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Electrodynamics and thermodynamics of a classical electron surface layer*

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The properties of a classical charge-compensated electron surface layer on a dielectric (for example, liquid He, or liquid or solid Ne) are studied with many-body theory. The electrodynamic properties (static screening, plasma oscillations, and inelastic electron scattering) are obtained with the random-phase approximation. A "Debye-Hückel" expansion provides the corresponding thermodynamic functions in the nearly ideal limit.

I. INTRODUCTION

The prediction of bound-electron surface states[1] on certain bulk dielectrics such as He or Ne has stimulated several experimental searches for such behavior. Although early observations were contradictory,[2–4] recent data strongly support the original theory. For example, studies by Brown and Grimes[5] of cyclotron resonance in a tipped magnetic field have verified that the electron motion is indeed two dimensional, and detection of resonant transitions to excited surface states has confirmed the fundamental mechanism of image-charge binding in considerable detail.[6] These experiments raise the intriguing possibility of studying other properties of a two-dimensional electron gas, and a theoretical calculation of such effects becomes of great interest.

Consider a charge-compensated one-component system of N electrons confined to an area A at a temperature T. Three energies are significant: thermal ($\approx k_BT$), electrostatic ($\approx e^2n^{1/2}$), and zero-point ($\approx \hbar^2 n/m$), where $-e$ and m are the electron's charge and mass, and $n=N/A$ is the areal density. If the interparticle spacing $n^{-1/2}$ is much greater than the Bohr radius $\hbar^2/me^2$, then electrostatic effects dominate those of quantum degeneracy. This limit always applies in the present problem, because the electron density is usually in the range[5] $10^6 \lesssim n \lesssim 10^9$ cm^{-2}. The corresponding electrostatic energy per particle lies between 2×10^{-16} and 7×10^{-15} erg, which is roughly comparable with typical thermal energies ($\approx 1.4\times10^{-16}$ erg at 1 K). Thus, quantum effects are negligible, and the present two-dimensional surface electrons always exhibit classical behavior.

For electrons on the surface of liquid He, the low temperature ($T \lesssim 2$ K) implies that the Coulomb energy exceeds the thermal energy except at very low density ($n \lesssim 4\times10^5$ cm^{-2} at 1 K). Consequently, such a system may form a two-dimensional Wigner lattice.[7,8] Unfortunately, a reliable estimate of the critical density $n_c(T)$ for melting this lattice has proved elusive even in three dimensions, so that the existence and stability of the ordered two-dimensional array remains conjectural. Independent of the detailed configuration, however, these electron surface layers should exhibit two-dimensional analogs of the plasma oscillations and screening that characterize a bulk charged medium.

The surface of liquid or solid Ne (critical temperature ≈ 44 K, normal boiling temperature ≈ 27 K[9]) can offer a quite different environment. At these higher temperatures (note that ^{4}He and Ne attain a vapor pressure of 1 mm at 1.5 and 15.9 K, respectively[9]) an electron surface layer would better approximate a classical dilute plasma, analogous to the three-dimensional Debye-Hückel limit.[10] Indeed, a charged layer on (solid) Ne at (say) 10 K might be made to pass continuously from a Coulomb-dominated regime to a thermal one; such an experimental investigation could illuminate the analogous behavior of a degenerate electron gas, where direct variation of the electron density has proved impractical. Neon offers the added advantage that its larger dielectric constant binds the electrons more tightly to the surface. (Note that liquid ^{4}He and Ne have dielectric constants[9] 1.056 and 1.167, respectively.) The corresponding scale of length perpendicular to the surface varies approximately as $(\epsilon-1)^{-1}$ and is therefore smaller in Ne than in ^{4}He by a factor ≈ 3.

To clarify the properties of such a system, we have present a many-body theory of a dilute two-dimensional classical charged gas, placed at the interface between two semi-infinite dielectrics and neutralized by a rigid background charge. Sec. II analyzes the electrodynamic effects, including the screening of a static impurity, the spectrum and damping of plasma oscillations, and inelastic electron scattering. The equation of state and other thermodynamic functions are determined in Sec. III.

II. LINEAR RESPONSE TO ELECTRODYNAMIC PERTURBATIONS

Although the present problem is wholly classical, it will be convenient to formulate it as a many-body quantum-field theory. This approach permits a unified description that is valid for all temperature; at low density and high temperature, it reduces to the classical limit applicable to electrons on the surface of He and Ne, whereas the same theory at zero temperature describes a dense two-dimensional degenerate electron gas typified by inversion layers in certain semiconductors[11] and by electrons on the surface of ZnO.[12] Since the latter problem has already received considerable attention,[11,13-15] the present work will largely ignore the low-temperature limit.

The system of interest is N electrons confined to an area A in the xy plane with periodic boundary conditions and neutralized by a positive rigid background layer. Moreover, we assume that uniform media with dielectric constants ϵ_1 and ϵ_2 fill the adjacent half spaces. When a single test charge Ze is placed at the origin of the otherwise charge-free system, the electrostatic potential at the point $(\vec{r}, z)$ has the value

$$\phi(\vec{r}, z) = \frac{2Ze}{(\epsilon_1+\epsilon_2)(r^2+z^2)^{1/2}} . \tag{1}$$

This result follows immediately as the limit of standard textbook example[16]; it has the corollary that the electric field $\vec{E}$ is radial and isotropic, whereas the displacement field $\vec{D}$ is radial but concentrated on the side with the larger dielectric constant. In addition, Eq. (1) yields the potential energy $V(r)$ of two electrons in the xy plane a distance r apart

$$V(r) = \bar{e}^2/r , \tag{2}$$

where the "renormalized" charge

$$\bar{e} = e[2(\epsilon_1+\epsilon_2)^{-1}]^{1/2} \tag{3}$$

is the only remnant of the surrounding dielectric. This same reduced charge $\bar{e}$ determines the potential energy of the neutralizing background and its interaction with the mobile electrons; hence the subsequent reduction of the total Hamiltonian is virtually identical with that in three dimensions.[17] A straightforward calculation leads to the second-quantized expression

$$\hat{H} = \sum_{\vec{k}\lambda} \epsilon_k a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} + \frac{1}{2A} \sum_{\vec{k}\vec{p}\vec{q}}{}' \sum_{\lambda_1\lambda_2} V(q) \times a^\dagger_{\vec{k}+\vec{q},\lambda_1} a^\dagger_{\vec{p}-\vec{q},\lambda_2} a_{\vec{p}\lambda_2} a_{\vec{k}\lambda_1} , \tag{4}$$

where $a^\dagger_{\vec{k}\lambda}$ $(a_{\vec{k}\lambda})$ is a creation (destruction) operator for an electron with momentum $\hbar\vec{k}$ in spin state λ and $\epsilon_k = \hbar^2k^2/2m$. Here the primed sum means omit the terms with $\vec{q}=0$, and $V(q)$ is the two-dimensional Fourier transform of Eq. (2):

$$V(q) = \bar{e}^2 \int d^2r\, e^{i\vec{q}\cdot\vec{r}} r^{-1} = 2\pi\bar{e}^2 q^{-1} . \tag{5}$$

As our first example of electrodynamic response, consider the effect of an external scalar potential $\phi_{ex}(\vec{r}, t)$ on the two-dimensional electron gas. In this case, the response may be characterized by the induced particle density $\delta n(\vec{r}, t)$. Application of the standard theory[18] yields the relation

$$\delta n(\vec{q}, \omega) = -\Pi^R(\vec{q}, \omega)\, e\,\phi_{ex}(\vec{q}, \omega) \tag{6}$$

between the Fourier transforms of ϕ_{ex}, n, and the retarded polarization Π^R. In the simplest approximation of keeping only the ring diagrams [random-phase approximation (RPA)], this latter quantity reduces to

$$\Pi^R(\vec{q}, \omega) = \Pi^{0R}(\vec{q}, \omega)[1 - V(q)\Pi^{0R}(\vec{q}, \omega)]^{-1} , \tag{7}$$

where

$$\Pi^{0R}(\vec{q}, \omega) = -\lim_{\eta\to 0} 2\int \frac{d^2p}{(2\pi)^2}\left(\frac{n_{\vec{p}+\vec{q}} - n_p}{\hbar\omega + i\eta - \epsilon_{\vec{p}+\vec{q}} + \epsilon_p}\right) \tag{8}$$

is the retarded proper polarization of an ideal two-dimensional Fermi gas and n_p is the Fermi-Dirac distribution that reduces to $\exp\beta(\mu-\epsilon_p)$ in the classical limit.

If $\phi_{ex}(\vec{r}, t)$ arises from a static point charge Ze located at the origin of the xy plane, the Fourier transform of Eq. (1) yields

$$\phi_{ex}(\vec{q}, \omega) = [(2\pi)^2 2Ze/(\epsilon_1+\epsilon_2)q]\,\delta(\omega) . \tag{9}$$

An inverse transform of (6) gives

$$\delta n(r) = -Z\int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot\vec{r}} \frac{2\pi\bar{e}^2\Pi^{0R}(q,0)}{q - 2\pi\bar{e}^2\Pi^{0R}(q,0)} , \tag{10}$$

where $\bar{e}$ is again the "renormalized" charge from Eq. (3). The evaluation of $\Pi^{0R}(q, 0)$ proceeds exactly as in the three-dimensional classical limit and even gives the same answer when expressed in terms of the variables (T, A, N):

$$\Pi^{0R}(q, 0) = -n(k_BT)^{-1} g_1(q\lambda) , \tag{11}$$

where $\lambda = (2\pi\hbar^2/mk_BT)^{1/2}$ is the thermal wavelength,

$$g_1(x) = 2\pi^{1/2} x^{-1}\,\Phi(x/4\pi^{1/2}) , \tag{12}$$

and Φ denotes the real part of the plasma-dispersion function.[19] As a result, Eq. (10) may be written

$$\delta n(r) = \frac{Z}{2\pi}\int_0^\infty q\,dq\,J_0(qr)\,\frac{k_D g_1(q\lambda)}{q + k_D g_1(q\lambda)} , \tag{13}$$

where

$$k_D = 2\pi n\bar{e}^2/k_BT \tag{14}$$

is the two-dimensional analog of the Debye-Hückel[10] screening constant and J_0 is the usual Bessel function of order zero.

Equation (13) has several desirable features, such as a finite screening density $\delta n(r)$ even at $r=0$, and a total integrated screening charge $-Ze$.

For large r, the behavior at small q dominates the integral, and $g_1(q\lambda)$ may be replaced by its leading term $g_1(0)=1$:

$$\delta n(r) \sim \frac{Zk_D}{2\pi}\int_0^\infty q\,dq\,\frac{J_0(qr)}{q+k_D}\,. \tag{15}$$

This long-wavelength approximation evidently requires $r \gg \lambda$, where it reproduces the Thomas-Fermi model of Stern and Howard,[20] but with the modified screening constant (14).

The same methods also can describe a time-dependent perturbation with frequency ω. In the random-phase approximation, the resonant response (plasma oscillation) occurs at the solution $\Omega_q - i\gamma_q$ of the equation

$$1 = V(q)\Pi^{0R}(q, \Omega_q - i\gamma_q)\,, \tag{16}$$

which, in general, requires numerical evaluation of Π^{0R}. For small damping ($\gamma_q \ll \Omega_q$), however, an expansion to first order in γ_q uncouples the real and imaginary parts

$$1 = V(q)\mathrm{Re}\Pi^{0R}(q, \Omega_q)\,, \tag{17a}$$

$$\gamma_q = \mathrm{Im}\Pi^{0R}(q, \Omega_q)\bigg/\left(\frac{\partial \mathrm{Re}\Pi^{0R}(q,\omega)}{\partial\omega}\right)_{\omega=\Omega_q}\,. \tag{17b}$$

The evaluation of these quantities is very similar to that in three dimensions; after some rearrangement, we find

$$\Omega_q^2 \approx \frac{2\pi n\bar{e}^2 q}{m} + \frac{3q^2 k_B T}{m} = \frac{2\pi n \bar{e}^2 q}{m}\left(1+\frac{3q}{k_D}\right), \tag{18a}$$

$$\gamma_q \approx \left(\frac{2\pi n\bar{e}^2 q}{m}\right)^{1/2}\left(\frac{k_D}{q}\right)^{3/2}\left(\frac{\pi}{8}\right)^{1/2}\exp\left(-\frac{k_D}{2q}-\frac{3}{2}\right), \tag{18b}$$

which holds for $q \lesssim k_D$. The leading term of (18a) exhibits the usual $q^{1/2}$ behavior,[11,15,21,22] and the correction term confirms that obtained with the hydrodynamic model.[22] The present many-body treatment introduces one new feature, however, for the long-wavelength plasma oscillations undergo Landau damping[23] like that in three dimensions; in fact, the long-wavelength ratio $|\gamma_q/\Omega_q|$ is virtually unchanged by the altered dimensionality, apart from the substitution of k_D/q for $(k_D/q)^2$.

As another example of linear response, we study inelastic scattering of nonrelativistic electrons from the charged surface layer. If $\vec{K}$ and $\vec{K}'$ denote the incident and final three-dimensional wave vectors of the projectile and $\vec{q}$ is the projection of $\vec{K}-\vec{K}'$ onto the xy plane, then the double differential cross section for scattering with projectile energy loss $\hbar\omega$ and a two-dimensional momentum transfer $\hbar q$ to the layer is given by[24]

$$\frac{d^2\sigma}{d\Omega d\omega} = \frac{NK'}{K}\left(\frac{2m\bar{e}^2}{\hbar^2|\vec{K}-\vec{K}'|^2}\right)^2 S(\vec{q},\omega)\,, \tag{19}$$

where N is the number of electrons in the target layer and $S(\vec{q},\omega)$ is the corresponding dynamic structure factor.[25] It takes the usual form

$$S(\vec{q},\omega) = N^{-1}\sum_{if}\exp[\beta(\Omega - E_i + \mu N_i)] \times \delta[\omega - \hbar^{-1}(E_f - E_i)]\,|\langle\Psi_f|\tilde{n}(-\vec{q})|\Psi_i\rangle|^2\,. \tag{20}$$

Here $n \equiv \langle\hat{n}\rangle$ is the average electron surface density in the grand canonical ensemble, $\tilde{n}(-q)$ is the two-dimensional Fourier transform of the fluctuation surface density $\tilde{n} = \hat{n} - n$, and the sum runs over the complete set of initial and final states of the target. The Lehmann representation relates Eq. (20) to the retarded polarization introduced previously in Eq. (6), and a straightforward analysis yields

$$S(\vec{q},\omega) = -\frac{\hbar}{n\pi}\,\frac{\mathrm{Im}\Pi^R(\vec{q},\omega)}{1-\exp(-\beta\hbar\omega)}\,. \tag{21}$$

In the lowest (RPA) approximation, where the proper polarization is taken as that for an ideal gas, the dynamic structure factor assumes the simple form

$$S(\vec{q},\omega) \approx -\frac{\hbar}{n\pi(1-e^{-\beta\hbar\omega})} \times \frac{\mathrm{Im}\Pi^{0R}(\vec{q},\omega)}{[1-V(\vec{q})\mathrm{Re}\Pi^{0R}(\vec{q},\omega)]^2 + [V(\vec{q})\mathrm{Im}\Pi^{0R}(\vec{q},\omega)]^2}\,, \tag{22}$$

with $V(q)$ the interparticle potential from Eq. (5).

Consider first an ideal gas ($V=0$), when the denominator reduces to 1, and $S(\vec{q},\omega)$ then is proportional to $\mathrm{Im}\Pi^{0R}(\vec{q},\omega)$. For fixed q, the resulting ideal-gas structure factor has a maximum at the "quasielastic" value $\omega_0(q) = \hbar q^2/2m$, with a width of order $(k_B T q^2/m)^{1/2}$. The inclusion of interactions affects this behavior significantly, for Eq. (22) is large not only where $\mathrm{Im}\Pi^{0R}$ is large but also where $1 = V(q)\mathrm{Re}\Pi^{0R}$. Comparison with (17a) indicates that $S(\vec{q},\omega)$ for a two-dimensional charged layer should acquire a peak at the plasma frequency Ω_q (18a) with a width of order γ_q (18b). Indeed, an expansion of (22) near $\omega \approx \Omega_q$ yields a Lorentzian form

$$S(\vec{q},\omega) \approx \left(\frac{q}{2\pi k_D}\right)\frac{\beta\hbar\Omega_q}{1-\exp(-\beta\hbar\Omega_q)} \times \left(\frac{m\Omega_q^2}{2\pi n\bar{e}^2 q}\right)^2 \frac{\gamma_q}{(\omega-\Omega_q)^2+\gamma_q^2}\,. \tag{23}$$

Observation of such a plasma peak would verify the detailed form of the dispersion relation, which has previously been deduced from purely theoretical considerations.

III. THERMODYNAMIC FUNCTIONS

Section II studied the electrodynamic response of a classical two-dimensional electron gas in the random-phase approximation. Although this approach is rigorous only in the low-density limit ($n^{1/2} \ll k_B T\, e^{-2}$), its qualitative predictions are likely to have a wider range of validity. In particular, the essential features of the screening (15) and plasma oscillations (18) reflect the long range of the Coulomb interactions and occur in other more approximate descriptions.[8,15,20–22]

On the other hand, the situation is less favorable for the thermodynamic properties of the charged surface layer, because quantities like the equation of state can be obtained only as expansions in the small parameter $e^2 n^{1/2}/k_B T$. Consequently, the resulting theory is limited to low density ($n \lesssim 4 \times 10^5$ cm^{-2} for ^{4}He at 1 K and $n \lesssim 4\times 10^7$ cm^{-2} for Ne at 10 K). Nevertheless, it may have experimental application to these charged surface layers, and, moreover, the expressions exhibit an interesting nonanalyticity in the coupling constant. As the calculations are standard, the derivations are greatly abbreviated.

It is most convenient to evaluate the thermodynamic potential $\Omega(T, A, \mu)$,[26] whose derivatives yield the various thermodynamic functions like the entropy S, the pressure p, and the number of particles N:

$$S = -\left(\frac{\partial\Omega}{\partial T}\right)_{A\mu}, \quad p = -\left(\frac{\partial\Omega}{\partial A}\right)_{T\mu}, \quad N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{TA}. \tag{24}$$

Note that p here has the dimensions of a force per unit length, appropriate for a two-dimensional configuration. If the interaction potential $V(r)$ has a short range and a well-defined Fourier transform, then Ω may be constructed with perturbation theory. In the present situation of a long-range Coulomb potential, however, some of the terms diverge, and it becomes necessary to sum an infinite subset of all the contributions. This procedure is well known in three dimensions, where the divergence first appears in second order. There, the resulting zero-temperature correlation energy[27] varies like $e^4 \ln e$, whereas the first Coulomb correction to an ideal classical gas[10] varies like e^3. Each of these regimes involves nonanalytic behavior in e^2, with the classical limit more singular than the degenerate zero-temperature one. As shown below, the two-dimensional electron gas behaves similarly.

The evaluation of Ω from the Hamiltonian (4) is a standard problem in many-body theory, and the altered dimensions and presence of adjacent dielectrics affect the Feynman rules[28] only in that the momentum differentials become $A(2\pi)^{-2}d^2k$, and the Fourier transform of the potential is given by Eq. (5), which varies like q^{-1} for small q. As a result, the leading contribution to the two-dimensional correlation energy is finite at zero temperature[15] owing to the altered phase space and less singular form of $V(q)$. On the other hand, the first classical correction to the ideal gas in two dimensions turns out to vary like $e^4 \ln e$, which is less singular than the e^3 dependence in three-dimensions, but it still requires a summation of the ring diagrams to infinite order. A detailed calculation (see the Appendix) yields the approximate expression

$$\Omega(T, A, \mu) \approx -\frac{2Ae^{\beta\mu}}{\beta\lambda^2} - \frac{2\pi\beta A e^{2\beta\mu}\bar{e}^4}{\lambda^4} \times\left[\ln\left(\frac{\lambda}{4\pi\beta\bar{e}^2\exp(\beta\mu)}\right) + \frac{3}{2} + c\right], \tag{25}$$

where

$$c = \tfrac{1}{2}\ln(2\pi) - \tfrac{1}{2}\gamma - G \approx -0.2856 \tag{26}$$

and $G \approx 0.91597$ is Catalan's constant.

The first step in deriving the thermodynamic functions is to eliminate μ in favor of N with Eq. (24). The resulting thermodynamic potential is just $-pA$, which therefore yields the equation of state

$$p(T, A, N) \approx n k_B T\left\{1 - \frac{\pi}{2}\frac{n\bar{e}^4}{(k_B T)^2} \times\left[\ln\left(\frac{k_B T}{2\pi n^{1/2}\bar{e}^2}\right) + \frac{1}{2}\ln\left(\frac{m k_B T}{2\pi n \hbar^2}\right) + \frac{1}{2} + c\right]\right\}. \tag{27}$$

Note that the leading correction is indeed small in the limit $\hbar^2 n/m \ll n^{1/2}\bar{e}^2 \ll k_B T$; nevertheless, it is not permissible to neglect the zero-point energy entirely, for the expression diverges logarithmically as $\hbar \to 0$. The isothermal bulk modulus B_T exhibits a similar behavior.

To evaluate thermal properties, it is preferable to introduce the Helmholtz free energy $F(T, A, N) = \Omega + \mu N$. An elementary calculation gives the corresponding entropy

$$S(T, A, N) = N k_B\left\{\ln\left(\frac{m k_B T}{\pi n \hbar^2}\right) + 2 - \frac{\pi n \bar{e}^4}{2(k_B T)^2} \times\left[\ln\left(\frac{k_B T}{2\pi n^{1/2}\bar{e}^2}\right) + \frac{1}{2}\ln\left(\frac{m k_B T}{2\pi n\hbar^2}\right) + c\right]\right\} \tag{28}$$

along with the specific heat at constant area

$$C_A = N k_B\left\{1 + \frac{\pi n \bar{e}^4}{(k_B T)^2}\left[\ln\left(\frac{k_B T}{2\pi n^{1/2}\bar{e}^2}\right) + \frac{1}{2}\ln\left(\frac{m k_B T}{2\pi n\hbar^2}\right) + c - \frac{3}{4}\right]\right\} \tag{29}$$

and constant pressure

$$C_p = 2N k_B\left\{1 + \frac{3\pi n \bar{e}^4}{2(k_B T)^2}\left[\ln\left(\frac{k_B T}{2\pi n^{1/2}\bar{e}^2}\right)\right.\right.$$

$$+\frac{1}{2}\ln\left(\frac{mk_BT}{2\pi n\hbar^2}\right)+c-\frac{7}{12}\Bigg]\Bigg\}. \qquad (30)$$

An experimental study of some of these quantities for electron surface layers on liquid He or Ne would be valuable.

The appearance of $\hbar$ in these apparently classical expressions can be explained in various ways. First and most trivial, the thermodynamic functions of classical gases depend explicitly on $\hbar$ because it fixes the allowed cell size in classical phase space. More significant, however, is the factor $\ln(k_BT/2\pi n\bar{e}^2\lambda)$ in the second-order corrections. As in the case of a three-dimensional charged medium, this factor arises from a quantum-mechanical cutoff on the maximum allowed momentum transfer, or, equivalently, on the minimum allowed impact parameter.[29–31] In the present two-dimensional case, the minimum impact parameter is of order $\hbar/m\langle v\rangle=(\hbar^2/2mk_BT)^{1/2}$ ≈ 200 Å at 1 K, which is comparable with the thickness of the electron surface layer on ^{4}He. Thus it is possible that the finite perpendicular extension of the wave functions might alter the logarithmic dependence on λ. This effect requires further analysis; it would be less significant for electrons on Ne, owing to the enhanced binding to the surface.

IV. CONCLUSIONS

The present paper has analyzed the electrodynamic and thermodynamic properties of a classical two-dimensional electron gas, which serves as a model for surface electrons on ^{4}He or Ne at low temperatures. Most importantly, such a charged layer responds in a characteristic manner to an external electrostatic perturbation. It screens an external charge placed in its plane with a long-range algebraic tail[20,22] in contrast to the exponential Debye-Hückel form in three dimensions. Moreover, the dispersion relation Ω_q (18a) for two-dimensional plasma oscillations is virtually identical with that obtained previously.[11,15,22] Finally, inelastic electron scattering from the charged layer might verify the existence of the two-dimensional plasmons, for the cross section should peak sharply at an energy loss $\hbar\Omega_q$.

It is interesting briefly to compare these features with those for a degenerate two-dimensional electron gas, which serves as a model for semiconductor inversion layers[11] and for electrons on the surface of ZnO.[12] The present many-body formalism remains valid if n_k is taken as the general Fermi-Dirac distribution function, and most of the conclusions remain qualitatively correct. There are a few new features, however, which reflect the presence of a sharp Fermi surface. First, the correlation energy (terms of order e^4 and higher) is finite,[15] in contrast to the logarithmic singularity found here in the classical limit. Second, the screening around an external impurity exhibits long-range Friedel oscillations.[11] Third, if $\mathrm{Im}\Pi^{0R}(q,\omega)$ is considered a function of ω for fixed $q<2k_F$, it acquires a cusp at a characteristic frequency $(\hbar q/m)(k_F-\frac{1}{2}q)$, which might be detectable as an additional structure in inelastic electron scattering from degenerate surface layers. This last possibility merits further study.

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APPENDIX

This Appendix computes the approximate thermodynamic potential for a classical two-dimensional electron gas in the low-density limit ($e^2n^{1/2}\ll k_BT$). The leading contribution is that for an ideal spin-$\frac{1}{2}$ Fermi gas[26]

$$\Omega_0=-2k_BT\sum_{\vec{k}}\ln(1+e^{\beta(\mu-\epsilon_k)})$$
$$\approx -2Ae^{\beta\mu}/\beta\lambda^2, \qquad (\mathrm{A.1})$$

where $\lambda=(2\pi\hbar^2/mk_BT)^{1/2}$ is the thermal wavelength. Quantum corrections to (A.1) are smaller by a factor of order $e^{\beta\mu}\approx\frac{1}{2}n\lambda^2$, which is negligible in the present case. Moreover, the first-order correction to Ω_0 also vanishes in the classical limit,[32] because it arises solely from quantum-mechanical exchange. As a result, the presence of interactions affects Ω only in second order, and these quantities will now be examined in detail.

The small classical occupation of any particular single-particle state permits a simple classification of the various perturbation terms. The only relevant second-order contributions are the "direct" and "exchange" ones

$$\Omega_{2a}=-2A\mathcal{P}\int\frac{d^2k\,d^2p\,d^2q}{(2\pi)^6}\,\frac{[V(q)]^2n_kn_p}{\epsilon_{\vec{k}+\vec{q}}+\epsilon_{\vec{p}+\vec{q}}-\epsilon_k-\epsilon_p}, \qquad (\mathrm{A.2a})$$

$$\Omega_{2b}=A\mathcal{P}\int\frac{d^2k\,d^2p\,d^2q}{(2\pi)^6}\,\frac{V(q)V(\vec{k}+\vec{p}+\vec{q})n_kn_p}{\epsilon_{\vec{k}+\vec{q}}+\epsilon_{\vec{p}+\vec{q}}-\epsilon_k-\epsilon_p}, \qquad (\mathrm{A.2b})$$

where $\mathcal{P}$ denotes a Cauchy principal value and we have passed to the classical limit with $n_k=\exp\beta(\mu-\epsilon_k)\ll 1$. The term Ω_{2b} is finite, but Ω_{2a} diverges logarithmically, necessitating the inclusion of higher-order ring diagrams. The analysis is similar to that for a three-dimensional electron gas,[32] combining features of both the classical and degenerate limits. The net effect is to replace Ω_{2a} by $\Omega_{r1}+\Omega_{r2}$, where

$$\Omega_{r1}=\frac{A}{4\pi\beta}\int_0^{q_0} q\,dq\,\{\ln[1-V(q)\Pi^{0R}(q,0)] + V(q)\Pi^{0R}(q,0)\}, \quad \text{(A.3a)}$$

$$\Omega_{r2}=\frac{-2A}{(2\pi)^5}\,\mathcal{P}\int_{q_0}^{\infty} q\,dq\,[V(q)]^2\int\frac{d^2k\,d^2p\,n_k n_p}{\epsilon_{\vec{k}+\vec{q}}+\epsilon_{\vec{p}+\vec{q}}-\epsilon_k-\epsilon_p}\;. \quad \text{(A.3b)}$$

Here $\Pi^{0R}(q,0)$ is given in Eq. (11) and the cutoff q_0 must satisfy $q_0\lambda \ll 1$.

The first step in evaluating Ω_{2b} is the introduction of dimensionless variables

$$\Omega_{2b}=\frac{\beta A e^{2\beta\mu}}{2\pi^2}\,\frac{\bar{e}^4}{\lambda^4}\,\mathcal{P}\int\frac{d^2w\,d^2x\,d^2y\exp(-x^2-y^2)}{w\,|\vec{w}+\vec{x}+\vec{y}\,|[\vec{w}\cdot(\vec{w}+\vec{x}+\vec{y})]}\;. \quad \text{(A.4)}$$

A linear substitution $\vec{y}=-\vec{v}-\vec{w}-\vec{x}$ then permits a direct evaluation of the $\vec{x}$ integration

$$\Omega_{2b}=-\frac{\beta A e^{2\beta\mu}}{4\pi}\,\frac{\bar{e}^4}{\lambda^4}\,\mathcal{P}\times\int\frac{d^2v\,d^2w\exp(-\tfrac{1}{2}v^2-\tfrac{1}{2}w^2-\vec{v}\cdot\vec{w})}{vw(\vec{v}\cdot\vec{w})}\;. \quad \text{(A.5)}$$

The remaining fourfold integral may be simplified by observing that

$$\mathcal{P}\int d^2v\,d^2w\,\frac{\exp(-\tfrac{1}{2}v^2-\tfrac{1}{2}w^2)}{vw(\vec{v}\cdot\vec{w})}=0 \quad \text{(A.6)}$$

because the integrand is an odd function of its arguments $\vec{v}$ and $\vec{w}$. Consequently, a combination of Eqs. (A.5) and (A.6) with the identity

$$-\int_0^1 d\zeta\exp(-\zeta\,\vec{v}\cdot\vec{w})=(\vec{v}\cdot\vec{w})^{-1}[\exp(-\vec{v}\cdot\vec{w})-1] \quad \text{(A.7)}$$

yields

$$\Omega_{2b}=\frac{\beta A e^{2\beta\mu}}{4\pi}\,\frac{\bar{e}^4}{\lambda^4}\int_0^1 d\zeta\times\int\frac{d^2v\,d^2w}{vw}\exp(-\tfrac{1}{2}v^2-\tfrac{1}{2}w^2-\zeta\,\vec{v}\cdot\vec{w})\;, \quad \text{(A.8)}$$

where the principal-value prescription is now superfluous. If the double integrals are expressed in polar coordinates, the angular integrations produce a Bessel function $I_0(\zeta vw)\equiv J_0(i\zeta vw)$ of zero order and imaginary argument[33]

$$\Omega_{2b}=\pi\beta A e^{2\beta\mu}\,\frac{\bar{e}^4}{\lambda^4}\int_0^1 d\zeta\times\iint_0^\infty dv\,dw\exp(-\tfrac{1}{2}v^2-\tfrac{1}{2}w^2)\,I_0(\zeta vw)\;. \quad \text{(A.9)}$$

The remaining integrations are expressible in terms of known functions[33] and may eventually be rewritten in terms of an integral of the complete elliptic integral[34]

$$\Omega_{2b}=\pi\beta A e^{2\beta\mu}\,\frac{\bar{e}^4}{\lambda^4}\int_0^1 d\zeta\,K(\zeta) = 2\pi G\beta A e^{2\beta\mu}\,\bar{e}^4/\lambda^4\;, \quad \text{(A.10)}$$

where $G=1-9^{-1}+25^{-1}-\cdots\approx 0.91597$ is Catalan's constant.[35]

The next contribution Ω_{r1} requires $\Pi^{0R}(q,0)$ only for $q_0\lambda\ll 1$; a simple analysis [compare Eq. (11)] gives

$$\Pi^{0R}(q,0)\approx-2\beta e^{\beta\mu}\lambda^{-2} \quad \text{(A.11)}$$

apart from corrections of order $(q\lambda)^2$. Equation (A.3a) thus reduces to

$$\Omega_{r1}=\frac{A}{4\pi\beta\lambda^2}\int_0^{q_0\lambda} x\,dx[\ln(1+\alpha x^{-1})-\alpha x^{-1}]$$

$$=\frac{A}{8\pi\beta\lambda^2}\left\{(q_0\lambda)^2\ln\left[1+\left(\frac{\alpha}{q_0\lambda}\right)\right]-\alpha^2\ln\left[\left(\frac{q_0\lambda}{\alpha}\right)+1\right]-q_0\lambda\alpha\right\}$$

$$\approx\frac{A\alpha^2}{8\pi\beta\lambda^2}[-\ln(q_0\lambda)+\ln\alpha-\tfrac{1}{2}]\;, \quad \text{(A.12)}$$

where $\alpha=4\pi\beta e^{\beta\mu}\,\bar{e}^2/\lambda$ is a dimensionless parameter, and the last line omits terms that vanish faster than α^2 for small α.

The other contribution of interest is Ω_{r2}, given in Eq. (A.3b). The dimensionless variables used for Ω_{2b} reduce this quantity to

$$\Omega_{r2}=-\frac{2\beta A e^{2\beta\mu}}{\pi}\,\frac{\bar{e}^4}{\lambda^4}\,\mathcal{P}\int_{w_0}^\infty\frac{dw}{w}\times\int d^2x\,d^2y\,\frac{\exp(-x^2-y^2)}{\vec{w}\cdot(\vec{w}+\vec{x}+\vec{y})}\;, \quad \text{(A.13)}$$

where $w_0=q_0(\beta\hbar^2/2m)^{1/2}$. An orthogonal transformation $\vec{x}+\vec{y}=\sqrt{2}\,\vec{s}$, $\vec{x}-\vec{y}=\sqrt{2}\,\vec{t}$ renders two of the integrations elementary, leaving

$$\Omega_{r2}=-2\beta A e^{2\beta\mu}\,\frac{\bar{e}^4}{\lambda^4}\,\mathcal{P}\int_{w_0}^\infty\frac{dw}{w}\times\int d^2s\,\frac{\exp(-s^2)}{\vec{w}\cdot(\vec{w}+\sqrt{2}\,\vec{s})}\;. \quad \text{(A.14)}$$

If $\vec{s}$ is resolved into its Cartesian components $s_\parallel$ and $s_\perp$ parallel and perpendicular to $\vec{w}$, a straightforward integration over $s_\perp$ yields the two-dimensional integral

$$\Omega_{r2}=-\pi^{1/2}\beta A e^{2\beta\mu}\,\frac{\bar{e}^4}{\lambda^4}\int_{z_0}^\infty\frac{dz}{z^2}\,\mathcal{P}\int_{-\infty}^\infty ds\,\frac{\exp(-s^2)}{z+s}\;, \quad \text{(A.15)}$$

where the new variables $z=w/\sqrt{2}$ and $s=s_\parallel$ have been introduced and $z_0 \equiv w_0/\sqrt{2}$. Partial integration transforms the s integral as follows:

$$\mathcal{P}\int_{-\infty}^{\infty} ds \frac{\exp(-s^2)}{z+s} = 2z\mathcal{P}\int_0^{\infty} ds \frac{\exp(-s^2)}{z^2-s^2}$$

$$= 2\int_0^{\infty} ds\, s \exp(-s^2) \ln\left|\frac{z+s}{z-s}\right| , \quad (A.16)$$

and a combination with (A.15) gives

$$\Omega_{r2} = -2\pi^{1/2}\beta A e^{2\beta\mu}\frac{\bar{e}^4}{\lambda^4}$$

$$\times \int_0^{\infty} ds\, s\exp(-s^2)\int_{z_0}^{\infty}\frac{dz}{z^2}\ln\left|\frac{z+s}{z-s}\right| , \quad (A.17)$$

where the order of integration has been reversed. The z integral may be done exactly, and an expansion for small z_0 leads to

$$\int_{z_0}^{\infty}\frac{dz}{z^2}\ln\left|\frac{z+s}{z-s}\right| \approx s^{-1}[2+2\ln s+\ln(2/w_0^2)] . \quad (A.18)$$

Substitution into Eq. (A.17) then yields

$$\Omega_{r2} = 2\pi\beta A\, e^{2\beta\mu}(\bar{e}^4/\lambda^4)[\ln(q_0\lambda) - 1 - \tfrac{1}{2}\ln(2\pi) + \tfrac{1}{2}\gamma] , \quad (A.19)$$

where we used the definite integral

$$\int_0^{\infty} ds \ln s \exp(-s^2) = -\tfrac{1}{4}\pi^{1/2}(2\ln 2+\gamma) , \quad (A.20)$$

with $\gamma \approx 0.5772$ being Euler's constant. Equations (A.1), (A.10), (A.12), and (A.19) together give Eq. (25).

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