), that we can write

$$A(\xi) = \xi^{-2} \sum_{j=0}^{4} A_j \xi^j . \tag{4.39}$$

Using Eqs. (4.37) and (4.38), we find

$$A_{1l}(\xi) = (1-\alpha_1\xi^{-1})(1-\alpha_2\xi^{-1})(1-\alpha_3\xi^{-1}) \times (1-\alpha_4\xi^{-1}) B_{1l}\xi^2 , \tag{4.40}$$

where B_{1l} ($l=1, 2$) are two constants. Let

$$\begin{aligned} B_{1l}(\xi) &= B_{1l} , \quad & l = 1, 2 \\ B_{2l}(\xi) &= \xi^2 A(\xi)_{2l} , \quad & l = 1, 2 . \end{aligned} \tag{4.41}$$

It is clear from Eqs. (4.39) and (4.41), that $B(\xi)$ is a matrix analytic for $|\xi| < 1$. Using (4.40) and (4.41), we can rewrite Eq. (4.34) as

$$b(\xi) = R \begin{pmatrix} (1-\alpha_1\xi^{-1})(1-\alpha_2\xi^{-1})(1-\alpha_3\xi^{-1})(1-\alpha_4\xi^{-1}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^2 & 0 \\ 0 & \bar{\xi}^2 \end{pmatrix} B(\xi) . \tag{4.42}$$

Since $\det b(\xi)$ has exactly four roots (α_i, $i=1, 4$) inside the unit circle, we find, from Eq. (4.42), $\det B(\xi) \neq 0$ for $|\xi| < 1$. Clearly, Eq. (4.42) is the right standard factorization of the matrix $b(\xi)$. This means when Eq. (4.33) holds, the matrix $b(\xi)$ of Eq. (4.31) has nonvanishing right exponents

$$\nu_1 = 2 \quad \text{and} \quad \nu_2 = -2 .$$

Case 2. Consider now the case

$$K_{12} = \frac{b_{11}(\alpha_1)}{b_{21}(\alpha_1)} \neq K_{21}^{-1} = \frac{b_{11}(\alpha_2)}{b_{21}(\alpha_2)} . \tag{4.43}$$

According to (a), the matrix

$$B(\xi) = \begin{pmatrix} (1-\alpha_1\xi^{-1})^{-1} & 0 \\ 0 & (1-\alpha_2\xi^{-1})^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & -K_{12} \\ -K_{21} & 1 \end{pmatrix} b(\xi)(1-K_{12}K_{21})^{-1/2} \tag{4.44}$$

is of the form

$$B(\xi) = \sum_{j=-1}^{2} B_j \xi^j . \tag{4.45}$$

In order to violate Eq. (4.23), we assume

$$\frac{B_{11}(\alpha_3)}{B_{21}(\alpha_3)} = \frac{B_{11}(\alpha_4)}{B_{21}(\alpha_4)} \tag{4.46}$$

We multiply $B(\xi)$, defined by Eq. (4.44), from the left by a constant matrix

$$S^{-1} = \begin{pmatrix} 1 & -B_{11}(\alpha_3)/B_{21}(\alpha_3) \\ 0 & 1 \end{pmatrix} , \tag{4.47}$$

and let

$$C(\xi) = S^{-1} B(\xi) . \tag{4.48}$$

When Eq. (4.46) holds, we have

$$C_{1l}(\alpha_3) = 0 \quad \text{for } l = 1, 2 , \tag{4.49a}$$

and also

$$C_{1l}(\alpha_4)=0 \quad \text{for } l=1,\,2\,. \tag{4.49b}$$

According to Eqs. (4.45), (4.48), and (4.49), we can write

$$C_{1l}(\xi)=\xi(1-\alpha_3\xi^{-1})(1-\alpha_4\xi^{-1})D_{1l}(\xi),\quad l=1,\,2, \tag{4.50}$$

where $D_{1l}(\xi)\,(l=1,\,2)$ are linear polynomials in ξ; and also

$$C_{2l}(\xi)=B_{2l}(\xi)=\xi^{-1}D_{2l}(\xi), \tag{4.51}$$

where $D_{2l}(\xi)$ is a cubic polynomial in ξ. Consequently,

$$B(\xi)=S\begin{pmatrix}(1-\alpha_3\xi^{-1})(1-\alpha_4\xi^{-1}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}D(\xi)\,. \tag{4.52}$$

Let

$$K^{-1}=\begin{pmatrix}1 & -K_{12}\\ -K_{21} & 1\end{pmatrix}(1-K_{12}K_{21})^{-1/2} \tag{4.53}$$

and

$$F(\xi)=K\begin{pmatrix}(1-\alpha_1\xi^{-1}) & 0\\ 0 & (1-\alpha_2\xi^{-1})\end{pmatrix}\times S\begin{pmatrix}(1-\alpha_3\xi^{-1})(1-\alpha_4\xi^{-1}) & 0\\ 0 & 1\end{pmatrix}. \tag{4.54}$$

We can rewrite Eq. (4.44) as

$$B(\xi)=F(\xi)\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}D(\xi)\,. \tag{4.55}$$

By Eq. (4.54), we find $F(\xi)$ analytic and invertible for $|\xi|>1$. Since the elements of $D(\xi)$ are polynomials in ξ, $D(\xi)$ is analytic inside the unit circle. According to Eqs. (4.54) and (4.55), we find the four roots of $|D(\xi)|$ lie outside the unit circle. Therefore, Eq. (4.55) is a standard factorization of $b(\xi)$. We conclude, when Eq. (4.20) holds, but Eq. (4.23) does not hold, the matrix $b(\xi)$ of (4.26) has nonvanishing right exponents

$$\nu_1=1 \text{ and } \nu_2=-1\,.$$

This shows that the conditions

$$\frac{b_{11}(\alpha_1)}{b_{21}(\alpha_1)}\neq\frac{b_{11}(\alpha_2)}{b_{21}(\alpha_2)}$$

and (4.56)

$$\frac{B_{11}(\alpha_3)}{B_{21}(\alpha_3)}\neq\frac{B_{11}(\alpha_4)}{B_{21}(\alpha_4)}$$

are the necessary and sufficient conditions that a matrix $b(\xi)$ of the form (4.26) admits a right factorization.

In the next paragraph, we shall describe the procedure that enables one to calculate the block Toeplitz determinant $D_N[a]$, generated by the matrix of the form (3.22), in the limit $N\to\infty$.[14] According to (b), there exist a matrix $b^{(1)}(\xi)$ which has the form (4.21), and whose determinant has $2n-2$ roots inside the unit circle, such that

$$b(\xi)=b^{(1)}(\xi)\begin{pmatrix}1-\alpha_1\xi^{-1} & 0\\ 0 & 1-\alpha_2\xi^{-1}\end{pmatrix}K^{(1)}, \tag{4.57}$$

where $K^{(1)}$ is a 2×2 constant matrix whose determinant is unity.

Let

$$a^{(1)}(\xi)=b^{(1)}(\xi)/\chi(\xi)\,. \tag{4.58}$$

Thus the matrix $a(\xi)$ of Eq. (4.3) is related to $a^{(1)}(\xi)$ as

$$a(\xi)=a^{(1)}(\xi)\begin{pmatrix}1-\alpha_1\xi^{-1} & 0\\ 0 & 1-\alpha_2\xi^{-1}\end{pmatrix}K^{(1)}. \tag{4.59}$$

Now we can use identity Eq. (2.15) and Theorem V to calculate the ratio

$$r_1=\lim_{N\to\infty}D_N[a]/D_N[a^{(1)}]\,. \tag{4.60}$$

Repeating these steps n times, we obtain the ratio

$$r_n=\lim_{N\to\infty}D_N[a]/D_N[a^{(n)}]\,, \tag{4.61}$$

where $a^{(n)}(\xi)$ is equal to a scalar function $1/\chi(\xi)$ times a matrix function $b^{(n)}(\xi)$ which is an invertible and analytic function for $|\xi|<1$. For such a matrix $a^{(n)}(\xi)$, we can use Theorem III to calculate the block Toeplitz determinant $D_N[a^{(n)}]$ in the limit $N\to\infty$. Consequently, we find the block Toeplitz determinant $D_N[a]$, when $N\to\infty$, is

$$\lim_{N\to\infty}D_N[a]=r_n\lim_{N\to\infty}D_N[a^{(n)}]\,.$$

To calculate the asymptotic behavior of $D_N[a]$, for large N, generated by a matrix (4.3) that admit a factorization, we can simply substitute into Eq. (2.12) the left factor obtained by the procedure described in (c).

V. GENERATING FUNCTION AND SPONTANEOUS MAGNETIZATION FOR THE CASE $n=2$

In the following sections, we shall study in some detail the simplest nontrivial problem of the layered Ising model; we consider the lattice whose horizontal bonds are all equal to E_1, and whose vertical bonds can take two alternate values: $E_2(2K)=E_2$ and $E_2(2K+1)=E_2'$. We have shown in Sec. III that the spin-spin correlation function $\langle\sigma_{00}\sigma_{0N}\rangle^2$ is a block Toeplitz determinant $D_N[a]$ generated by a 2×2 matrix $a(e^{i\theta})$ given by Eq. (3.3). It was also shown that the matrix $a(\xi)$ can be written in the form (3.11), i.e.,

$$a(\xi)=b(\xi)/\chi(\xi)\,. \tag{5.1}$$

Here, in this simple case, the 2×2 matrices T and $\overline{T}$ defined by (3.8) become

$$T=\begin{pmatrix}(a^2+b^2)^2+a^2z_2'^2 & az_2^2(a^2+b^2+z_2'^2)\\ a(a^2+b^2+z_2'^2) & a^2z_2^2+z_2^2z_2'^2\end{pmatrix} \tag{5.2a}$$

and

$$\overline{T}=\begin{pmatrix}(a^2+b^2)^2+a^2z_2^2 & az_2'^2(a^2+b^2+z_2^2)\\ a(a^2+b^2+z_2^2) & a^2z_2'^2+z_2^2z_2'^2\end{pmatrix}. \tag{5.2b}$$

Consequently, Eq. (3.33a) yields

$$b_{11}(e^{i\theta})=b_{22}(e^{i\theta})=z_1(1-z_1^2)(z_2^2-z_2'^2)(e^{i\theta}-e^{-i\theta})\,. \tag{5.3a}$$

by Eqs. (3.29) and (3.33b), we have

$$b_{21}(e^{i\theta})=-b_{12}(e^{-i\theta})=\{[(1+z_1^2)(1+z_2z_2')-z_1(1-z_2z_2')(e^{i\theta}+e^{-i\theta})]\times[2z_1(1+z_2z_2')-(1-z_2z_2')(e^{i\theta}+z_1^2e^{-i\theta})]-z_1(z_2-z_2')^2(e^{i\theta}-e^{-i\theta})(e^{i\theta}-z_1^2e^{-i\theta})\}\,. \tag{5.3b}$$

Likewise, Eqs. (3.23) and (3.32) yield

$$\chi^2(e^{i\theta})=|1+z_1e^{i\theta}|^8[(a^2+b^2-z_2z_2')-ia(z_2+z_2')][(a^2+b^2-z_2z_2')+ia(z_2+z_2')]\times[(a^2+b^2+z_2z_2')-ia(z_2-z_2')][a^2+b^2+z_2z_2')+ia(z_2-z_2')]\,. \tag{5.4}$$

Let

$$P_1(e^{i\theta})=|1+z_1e^{i\theta}|^2[(a^2+b^2-z_2z_2')-ia(z_2+z_2')]=(1+z_1^2)(1-z_2z_2')-z(1-z_2)(1-z_2')e^{i\theta}-z_1(1+z_2)(1+z_2')e^{-i\theta} \tag{5.5}$$

and

$$P_2(e^{i\theta})=|1+z_1e^{i\theta}|^2[(a^2+b^2+z_2z_2')+ia(z_2-z_2')]=(1+z_1^2)(1+z_2z_2')-z_1(1+z_2)(1-z_2')e^{i\theta}-z_1(1-z_2)(1+z_2')e^{-i\theta}\,. \tag{5.6}$$

Clearly, Eq. (5.4) becomes

$$\chi^2(\xi)=P_1(\xi)P_1(\xi^{-1})P_2(\xi)P_2(\xi^{-1})\,. \tag{5.7}$$

Let γ_1 and γ_2^{-1} be the two roots of the quadratic polynomial $P_1(\xi)$, and let $\gamma_1<\gamma_2^{-1}$. We find

$$\gamma_1=\{(1+z_1^2)(1-z_2z_2')-[(1+z_1^2)^2(1-z_2z_2')^2-4z_1^2(1-z_2^2)(1-z_2'^2)]^{1/2}\}/[2z_1(1-z_2)(1-z_2')]\,, \tag{5.8}$$

and

$$\gamma_2=[(1-z_2)(1-z_2')/(1+z_2)(1+z_2')]\gamma_1\,. \tag{5.9}$$

Here, we choose $(x)^{1/2}>0$ for real and positive x. Let γ_3 and γ_4^{-1} be the two roots of $P_2(\xi)$, and let $\gamma_3<\gamma_4^{-1}$. We find

$$\gamma_3=\{(1+z_1^2)(1+z_2z_2')-[(1+z_1^2)^2(1+z_2z_2')^2-4z_1^2(1-z_2^2)(1-z_2'^2)]^{1/2}\}/[2z_1(1+z_2)(1-z_2')] \tag{5.10}$$

and

$$\gamma_4=\frac{(1+z_2)(1-z_2')}{(1-z_2)(1+z_2')}\gamma_3\,. \tag{5.11}$$

It is evident from Eq. (5.7), that $\chi^2(\xi)$ has eight roots ($\gamma_i^{\pm 1}$, $i=1, 4$). Therefore,

$$\chi(\xi)=z_1^2(1-z_2^2)(1-z_2'^2)[(\xi-\gamma_1^{-1})(\xi-\gamma_2^{-1})\times(\xi-\gamma_3^{-1})(\xi-\gamma_4^{-1})(1-\gamma_1\xi^{-1})(1-\gamma_2\xi^{-1})\times(1-\gamma_3\xi^{-1})(1-\gamma_4\xi^{-1})]^{1/2}\,. \tag{5.12}$$

It is easy to verify that, when $0\le z_1\le 1$, $0\le z_2\le 1$, and $0\le z_2'\le 1$, we have (i) $\gamma_i\ge 0$, $i=1, 2, 3, 4$; (ii) $\gamma_3(z_2', z_2)=\gamma_4(z_2, z_2')$; $\gamma_3=\gamma_4$ when $z_2=z_2'$; (iii) $0<\gamma_2<\gamma_4\le\gamma_3<\gamma_1$ for $z_2\le z_2'$; (iv) γ_2, γ_3, and $\gamma_4<1$; (v) when $T\le T_c$ [that is, when $(1-z_1)^2/(1+z_1)^2\le z_2z_2'$], we have $\gamma_1\ge 1$ and when $T>T_c$ [$(1-z_1)^2/(1+z_1)^2>z_2z_2'$], we have $\gamma_1<1$. Since

$$b_{11}(\xi)=b_{22}(\xi)=-b_{11}(\xi^{-1}) \tag{5.13}$$

and

$$b_{12}(\xi)=-b_{21}(\xi^{-1})\,, \tag{5.14}$$

we find

$$\frac{b_{11}(\gamma_i)}{b_{21}(\gamma_i)}=\frac{b_{12}(\gamma_i)}{b_{11}(\gamma_i)}=\frac{b_{21}(\gamma_i^{-1})}{b_{11}(\gamma_i^{-1})}\quad\text{for all }\gamma_i\,. \tag{5.15}$$

It can be shown that

$$\frac{b_{11}(\gamma_1)}{b_{21}(\gamma_1)}=\frac{-b_{11}(\gamma_2)}{b_{21}(\gamma_2)}=\frac{-(1+z_2)(1+z_2')\gamma_1^{-1}+z_1(1-z_2z_2')}{z_2-z_2'} \tag{5.16}$$

and

$$\frac{b_{11}(\gamma_3)}{b_{21}(\gamma_3)}=\frac{-b_{11}(\gamma_4)}{b_{21}(\gamma_4)}=\frac{(1-z_2)(1+z_2')\gamma_3^{-1}-z_1(1+z_2z_2')}{z_2+z_2'}. \quad (5.17)$$

Since γ_3, $\gamma_4<1$ for any temperature, Eq. (5.17) implies that condition (4.37) holds. This means that $b(\xi)$ belongs to the class described by case 2 in Sec. IV, case (d).

Let

$$\nu=b_{11}(\gamma_1)/b_{21}(\gamma_1), \quad (5.18)$$

$$\omega=b_{11}(\gamma_3)/b_{21}(\gamma_3). \quad (5.19)$$

Consider the matrix

$$B(\xi)=\begin{pmatrix}(1-\gamma_3\xi^{-1})^{-1} & 0\\ 0 & (1-\gamma_4\xi^{-1})^{-1}\end{pmatrix}K^{-1}b(\xi), \quad (5.20)$$

where

$$K^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\omega\\ \omega^{-1} & 1\end{pmatrix}. \quad (5.21)$$

From (5.20), we find

$$\frac{B_{11}(\gamma_1)}{B_{21}(\gamma_1)}=\omega\frac{\nu-\omega}{\nu+\omega}\,\frac{1-\gamma_4\gamma_1^{-1}}{1-\gamma_3\gamma_1^{-1}}, \quad (5.22)$$

$$\frac{B_{11}(\gamma_2)}{B_{21}(\gamma_2)}=\omega\frac{\nu+\omega}{\nu-\omega}\,\frac{1-\gamma_4\gamma_2^{-1}}{1-\gamma_3\gamma_2^{-1}}, \quad (5.23)$$

and

$$\frac{B_{11}(\gamma_1^{-1})}{B_{21}(\gamma_1^{-1})}=\omega\frac{1-\omega\nu}{1+\omega\nu}\,\frac{1-\gamma_4\gamma_1}{1-\gamma_3\gamma_1}. \quad (5.24)$$

It can be verified that

$$\frac{B_{11}(\gamma_1)}{B_{21}(\gamma_1)}=\frac{B_{11}(\gamma_2)}{B_{21}(\gamma_2)} \quad (5.25)$$

and

$$\frac{B_{11}(\gamma_1^{-1})}{B_{21}(\gamma_1^{-1})}\neq\frac{B_{11}(\gamma_2)}{B_{21}(\gamma_2)}. \quad (5.26)$$

When $T>T_c$, γ_1 and $\gamma_2<1$. Since Eq. (5.25) holds, the matrix $b(\xi)$ admits a right standard factorization with nonvanishing right exponents $\nu_1=1$ and $\nu_2=-1$. Whereas, γ_1^{-1} and $\gamma_2<1$ for $T<T_c$. Because of Eq. (5.26), the matrix $b(\xi)$ admits a right factorization.

Let

$$Q_{12}=B_{11}(\gamma_1^{-1})/B_{21}(\gamma_1^{-1}) \quad (5.27a)$$

and

$$Q_{21}=B_{21}(\gamma_2)/B_{11}(\gamma_2). \quad (5.27b)$$

Furthermore, define

$$Q=(1-Q_{12}Q_{21})^{-1/2}\begin{pmatrix}1 & Q_{12}\\ Q_{21} & 1\end{pmatrix}. \quad (5.27c)$$

According to Sec. IV, case (c), we can write

$$b(\xi)=K\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}\times Q\begin{pmatrix}1-\gamma_1^{-1}\xi^{-1} & 0\\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}C(\xi), \quad (5.28)$$

where $C(\xi)$ is a 2×2 matrix whose elements are quadratic polynomials in ξ, and whose determinant does not vanish for $|\xi|<1$, at $T<T_c$.

Let us rewrite (5.12) as

$$\chi(\xi)=\chi_-(\xi)\chi_+(\xi), \quad (5.29a)$$

where

$$\chi_-(\xi)=[(1-\gamma_1^{-1}\xi^{-1})(1-\gamma_2\xi^{-1})\times(1-\gamma_3\xi^{-1})(1-\gamma_4\xi^{-1})]^{1/2} \quad (5.29b)$$

and

$$\chi_+(\xi)=z_1^2(1-z_2^2)(1-z_2'^2)\times[(\xi-\gamma_1)(\xi-\gamma_2^{-1})(\xi-\gamma_3^{-1})(\xi-\gamma_4^{-1})]^{1/2}. \quad (5.29c)$$

The matrix $a(\xi)$ of Eq. (5.1) admits a right factorization

$$a(\xi)=a_-(\xi)a_+(\xi), \quad |\xi|=1, \quad (5.30a)$$

where

$$a_+(\xi)=KQC(\xi)/\chi_+(\xi) \quad (5.30b)$$

and

$$a_-(\xi)=K\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}\times Q\begin{pmatrix}1-\gamma_1^{-1}\xi^{-1} & 0\\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}Q^{-1}K^{-1}/\chi_-(\xi). \quad (5.30c)$$

From Eqs. (5.13) and (5.14), we find

$$a(\xi^{-1})=\begin{pmatrix}-a_{11}(\xi) & -a_{12}(\xi)\\ -a_{21}(\xi) & -a_{22}(\xi)\end{pmatrix}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}a(\xi)\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \quad (5.31)$$

Therefore, $a(\xi)$ admits a left factorization

$$a(\xi)=\tilde{a}_+(\xi)\tilde{a}_-(\xi), \quad (5.32a)$$

with

$$\tilde{a}_+(\xi)=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}a_-(\xi^{-1})\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} \quad (5.32b)$$

and

$$\tilde{a}_-(\xi)=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}a_+(\xi^{-1})\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \quad (5.32c)$$

As mentioned before, the generating function $a'(\xi)$ of $\langle\sigma_{10}\sigma_{1N}\rangle^2$ can be obtained from $a(\xi)$ by interchanging z_2 and z_2'. Hence we have

$$a'(\xi)=\begin{pmatrix}-a_{11}(\xi) & a_{12}(\xi)\\ a_{21}(\xi) & -a_{22}(\xi)\end{pmatrix}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}a(\xi)\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

Consequently, using (2.15), one obtains

$$\langle\sigma_{00}\sigma_{0N}\rangle=\langle\sigma_{10}\sigma_{1N}\rangle\,. \tag{5.33}$$

By considering the symmetry of the system, this identity (5.33) is also evident.

Accordingly,

$$M^2=\lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle \tag{5.34}$$

is the spontaneous magnetization of the second-order layered Ising lattice.

A. Spontaneous magnetization for $T<T_c$ $(\gamma_1^{-1}<1)$

Consider the matrix $B(\xi)$ defined by Eq. (5.22). It is easy to show that

$$\frac{B_{11}(\gamma_1^{-1})}{B_{12}(\gamma_1^{-1})}=\frac{b_{11}(\gamma_1^{-1})}{b_{12}(\gamma_1^{-1})}=\nu \tag{5.35a}$$

and

$$\frac{B_{11}(\gamma_2)}{B_{13}(\gamma_2)}=\frac{b_{11}(\gamma_2)}{b_{12}(\gamma_2)}=\frac{-1}{\nu}\,. \tag{5.35b}$$

Therefore,

$$\frac{B_{11}(\gamma_1^{-1})}{B_{12}(\gamma_1^{-1})}\neq\frac{B_{12}(\gamma_2)}{B_{12}(\gamma_2)}\,.$$

Hence, from Sec. IV, case (b), we can write

$$B(\xi)=G(\xi)\begin{pmatrix}1-\gamma_1^{-1}\xi^{-1} & 0\\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}K_2\,, \tag{5.36}$$

where

$$K_2^{-1}=(1+\nu^2)^{-1/2}\begin{pmatrix}1 & +\nu\\ -\nu & 1\end{pmatrix}$$

and $G(\xi)$ is a 2×2 matrix whose elements are quadratic polynomials in ξ, and $|G(\xi)|\neq 0$ for $|\xi|<1$. From Eqs. (5.20) and (5.36), we obtain

$$b(\xi)=K\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}\times G(\xi)\begin{pmatrix}1-\gamma_1^{-1}\xi^{-1} & 0\\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}K_2\,. \tag{5.37}$$

Let

$$P(\xi)=G(\xi)/\chi(\xi) \tag{5.38}$$

and

$$Q(\xi)=K\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}G(\xi)/\chi(\xi)\,. \tag{5.39}$$

By Eqs. (5.1), (5.37), and (5.39), we have

$$a(\xi)=Q(\xi)\begin{pmatrix}1-\gamma_1^{-1}\xi^{-1} & 0\\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}K_2\,. \tag{5.40}$$

Using identity (2.15) and theorem V (in Sec. II), we get

$$\lim_{N\to\infty}\frac{D_N[a]}{D_N[Q]}=\frac{1}{\det Q_+(\gamma_1^{-1})\det Q_+(\gamma_2)}\begin{vmatrix}Q_+(\gamma_1^{-1})_{22} & Q_+(\gamma_1^{-1})_{12}\\ Q_+(\gamma_2)_{22} & Q_+(\gamma_2)_{11}\end{vmatrix}\,. \tag{5.41}$$

It is obvious from Eq. (5.39), that $Q(\xi)$ admits a right factorization

$$Q(\xi)=Q_-(\xi)Q_+(\xi)\,, \tag{5.42a}$$

with

$$Q_+(\xi)=KG(\xi)/\chi_+(\xi)\,, \tag{5.42b}$$

where $\chi_+(\xi)$ is given by (5.29c). Substituting Eq. (5.42b) into (5.41), we get

$$\lim_{N\to\infty}\frac{D_N[a]}{D_N[Q]}=\frac{1}{\chi_+(\gamma_1^{-1})\chi_+(\gamma_2)}\times\begin{vmatrix}G(\gamma_1^{-1})_{22} & G(\gamma_1^{-1})_{12}\\ G(\gamma_2)_{21} & G(\gamma_2)_{11}\end{vmatrix}\,. \tag{5.43}$$

Next, we compare Eqs. (5.38) and (5.39), and find

$$Q(\xi)=K\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}P(\xi)\,. \tag{5.44}$$

Therefore,

$$Q^T(\xi)=P^T(\xi)\begin{pmatrix}1-\gamma_3\xi^{-1} & 0\\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}K^T\,. \tag{5.45}$$

Again, making use of Eq. (2.15) and theorem V, we find

$$\lim_{N\to\infty}\frac{D[Q^T]}{D[P^T]}=\frac{1}{\chi_+(\gamma_3)\chi_+(\gamma_4)}\times\begin{vmatrix}G(\gamma_3)_{22} & G(\gamma_3)_{21}\\ G(\gamma_4)_{12} & G(\gamma_4)_{11}\end{vmatrix}\,, \tag{5.45}$$

where we have used the fact that $P^T(\xi)$ admits a right factorization

$$P^T(\xi)=[1/\chi_-(\xi)][G^T(\xi)/\chi_+(\xi)]\,,$$

with

$$(P^T)_+=G^T(\xi)/\chi_+(\xi)\,.$$

Combining Eqs. (2.18), (5.45), and (5.43), we have

$$\lim_{N\to\infty}\frac{D_N[a]}{D_N[P]}=\frac{[G(\gamma_2)_{11}G(\gamma_1^{-1})_{22}-G(\gamma_2)_{21}G(\gamma_1^{-1})_{12}][G(\gamma_4)_{11}G(\gamma_3)_{22}-G(\gamma_4)_{12}G(\gamma_3)_{21}]}{\chi_+(\gamma_3)\chi_+(\gamma_4)\chi_+(\gamma_1^{-1})\chi_+(\gamma_2)}\,. \tag{5.46}$$

According to theorem I in Sec. II, we have

$$\mu = \lim_{N\to\infty} D_{N+1}(P)/D_N(P) = 1 .$$

Since $P(\xi)$ is a scalar function $1/\chi(\xi)$ times a matrix function which is invertible and analytic for $|\xi|<1$, we can use theorem III.

Since

$$\det P(\xi) = [(1-\gamma_1^{-1}\xi^{-1})(1-\gamma_2\xi^{-1})(1-\gamma_3\xi^{-1})(1-\gamma_4\xi^{-1})]^{-1},$$

we have $k_n = 0$. Therefore

$$\lim_{N\to\infty} D_N[P] = 1 . \tag{5.47}$$

From Eq. (5.37), we can compute $G(\gamma_i)_{ij}$, and find

$$G_{11}(\gamma_2) = b_{11}(\gamma_2)(1+\omega\nu^{-1})(1+\nu^2)^{1/2}2^{-1/2}/(1-\gamma_3\gamma_2^{-1})(1-\gamma_1^{-1}\gamma_2^{-1}),$$
$$G_{21}(\gamma_2) = b_{11}(\gamma_2)(\omega^{-1}-\nu^{-1})(1+\nu^2)^{1/2}2^{-1/2}/(1-\gamma_4\gamma_2^{-1})(1-\gamma_1^{-1}\gamma_2^{-1}),$$
$$G_{22}(\gamma_1^{-1}) = b_{11}(\gamma_1^{-1})(1+\omega^{-1}\nu^{-1})(1+\nu^2)^{1/2}2^{-1/2}/(1-\gamma_4\gamma_1)(1-\gamma_2\gamma_1),$$
$$G_{12}(\gamma_1^{-1}) = b_{11}(\gamma_1^{-1})(\nu^{-1}-\omega)(1+\nu^2)^{1/2}2^{-1/2}/(1-\gamma_3\gamma_1)(1-\gamma_2\gamma_1)$$

and (5.48)

$$G_{11}(\gamma_4) = b_{11}(\gamma_4)(1+\omega\nu)2^{1/2}(1+\nu^2)^{-1/2}/(1-\gamma_3\gamma_4^{-1})(1-\gamma_1^{-1}\gamma_4^{-1}),$$
$$G_{12}(\gamma_4) = b_{11}(\gamma_4)(\nu-\omega)2^{1/2}(1+\nu^2)^{-1/2}/(1-\gamma_3\gamma_4^{-1})(1-\gamma_2\gamma_4^{-1}),$$
$$G_{22}(\gamma_3) = b_{11}(\gamma_3)(1+\nu\omega^{-1})2^{1/2}(1+\nu^2)^{-1/2}/(1-\gamma_4\gamma_3^{-1})(1-\gamma_2\gamma_3^{-1}),$$
$$G_{21}(\gamma_3) = b_{11}(\gamma_3)(\omega^{-1}-\nu)2^{1/2}(1+\nu^2)^{-1/2}/(1-\gamma_4\gamma_3^{-1})(1-\gamma_1^{-1}\gamma_3^{-1}).$$

Substituting Eqs. (5.47) and (5.48) into (5.46), we obtain the remarkably simple result

$$M^4 = \lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle^2 = \lim_{N\to\infty} D_N[a] = \frac{(z_2+z_2')^2}{4z_2z_2'}\,\frac{[1-\gamma_1^{-2})(1-\gamma_2^2)(1-\gamma_3^2)(1-\gamma_4^2)]^{1/2}}{(1-\gamma_1^{-1}\gamma_2)(1-\gamma_3\gamma_4)} . \tag{5.49}$$

As $T\to T_c$,

$$\gamma_1 \sim 1 + \frac{(\beta-\beta_c)(1-z_{1c})^2}{2z_{1c}(z_{2c}+z_{2c}')}\{4E_1 + [z_{2c}'^{-1}(1-z_{2c}'^2)E_2' + z_{2c}^{-1}(1-z_{2c}^2)E_2]\} . \tag{5.50}$$

Therefore, for $T\sim T_c$,

$$M \sim A(E_1, E_2, E_2')(1-T/T_c)^{1/8}, \tag{5.51}$$

where

$$A^4(E_1, E_2, E_2') = \frac{(z_{2c}+z_{2c}')^2}{4z_2z_2'}\,\frac{[(1-\gamma_{2c}^2)(1-\gamma_{3c}^2)(1-\gamma_{4c}^2)]^{1/2}}{(1-\gamma_{2c})(1-\gamma_{3c}\gamma_{4c})}$$
$$\times\left\{\frac{(1-z_{1c})^2}{2z_{1c}(z_{2c}+z_{2c}')}\beta_c[4E_1 + (z_{2c}'^{-1}-z_{2c}')E_2' + (z_{2c}^{-1}-z_{2c})E_2]\right\}^{1/2} . \tag{5.52}$$

In Fig. 2 we plot $A(E_1, E_2, E_2')A(E_1, E_1, E_1)^{-1}$ as a function of E_2/E_2', for E_1 and T_c fixed.

B. Spontaneous magnetization for $T>T_c$ $(\gamma_1<1)$

Because

$$B_{11}(\gamma_1)/B_{12}(\gamma_1) - b_{11}(\gamma_1)/b_{12}(\gamma_1) = \nu^{-1}, \tag{5.53a}$$
$$B_{11}(\gamma_2)/B_{12}(\gamma_2) = b_{11}(\gamma_2)/b_{12}(\gamma_2) = -\nu^{-1}, \tag{5.53b}$$

we find

$$B_{11}(\gamma_1)/B_{12}(\gamma_1) \neq B_{11}(\gamma_2)/B_{12}(\gamma_2) . \tag{5.53c}$$

Therefore, we can write

$$b(\xi) = K\begin{pmatrix} 1-\gamma_3\xi^{-1} & 0 \\ 0 & 1-\gamma_4\xi^{-1}\end{pmatrix}$$
$$\times G^*(\xi)\begin{pmatrix} 1-\gamma_1\xi^{-1} & 0 \\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}K', \tag{5.54a}$$

where

$$K'^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \nu \\ -\nu^{-1} & 1\end{pmatrix} \tag{5.54b}$$

and $G^*(\xi)$ is a 2×2 matrix whose elements are quadratic polynomials and whose determinant $|G^*(\xi)| \neq 0$ for $|\xi| < 1$.

Note the matrix $B(\xi)$ given by Eq. (5.22) admits a right factorization because of Eq. (5.53c), and does not admit a left factorization because of Eq. (5.25). Similar to the derivation of Eq. (5.46), we can derive the following formula for $T > T_c$:

$$\lim_{N\to\infty} D_N[a] \equiv [G^*(\gamma_2)_{11}G^*(\gamma_1)_{22} - G^*(\gamma_2)_{21}G^*(\gamma_1)_{12}] \times [G^*(\gamma_4)_{11}G^*(\gamma_3)_{22} - G^*(\gamma_4)_{12}G^*(\gamma_3)_{21}/ \chi_+(\gamma_1)\chi_+(\gamma_2)\chi_+(\gamma_3)\chi_+(\gamma_4)\,. \tag{5.55a}$$

Here

$$\chi(\xi) = z_1^2(1-z_2^2)(1-z_2'^2) \times [(\xi-\gamma_1^{-1})(\xi-\gamma_2^{-1})(\xi-\gamma_3^{-1})(\xi-\gamma_4^{-1})]^{1/2}\,. \tag{5.55b}$$

Since

$$B(\xi) = G^*(\xi)\begin{pmatrix} 1-\gamma_1\xi^{-1} & 0 \\ 0 & 1-\gamma_2\xi^{-1}\end{pmatrix}K'\,,$$

we can show

$$B(\gamma_1)_{11}/B(\gamma_1)_{21} = G^*(\gamma_1)_{12}/G^*(\gamma_1)_{22}$$

and

$$B(\gamma_2)_{11}/B(\gamma_2)_{21} = G^*(\gamma_2)_{11}/G^*(\gamma_2)_{21}\,.$$

Because of Eq. (5.25), we find,

$$M^4 = \lim_{N\to\infty} D_N[a] = 0 \quad \text{for } T > T_c\,.$$

VI. ASYMPTOTIC BEHAVIOR OF $\langle\sigma_{00}\sigma_{0N}\rangle$ FOR $T < T_c$

We learn from Sec. V, that the generating function $a(\xi)$ of $\langle\sigma_{00}\sigma_{0N}\rangle^2 = D_N[a]$ admits a factorization for $T < T_c$; but it does not admit a factorization for $T > T_c$. Therefore, we can use theorem IV to calculate the asymptotic behavior of $\langle\sigma_{00}\sigma_{0N}\rangle$ for $T < T_c$, but not for $T > T_c$. We shall restrict ourselves to the case of $T < T_c$ in this section, and defer the discussion on $T > T_c$ to Sec. VII.

Rewriting Eq. (5.30c), we have

$$a_-(\xi) = K\begin{pmatrix} g^{-1}(\xi^{-1}) & 0 \\ 0 & g(\xi^{-1})\end{pmatrix} \times Q\begin{pmatrix} h^{-1}(\xi^{-1}) & 0 \\ 0 & h(\xi^{-1})\end{pmatrix}Q^{-1}K^{-1}\,, \tag{6.1}$$

where

$$h(\xi) = [(1-\gamma_2\xi)/(1-\gamma_1^{-1}\xi)]^{1/2} \tag{6.2a}$$

and

$$g(\xi) = [(1-\gamma_4\xi)/(1-\gamma_3\xi)]^{1/2}\,. \tag{6.2b}$$

By Eq. (5.32b), we have

$$\tilde{a}_+(\xi) = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}K\begin{pmatrix} g^{-1}(\xi) & 0 \\ 0 & g(\xi)\end{pmatrix} \times Q\begin{pmatrix} h^{-1}(\xi) & 0 \\ 0 & h(\xi)\end{pmatrix}Q^{-1}K^{-1}\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}. \tag{6.3}$$

We substitute Eqs. (6.1) and (6.3) into (2.12c), and let

$$D(\xi) = Q^{-1}\begin{pmatrix} g(\xi^{-1}) & 0 \\ 0 & g^{-1}(\xi^{-1})\end{pmatrix}K^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \times K\begin{pmatrix} g^{-1}(\xi) & 0 \\ 0 & g(\xi)\end{pmatrix}Q\,, \tag{6.4}$$

to obtain

$$M(\xi^{-1}) = L^{-1}(\xi) = [a_-(\xi)]^{-1}\tilde{a}_+(\xi) = KQ\begin{pmatrix} h(\xi^{-1}) & 0 \\ 0 & h^{-1}(\xi^{-1})\end{pmatrix}D(\xi)\begin{pmatrix} h^{-1}(\xi) & 0 \\ 0 & h(\xi)\end{pmatrix} \times Q^{-1}K^{-1}\begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}. \tag{6.5}$$

Consequently,

$$M(\xi^{-1})L(\eta) = KQ\begin{pmatrix} h(\xi^{-1}) & 0 \\ 0 & h^{-1}(\xi^{-1})\end{pmatrix} \times D(\xi)\begin{pmatrix} h^{-1}(\xi)h(\eta) & 0 \\ 0 & h(\xi)h^{-1}(\eta)\end{pmatrix} \times D^{-1}(\eta)\begin{pmatrix} h^{-1}(\eta^{-1}) & 0 \\ 0 & h(\eta^{-1})\end{pmatrix}Q^{-1}K^{-1} \tag{6.6}$$

and

$$\begin{aligned}\mathrm{Tr}[M(\xi^{-1})L(\eta)] &= h(\xi^{-1})D(\xi)_{11}h^{-1}(\xi)h(\eta)D^{-1}(\eta)_{11}h^{-1}(\eta^{-1}) \\ &+ h(\xi^{-1})D(\xi)_{12}h(\xi)h^{-1}(\eta)D^{-1}(\eta)_{21}h^{-1}(\eta^{-1}) \\ &+ h^{-1}(\xi^{-1})D(\xi)_{21}h^{-1}(\xi)h(\eta)D^{-1}(\eta)_{12}h(\eta^{-1}) \\ &+ h^{-1}(\xi^{-1})D(\xi)_{22}h(\xi)h^{-1}(\eta)D^{-1}(\eta)_{22}h(\eta^{-1})\,.\end{aligned} \tag{6.7}$$

Consider now the integral

$$\mathrm{Tr}\Omega = \frac{1}{(2\pi i)^2}\oint_{|\xi|=1} d\xi \oint_{|\eta|=1} d\eta\left(\frac{\xi}{\eta}\right)^N(\eta-\xi)^{-2} \times \mathrm{Tr}[M(\xi^{-1})L(\eta)]\,. \tag{6.8}$$

Since the singularities of the function $g^{\pm1}(\xi_{\pm1})$ and $h^{\pm1}(\xi^{\pm1})$ are the eight branch points at $\gamma_1^{\pm1}$, $\gamma_2^{\pm1}$, $\gamma_3^{\pm1}$, and $\gamma_4^{\pm1}$, the integrand which is a product of these functions, has cuts from γ_2 to γ_3, and from γ_4 to γ_1^{-1}, inside the unit circle and cuts from γ_1 to γ_4^{-1}, and γ_3^{-1} to γ_2^{-1} outside the unit circle. We deform the contour of the integration variable η to be the lines above and below the cuts outside the unit circle, and deform the contour of integration of ξ to be the lines above and below the cuts inside the unit circle.

One can see readily that the integration of η from γ_3^{-1}, to γ_4^{-1} gives rise to a term proportional to γ_3^N, while the integration of η from γ_1 to γ_4^{-1} gives rise to a term proportional to γ_1^{-N}. Hence for large N, we can neglect the former. For the same reason the integration of ξ from γ_2 to γ_3 is neglected. Now the integral (6.8) becomes

$$\mathrm{Tr}\Omega = -\frac{1}{\pi^2}\int_{\gamma_4}^{\gamma_1^{-1}} d\xi \int_{\gamma_1}^{\gamma_4^{-1}} d\eta\left(\frac{\xi}{\eta}\right)^N(\eta-\xi)^{-2} \times \mathrm{Tr}[M(\xi^{-1})L(\eta)]. \qquad (6.9)$$

Note that $h(\xi^{-1})$ is divergent at $\xi=\gamma_1^{-1}$ and $h(\eta)$ is divergent at $\eta=\gamma_1$.

One can show that

$$D^{-1}(\xi)=D(\xi^{-1}) \qquad (6.10)$$

and

$$D_{11}(\gamma_1^{-1})=D_{11}^{-1}(\gamma_1)=0. \qquad (6.11)$$

Therefore, there exists a function $R_{11}(\xi)$ such that

$$D_{11}(\xi)=(1-\gamma_1\xi)R_{11}(\xi) \qquad (6.12a)$$

and

$$D_{11}^{-1}(\eta)=D_{11}(\eta^{-1})=(1-\gamma_1\eta^{-1})R_{11}(\eta^{-1}). \qquad (6.12b)$$

Substituting Eq. (6.12) into (6.7), we find that the second and third terms in Eq. (6.7) are the most singular terms of the integrand.

Let

$$R(\xi,\eta)=(1-\gamma_1\xi)^{1/2}(1-\gamma_1\eta^{-1})^{1/2}(\eta-\xi)^{-2} \times[h(\xi^{-1})D_{12}(\xi)h(\xi)h^{-1}(\eta)D_{21}(\eta^{-1})h^{-1}(\eta^{-1}) + h^{-1}(\xi^{-1})D_{21}(\xi)h^{-1}(\xi)h(\eta)D_{12}(\eta^{-1})h(\eta^{-1})]. \qquad (6.13)$$

The integral I_2 due to the second and third terms of Eq. (6.7) is

$$I_2=\frac{-1}{\pi^2}\int_{\gamma_4}^{\gamma_1^{-1}} d\xi\int_{\gamma_1}^{\gamma_4^{-1}} d\eta\left(\frac{\xi}{\eta}\right)^N \times R(\xi,\eta)(1-\gamma_1\xi)^{-1/2}(1-\gamma_1\eta^{-1})^{-1/2}. \qquad (6.14)$$

Expanding $R(\xi,\eta)$ about $\xi=\gamma_1^{-1}$ and $\eta^{-1}=\gamma_1^{-1}$, we have

$$R(\xi,\eta)=\sum_{l,m=0}^{\infty}R_{lm}(1-\gamma_1\xi)^l(1-\gamma_1\eta^{-1})^m, \qquad (6.15)$$

where $(-\gamma_1)^{-l-m}R_{lm}$ are the Taylor coefficient of the above expansion. We find

$$R_{00}=0, \qquad (6.16a)$$

$$R_{10}=R_{01}=D_{12}(\gamma_1^{-1})D_{21}(\gamma_1^{-1})(\gamma_1-\gamma_1^{-1})^{-2}. \qquad (6.16b)$$

Consequently, Eq. (6.14) becomes

$$I_2=\frac{-1}{\pi^2}\sum_{l,m=0}^{\infty}R_{lm}I_l\tilde{I}_m, \qquad (6.17)$$

where

$$I_l=\int_{\gamma_4}^{\gamma_1^{-1}} d\xi\,\xi^N(1-\gamma_1\xi)^{l-1/2} \sim\gamma_1^{-(N+1)}\beta(l+\tfrac{1}{2},N+1) \qquad (6.18a)$$

and

$$\tilde{I}_m=\int_{\gamma_1}^{\gamma_4^{-1}} d\eta\,\eta^{-N}(1-\gamma_1\eta^{-1})^{m-1/2} \sim\gamma_1^{-(N-1)}\beta(m+\tfrac{1}{2},N-1). \qquad (6.18b)$$

For large N, the β functions in Eqs. (6.18) behave as

$$\beta(m+\tfrac{1}{2},N+1)\sim\beta(m+\tfrac{1}{2},N-1)\sim\Gamma(m+\tfrac{1}{2})/N^{1/2+m}. \qquad (6.19)$$

Substituting Eqs. (6.16), (6.18), and (6.19) into (6.17), we get

$$I_2\sim(1/\pi^2)D_{12}(\gamma_1^{-1})D_{21}(\gamma_1^{-1})(\gamma_1-\gamma_1^{-1})^{-2} \times(I_1\tilde{I}_0+I_0\tilde{I}_1)+\cdots\sim(-1/\pi)D_{12}(\gamma_1^{-1})D_{21}(\gamma_1^{-1}) \times(\gamma_1-\gamma_1^{-1})^{-2}\gamma_1^{-2N}/N^2+\gamma_1^{-2N}\times O(1/N^3). \qquad (6.20)$$

Since $\det D(\xi)=-1$, and $D_{11}(\gamma_1^{-1})=0$, we find

$$D_{12}(\gamma_1^{-1})D_{21}(\gamma_1^{-1})=1. \qquad (6.21)$$

Clearly, because $D_{11}(\gamma_1^{-1})=0$, the first term in Eq. (6.7) gives rise to a term of the order γ_1^{-2N}/N^3; and the integral due to the last term of Eq. (6.7) is also of the order γ_1^{-2N}/N^3. Hence we get

$$\mathrm{Tr}\Omega=\frac{(-1/\pi N^2)\gamma_1^{-2N}}{(\gamma_1-\gamma_1^{-1})^2}+\gamma_1^{-2N}\times O\left(\frac{1}{N^3}\right).$$

Therefore, the spin correlation for large N at $T<T_c$ is

$$\langle\sigma_{00}\sigma_{0N}\rangle=D_N[a]^{1/2}\sim M^2(1-\tfrac{1}{2}\mathrm{Tr}\Omega) \sim M^2\left(1+\frac{1}{2\pi N^2}\frac{\gamma_1^{-2N}}{(\gamma_1-\gamma_1^{-1})^2}+\gamma_1^{-2N}\times O\left(\frac{1}{N^3}\right)\right), \qquad (6.22)$$

where M is the spontaneous magnetization given by Eq. (5.49).

VII. SPIN-CORRELATION FUNCTION FOR $T>T_c$

When $T>T_c$, $\gamma_i<1$ $(i=1,\ldots,4)$, the matrix $a(\xi)$ is not factorizable. To calculate the asymptotic behavior of $\langle\sigma_{00}\sigma_{0N}\rangle$, let us consider the matrix

$$s(\xi)=\begin{pmatrix}\xi b(\xi)_{11} & b_{12}(\xi)\\ b_{21}(\xi) & \xi^{-1}b(\xi)_{22}\end{pmatrix}. \qquad (7.1)$$

Let

$$r(\xi)=s(\xi)/\chi(\xi). \qquad (7.2)$$

Clearly,

$$r(\xi)=\begin{pmatrix}\xi a_{11}(\xi) & a_{12}(\xi)\\ a_{21}(\xi) & \xi^{-1}a_{22}(\xi)\end{pmatrix}. \qquad (7.3)$$

The matrix $s(\xi)$ still has the form

$$s(\xi)=\sum_{j=-2}^{2}s_j\xi^j \qquad (7.4)$$

since $\det s(\xi)=\det b(\xi)$, $\det s(\xi)$ also has the eight roots $(\gamma_i^{\pm 1};\ i=1,\ldots,4;\ \gamma_i<1)$.

It shall be shown that the matrix $r(\xi)$ admits a factorization. Therefore we can use the method described in Sec. IV, case (d) to calculate the block Toeplitz determinant $D_N[r]$.

In the following paragraph we shall derive the relation of the block Toeplitz determinant $D_N[a]$ to the block Toeplitz determinant $D_N[r]$.

Since

$$r_l=\frac{1}{2\pi}\int_0^{2\pi} e^{-il\theta} r(e^{i\theta})\,d\theta \tag{7.5}$$

and

$$a_l=\frac{1}{2\pi}\int_0^{2\pi} e^{-il\theta} a(e^{i\theta})\,d\theta \tag{7.6}$$

by Eq. (7.3) we have

$$(a_l)_{11}=(r_{l+1})_{11},\quad (a_l)_{12}=(r_l)_{12}$$

and (7.7)

$$(a_l)_{21}=(r_l)_{21},\quad (a_l)_{22}=(r_{l-1})_{22}\,.$$

Accordingly,

$$D_N[a]=\begin{vmatrix} (r_1)_{11} & (r_0)_{12} & (r_0)_{11} & (r_{-1})_{12} & \cdots & (r_{-N+2})_{11} & (r_{-N+1})_{12}\\ (r_0)_{21} & (r_{-1})_{22} & (r_{-1})_{21} & (r_{-2})_{22} & \cdots & (r_{-N+1})_{21} & (r_{-N})_{22}\\ (r_2)_{11} & (r_1)_{12} & (r_1)_{11} & (r_0)_{12} & \cdots & (r_{-N+3})_{11} & (r_{-N+2})_{12}\\ (r_1)_{21} & (r_0)_{22} & (r_0)_{21} & (r_{-1})_{22} & \cdots & (r_{-N+2})_{21} & (r_{-N+1})_{22}\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ (r_N)_{11} & (r_{N-1})_{12} & (r_{N-1})_{11} & (r_{N-2})_{12} & \cdots & (r_1)_{11} & (r_0)_{12}\\ (r_{N-1})_{21} & (r_{N-2})_{22} & (r_{N-2})_{21} & (r_{N-3})_{22} & \cdots & (r_0)_{21} & (r_{-1})_{22} \end{vmatrix}\,. \tag{7.8}$$

Interchange the $(2j-1)$th row with the $2j$th row for $j=1,2,\ldots,N$, and the interchange the $2j$th column with the $(2j+1)$th column for $j=1,2,\ldots,N-1$, the $2N\times 2N$ determinant in Eq. (7.8) becomes

$$D_N[a]=(-1)^N(-1)^{N-1}\begin{vmatrix} (r_0)_{21} & (r_{-1})_{21} & (r_{-1})_{22} & \cdots & (r_{-N+1})_{21} & (r_{-N+1})_{22} & (r_{-N})_{22}\\ (r_1)_{11} & (r_0)_{11} & (r_0)_{12} & \cdots & (r_{-N+2})_{11} & (r_{-N+2})_{12} & (r_{-N+1})_{12}\\ (r_1)_{21} & (r_0)_{21} & (r_0)_{22} & \cdots & (r_{-N+2})_{21} & (r_{-N+2})_{22} & (r_{-N+1})_{22}\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ (r_{N-1})_{11} & (r_{N-2})_{11} & (r_{N-2})_{12} & \cdots & (r_0)_{11} & (r_0)_{12} & (r_{-1})_{12}\\ (r_{N-1})_{21} & (r_{N-2})_{21} & (r_{N-2})_{22} & \cdots & (r_0)_{21} & (r_0)_{12} & (r_{-1})_{22}\\ (r_N)_{11} & (r_{N-1})_{11} & (r_{N-1})_{12} & \cdots & (r_1)_{11} & (r_1)_{12} & (r_0)_{12} \end{vmatrix}\,. \tag{7.9}$$

Comparing with the $2(N+1)\times 2(N+1)$ determinant $D_{N+1}[r]=\det T$, [T denotes the $2(N+1)\times 2(N+1)$ matrix generated by $r(\xi)$], we find that $(-1)D_N[a]$ is a co-factor of $|T_{1,2}T_{2N,2N-1}|$. [That is, $(-1)D_N[a]$ can be obtained from $D_{N+1}[r]$ by striking out the first row, the last row ($2N$th row), the second column and the $(2N-1)$th column of the matrix T.]

Consider the following systems of linear equations:

$$\sum_{m=0}^{N} r_{l,m}\begin{pmatrix} x_m^1\\ x_m^2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\delta_{l,0},\quad l=0,\ldots,N, \tag{7.10}$$

and

$$\sum_{m=0}^{N} r_{l,m}\begin{pmatrix} y_m^1\\ y_m^2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\delta_{l,N},\quad l=0,\ldots,N. \tag{7.11}$$

Using Cramer's rule, we find from Eq. (7.10),

$$x_0^2=\operatorname{cof}|T_{1,2}|/\det T \tag{7.12a}$$

and

$$x_N^1=\operatorname{cof}|T_{1,2N+1}|/\det T, \tag{7.12b}$$

and from (7.11), we have

$$y_0^2=\operatorname{cof}|T_{2N+2,2}|/\det T \tag{7.12c}$$

and

$$y_N^1=\operatorname{cof}|T_{2N,2N-1}|/\det T. \tag{7.12d}$$

By Jacobi's theorem[13] on minors of the adjugate, we find

$$X=\begin{vmatrix} x_2^0 & x_N^1\\ y_0^2 & y_N^1\end{vmatrix}=\frac{-D_N[a]}{\det T}=\frac{-D_N[a]}{D_{N+1}[r]}\,. \tag{7.13}$$

We now may proceed to factorize the matrix $r(\xi)$. Note that

$$s_{11}(\gamma_3)/s_{21}(\gamma_3)=\gamma_3\omega$$

and

$$s_{11}(\gamma_4)/s_{21}(\gamma_4)=-\gamma_4\omega\,;$$

therefore,

$$s_{11}(\gamma_3)/s_{21}(\gamma_3) \neq s_{11}(\gamma_4)/s_{21}(\gamma_4).$$

Let

$$v(\xi) = \begin{pmatrix} (1-\gamma_3\xi^{-1})^{-1} & 0 \\ 0 & (1-\gamma_4\xi^{-1})^{-1} \end{pmatrix} R^{-1} s(\xi), \quad (7.14)$$

where

$$R^{-1} = (1+\gamma_3\gamma_4^{-1})^{-1/2} \begin{pmatrix} 1 & -\gamma_3\omega \\ (\gamma_4\omega)^{-1} & 1 \end{pmatrix}. \quad (7.15)$$

One finds

$$\frac{v_{11}(\gamma_1)}{v_{21}(\gamma_1)} = \frac{1-\gamma_4\gamma_1^{-1}}{1-\gamma_3\gamma_1^{-1}} \frac{\gamma_1\nu - \gamma_3\omega}{\gamma_1\nu+\gamma_4\omega} (\gamma_4\omega), \quad (7.16a)$$

and

$$\frac{v_{21}(\gamma_2)}{v_{11}(\gamma_2)} = \frac{1-\gamma_3\gamma_2^{-1}}{1-\gamma_4\gamma_2^{-1}} \frac{\gamma_2\nu - \gamma_4\omega}{\gamma_2\nu+\gamma_3\omega} (\gamma_4\omega)^{-1} \quad (7.16b)$$

and it can be shown that

$$v_{11}(\gamma_1)/v_{21}(\gamma_1) \neq v_{11}(\gamma_2)/v_{21}(\gamma_2). \quad (7.17)$$

Let us denote

$$W_{12} = v_{11}(\gamma_1)/v_{21}(\gamma_1), \quad (7.16c)$$

$$W_{21} = v_{21}(\gamma_2)/v_{11}(\gamma_2), \quad (7.16d)$$

and

$$W^{-1} = (1-W_{12}W_{21})^{-1/2} \begin{pmatrix} 1 & -W_{12} \\ -W_{21} & 1 \end{pmatrix}. \quad (7.18)$$

According to Sec. IV, case (a), the matrix

$$V(\xi) = \begin{pmatrix} (1-\gamma_1\xi^{-1})^{-1} & 0 \\ 0 & (1-\gamma_2\xi^{-1})^{-1} \end{pmatrix} W^{-1} v(\xi) \quad (7.19)$$

is a 2×2 matrix whose elements are quadratic polynomials in ξ and $\det V(\xi) \neq 0$ for $|\xi| < 1$.

Consequently, the matrix $r(\xi)$ admits a right factorization

$$r(\xi) = r_-(\xi) r_+(\xi), \quad (7.20)$$

where

$$r_+(\xi) = RWV(\xi)/\chi_+(\xi) \quad (7.21a)$$

and

$$r_-(\xi) = R \begin{pmatrix} 1-\gamma_3\xi^{-1} & 0 \\ 0 & 1-\gamma_4\xi^{-1} \end{pmatrix} \times W \begin{pmatrix} 1-\gamma_1\xi^{-1} & 0 \\ 0 & 1-\gamma_2\xi^{-1} \end{pmatrix} W^{-1}R^{-1}/\chi_-(\xi), \quad (7.21b)$$

with

$$\chi_+(\xi) = z_1^2(1-z_2^2)(1-z_2'^2) \times [(\xi-\gamma_1^{-1})(\xi-\gamma_2^{-1})(\xi-\gamma_3^{-1})(\xi-\gamma_4^{-1})]^{1/2} \quad (7.22)$$

and

$$\chi_-(\xi) = [(1-\gamma_1\xi^{-1})(1-\gamma_2\xi^{-1})(1-\gamma_3\xi^{-1})(1-\gamma_4\xi^{-1})]^{1/2}.$$

From Eqs. (7.3) and (5.31), we have

$$r(\xi^{-1}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} r(\xi) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (7.23)$$

Hence, $r(\xi)$ admits a left factorization

$$r(\xi) = \tilde{r}_+(\xi)\tilde{r}_-(\xi), \quad (7.24a)$$

where

$$\tilde{r}_+(\xi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} r_-(\xi^{-1}) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (7.24b)$$

and

$$\tilde{r}_-(\xi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} r_+(\xi^{-1}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (7.24c)$$

We can now use the method of Ref. 8 to obtain the asymptotic expansion of x_0^2, x_N^1, y_0^2, and y_N^1 for large N. We find

$$x_0^2 = -y_N^1 = [r_+^{-1}(0)]_{21} - \{[r_+^{-1}(0)]_{21}\Lambda_{11} + [r_+^{-1}(0)]_{22}\Lambda_{21}\}, \quad (7.25)$$

where

$$\Lambda = \frac{1}{(2\pi i)^2} \oint_{|\xi|=1} M(\xi^{-1})\xi^N d\xi \times \oint_{|\eta|=1} (\eta-\xi)^{-1}\eta^{-N-1}L(\eta)\,d\eta, \quad (7.26)$$

with

$$M(\xi^{-1}) = L^{-1}(\xi) = [r_-(\xi)]^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_-(\xi^{-1}); \quad (7.27)$$

and

$$x_N^1 = -y_0^2 = [r_+^{-1}(0)]_{21}\Sigma_{11} + [r_+^{-1}(0)]_{22}\Sigma_{21}, \quad (7.28)$$

where

$$\Sigma = -[1/(2\pi i)]\oint_{|\xi|=1} \xi^{N-1}L(\xi^{-1})\,d\xi. \quad (7.29)$$

From Eq. (7.20), we have

$$r_+^{-1}(0) = r^{-1}(0) r_-(0). \quad (7.30a)$$

Using Eqs. (7.21b) and (7.30), one can show

$$[r_+^{-1}(0)]_{21} \equiv (-z_1)(1+\gamma_3\gamma_4^{-1})^{-1} \times (1-W_{12}W_{21})^{-1}(\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1})^{1/2} \times [(1+\gamma_1^{-1}\gamma_2)(1-W_{12}W_{21}) + (1-\gamma_1\gamma_2^{-1})(\gamma_4\omega W_{21} - W_{12}/\gamma_4\omega)] = 0 \quad (7.30b)$$

and

$$[r_+^{-1}(0)]_{22} \equiv z_1(\gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1})^{1/2}$$

$$\times(1+\gamma_3\gamma_4^{-1})^{-1}(1-W_{12}W_{21})^{-1}$$
$$\times[(1-\gamma_1^{-1}\gamma_2\gamma_3^{-1}\gamma_4)+(\gamma_3^{-1}\gamma_4-\gamma_1\gamma_2^{-1})W_{12}W_{21}$$
$$+(1-\gamma_1\gamma_2^{-1})(\gamma_4\omega W_{21}+W_{12}/\gamma_4\omega)]. \quad (7.30c)$$

Therefore, Eqs. (7.25) and (7.28) become

$$x_0^2=-y_N^1=-[r_+^{-1}(0)]_{22}\Lambda_{21} \quad (7.31a)$$

and

$$x_N^1=-y_0^2=[r_+^{-1}(0)]_{22}\Sigma_{21}. \quad (7.31b)$$

It is evident from Sec. VI, that Λ_{ij} (which is a double integral) is of the order γ_1^{2N}/N; whereas Σ_{ij} (which is a single integral) is of the order $\gamma_1^N/\sqrt{N}$. Hence X, given by Eq. (7.13), can be written

$$X\sim(x_N^1)^2+O(\gamma_1^{4N}/N^2). \quad (7.32)$$

It is also clear, from Sec. VI, that

$$D_N[r]\sim D[r][1-O(\gamma_1^{2N}/N)], \quad (7.33)$$

where

$$D(r)=\lim_{N\to\infty}D_N[r]. \quad (7.34)$$

Consequently, Eq. (7.13) becomes

$$D_N[a]\sim-D[r](x_N^1)^2+O(\gamma_1^{4N}/N^2). \quad (7.35)$$

The spin-correlation function for large N at $T>T_c$ can now be written

$$\langle\sigma_{00}\sigma_{0N}\rangle\sim(-D[r])^{1/2}|x_N^1|$$
$$\sim(-D[r])^{1/2}[r_+^{-1}(0)]_{22}\Sigma_{21}. \quad (7.36)$$

We now calculate the block Toeplitz determinant generated by $r(\xi)$, in the limit $N\to\infty$.

Because

$$\frac{v_{11}(\gamma_1)}{v_{12}(\gamma_1)}=\frac{s_{11}(\gamma_1)}{s_{12}(\gamma_1)}=\frac{\gamma_1}{\nu}$$

and

$$\frac{v_{11}(\gamma_2)}{v_{12}(\gamma_2)}=\frac{s_{11}(\gamma_2)}{s_{12}(\gamma_2)}=\frac{-\gamma_2}{\nu},$$

we find

$$\frac{v_{11}(\gamma_1)}{v_{12}(\gamma_1)}\neq\frac{v_{11}(\gamma_2)}{v_{12}(\gamma_2)}.$$

Therefore,

$$Z(\xi)=v(\xi)R_2^{-1}\begin{pmatrix}(1-\gamma_1\xi^{-1})^{-1} & 0\\ 0 & (1-\gamma_2\xi^{-1})^{-1}\end{pmatrix}, \quad (7.37a)$$

with

$$R_2^{-1}=(1+\gamma_1\gamma_2^{-1})^{-1/2}\begin{pmatrix}1 & \nu/\gamma_2\\ -\gamma_1/\nu & 1\end{pmatrix}, \quad (7.37b)$$

is a matrix function whose determinant $\det Z(\xi)\neq0$ for $|\xi|<1$ and whose elements are quadratic polynomials in ξ.

Combining Eqs. (7.14) and (7.36), we get

$$s(\xi)=R\begin{pmatrix}(1-\gamma_3\xi^{-1}) & 0\\ 0 & (1-\gamma_4\xi^{-1})\end{pmatrix}$$
$$\times Z(\xi)\begin{pmatrix}(1-\gamma_1\xi^{-1}) & 0\\ 0 & (1-\gamma_2\xi^{-1})\end{pmatrix}R_2. \quad (7.38)$$

Similar to Eq. (5.46), we find

$$D[r]=\lim_{N\to\infty}D_N[r]$$
$$=[Z_{11}(\gamma_2)Z_{22}(\gamma_1)-Z_{21}(\gamma_2)Z_{12}(\gamma_1)]$$
$$\times[Z_{11}(\gamma_4)Z_{22}(\gamma_3)-Z_{12}(\gamma_4)Z_{21}(\gamma_3)]/$$
$$\chi_+(\gamma_1)\chi_+(\gamma_2)\chi_+(\gamma_3)\chi_+(\gamma_4). \quad (7.39)$$

One can easily show that

$$1-\frac{Z_{12}(\gamma_1)}{Z_{22}(\gamma_1)}\frac{Z_{21}(\gamma_2)}{Z_{11}(\gamma_2)}=W_{12}W_{21}, \quad (7.40)$$

and also show

$$1-\frac{Z_{22}(\gamma_3)}{Z_{21}(\gamma_3)}\frac{Z_{12}(\gamma_4)}{Z_{11}(\gamma_4)}$$
$$=\frac{(\gamma_1\nu+\gamma_4\omega)(\gamma_2\nu+\gamma_3\omega)}{(-z_1^2)(\gamma_1\omega+\gamma_4\nu)(\gamma_2\omega+\gamma_3\nu)}(1-W_{12}W_{21}). \quad (7.41)$$

From Eq. (7.38), we find

$$Z_{11}(\gamma_2)Z_{22}(\gamma_1)Z_{11}(\gamma_4)Z_{22}(\gamma_3)\equiv[(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}b_{11}(\gamma_1)b_{11}(\gamma_2)b_{11}(\gamma_3)b_{11}(\gamma_4)$$
$$\times(\gamma_2\nu+\gamma_3\omega)(\gamma_1\nu+\gamma_4\omega)(\gamma_2\omega+\gamma_3\nu)(\gamma_1\omega+\gamma_4\nu)\nu^{-2}\omega^{-2}]/[\gamma_2\gamma_3^{-1}(1-\gamma_1\gamma_2^{-1})(1-\gamma_3\gamma_4^{-1})(1-\gamma_3\gamma_2^{-1})(1-\gamma_4\gamma_1^{-1})]^2. \quad (7.42)$$

Substituting Eqs. (7.40), (7.41), and (7.42) into (7.39), we get

$$(-D[r])^{1/2}\equiv\left(\prod_{i=1}^{4}\frac{\gamma_i^{-1}b_{11}(\gamma_i)}{\chi_+(\gamma_i)}\right)^{1/2}\frac{(\gamma_1\nu+\gamma_4\omega)(\gamma_2\nu+\gamma_3\omega)(1-W_{12}W_{21})}{z_1(1-\gamma_1\gamma_2^{-1})(1-\gamma_3\gamma_4^{-1})(1-\gamma_3\gamma_2^{-1})(1-\gamma_4\gamma_1^{-1})\nu\omega\gamma_2\gamma_3^{-1}}. \quad (7.43)$$

When $z_2\to z_2'$, we find from Eqs. (7.30c) and (7.43) that[7]

$$(-D[r])^{1/2}[(r_+^{-1}(0)]_{22}=[(1-\gamma_1^2)(1-\gamma_2^2)]^{1/4}(1-\gamma_1\gamma_2)^{1/2}. \quad (7.44)$$

We shall now calculate Σ_{21} in Eq. (7.36), using the same method we have used in Sec. VI. We deform the contour of integration to be the lines above and below the cuts inside the unit circle, and neglect the integration of ξ from γ_2 to γ_3. Now Eq. (7.29) becomes

$$\Sigma_{21} \sim \frac{-i}{\pi i} \int_{\gamma_4}^{\gamma_1} d\xi\, \xi^{N-1} L(\xi^{-1})_{21}. \tag{7.45}$$

It is evident from Eqs. (7.27) and (7.21b), that $[i(1-\gamma_1^{-1}\xi)^{1/2}L(\xi^{-1})_{21}]$ is finite in the interval $\xi\epsilon(\gamma_4, \gamma_1)$. We can expand this function about $\xi=\gamma_1$, so that

$$L(\xi^{-1})_{21} = -i(1-\gamma_1^{-1}\xi)^{-1/2}[A_0 + A_1(\xi-\gamma_1)+\cdots]. \tag{7.46}$$

Substituting into Eq. (7.45), we obtain,

$$\Sigma_{21} \sim A_0\gamma_1^N/(\pi N)^{1/2} + \gamma_1^N \times O(N^{-3/2}), \tag{7.47}$$

where

$$A_0 = \frac{(1-\gamma_2\gamma_1^{-1})^{1/2}}{(1-\gamma_1^2)^{1/2}(1-\gamma_1\gamma_2)^{1/2}} \frac{1-\gamma_4\omega W_{21}}{(1+\gamma_3\gamma_4^{-1})^2(1-W_{12}W_{21})^2} \times\{(1-\gamma_1^2)[1-(\gamma_4\omega)^{-1}W_{12}]M_{11}(\gamma_1) + (1-\gamma_1\gamma_2)[1-\gamma_4\omega W_{21}]M_{12}(\gamma_1)\}, \tag{7.48}$$

with

$$M_{11}(\gamma_1) = \{g^{-1}(\gamma_1)g(\gamma_1^{-1})[\gamma_3\gamma_4^{-1}+(\gamma_4\omega)^{-2}] + g^{-1}(\gamma_1)g^{-1}(\gamma_1^{-1})(\gamma_4\omega)^{-1}W_{12}[1-(\gamma_4\omega)^{-2}] + g(\gamma_1)g(\gamma_1^{-1})[\gamma_3^2\gamma_4^{-2}-(\gamma_4\omega)^{-2}](\gamma_4\omega)W_{21} + g(\gamma_1)g^{-1}(\gamma_1^{-1})W_{21}W_{12}[\gamma_3\gamma_4^{-1}+(\gamma_4\omega)^{-2}]\} \tag{7.49a}$$

and

$$M_{12}(\gamma_1) = \{g^{-1}(\gamma_1)g(\gamma_1^{-1})(\gamma_4\omega)^{-1}W_{12}[\gamma_3\gamma_4^{-1}+(\gamma_4\omega)^{-2}] + g^{-1}(\gamma_1)g^{-1}(\gamma_1^{-1})(\gamma_4\omega)^{-2}W_{12}^2[1-(\gamma_4\omega)^{-2}] + g(\gamma_1)g(\gamma_1^{-1})[\gamma_3^2\gamma_4^{-2}-(\gamma_4\omega)^{-2}] + g(\gamma_1)g^{-1}(\gamma_1^{-1})(\gamma_4\omega)^{-1}W_{12}[\gamma_3\gamma_4^{-1}+(\gamma_4\omega)^{-2}]\}. \tag{7.49b}$$

The function $g(\xi)$ in Eq. (7.49) is given by

$$g(\xi) = [(1-\gamma_4\xi)/(1-\gamma_3\xi)]^{1/2}. \tag{7.50}$$

In Eqs. (7.48) and (7.49), W_{12} and W_{21} is given by Eq. (7.16); while ω and ν are given by Eqs. (5.16)–(5.19).

Substituting Eq. (7.47) into (7.36), we have

$$\langle\sigma_{00}\sigma_{0N}\rangle \sim \frac{(-D[r])^{1/2}[r_+^{-1}(0)]_{22}A_0\gamma_1^N}{(\pi N)^{1/2}} + O\left(\frac{\gamma_1^N}{N^{3/2}}\right) \tag{7.51}$$

for $T > T_c$. When $E_2 \to E_2'$ $(z_2 \to z_2')$, we find

$$M_{11}(\gamma_1) = M_{12}(\gamma_1) = 4$$

and

$$A_0 = (1-\gamma_1^2)^{-1/2}(1-\gamma_1\gamma_2)^{-1/2}(1-\gamma_1^{-1}\gamma_2)^{-1/2}. \tag{7.52}$$

Combining with Eq. (7.44), we get

$$\langle\sigma_{00}\sigma_{0N}\rangle \sim [1-\gamma_2^2]^{1/4}[1-\gamma_1^2]^{-1/4} \times[1-\gamma_1^{-1}\gamma_2]^{-1/2}\gamma_1^N/(\pi N)^{1/2} \tag{7.53}$$

for $E_2 = E_2'$. This is exactly the result of Wu.[7]

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APPENDIX

Here we shall prove the following lemma.

Lemma. Let $f(e^{i\theta})$ be a function whose Fourier coefficients f_{-j} vanish for $j>n$, and $f_{-n}\neq 0$. If $f(\xi)$ admit a factorization, $f(\xi)$ has exactly n roots inside the unit circle.

Proof. If $f(\xi)$ admits a factorization

$$f(\xi) = f_+(\xi)f_-(\xi), \quad |\xi|=1, \tag{A1}$$

where $f_+(\xi)$ $(f_-(\xi))$ is an analytic and invertible function for $|\xi|<1$ $(|\xi|>1)$, then we can write,

$$f_+(\xi) = \sum_{l=0}^{\infty}(f_+)_l\xi^l, \quad |\xi|\leq 1 \tag{A2}$$

and

$$f_-(\xi) = \sum_{l=0}^{\infty}(f_-)_l\xi^{-l}, \quad |\xi|\geq 1. \tag{A3}$$

Furthermore,

$$f_+(\xi)\neq 0 \ \text{ for } |\xi|<1 \tag{A4}$$

and

$$f_-(\xi)\neq 0 \ \text{ for } |\xi|>1. \tag{A5}$$

In particular, $f_+(0)\neq 0$, therefore $(f^+)_0\neq 0$.

Since $f_{-j}\equiv 0$ for $j>n$, we can write

$$f(e^{i\theta}) = \sum_{j=-n}^{\infty} f_j e^{ij\theta}. \tag{A6}$$

Hence Eq. (A1) becomes

$$\sum_{j=-n}^{\infty} f_j\xi^j = \sum_{l=0}^{\infty}(f_+)_l\xi^l \sum_{m=0}^{\infty}(f_-)_m\xi^{-m}. \tag{A7}$$

As the coefficients of ξ^{-k} must vanish for $k>n$ on the right-hand side also, we have

$$\sum_{l \geq k}^{\infty} (f_+)_{l-k}(f_-)_l \equiv 0 \quad \text{for } k \geq n+1 . \tag{A8}$$

Consider $(f_-)_l$ $(l = n+1,\ n+2, \ldots, M)$ as solutions of the following system of homogeneous linear equations:

$$\sum_{l \geq k}^{M} (f_+)_{l-k}(f_-)_l = 0 \quad \text{for } M \geq k \geq n+1 , \tag{A9}$$

where M is any integer greater than n.

To clarify, let us rewrite Eq. (A9)

$$\begin{aligned}(f_+)_0(f_-)_{n+1} + (f_+)_1(f_-)_{n+2} + \cdots + (f_+)_{M-n-1}(f_-)_M &= 0 ,\\ (f_+)_0(f_-)_{n+2} + \cdots + (f_+)_{M-n-2}(f_-)_M &= 0 ,\\ &\ \vdots \\ (f_+)_0(f_-)_M &= 0 ,\end{aligned} \tag{A10}$$

Since the coefficients of Eq. (A10) form a triangular matrix whose diagonal elements are all equal to $(f_+)_0$, because $(f_+)_0 \neq 0$, the only solution to this homogeneous equations are the trivial solution; i.e., $(f_-)_{n+1} = (f_-)_{n+2} = \cdots = (f_-)_M = 0$. This is true for any integer $M \geq n+1$. Therefore, we conclude that the only solution of the homogeneous equation (A8) is the trivial solution $(f_-)_{n+l} = 0\,(1 \leq l < \infty)$.

Consequently,

$$f_-(\xi) = \sum_{l=0}^{n} (f_-)_l \xi^{-l} . \tag{A11}$$

Now we rewrite Eq. (A7)

$$\sum_{j=-n}^{\infty} f_j \xi^j = (f_+)_0 \sum_{m=0}^{n} (f_-)_m \xi^{-m} + (f_+)_1 \xi \sum_{m=0}^{n} (f_-)_m \xi^{-m} + \cdots .$$

Since $f_{-n} \neq 0$, we must have $(f_-)_n \neq 0$. This implies that $f_-(\xi)$ has n roots. Because $f_-(\xi) \neq 0$ for $|\xi| > 1$, these n roots must lie inside the unit circle. Therefore $f(\xi) = f_+(\xi) f_-(\xi)$ must have n roots inside the circle. Since $f_+(\xi) \neq 0$ for $|\xi| < 1$, $f(\xi)$ has exactly n roots inside the unit circle.

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$$\sum_{m=0}^{\infty} a_{l-m} X_m = \delta_{l,0} I, \quad l = 0,\ \infty,$$

where X_m $(m = 0,\ \infty)$ are $r \times r$ matrices; one can show Eq. (2.7) holds whenever $a(\xi)$ admits a right factorization.

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[14]The authors have been recently informed by Prof. H. Widom that the spontaneous magnetization $D_\infty[a]$ for matrix of the form (1.4) can be evaluated without factorization. We are most grateful to him for this information.