Spin correlations in an electron liquid at metallic densities*

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A nonlinear integral equation for the ground-state magnetic structure factor $\bar{S}(k)$ of an electron liquid has been obtained by using a method originally due to Mihara and Puff. This equation is solved self-consistently to get the values of $\bar{S}(k)$ which are used to estimate the pair correlation functions of electrons with parallel and antiparallel spins. The results obtained compare favorably with that of earlier sophisticated calculations.

I. INTRODUCTION

The study of the effect of electron-electron interactions on metallic properties is quite an old problem, but still it remains to be solved. An interesting and quite successful approach to investigate this problem has been to study the system of a uniform electron gas immersed in a uniform compensating background of positive charge and to apply the results directly to calculations in real metals, particularly free-electron-like metals. Recently, Mihara and Puff[1] (MP) derived a nonlinear integral equation for the ground-state structure factor of interacting bosons by using an inequality and the sum rules, obeyed by the spectral function of the density response function. They found reasonably good results for some ground-state properties of liquid He[4]. Based on that work, Kugler[2] obtained an integral equation for the static structure factor of an electron liquid. Very recently we have numerically solved[3] the above integral equation and found that it gives quite reasonable results for the static structure factor and pair correlation function at metallic densities.

In this paper we derive a nonlinear integral equation for the magnetic structure factor $\bar{S}(k)$ of an electron liquid by using the approach of MP.[1] This integral equation is then solved self-consistently to get the values of $\bar{S}(k)$. These values are used to calculate the spin pair correlation function $\bar{g}(r)$. Knowledge of this function when combined with the spin-symmetric pair correlation function $g(r)$ obtained by us earlier,[3] yields separately the correlation functions for the two types of spin pairs. Our results for the pair correlation function of electrons with opposite spins satisfy the positive-definiteness requirement over the whole range of metallic densities. The pair correlation function for electrons of parallel spins is negative throughout, like in most of the other available theories. However, the results of Lobo et al.[4] are comparatively less negative. The results for the pair correlation functions are compared with other available theories.[4-6]

In Sec. II relevant quantities are introduced which are to be used in future discussion. The nonlinear integral equation for the magnetic structure factor is set up in Sec. III and the small- and large-wave-vector limit of $\bar{S}(k)$ is then discussed. In Sec. IV we solve the integral equation self-consistently and present and discuss the numerical results for the magnetic structure factor and various spin pair correlation functions. Section V contains our concluding remarks. In Appendix A we present a brief derivation of the third moment of the spin-density response function and the potential contribution to this is worked out in the Hartree-Fock approximation in Appendix B.

II. SPIN-DENSITY RESPONSE FUNCTION AND THE SUM RULES

The spin susceptibility of an electron liquid is a quantity of fundamental interest for discussing the spin correlations in an electron liquid. The Fourier transform of the z component of the spin-density fluctuation operator is defined as

$$S_z(\vec{k}) = \frac{1}{2}\sum_{\vec{q}\sigma}\eta_\sigma a^\dagger_{\vec{q}\sigma} a_{\vec{q}+\vec{k}\sigma}$$

$$= \frac{1}{2}\sum_{\vec{q}\sigma}\eta_\sigma \rho_{\vec{q}\sigma}(\vec{k}); \quad \eta_\sigma = \begin{cases} 1 & \text{for } \sigma = \uparrow \\ -1 & \text{for } \sigma = \downarrow \end{cases} \quad (1)$$

where $a^\dagger_{k\sigma}$ and $a_{k\sigma}$ are the electron creation and annihilation operators, respectively. The spin susceptibility is determined by the linear response of the electron liquid to an infinitesimal external magnetic field. We define the spin susceptibility (in units of $g^2\mu_B^2$, where g is the electron Lande factor and μ_B is the Bohr magneton) as the retarded spin-density response function[7,8]

$$\chi(\vec{k}, t) = i\theta(t)\langle[S_z(\vec{k}, t), S_z^\dagger(\vec{k}, 0)]\rangle, \quad (2)$$

where $\theta(t)$ is the Heaviside unit step function. The angular brackets denote the equilibrium ensemble average with respect to familiar electron liquid Hamiltonian

$$H = H_0 + H_1 = \sum_{\vec{q}\sigma}\omega(\vec{q})a^\dagger_{\vec{q}\sigma}a_{\vec{q}\sigma}$$

$$+\frac{1}{2}\sum_{\vec{q}\neq 0}\phi(\vec{q})\{\rho(\vec{q})\rho^\dagger(\vec{q})-n\}, \qquad (3)$$

where

$$\omega(\vec{q})=\hbar q^2/2m\ ;\quad \phi(\vec{q})=4\pi e^2/q^2$$

and n is the uniform electron density. $\rho(\vec{q})$ $=\sum_{\vec{k}\sigma}\rho_{\vec{k}\sigma}(\vec{q})$ is the Fourier transform of the electron density fluctuation operator.

The spectral function of the spin-density response function is defined as[8]

$$\chi''(\vec{k},\ \omega)=\int_{-\infty}^{\infty}dt\,\tfrac{1}{2}\langle[S_z(\vec{k},\ t),\ S_z^\dagger(\vec{k},\ 0)]\rangle\,e^{i\omega t}. \qquad (4)$$

This function satisfies f-sum rule[9]

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\,\omega\chi''(\vec{k},\ \omega)=\tfrac{1}{2}n\omega(\vec{k}). \qquad (5)$$

Another useful and exact relation satisfied by $\chi''(\vec{k},\ \omega)$ is the third-frequency-moment sum rule:

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\,\omega^3\chi''(\vec{k},\ \omega)=\tfrac{1}{2}n\omega(\vec{k})\omega_3^2(\vec{k}), \qquad (6)$$

where

$$\omega_3^2(\vec{k})=\omega^2(\vec{k})+4\omega(\vec{k})\langle E_{KE}\rangle/\hbar+I_3(\vec{k}). \qquad (7)$$

Here $\langle E_{KE}\rangle$ denotes the exact kinetic energy per electron and

$$I_3(\vec{k})=\frac{1}{m}\sum_{\vec{q}\neq 0}\phi(q)(\hat{k}\cdot\vec{q})^2[\tilde{S}(\vec{k}-\vec{q})-S(\vec{q})]$$

$$=\frac{e^2}{m\pi}\int_0^\infty dq\,q^2\left\{[\tilde{S}(q)-1]\left[\frac{3}{2}-\frac{k^2}{2q^2}+\frac{k}{4q}\left(\frac{q^2}{k^2}-1\right)^2\ln\left|\frac{k+q}{k-q}\right|\right]-\tfrac{2}{3}[S(q)-1]\right\}, \qquad (8)$$

where

$$\tilde{S}(k)=\frac{2}{n}\left[\langle\rho(k\uparrow)\rho^\dagger(k\uparrow)\rangle-\langle\rho(k\uparrow)\rho^\dagger(k\downarrow)\rangle\right] \qquad (9)$$

is the magnetic structure factor and

$$S(k)=\frac{2}{n}\left[\langle\rho(k\uparrow)\rho^\dagger(k\uparrow)\rangle+\langle\rho(k\uparrow)\rho^\dagger(k\downarrow)\rangle\right] \qquad (10)$$

is the usual structure factor in the paramagnetic state. It may be noted that the third moment of the spin-density response function is not expressible in terms of $\tilde{S}(k)$ alone. Our derivation of Eq. (6) is given in Appendix A.[10] The result for the third moment [i.e., Eq. (6)] for the electron liquid was obtained by Goodman,[11] and its physical content has recently been analyzed in detail by Goodman and Sjölander.[12] A discussion of the third moment of the spin-density response function for Fermi liquid with short-range interaction has also been given by Safir and Widom.[13]

We now give the small- and large-k behavior of the third moment which shall be used in Sec. III. It can be obtained by noting that

$$I_3(k\to 0)=\tfrac{1}{3}\omega_p^2(1-g_{\uparrow\uparrow}(0)) \qquad (11)$$

and

$$I_3(k\to\infty)=\tfrac{1}{3}\omega_p^2(1+g_{\uparrow\uparrow}(0)-2g_{\uparrow\downarrow}(0)) \qquad (12)$$

where $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency; $g_{\uparrow\uparrow}(0)$ and $g_{\uparrow\downarrow}(0)$ are the values at the origin of the pair correlation functions for electrons with parallel and antiparallel spins, respectively. It can now be seen from Eq. (7) that $\omega_3^2(k)$ behaves as $\tfrac{1}{3}\omega_p^2(1-g_{\uparrow\uparrow}(0))$ and $\omega^2(k)$ in the small- and large-k limit, respectively. Thus in the limiting cases of small and large k, the third moments of the spectral functions of spin-density and density response func-

tion behave in the same way.[14]

It can be seen from Eq. (4), together with the translational and rotational invariance of the system that $\chi''(\vec{k},\ \omega)$ satisfies the following relations[15]

$$\chi''(\vec{k},\ \omega)=-\chi''(\vec{k},\ -\omega)=\chi''(k,\ \omega);\quad \omega\chi''(k,\ \omega)\geq 0. \qquad (13)$$

Further by using the fluctuation dissipation theorem[16] it can be shown that $\chi''(k,\ \omega)$ is related to $\tilde{S}(k)$ through the exact relation

$$\tilde{S}(k)=\frac{2}{\pi n}\int_{-\infty}^\infty d\omega\chi''(k,\ \omega)\coth(\tfrac{1}{2}B\hbar\omega), \qquad (14)$$

where

$$B=(k_B T)^{-1}.$$

III. INTEGRAL EQUATION FOR $\tilde{S}(k)$

The results given in Sec. II shall be used here to set up a nonlinear integral equation for $\tilde{S}(k)$. We closely follow MP[1] in deriving an inequality for $\tilde{S}(k)$. Thus we get from Eqs. (6) and (14)

$$\left[\omega_3^2(k)\left(\frac{\tilde{S}(k)}{\omega(k)}\right)^2\right]^{1/3}=\left[\int_{-\infty}^\infty d\omega\,\omega^3 f_s(k,\ \omega)\right]^{1/3}$$

$$\times\left[\int_{-\infty}^\infty d\omega f_s(k,\ \omega)\coth(\tfrac{1}{2}B\hbar\omega)\right]^{2/3}, \qquad (15)$$

where $f_s(k,\ \omega)=(2/\pi n)\chi''(k,\ \omega)/\omega(k)$, is a real positive-definite function on the positive ω axis. The application of Holder's inequality[17] to Eq. (15) gives

$$\left[\omega_3^2(k)\left(\frac{\tilde{S}(k)}{\omega(k)}\right)^2\right]^{1/3}\geq\int_{-\infty}^\infty d\omega\left\{[\omega^3 f_s(k,\ \omega)]^{1/3}\right.$$

$$\left.\times[f_s(k,\ \omega)\coth(\tfrac{1}{2}B\hbar\omega)]^{2/3}\right\}$$

$$\geq\int_{-\infty}^\infty d\omega\,\omega f_s(k,\ \omega)=1, \qquad (16)$$

where the last equality follows from the f-sum rule. Thus we obtain an exact inequality

$$[\bar{S}(k)/\omega(k)]^2 \geq 1/\omega_3^2(k) . \tag{17}$$

This can be written[1] as

$$[\omega(k)/\bar{S}(k)]^2 = \omega_3^2(k) - t(k) , \tag{18}$$

where $t(k) \geq 0$ for all k.

As we are considering a fermion system, so $\bar{S}(k)$ in Eq. (18) must reduce to $S_0(k)$ in the case of free-fermion gas where

$$S_0(k) = \begin{cases} \frac{3}{4} k/k_F - \frac{1}{16} k^3/k_F^3 , & 0 < k < 2k_F \\ 1, & 2k_F < k \end{cases} \tag{19}$$

is the static structure factor of the noninteracting fermions in the ground state. Thus from Eq. (18), we obtain

$$t_f(k) = \omega_{3f}^2(k) - \omega^2(k)/S_0^2(k) \tag{20}$$

where $t_f(k)$ is the value of $t(k)$ for the noninteracting electron gas and

$$\omega_{3f}^2(k) = \omega^2(k) + 4\omega(k)\langle E_{KE}\rangle_f /\hbar . \tag{21}$$

Here $\langle E_{KE}\rangle_f$ is the kinetic energy per electron for a free-electron gas. Using Eq. (20), one can write Eq. (18) as

$$[\omega(k)/\bar{S}(k)]^2 = [\omega(k)/S_0(k)]^2 + I_3(k)$$
$$+ 4\omega(k)\Delta\langle E_{KE}\rangle/\hbar - h(k) , \tag{22}$$

where $\Delta\langle E_{KE}\rangle = \langle E_{KE}\rangle - \langle E_{KE}\rangle_f$ and

$$h(k) = t(k) - t_f(k) . \tag{23}$$

The integral Eq. (22) now gives the structure factor for the noninteracting case in the appropriate limit. It can be seen from the integral equation that in order to have a finite spin pair correlation function at the origin, the leading term of the function $h(k)$

must have the value $4\omega(k)\Delta\langle E_{KE}\rangle/\hbar$ for large k. In view of this we assume $h(k) = 4\omega(k)\Delta\langle E_{KE}\rangle/\hbar$ for all k. So the nonlinear integral equation for the ground-state magnetic structure factor in the final form is given as

$$[\omega(k)/\bar{S}(k)]^2 = [\omega(k)/S_0(k)]^2 + I_3(k) . \tag{24}$$

It may be noted that this integral equation is similar to the integral equation for $S(k)$, obtained by Kugler.[2]

The interesting features of this integral equation are that it is consistent with the frequency moment sum rules and gives a finite value of $\tilde{g}(r)$. It can be seen from Eqs. (24) and (11) that for small k, $\bar{S}(k)$ behave as

$$\bar{S}(k) = \left(\frac{3}{1-g_{\uparrow\downarrow}(0)}\right)^{1/2} \frac{\hbar k^2}{2m\omega_p} \tag{25}$$

which is similar to the corresponding exact result for $S(k)$. It is to be noted that this result is not valid in the Hartree-Fock approximation. In this case our $\bar{S}(k) = S_0(k)$, by the very construction of the integral equation. Equation (24) together with Eq. (12) predicts the large-k behavior of $\bar{S}(k)$ to be of the form

$$\bar{S}(k) - 1 = b/k^4 , \tag{26}$$

where k is measured in units of Fermi wave vector k_F and

$$b = -2(4/9\pi)^{4/3} r_s(1 + g_{\uparrow\uparrow}(0) - 2g_{\uparrow\downarrow}(0)) . \tag{27}$$

Here r_s in the electron-density parameter. The similar asymptotic limit was also found for $S(k)$.[3]

IV. SELF-CONSISTENT SOLUTION OF THE INTEGRAL EQUATION

In order to solve the integral Eq. (24) numerically, we write it in the dimensionless form

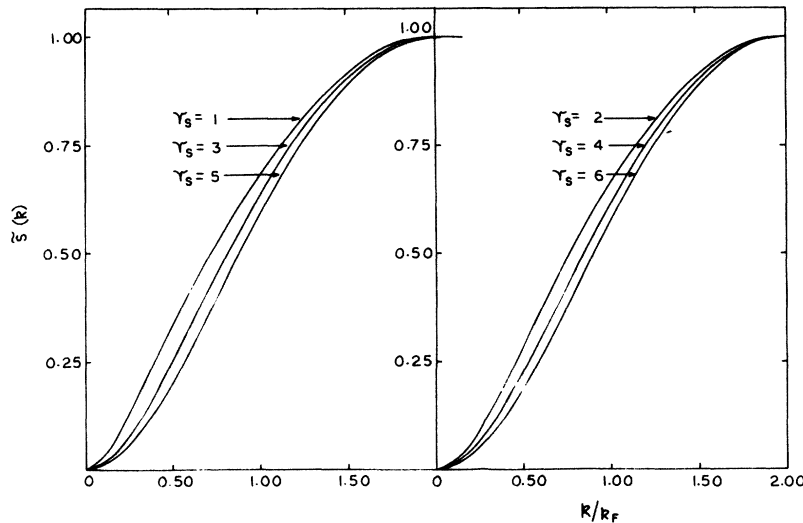FIG. 1. Self-consistent values of the ground-state magnetic structure factor $\bar{S}(k)$ of the electron liquid versus k for $r_s = 1$ to 6.
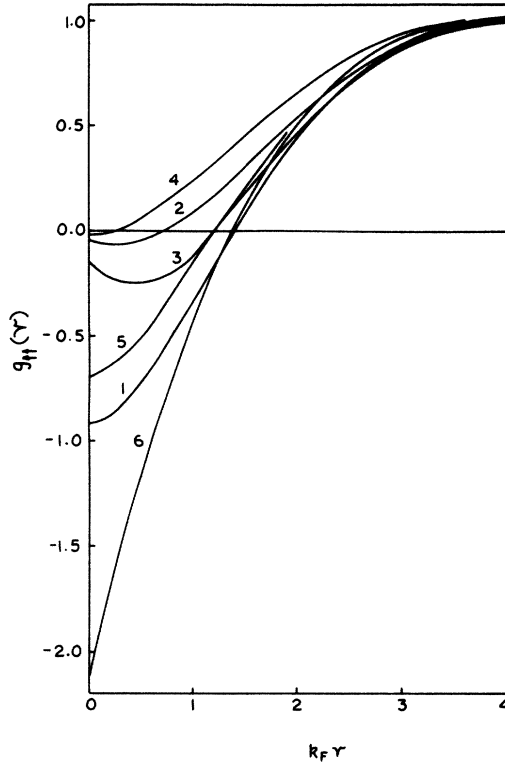
FIG. 2. The pair correlation function $g_{\uparrow\uparrow}(r)$ versus $k_F r$ for $r_s = 4$. Curve 1 is the present result; curve 2 is from Lobo et al.; curve 3 is from Singwi et al.; curve 4 is from Hubbard; curve 5 is from Hubbard as modified by Sham; curve 6 is from RPA.

$$\tilde{S}(k) = \frac{c k^4}{[c k^4/S_0^2(k)]^2 + I_3(k)},\tag{28}$$

where

$$c = \tfrac{3}{16}\pi(\tfrac{9}{4}\pi)^{4/3}\, r_s\tag{29}$$

and

$$I_3(k) = \frac{3}{4}\int_0^\infty dq\, q^2 \left\{[\tilde{S}(q)-1]\right.$$

$$\times \left[\frac{3}{2} - \frac{q^2}{2k^2} + \frac{k}{4q}\left(\frac{q^2}{k^2} - 1\right)^2 \ln\left|\frac{k+q}{k-q}\right|\right]$$

$$\left. - \tfrac{2}{3}[S(q)-1]\right\}.\tag{30}$$

Obviously $\tilde{S}(k)$ is a function of $I_3(k)$ and $I_3(k)$ is a functional of $\tilde{S}(k)$ and $S(k)$; so in order to calculate $\tilde{S}(k)$ from this integral equation, we must know the values for $S(k)$. To remain consistent with our present formulism, we make use of [in order to know the values of $S(k)$] an integral equation for $S(k)$ which was derived earlier by Kugler.[2] So we have, in fact, two coupled nonlinear integral equations—first the present one for $\tilde{S}(k)$ and $S(k)$, and the second for $S(k)$ alone—which are to be solved

self-consistently. We have already solved[3] the integral equation for $S(k)$ self-consistently. The self-consistent values of $S(k)$ obtained there are used in Eq. (30). One can now solve the integral equation (28) self-consistently by making use of Eq. (30). After six iterations we find that the values of $\tilde{S}(k)$ are self-consistent within 0.05% for small r_s and within 0.5% for large r_s values.

In Fig. 1 are plotted the self-consistent values of $\tilde{S}(k)$ for $r_s = 1$ to 6. It behaves the same way as $S(k)$ except that the values of $\tilde{S}(k)$ for small k are larger than the corresponding values of $S(k)$. This is also expected from Eq. (25). The values of the magnetic structure factor are used to estimate the spin pair correlation function, $\tilde{g}(r)$ which is defined

$$\tilde{g}(r) = \frac{3}{2r}\int_0^\infty dq\, q \sin(qr)[\tilde{S}(q) - 1],\tag{31}$$

where q and r are expressed in units of k_F and k_F^{-1}, respectively. Knowledge of this function when combined with the knowledge of the spin-symmetric pair correlation function $g(r)$, which was calculated by us[3] earlier, enables us to determine $g_{\uparrow\uparrow}(r)$ and $g_{\uparrow\downarrow}(r)$ separately. We have estimated these pair correlation functions for the whole range of metallic densities, but the results for $g_{\uparrow\uparrow}(r)$ and $g_{\uparrow\downarrow}(r)$ are presented only for $r_s = 4$ in Figs. 2 and 3, respectively. As usual the curves for other r_s values follows a similar pattern, so that for $r_s < 4$, they are better, and for $r_s > 4$ they are worse in quality as
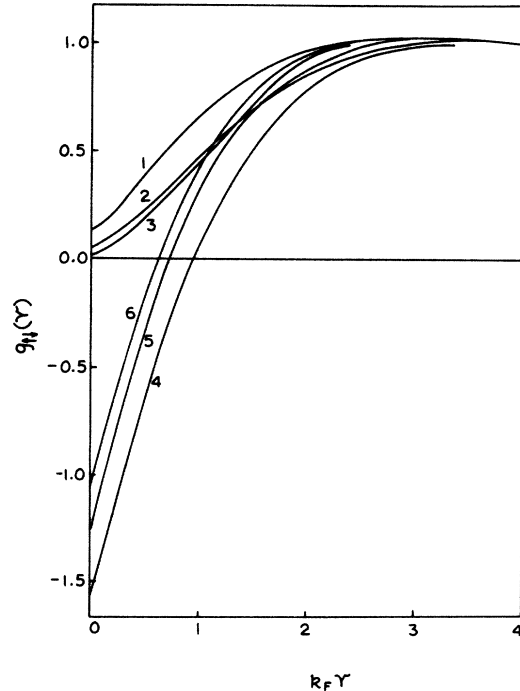

FIG. 3. The pair correlation function $g_{\uparrow\downarrow}(r)$ versus $k_F r$ for $r_s = 4$. The curves are labeled as in Fig. 2.
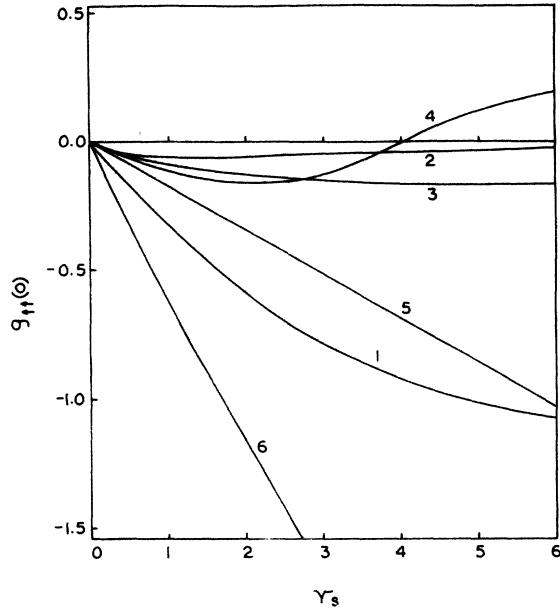
FIG. 4. The values of the pair correlation function $g_{11}(0)$ versus r_s. The curves are labeled as in Fig. 2.

compared to the curves for $r_s = 4$. The results are compared with those based on earlier theories, which includes the random-phase approximation (RPA), Hubbard's approximation[18] and its modification by Sham,[5] the generalized-random-phase approximation (GRPA) of Lobo et al.[4] and the GRPA including screening of Coulomb potential as put forward by Singwi et al.[19] Our $g_{11}(r)$ does depend on r_s values as in all the said theories except that of Lobo et al., where it is almost constant for whole of the metallic density range, thus reflecting a Hartree-Fock-like behavior.

To give an idea about the quality of these pair correlation functions over the range of metallic densities, we have plotted $g_{11}(0)$ and $g_{11}(0)$ as function of r_s in Figs. 4 and 5, respectively. It can be seen that like most of the available theories, $g_{11}(0)$ is negative for all r_s values. It is clear from Fig. 5 that all the available theories, except that of Lobo et al., violate strongly the requirement of positive definiteness on $g_{11}(0)$. Even in their theory $g_{11}(0)$ becomes slightly negative at $r_s \simeq 6$. Our $g_{11}(0)$ is positive even well beyond $r_s \simeq 6$. During the last stage of writing this paper we saw a paper by Hasegawa and Shimzu,[8] who have also estimated similar things. Their approach is very similar to that of Vashishta and Singwi[20] except that they have not introduced any adjustable parameter while considering the effect of local density fluctuations of electrons on the pair correlation function. Their $g_{11}(r)$ behaves the same way as in the theory of Lobo et al., except for small r, where it is about twice as negative as that of Lobo et al. Their

$g_{11}(r)$ is positive up to $r_s = 4$, after which it seems to become negative.

V. CONCLUDING REMARKS

The main results of the present paper are the inequality [Eq. (17)] and the nonlinear integral equation (24) for the ground-state static magnetic structure factor, which are obtained by using the sum rules. The important features of this integral equation are that it gives finite spin pair correlation function at the origin and is consistent with the frequency moment sum rules. It also predicts the small- and large-k behavior of $\tilde{S}(k)$. The integral equation is solved self-consistently and the values of $\tilde{S}(k)$ thus obtained are used to study the spin correlations in the paramagnetic stage of an electron liquid. The results obtained are in agreement with other recent theories.

The starting equation [i.e., Eq. (18)] in deriving the integral equation is free from any approximations. But thereafter enters some nonuniqueness in the derivation, because there may be more than one way of obtaining the functions $t(k)$ and $h(k)$. We have obtained these by demanding that our integral equation must give some well-established results in certain limiting cases. The limitation of the integral equation is that it does not tell us anything about the dynamical properties of the electron liquid.
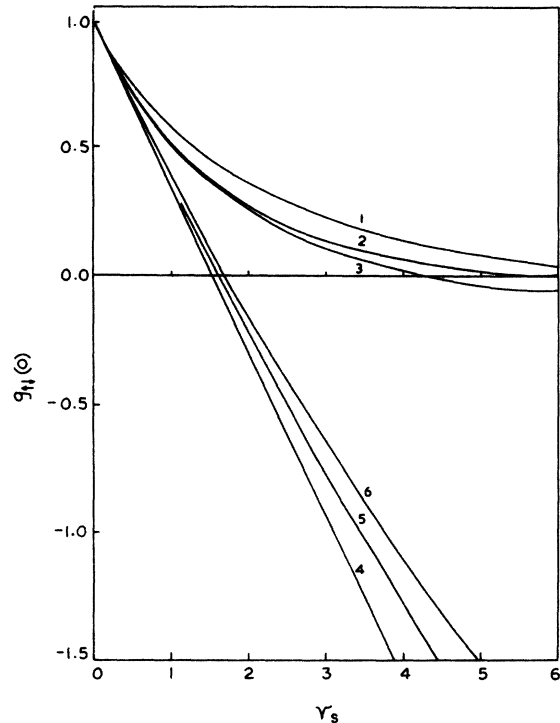

FIG. 5. The values of the pair correlation function $g_{11}(0)$ versus r_s. The curves are labeled as in Fig. 2.

APPENDIX A: DERIVATION OF THE THIRD MOMENT OF THE SPIN-DENSITY RESPONSE FUNCTION

It can be seen that the third moment is given by[9]

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^3 \chi''(\vec{k}, \omega) = \langle [[[[S_x(\vec{k}), H], H], H], S_x^\dagger(\vec{k})] \rangle. \tag{A1}$$

As the second frequency moment of $\chi''(\vec{k}, \omega)$ is zero, the above expression can be rewritten as

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^3 \chi''(\vec{k}, \omega) = \frac{1}{2} \sum_{\vec{q}\sigma} \eta_\sigma \left(\frac{\vec{k}\cdot\vec{q}}{m} \right)$$
$$\times \langle \{ [[[\rho_{\vec{q}\sigma}(\vec{k}), H_1], H], S_x^\dagger(\vec{k})]$$
$$+ [[[\rho_{\vec{q}\sigma}(\vec{k}), H_0], H_0], S_x^\dagger(\vec{k})]$$
$$+ [[[\rho_{\vec{q}\sigma}(\vec{k}), H_0], H_1], S_x^\dagger(\vec{k})] \} \rangle. \tag{A2}$$

Now making use of the stationarity property[7,21] of the correlation function and the fact that $S_x(\vec{k})$ commutes with H_1, it can be shown that

$$\langle [[\rho_{\vec{q}\sigma}(\vec{k}), H_1], S_x^\dagger(\vec{k})] \rangle = \langle [[S_x^\dagger(\vec{k}), H_0], \rho_{\vec{q}\sigma}(\vec{k})] \rangle$$
$$- \langle [[\rho_{\vec{q}\sigma}(\vec{k}), H_0], S_x^\dagger(\vec{k})] \rangle$$
$$= 0. \tag{A3}$$

Then the last term in Eq. (A2) becomes zero and we get

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^3 \chi''(\vec{k}, \omega) = \frac{1}{2} \sum_{\vec{q}\sigma} \eta_\sigma \left(\frac{\vec{k}\cdot\vec{q}}{m} \right)$$
$$\times \langle \{ [[[\rho_{\vec{q}\sigma}(\vec{k}), H_0], H_0], S_x^\dagger(\vec{k})]$$
$$- [[\rho_{\vec{q}\sigma}(\vec{k}), H_1], [S_x^\dagger(\vec{k}), H]] \} \rangle. \tag{A4}$$

It is now straightforward to evaluate these commutators and after a brief simplification we arrive at the result (6) quoted in the text.

APPENDIX B: EVALUATION OF $I_3(k)$ IN HARTREE-FOCK APPROXIMATION

It can be seen from Eq. (8) that the potential contribution to the third moment of the spin-density response function in the Hartree-Fock (HF) approximation is given as

$$I_3(k) = \frac{e^2}{m\pi} \int_0^2 dq\, q^2 \left\{ [S_0(q) - 1] \right.$$
$$\left. \times \left[\frac{5}{6} - \frac{q^2}{2k^2} + \frac{k}{4q}\left(\frac{q^2}{k^2} - 1 \right)^2 \ln\left| \frac{k+q}{k-q} \right| \right] \right\}. \tag{B1}$$

It is thus the same[2] as the potential contribution to the third moment of the density response function in the HF approximation. Sometimes it is useful to know these moments in this approximation. Therefore we present here its calculation for all k. The analytic evaluation of the integral is quite lengthy but it can be worked out exactly. The result is

$$I_3(k) = -\frac{\omega_p^2}{840} k^2 \left[\left(71 + k^2 + \frac{304}{k^2} - \frac{80}{k^4} \right) \right.$$
$$+ k^2 \left(\frac{k^2}{4} - 9 \right) \ln\left| 1 - \frac{4}{k^2} \right|$$
$$\left. + \frac{3}{k}\left(21 - \frac{35k^2}{4} - \frac{36}{k^2} + \frac{80}{3k^4} \right) \ln\left| \frac{k+2}{k-2} \right| \right], \tag{B2}$$

where k is measured in units of k_F. For $k \to 0$, Eq. (B2) yields

$$\lim_{k\to 0} I_3(k) = -\tfrac{3}{20} \omega_p^2 k^2. \tag{B3}$$

The k dependence of this result is quite different from the result given in Eq. (11). On the other hand, for large k, we obtain

$$\lim_{k\to\infty} I_3(k) = -\tfrac{1}{3} \omega_p^2, \tag{B4}$$

which behaves the same way as the result (12).

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