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Extension of the Callen-Symanzik approach to critical phenomena

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We derive and discuss an extension of the Callen-Symanzik equation appropriate to the study of critical phenomena. By applying our result to an Ising Hamiltonian with both fourth- and sixth-power interactions (pseudo-spin-1), we find that its correlation exponent η is the same as the correlation exponent of a system with only fourth-power interaction (spin-1/2). We also show how our result may be used to discuss the correlation functions of spin-1/2 systems above the critical point.

I. INTRODUCTION

One of the most interesting results of Wilson's investigations of critical phenomena has been the construction of a theoretical framework within which one can understand how different Hamiltonians give rise to the same critical behavior.[1] The insightfulness and novelty of his use of the language of field theory compel one to ask if other field-theoretic formalisms can be shown, even if only with the help of hindsight, to yield comparable results. Such demonstrations, in their turn, may stimulate a clarification and extension of both the old and new approaches.

Since the work of Brezin, LeGuillou, and Zinn-Justin it has been known that the line of thought embodied in the Callen-Symanzik (CS) equation and its attendant approximations is capable of deriving the same results for the spin-$\frac{1}{2}$ Ising model as Wilson's procedure of integrating over successive degrees of freedom as implemented by the Wilson-Fisher ϵ expansion.[2,3] In fact, a number of authors have discussed the application of the CS equation to the spin-$\frac{1}{2}$ Ising problem.[4] Nonetheless, a simple derivation, within the context of statistical mechanics, of this equation has not appeared. Nor has a detailed application of it to higher-spin systems been published.

In Sec. II of this paper we discuss the field-theoretic formulation of higher-spin Ising models and find in particular that a precise formulation of the spin-1 problem involves interactions of up to the fourteenth power in the spin. Nonetheless, later we take seriously the possibility that a model we dub pseudo-spin-1, which involves only interactions in the fourth and sixth power of the spin, may serve as a qualitative guide to spin-1 systems. In Sec. III and IV we give a simple derivation of a generalization of the CS equation for a Hamiltonian containing interaction terms of a finite but arbitrary degree in the spins. While our equation reduces to the CS equation at the critical point, it is more general than the CS equation because it explicitly involves the ratio of the effective lattice spacing to the correlation length. In Sec. V we illustrate the use of our equation at the critical point by discussing the critical behavior of the pseudo-spin-1 model. We find that, to the order of approximation we work, the pseudo-spin-1 model has the same critical behavior as the spin-$\frac{1}{2}$ Ising model. In Sec. VI, we illustrate the use of our equation by finding a general form for the Green's functions of the spin-$\frac{1}{2}$ Ising problem not at, but above the crtical point. Finally, we summarize our results.

II. HIGHER SPIN, MOTIVATION AND NOTATION

Let us recall our reasons for studying σ^n models and at the same time establish some of our notation. Wilson begins his ingenious studies of the spin-$\frac{1}{2}$ Ising model with nearest-neighbor interactions by approximating it with a cutoff bare σ^4 field theory.[1] There are two crucial steps in formulating the Ising problem in this manner. The first involves the recognition that because the partition function Z_I may be written thus:

$$Z_I = \prod_{\vec{n}} \int ds(a\vec{n}) \, [\delta(s(a\vec{n}) - \tfrac{1}{2}) + \delta(s(a\vec{n}) + \tfrac{1}{2})] \times \exp(-\beta H_I[s]) \quad (1)$$

and because

$$\delta(x - \tfrac{1}{2}) + \delta(x + \tfrac{1}{2}) = \delta(x^2 - \tfrac{1}{4}) = \lim_{b\to\infty} \frac{b}{\sqrt{2\pi}} \exp[-\tfrac{1}{2}b^2(x^2 - \tfrac{1}{4})^2] ,$$

one may write

$$Z_I = \lim_{b\to\infty} \prod_{\vec{n}} \int ds(a\vec{n}) \, \frac{b}{\sqrt{2\pi}} \times \exp(-\beta H_I[s] - \tfrac{1}{2}b^2 \sum_{\vec{n}} \{[s(a\vec{n})]^2 - \tfrac{1}{4}\}^2) , \quad (2)$$

where a is the lattice spacing, $\vec{n}$ is a d-tuple of integers which specify the lattice site in d dimensions, and $s(a\vec{n})$ is the dimensionless spin variable. The second step involves the transition to momentum space, Eq. (4a), the replacement of the hypercubic Brillouin zone of edge length 2π by a smaller spherical one of radius c, and the disregard of some temperature-independent factors. The result of this manipulation is that the partition function is represented by the functional integral:

$$Z_I = \lim_{b\to\infty} \int \delta\sigma(q) \exp(H[\sigma]) , \quad (3)$$

where

$$s(a\vec{n}) = \int dq\, e^{iq\vec{n}} \sigma(q) , \quad (4a)$$

$$dq = (2\pi)^{-d} d^d q\, \Theta(c^2 - q^2) , \quad (4b)$$

$$\delta\left(\sum_j q_j\right) = (2\pi)^d \delta^d \left(\sum_j q_j\right) , \quad (4c)$$

$$H[\sigma] = -\frac{1}{2}\int dq\,(q^2+\mu_0^2)\,|\sigma(q)|^2 - (4!)^{-1}\mathfrak{U}_4 \int \prod_{j=1}^{4} dq_j\, \delta\left(\sum_{j=1}^{4} q_j\right) \prod_{j=1}^{4} \sigma(q_j) , \quad (4d)$$

and μ_0^2 and $\mathfrak{U}_4$ depend on b^2.

It is obvious that the σ^4 term arose because Wilson considered spin-$\frac{1}{2}$. If we wish to consider higher spins, to be specific let us say spin-1, we must find a polynomial P such that

$$\delta(P(x)) = \delta(x-1) + \delta(x+1) + \delta(x) . \quad (5)$$

A short computation shows that P must be at least of seventh degree. An example, which is not unique, of a suitable polynomial is

$$P(x) = \tfrac{1}{2} x(x^2-1)\,[\tfrac{1}{3}(x^2-2)^2 + \tfrac{2}{3}] . \quad (6)$$

By the method outlined, this P will lead to the study of a Hamiltonian with interaction terms of fourteenth degree and lower. Naturally, one would hope that because

$$\delta(\tfrac{1}{2}x(x^2-1)) = \delta(x-1) + \delta(x+1) + 2\delta(x) \quad (7)$$

the qualitative aspects of the spin-1 problem will emerge from a study of a Hamiltonian with interaction terms of sixth and fourth degree. We call this the pseudo-spin-1 problem. In any case, one sees that to formulate a problem with spin-1 or more, one must begin by considering Hamiltonians with many interaction terms.

There is another reason for studying such Hamiltonians. Experiment shows, and Wilson's theory suggests, that a wide class of physical systems exhibit the same behavior near criticality.[5] The interactions which differentiate two Hamiltonians with the same critical behavior are usually said to be "irrelevant."[6] Since one would like to understand this "irrelevance" from as many points of view as possible, we consider all such interactions in the same formalism.

For these reasons we begin by considering Hamiltonians of the form

$$H[\sigma] = -\frac{1}{2}\int dq\,(q^2+\mu_0^2)\,|\sigma(q)|^2 + \sum_{n=3}^{N} (n!)^{-1}\mathfrak{U}_n \times \int \prod_{j=1}^{n} dq_j\, \delta\left(\sum_{j=1}^{n} q_j\right) \prod_{j=1}^{n} \sigma(q_j) . \quad (8)$$

Let us denote the connected multipoint Green's functions obtained from such a theory by $G^{(m)}(\vec{q}\,|\,\mu_0^2, c^2, \vec{\mathfrak{U}})$, where $\vec{q}$ denotes its m-1 independent momentum arguments and $\vec{\mathfrak{U}}$ is the set of coupling constants whose members are U_n. Since we will be concerned with the comparison of the behavior of Green's functions at large and small values of their momentum arguments, we will later regard $G^{(m)}$ as the product of a momentum-independent factor $Z_3^{m/2}$ and a momentum-dependent one, $G_R^{(m)}$. That is to say, symbolically, $G^{(m)} = Z_3^{m/2} G_R^{(m)}$. Obviously with this definition of $G_R^{(m)}$ we have $G^{(m)}(e^\lambda\vec{q})/G^{(m)}(\vec{q}) = G_R^{(m)}(e^\lambda q)/G_R^{(m)}(q)$; thus while $G_R^{(m)}$ has the same momentum dependence as $G^{(m)}$ it is normalized differently; that is to say, it is renormalized. It will emerge, in Sec. IV, that when $G_R^{(m)}$ is parametrized in terms of the correlation length, $\mu^{-1}a$, and other quasithermodynamic variables, its dependence on them and on momentum is constrained to satisfy a certain partial differential equation.

III. DIMENSIONLESS ANALYSES

In order to carry out our program, we first establish a kind of dimensional analysis of dimensionless quantities or, with tongue in cheek, dimensionless analysis. This is important to us because although we deal with dimensionless quantities these may either involve the comparison of a length with the lattice spacing or with the correlation length. Our dimensionless analysis will restrict the dependence of $G^{(m)}$ on the parameters which appear in the Hamiltonian, and thus it restricts the dependences of the correlation length and the so-called "dimensionless" renormalized coupling constants on these parameters also. These restrictions are embodied in the following lemma and some related results:

Lemma

$$G^{(m)}(\vec{q}\,|\,\mu_0^2, c^2, \vec{\mathfrak{U}}) = e^{\mathfrak{D}(m,d)\lambda} G^{(m)} \times (e^\lambda\vec{q}\,|\,e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{U}}(\lambda)) , \quad (9)$$

$$\mathfrak{U}_n(\lambda) \equiv e^{\epsilon(n,d)\lambda}\mathfrak{U}_n , \quad (10)$$

$$\epsilon(n,d) \equiv n + d - \tfrac{1}{2}nd , \quad (11)$$

$$\mathfrak{D}(m,d) \equiv m - d + \tfrac{1}{2}md . \quad (12)$$

In order to prove this we consider the generating functional, C, of the cumulant averages in momentum space of the products of spins[7]:

$$C[E(q);\, \mu_0^2,\, c^2,\, \vec{\mathfrak{u}}\,] \equiv \ln \int \delta\sigma(q) \exp\left(H[\sigma] + \int dq\, E(q)\sigma(q)\right) . \tag{13}$$

By making the linear functional transformation

$$\sigma(x) \to \exp[-(\tfrac{1}{2}d+1)\lambda]\int dx\, \delta(e^{-\lambda}y - x)\sigma(x) ,$$

and noting that linearity implies that the Jacobian, J, of this transformation is independent of σ, we can show that

$$C[E(q);\, \mu_0^2,\, c^2,\, \vec{\mathfrak{u}}] = C[e^{-(d/2+1)\lambda}E(q);\; e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)] . \tag{14}$$

If we now apply $\Pi_{i=1}^m[\delta/\delta E(q_i)]$ to both sides of Eq. (14) and evaluate the result at $E=0$, we learn that

$$e^{-md\lambda}\delta\left(\sum_{i=1}^m e^{-\lambda}q_i\right) G^{(m)}(e^{-\lambda}\vec{q}\,|\,\mu_0^2, c^2, \vec{\mathfrak{u}}) = e^{m(1-d/2)\lambda}\delta\left(\sum_{i=1}^m q_i\right) G^{(m)}(\vec{q}\,|\,e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) , \tag{15}$$

or

$$G^{(m)}(\vec{q}\,|\,\mu_0^2, c^2, \vec{\mathfrak{u}}) = e^{\mathfrak{D}(m,d)\lambda}G^{(m)}(e^{\lambda}\vec{q}\,|\,e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) . \tag{9}$$

Since $G^{(m)}$ is a scalar, it can depend on its arguments q only through the variables

$$s \equiv \sum_{i=1}^{m-1} q_i^2 + \left(\sum_{i=1}^{m-1} q_i\right)^2 \tag{16a}$$

and

$$Q_{ij} \equiv (q_i \cdot q_j)s^{-1} . \tag{16b}$$

Hence the infinitesimal form of Eq. (9) is Eq. (17)[8]:

$$0 = \left(\mathfrak{D}(m,d) + 2s\frac{\partial}{\partial s} + 2c^2\frac{\partial}{\partial c^2} + 2\mu_0^2\frac{\partial}{\partial\mu_0^2} + \sum_{n=3}^N \epsilon(n,d)\mathfrak{u}_n\frac{\partial}{\partial\mathfrak{u}_n}\right)G^{(m)} . \tag{17}$$

Let us now introduce the dimensionless correlation length μ^{-1} in the standard manner:

$$\mu^2 \equiv -\tfrac{1}{2}\left(\frac{d}{ds}\ln G^{(2)}(s=0)\right)^{-1} . \tag{18}$$

It is easy to see, by referring to Eq. (9), that

$$\mu^2(\mu_0^2, c^2, \vec{\mathfrak{u}}) = e^{-2\lambda}\mu^2(e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) \tag{19a}$$

or

$$\mu^2 = \left(\mu_0^2\frac{\partial}{\partial\mu_0^2} + c^2\frac{\partial}{\partial c^2} + \frac{1}{2}\sum_{n=3}^N \epsilon(n,d)\mathfrak{u}_n\frac{\partial}{\partial\mathfrak{u}_n}\right)\mu^2 . \tag{19b}$$

We also introduce the quantity $Z_3 \equiv \mu^2 G^{(2)}(s=0)$, which is the susceptibility divided by the square of the dimensionless correlation length, and note that

$$Z_3(\mu_0^2 c^2, \vec{\mathfrak{u}}) = Z_3(e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) \tag{20a}$$

or

$$0 = \left(\mu_0^2\frac{\partial}{\partial\mu_0^2} + c^2\frac{\partial}{\partial c^2} + \frac{1}{2}\sum_{n=3}^N \epsilon(n,d)\mathfrak{u}_n\frac{\partial}{\partial\mathfrak{u}_n}\right)Z_3 . \tag{20b}$$

If we Legendre-transform C with respect to $\langle\sigma(q)\rangle$, we arrive at the generating functional for the generalized stiffness matrices or vertex functions, $\Gamma^{(m)}$.[9] An argument similar to that sketched in the second paragraph of this section shows that

$$\Gamma^{(m)}(s\,|\,\mu_0^2, c^2, \vec{\mathfrak{u}}) = e^{\epsilon(m,d)\lambda}\Gamma^{(m)}(e^{2\lambda}s\,|\,e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) . \tag{21}$$

Hence if we define the so-called dimensionless renormalized coupling constants g_n by Eq. (22), then Eqs. (23) follow:

$$g_n \equiv \mu^{-\epsilon(n,d)}\Gamma^{(n)}(s=0) , \tag{22}$$

$$g_n(\mu_0^2, c^2, \vec{\mathfrak{u}}) = g_n(e^{2\lambda}\mu_0^2, e^{2\lambda}c^2, \vec{\mathfrak{u}}(\lambda)) , \tag{23a}$$

$$0 = \left(\mu_0^2\frac{\partial}{\partial\mu_0^2} + c^2\frac{\partial}{\partial c^2} + \frac{1}{2}\sum_{k=3}^N \epsilon(k,d)\mathfrak{u}_k\frac{\partial}{\partial\mathfrak{u}_k}\right)g_n . \tag{23b}$$

It should be noted that our dimensionless analysis has lead to the natural appearance of a different epsilon, $\epsilon(n,d) \equiv n + d - \frac{1}{2}nd$, for each interaction term. Each $\epsilon(n,d)$ results from the comparison of the scaling behavior of the Gaussian term $(q^2+\mu_0^2)|\sigma(q)|^2$, to that of the interaction term, $\Pi_{j=1}^n \sigma(q_j)$, in the Hamiltonian.

IV. DERIVATION AND EXTENSION OF CALLEN-SYMANZIK EQUATION

Let us now define the renormalized Green's functions $G_R^{(m)}$:

$$G_R^{(m)} \equiv Z_3^{-m/2}G^{(m)} . \tag{24}$$

We note that if we regard $G_R^{(m)}$ as a function of μ^2, c^2, and $\vec{g}$ rather than as a function of μ_0^2, c^2, and $\vec{\mathfrak{u}}$, the work of the last section implies that

$$G_R^{(m)}(s\,|\,\mu^2, c^2, \vec{g}) = e^{\mathfrak{D}(m,d)\lambda}G_R^{(m)}(e^{2\lambda}s\,|\,e^{2\lambda}\mu^2, e^{2\lambda}c^2, \vec{g}) .$$

However, this relation does not exhaust the physics which may be gleaned by rewriting the infinitesimal forms of the homogeneity statements of the last section in terms of renormalized variables.

We begin by substituting $Z_3^{m/2}G_R^{(m)}$ for $G^{(m)}$ in Eq. (17) and make use of Eq. (20b) to eliminate explicit reference to cutoff c when differentiating Z_3. The result is Eq. (25):

$$\left[\mathfrak{D}(m,d) + 2s\frac{\partial}{\partial s} + \left(2c^2\frac{\partial}{\partial c^2} + \sum_{n=3}^N \epsilon(n,d)\mathfrak{u}_n\frac{\partial}{\partial\mathfrak{u}_n}\right) + \frac{m}{2}\left(-2\mu_0^2\frac{\partial}{\partial\mu_0^2}\ln Z_3\right)\right]G_R^{(m)} = -Z_3^{-m/2}2\mu_0^2\frac{\partial}{\partial\mu_0^2}G^{(m)} . \tag{25}$$

Equations (19b) and (23b) imply that the first three terms in the square brackets on the left-hand side of Eq. (25) may be replaced by

$$-2\mu_0^2 \frac{\partial \mu^2}{\partial \mu_0^2} \frac{\partial}{\partial \mu^2} - \sum_{n=3}^{N} \left(2\mu_0^2 \frac{\partial g_n}{\partial \mu_0^2}\right) \frac{\partial}{\partial g_n} .$$

We may then rewrite Eq. (25) and obtain Eq. (26):

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \sum_{n=3}^{N} \left(\mu^2 \frac{\partial g_n}{\partial \mu_0^2} \Big/ \frac{\partial \mu^2}{\partial \mu_0^2}\right) \frac{\partial}{\partial g_n} + \frac{m}{2}\left(-\mu^2 \frac{\partial}{\partial \mu_0^2} \ln Z_3 \Big/ \frac{\partial \mu^2}{\partial \mu_0^2}\right)\right] G_R^{(m)} = Z_3^{-m/2}\left(\mu^{-2} \frac{\partial \mu^2}{\partial \mu_0^2}\right)^{-1} \frac{\partial}{\partial \mu_0^2} G^{(m)} . \tag{26a}$$

If the coefficient of $\partial/\partial g_n$ is regarded as a function of g, μ^2, and c^2, then Eqs. (19b) and (23b) imply that it can only depend on g and $\Lambda^{-2} \equiv (\mu/c)^2$. We denote it by $\frac{1}{2}\beta_n(\vec{g}, \Lambda^{-2})$. It also follows that the coefficient of $m/2$ depends only on g and Λ^{-2}. We denote it by $\frac{1}{2}\mathrm{H}(\vec{g}, \Lambda^{-2})$. Finally, we note that since

$$G_R^{(m)}(s \,|\, \mu^2, c^2, \vec{g}) = \mu^{-\mathfrak{D}(m,d)} G_R^{(m)}(s/\mu^2 \,|\, 1, \Lambda^2, \vec{g})$$

we define $\mathrm{s} \equiv s/\mu^2$ and express Eq. (26a) more compactly thus:

$$\left(2\mathrm{s} \frac{\partial}{\partial \mathrm{s}} + \mathfrak{D}(m,d) - \frac{m}{2} \mathrm{H}(\vec{g}, \Lambda^{-2}) - \sum_{n=3}^{N} \beta_n(\vec{g}, \Lambda^{-2}) \frac{\partial}{\partial g_n} + 2\Lambda^{-2} \frac{\partial}{\partial \Lambda^{-2}}\right) G_R^{(m)} = -2 Z_3^{-m/2}\left(\mu^{-2} \frac{\partial \mu^2}{\partial \mu_0^2}\right)^{-1} \frac{\partial}{\partial \mu_0^2} G^{(m)} . \tag{26b}$$

Several comments are in order. Equation (26b) relates the dependence of $G_R^{(m)}$ on s to its dependence on the renormalized coupling constants g_n and to its dependence on the ratio Λ^{-1} of the effective lattice spacing a/c to the correlation length $a\mu^{-1}$. An equation similar to ours has been derived for quantum field theory by Callen and Symanzik.[10] However, in their work $\Lambda^{-1} = 0$ and the operator $2\Lambda^{-2}\partial/\partial\Lambda^{-2}$ does not appear, because the analog of c/a is the ultraviolet cutoff which is put equal to infinity. In our work, $\Lambda^{-1} = 0$ at the critical point because the correlation length $\mu^{-1}a$ is infinite there. The folklore of quantum field theory says that when s is large and $\Lambda^{-1} = 0$, the right-hand side of Eq. (26b) can be neglected.[11] We conjecture that there is a region near the critical point, characterized by $1 \ll \mathrm{s} \ll \Lambda^2$, where it may be dropped. Our working hypothesis then is

$$\left(2\mathrm{s} \frac{\partial}{\partial \mathrm{s}} + \mathfrak{D}(m,d) - \frac{m}{2} \mathrm{H}(\vec{g}, \Lambda^{-2}) - \sum_{n=3}^{N} \beta_n(\vec{g}, \Lambda^{-2}) \frac{\partial}{\partial g_n} + 2\Lambda^{-2} \frac{\partial}{\partial \Lambda^{-2}}\right) G_R^{(m)} = 0 . \tag{27}$$

We see that if we are at the critical point, $\Lambda^{-1} = 0$ and $\vec{g} = \vec{g}_*$ such that $\beta_n(\vec{g}_*, \Lambda^{-2} = 0) = 0$, then $G_R^{(m)}$ is proportional to $\mathrm{s}^{-1/2[\mathfrak{D}(m,d) - (m/2)\mathrm{H}(\vec{g}_*, 0)]}$. Since

$$\frac{G^{(2)}(e^{2\lambda} s)}{G^{(2)}(s)} = \frac{G_R^{(2)}(e^{2\lambda} s)}{G_R^{(2)}(s)} = e^{-\lambda[2 - \mathrm{H}(\vec{g}_*, 0)]} ,$$

one can identify $\mathrm{H}(\vec{g}_*, 0)$ with the correlation exponent η. This result is not astonishing if one recalls that Z_3 is the susceptibility divided by the square of dimensionless correlation length.

Obviously methods for calculating β_n and H are greatly desired. To date, the only method known is that of renormalized perturbation theory. We shall see, in the next section, that numerically satisfactory results can be obtained in this way.

V. PSEUDO-SPIN-1

In Sec. II we saw that different values of spin lead to different interactions when the Ising problem is formulated as a field theory. In order to explore this fact, using the formalism of the last section, we have performed a calculation to second order in the renormalized coupling constants of the zeroes of the β functions and η for the pseudo-spin-1 problem. The results of such a calculation have a double interest. They may serve as a guide to the critical behavior of spin-1 systems. They may be compared to Wilson's treatment of spin-$\frac{1}{2}$ where a σ^6 arises but is deemed irrelevant. In the section, we present the essence of the calculation and our results.

The calculation is straightforward but tedious.[12] One begins by writing the pseudo-spin-1 Hamiltonian $H[\sigma]$ in terms of the renormalized spins $\sigma_R(q) \equiv Z_3^{-1/2}\sigma(q)$ and the renormalized coupling constants:

$$H[\sigma] = H_R[\sigma_R] = -\frac{1}{2}\int dq\, Z_3(q^2 + \mu^2)\,|\sigma_R(q)|^2 + \left[-\tfrac{1}{2} Z_3 \delta\mu^2 \int dq\, |\sigma_R(q)|^2 + (4!)^{-1} g_4 \mu^{\epsilon(4,d)} \mathfrak{z}_4 \int \prod_{j=1}^{4} dq_j\, \delta\left(\sum_{j=1}^{4} q_j\right) \prod_{j=1}^{4} \sigma_R(q_j) - (6!)^{-1} g_6 \mu^{\epsilon(6,d)} \mathfrak{z}_6 \int \prod_{j=1}^{6} dq_j\, \delta\left(\sum_{j=1}^{6} q_j\right) \prod_{j=1}^{6} \sigma_R(q_j)\right], \tag{28}$$

the constants $\mathfrak{z}_4$ and $\mathfrak{z}_6$ have been introduced to relate the renormalized coupling constant to the unrenormalized ones according to $\mathfrak{U}_4 = g_4 \mu^{\epsilon(4,d)} \mathfrak{z}_4 Z_3^{-2}$ and $\mathfrak{U}_6 = g_6 \mu^{\epsilon(6,d)} \mathfrak{z}_6 Z_3^{-3}$. The constants $\mathfrak{z}_4$, $\mathfrak{z}_6$, and

Z_3 are fixed to each order of perturbation theory by the requirements $\Gamma_R^{(2)}(s=0)=\mu^2$, $(d/ds)\Gamma_R^{(2)}(s=0)=\frac{1}{2}$, $\Gamma_R^{(4)}(s=0)=g_4\mu^{\epsilon(4,d)}$, and $\Gamma_R^{(6)}(s=0)=g_6\mu^{\epsilon(6,d)}$.

We find that

$$Z_3^{-1}=1-[\tfrac{1}{6}F_3'g_4^2+(1/5!)F_5'g_6^2]+O(3)\ , \tag{29a}$$

$$\mathfrak{z}_4=(-1+\tfrac{1}{2}F_1g_6g_4^{-1})+(\tfrac{5}{6}F_3F_1g_4^{-1}g_6^2+\tfrac{15}{4}F_2F_1g_6-\tfrac{3}{2}F_2g_4)+O(2)\ , \tag{29b}$$

$$\mathfrak{z}_6=1+(\tfrac{5}{2}F_3g_6+\tfrac{15}{2}F_2g_4)+O(2)\ , \tag{29c}$$

where F_1, F_2, F_3, F_3', and F_5' are Feynman integrals (see Appendix A) which depend on Λ^2. The product F_1g_6 appears in some of the terms for $\mathfrak{z}_4$ because, in some graphs, the σ^6 interaction simulates the σ^4 interaction since two of its legs combine to form a closed loop.

The β functions whose definitions involved unrenormalized variables can be expressed in terms of renormalized variables. For pseudo-spin-1, we have shown (see Appendix B) that β_4 and β_6 are given by the following formula:

$$\beta_a=g_a\ \frac{(X_b+g_bX_{b,b})[\epsilon(a,d)X_a+\tilde{X}_a]-g_bX_{a,b}[\epsilon(b,d)X_b+\tilde{X}_b]}{g_ag_bX_{a,b}X_{b,a}-(X_a-g_aX_{a,a})(X_b-g_bX_{b,b})}\ , \tag{30}$$

where $a=4$, $b=6$, or $a=6$, $b=4$ and $X_a\equiv\mathfrak{z}_aZ_3^{-a/2}$, $\tilde{X}_a\equiv\mu\partial X_a/\partial\mu$, $X_{a,b}\equiv\partial X_a/\partial g_b$. By expanding β_4 and β_6 in a Maclaurin series in g_4 and g_6 and then substituting the results (see Appendix B) of calculation for $\mathfrak{z}_4$, $\mathfrak{z}_6$, and Z_3^{-1} in it, we find

$$\beta_4=[-\epsilon(4,d)g_4-F_2g_6+\tfrac{3}{2}\epsilon(4,d)F_2g_4^2-Bg_6^2-Cg_4g_6+O(3)] \tag{30a}$$

$$\beta_6=[-\epsilon(6,d)g_6+\beta g_6^2+\gamma g_4g_6+O(3)]\ , \tag{30b}$$

where the coefficients of the coupling constant are functions of Λ^{-1}. The points in the g_4-g_6 plane to which the roots of $\beta_4=\beta_6=0$ tend when $4>d\geq 3$ and Λ^{-2} tends to zero are $g_4=0$, $g_6=0$, and $g_4=(\frac{3}{2}F_2)^{-1}$, $g_6=0$. The point $g_4=g_6=0$ is an artifact of any perturbative calculation. It is analogous to the Gaussian fixed point in Wilson's approach in that it leads to mean field (free field theory) results for the exponents.[1] The point $g_4=(\frac{3}{2}F_2)^{-1}$, $g_6=0$ is much more interesting as we shall see.

The function $\mathrm{H}(g_4, g_6, \Lambda^{-2})$ for pseudo-spin-1 is (see Appendix B)

$$\mathrm{H}=-Z_3(\tilde{Z}_3^{-1}+\beta_4Z_{3,4}^{-1}+\beta_6Z_{3,6}^{-1})\ . \tag{31}$$

When H is expanded to second order in the coupling constants, Λ^{-1} is put equal to zero, and the result is evaluated at $g_4=(\frac{3}{2}F_2)^{-1}$, $g_6=0$, we find that

$$\mathrm{H}(g_4=(\tfrac{3}{2}F_2)^{-1},\ g_6=0,\Lambda^{-2}=0)=\eta=[\tfrac{1}{3}\epsilon(4,d)\,F_3'\,](\tfrac{3}{2}F_2)^{-2}\ . \tag{32}$$

This is the same result, to second order, which Brezin *et al.* found for the spin-$\frac{1}{2}$ Ising problem.[2] Moreover, when this result is expanded in powers of $\epsilon(4,d)$ it agrees with the results of the Wilson-Fisher ϵ-expansion technique.[13]

One sees that, to the order we have worked, the large-a behavior of the Green's functions at the critical point of the pseudo-spin-1 and the spin-$\frac{1}{2}$ problems are the same and that, in this sense, the σ^6 interaction is irrelevant.

Naturally one would like to extend this calculation to higher orders and to the specific-heat exponent α. Nonetheless, we expect that spin-$\frac{1}{2}$ critical behavior will always be a possibility for a pseudo-spin-1 system because we do not believe higher-order calculations will produce a term in β_6 which is not proportional to some positive power of g_6. Thus the condition $g_6=0$ will always imply the $\beta_6=0$ and we expect that perturbative calculations of β_4, η, and α for pseudo-spin-1 with $g_6=0$ will reduce to the appropriate calculations for spin-$\frac{1}{2}$. The reason for doing higher-order calculations is one's desire to know if there are any roots of $\beta_4=\beta_6=0$ for which $g_6\neq 0$, that is to say if spin-$\frac{1}{2}$ behavior is the only possible critical behavior of pseudo-spin-1.

VI. AN APPROACH TO CORRECTIONS TO SCALING

As we have said, it is plausible to assume that the solutions of Eq. (27) will be relevant to the study of critical phenomena as long as $1\ll s\ll\Lambda^2$, that is to say, not only at the critical point, but at a small though finite distance from it. If this is so, the solutions of Eq. (27) will provide a framework for discussing corrections to scaling. In order to illustrate both the promise and difficulty of this approach, we consider the spin-$\frac{1}{2}$ Ising model.

Since there is only one renormalized coupling constant, that of the σ^4 term, our equation is

$$\left(2s\frac{\partial}{\partial s}-\beta(g,\Lambda^{-2})\frac{\partial}{\partial g}+2\Lambda^{-2}\frac{\partial}{\partial\Lambda^{-2}}+\mathfrak{D}(m,d)-\frac{m}{2}\mathrm{H}(g,\Lambda^{-2})\right)G_R^{(m)}=0\ . \tag{33}$$

In order to find its solution, we must first solve the simultaneous ordinary differential equations, Eqs. (34), for its characteristics[14]:

$$\frac{dy}{ds}=s^{-1}y\ , \tag{34a}$$

$$\frac{dZ}{ds} = -\tfrac{1}{2}\,s^{-1}\beta(Z,y)\ , \tag{34b}$$

$$\frac{d}{ds}\,\mathcal{G} = \tfrac{1}{2}\,s^{-1}\left[\tfrac{1}{2}m\mathrm{H}(Z,y) - \mathfrak{D}(m,d)\right]\mathcal{G}\ . \tag{34c}$$

Since the solution of Eq. (34a) is $y = y_0 s_0^{-1} s$, the remaining equations assume the following form:

$$\frac{dZ}{ds} = -\tfrac{1}{2}\,s^{-1}\beta(Z, y_0 s_0^{-1} s)\ , \tag{35a}$$

$$\frac{d}{ds}\,\mathcal{G} = \tfrac{1}{2}\,s^{-1}\left[\tfrac{1}{2}m\mathrm{H}(Z, y_0 s_0^{-1} s) - \mathfrak{D}(m,d)\right]\mathcal{G}\ . \tag{35b}$$

If $C(s;s_0, y_0 s_0^{-1}, g_0)$ denotes the solution of Eq. (35a) which satisfies the initial condition $C(s_0;s_0, y_0 s_0^{-1}, g_0) = g_0$, then the solution of Eq. (35b) is

$$\mathcal{G} = \mathcal{G}_0 \exp\left[\frac{1}{2}\int_{s_0}^{s}\frac{dx}{x}\left(\frac{m}{2}\,\mathrm{H}(C(x), y_0 s_0^{-1}x) - \mathfrak{D}(m,d)\right)\right].$$

It follows that the general solution of Eq. (33) is

$$G_R^{(m)}(s\,|\,\mu^2, c^2, g) = \mu^{-\mathfrak{D}(m,d)}(s/s_0)^{-\mathfrak{D}(m,d)/2}F^{(m)}\left((s_0/s)\Lambda^{-2},\ \omega(s;s_0,\ \Lambda^{-2}s^{-1}, g)\right)$$
$$\times\exp\left(\frac{m}{4}\int_{s_0}^{s}\frac{dx}{x}\,\mathrm{H}(C(x;s_0,\ \Lambda^{-2}s_0^{-1}\omega(s;s_0,\ \Lambda^{-2}s^{-1}, g)),\ x\Lambda^{-2}s^{-1})\right)\ , \tag{36}$$

where ω is g_0 in terms of s, s_0, Λ^{-2}, and g; that is to say, $C(s;s_0,\Lambda^{-2}s^{-1},\omega(s;s_0,\Lambda^{-2}s^{-1},g)) = g$ is an identity. To see that this result for the behavior of the Green's functions near the critical point goes over to the one we obtained at the critical point, one recalls $G^{(m)}(s)/G^{(m)}(s_0) = G_R^{(m)}(s)/G_R^{(m)}(s_0)$, sets $\Lambda^{-2} = 0$, and notes that $C(s) = \text{const} = g_*$ is a solution of Eq. (35b) if $\beta(g_*, 0) = 0$ and replaces $\exp[\frac{1}{4}m\int_{s_0}^{s}dx/x]$ by $(s/s_0)^{m/4}$.

Obviously, the explicit solution of Eq. (35a) is as desired as it is difficult. Had we considered the many-coupling-constant problem, we would have been faced with many such coupled nonlinear ordinary differential equations. At present we have nothing to say about the solutions or approximate solutions to them. However, we believe that Eqs. (33) and (36) may constitute a formalism alternate to Wegner's within which one may discuss corrections to scaling.[15]

VII. SUMMARY

Let us summarize. By finding the dependence of the dimensionless correlation length, renormalized coupling constants, and susceptibility on the parameters which appear in the effective Hamiltonian, we established a partial differential equation which relates the dependence of the renormalized Green's functions on their momentum arguments to their dependence on their quasi-thermodynamic arguments, the correlation length, and the values of the generalized stiffness matrices. The partial differential equation is more general than the Callen-Symanzik equation to which it is analogous because it involves the ratio Λ^{-1} of the effective lattice spacing to the correlation length. We conjectured by analogy to the case in quantum field theory that near the critical point, and for fluctuations whose wavelength was much larger than the effective lattice, spacing the right-hand side of our equation could be neglected. The result was a linear first-order partial differential equation whose study can be reduced to the study of the ordinary differential equations which determine its characteristics. Our application of this result to the spin-$\frac{1}{2}$ Ising problem resulted in a general form for its Green's functions which may in the future be used to discuss corrections to scaling. We also applied our result to study the critical behavior of pseudo-spin-1 systems and found that, to the order of approximation we employ, the σ^6 interaction which is necessary to the formulation of the problem is irrelevant to its critical behavior.

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APPENDIX A

The Feynman integrals referred to in Sec. V are defined as follows:

$$F_1 \equiv (2\pi)^{-d}\int d^dK\,\Theta(\Lambda^2 - K^2)(K^2+1)^{-1}\ ,$$

$$F_2 \equiv (2\pi)^{-d}\int d^dK\,\Theta(\Lambda^2 - K^2)(K^2+1)^{-2}\ ,$$

$$F_3 \equiv (2\pi)^{-2d}\int\prod_{j=1}^{3}d^dK_i\,\delta^d(K_1+K_2+K_3)\times\prod_{i=1}^{3}\Theta(\Lambda^2 - K_i^2)(K_i^2+1)^{-1}\ ,$$

$$F_3' \equiv \frac{d}{dQ^2}\left[(2\pi)^{-2d}\left(\prod_{i=1}^{2}d^dK_i\ \prod_{i=1}^{2}\Theta(\Lambda^2 - K_i^2)(K_i^2+1)^{-1}\right)\times\Theta(\Lambda^2-(Q-K_1-K_2)^2)[(Q-K_1-K_2)^2+1]^{-1}\right]_{Q^2=0}\ ,$$

$$F_5' \equiv \frac{d}{dQ^2}\quad(2\pi)^{-4d}\int\prod_{i=1}^{4}d^dK_i\left(\prod_{i=1}^{4}\Theta(\Lambda^2 - K_i^2)(K_i^2+1)^{-1}\right)$$

$$\times\Theta(\Lambda^2-(Q-K_1-K_2-K_3-K_4)^2)$$
$$\times[(Q-K_1-K_2-K_3-K_4)^2+1]^{-1}\Big]_{Q^2=0}\,.$$

APPENDIX B

In this appendix we derive the expressions for the β and H functions that were given in Sec. IV. Recall that β_a was defined as

$$\beta_a \equiv 2\,\mu^2\left(\frac{\partial g_a}{\partial \mu_0^2}\right)\left(\frac{\partial \mu^2}{\partial \mu_0^2}\right)^{-1}, \tag{37}$$

where the partial derivatives are to be evaluated at constant $\vec{\mathfrak{U}}$ and c. Since we desire β_a as a function of μ, $\vec{g}$, and c, extensive use of the chain rule is called for. To arrive at our result, one notes that the definition of β_a implies that $\beta_a = \mu\partial g_a/\partial\mu$ where the partial derivative is evaluated at constant $\vec{\mathfrak{U}}$ and c. Next one recalls that since

$$\frac{\partial g_a}{\partial\mu}\;\frac{\partial \mathfrak{U}_a}{\partial g_a}\left(\frac{\partial\mathfrak{U}_a}{\partial\mu}\right)^{-1}=-1\,,$$

the β functions may be written as an Eq. (38)

$$\beta_a=-\,\mu\left(\frac{\partial\mathfrak{U}_a}{\partial\mu}\right)\Big/\left(\frac{\partial\mathfrak{U}_a}{\partial g_a}\right)\,, \tag{38}$$

where the numerator is evaluated at constant g_a, $\mathfrak{U}_b$ and c while the denominator is evaluated at constant μ, $\mathfrak{U}_b$ and c. Now $\mathfrak{U}_a$ is known (from the perturbative calculation of Sec. V) as a function of μ, g_a, g_b, and c, $\mathfrak{U}_a=f_a(\mu,g_4,g_6,c)$ so we must use the chain rule to reexpress Eq. (38) in terms of f_a. This leads to the result

$$\beta_a=\frac{\tilde{f}_a f_{b,b}-\tilde{f}_b f_{a,b}}{f_{a,b}f_{b,a}-f_{a,a}f_{b,b}}\quad. \tag{39}$$

When it is recalled that $f_a=g_a\mu^{\epsilon(a,d)}X_a$, one sees that Eq. (39) implies Eq. (29):

$$\beta_a g_a\left(\frac{(X_b+g_bX_{b,b})[\epsilon(a,d)X_a+X_a]-g_bX_{a,b}[\epsilon(b,d)X_b+\tilde{X}_b]}{g_ag_bX_{a,b}X_{b,a}-(X_a+g_aX_{a,a})(X_b+g_bX_{b,b})}\right)\,. \tag{29}$$

The expression H function is now easily derived. Its definition is

$$\mathrm{H}=2\mu^2\left(\frac{\partial}{\partial\mu_0^2}\ln Z_3\right)\left(\frac{\partial\mu^2}{\partial\mu_0^2}\right)^{-1}, \tag{40}$$

where, once again, the partial derivatives are to be evaluated at constant $\vec{\mathfrak{U}}$ and c. The definition is equivalent to $\mathrm{H}=-\,\mu(\partial/\partial\mu)\ln Z_3^{-1}$ at constant $\vec{\mathfrak{U}}$ and c, but this is the same as

$$\mathrm{H}=-\left(\mu\frac{\partial}{\partial\mu}\ln Z_3^{-1}+\mu\frac{\partial g_4}{\partial\mu}\;\frac{\partial}{\partial g_4}\ln Z_3^{-1}+\mu\frac{\partial g_6}{\partial\mu}\;\frac{\partial}{\partial g_6}\ln Z_3^{-1}\right),$$

when is H regarded as a function of g_4, g_6, μ, and c. Hence we obtain

$$\mathrm{H}=-\,Z_3(\tilde{Z}_3^{-1}+\beta_4Z_{3,4}^{-1}+\beta_6Z_{3,6}^{-1})\,. \tag{31}$$

[1]A thorough review of these investigations can be found in K. G. Wilson and J. B. Kogut, Phys. Rept. (to be published).

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[3]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[4]See G. Parisi, Columbia University report, 1973 (unpublished) and references cited therein. We also note that as this article was being finished, Professor B. Schroer was kind enough to call our attention to B. Schroer, Phys. Rev. B 8, 4200 (1973), and references cited therein.

[5]An excellent review of the experimental situation for magnets is L. J. deJongh and A. R. Miedema, Adv. Phys. 23, 1 (1974). For a comparison of fluids and magnets see E. Stanley, *Phase Transitions and Critical Phenomena* (Oxford U. P. New York, 1971).

[6]The appellation "irrelevant" also has a technical meaning within the language of Wilson's renormalization group. There it means that if the field, to which an irrelevant interaction couples, deviates from its fixed-point value (usually zero), then under repeated renormalization this field will converge to its fixed-point value. Hence the initial value of the field is irrelevant. The relation between the Callen-Symanzik approach and this definition of irrelevance has been expounded in Brezin *et al.*, Phys. Rev. D 6, 2418 (1973); Phys. Rev. B 8, 5330 (1973); and Zinn-Justin, Cargese Summer School Lectures, 1973 (unpublished).

[7]The essence of this argument does not depend on the field-theoretic formalism we use here. It was applied for fluids in A. M. Wolsky and M. S. Green, Phys. Rev. A 9, 957 (1974). Our argument and results also apply when the sharp cutoff, $q^2\le c^2$, is dropped and the Gaussian term $(q^2+\mu_0^2)\,|\,\sigma(q)\,|^2$ is replaced by $[q^2(1+c^{-2}q^2)^p+\mu_0^2]\,|\,\sigma(q)\,|^2$.

[8]From now on we suppress the dependence of $G^{(m)}$ on the variables Q_{ij} because it is irrelevant to the arguments which follow. We also suppress the dependence of s on m because it will be clear in context.

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