Critical exponents of ferromagnets with dipolar interactions: Second-order ϵ expansion*

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The zero-field critical behavior of ferromagnets above T_c, with both isotropic exchange coupling and dipolar interactions, is studied by renormalization-group techniques in $d = 4 - \epsilon$ dimensions with n $(= d)$ component spins. The critical exponents are calculated to order ϵ^2, and are found to be numerically very close to their nondipolar analogs. In particular the correlation length exponent is given by $2\nu = 1 + (9/34)\epsilon + (7013/58\,956)\epsilon^2 + O(\epsilon^3)$. The value of the specific-heat exponent α_s $(= 2 - d\nu)$ seems inconsistent with experimental data (at $d = 3$, $\epsilon = 1$). The critical scattering function is shown to have the form $\Gamma^{\alpha\beta}(\vec{q}) = Ct^{-\gamma}\hat{D}(\xi^2 q^2)(\delta_{\alpha\beta} - q^\alpha q^\beta/q^2)$ to all orders in ϵ. The exponents related to the leading correction to scaling, to crossovers to anisotropic exchange behavior, and to cubic dipolar behavior, are also calculated, and are found to differ significantly from their nondipolar counterparts.

I. INTRODUCTION

According to the universality hypothesis, the critical behavior of a system depends only on the spatial dimensionality d, on the number of "spin" components n, on the symmetry of the Hamiltonian, and on the range of the interactions in the system. This hypothesis has recently been beautifully vindicated within the framework of Wilson's theory of the renormalization group.[1-5]

Magnetic dipole-dipole interactions have both a long-range and a reduced symmetry, relative to isotropic short-range exchange interactions. Moreover, the standard techniques of studying critical behavior through high-temperature-series expansions become very tedious for long-range interactions. However, such interactions are relatively important for many magnetic materials with a low transition temperature.[6] For these reasons, a theoretical study of magnets with dipole-dipole interactions has been of great interest for a long time. Such a study became possible only recently, with the development of the methods of the renormalization group and the ϵ expansion.[4,5] In a series of recent papers, Fisher and Aharony,[6,7] and Aharony[8-12] studied the critical behavior of magnets with dipolar interactions. It was shown that, close to the critical temperature, a crossover occurs to a characteristic dipolar behavior which has new critical exponents and exhibits a special angular dependence of the correlation function. Due to the complicated nature of the integrals involved, all exponents (with the exception of η) were calculated only to leading order in ϵ $(\equiv 4 - d)$.

A surprising feature of the results was that the deviation of most of the exponents from their classical (mean-field) values was found to be *larger* than for the short-range exchange case. Numerically, however, with the exception of the specific-heat exponent, the two sets of exponents turned out to be so similar to this order (even at $\epsilon = 1$) that no experiment could distinguish between them.

In the case of the specific-heat exponent α_s, recent experiments on EuO[13,14] have shown that this ferromagnet exhibits a critical behavior quite distinct from that of nondipolar ferromagnets or antiferromagnets. Explicitly, the experiments indicate that $\alpha_s = -0.04 \pm 0.02$, compared with nondipolar values of $\alpha \simeq -0.1$ to -0.2. Salamon[14] has also observed the crossover from the nondipolar regime to dipolar behavior. The experimental values just quoted seemed to be in reasonable agreement with those calculated, to order ϵ in the dipolar case and to order ϵ^2 in the nondipolar case,[6] with $n = d = 4 - \epsilon$, at $\epsilon = 1$. Since the second-order contribution to α_s is quite important in the nondipolar case, it is clear that its value for the dipolar case is of great interest. Indeed, experience with the ϵ expansion for the short-range interactions suggests[15] that the results truncated at order ϵ^2 are, for $\epsilon = 1$, in quite good agreement with both experiments and high-temperature-series expansions. It is the purpose of this paper to describe a calculation of the dipolar exponents to second order.

Within the general framework of renormalization-group theory, critical exponents can be calculated in three ways: with the aid of the Callan-Symanzik equation,[16] through direct diagrammatic expansions,[5] and through an analysis of recursion relations.[1,4] Most of the published calculations that go beyond the first order use either the first or the second of these techniques. In the present work we prefer to adopt Aharony's[9,17] modified use of recursion relations, employed successfully in the analogous short-range problem. This approach, in our opinion, gives greater insight into questions of stability and crossover behavior.

In Sec. II we present the Hamiltonian adopted, and develop the recursion relations for the isotropic dipolar system. Possible fixed points and the crossover between them are then discussed. Section III is devoted to a detailed study of the isotropic dipolar fixed point and its critical exponents, together with a discussion of the form of the correlation function and the susceptibility. Section IV contains a separate study of the recursion relations appropriate for anisotropic cubic systems,[17,18] and for systems with anisotropic exchange interactions.[10,18,19] A discussion of the results is presented in Sec. V. Finally, in the Appendices, we give some details of the calculations of the integrals occurring in the recursion relations.

II. HAMILTONIAN AND RECURSION RELATIONS

The isotropic dipolar behavior, of principal interest in this work, results from a Hamiltonian of the form[6-12]

$$\mathcal{H} = \mathcal{H}_{SR} + \mathcal{H}_{DD} \ . \qquad (2.1)$$

The short-range exchange part is taken to be

$$\mathcal{H}_{SR} = -\tfrac{1}{2} J \sum_{\vec{R},\vec{\delta}} \vec{S}_{\vec{R}} \cdot \vec{S}_{\vec{R}+\vec{\delta}} \ , \qquad (2.2)$$

where $\vec{S}_{\vec{R}}$ is a classical spin vector of unit length, located at the site $\vec{R}$ in a d-dimensional lattice of cubic symmetry and coordination number c. The vectors $\vec{\delta}$ of length a (the lattice spacing, which we shall hereafter choose as unity) run over the c nearest-neighbor sites of the origin site. The dipole-dipole part of the interaction is then

$$\mathcal{H}_{DD} = -\tfrac{1}{2}(g_S \mu_B)^2 \sum_{\vec{R}\neq\vec{R}'} \sum_{\alpha,\beta} \frac{\partial^2}{\partial R_\alpha \partial R'_\beta} \times (|\vec{R}-\vec{R}'|^{2-d}) S^\alpha_{\vec{R}} S^\beta_{\vec{R}'} \ . \qquad (2.3)$$

Due to the combination of spin and space variables in Eq. (2.3), we must take the number of spin components $n = d = 4-\epsilon$.

We now follow closely the steps of Sec. II in I[7]: we move to a continuous-spin model with the usual weighting factor,[2] Fourier transform the spin variables, and rescale them. The system is finally described by a partition function of the form

$$Z = \int d^n\sigma \, e^{\overline{\mathcal{H}}_0} \ , \qquad (2.4)$$

with

$$\overline{\mathcal{H}}_0 = -\frac{1}{2}\sum_{\alpha\beta}\int_{\vec{q}} U_2^{0,\alpha\beta}(\vec{q})\sigma^\alpha_{\vec{q}}\sigma^\beta_{-\vec{q}} - u_0 \sum_{\alpha\beta}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} \sigma^\alpha_{\vec{q}}\sigma^\alpha_{\vec{q}'}\sigma^\beta_{\vec{q}''}\sigma^\beta_{-\vec{q}-\vec{q}'-\vec{q}''} \ , \qquad (2.5)$$

where

$$U_2^{0,\alpha\beta}(\vec{q}) = (r_0 + q^2)\delta_{\alpha\beta} + g_0(q^\alpha q^\beta/q^2) - h_0 q^\alpha q^\beta, \qquad (2.6)$$

while r_0 is linear in the temperature and g_0 and h_0 are of order $(g_S\mu_B)^2/J$. $\int_{\vec{q}}$ stands for $(2\pi)^{-d}\int d^dq$, over the range $|\vec{q}| < 1$.

In writing Eq. (2.5), we have ignored terms of only cubic symmetry. As discussed at length in I, such terms can lead to an instability of the dipolar fixed point; we shall discuss this instability separately in Sec. IV.

We adopt the usual form for the renormalization group transformation,[2,7] in which "spins" $\sigma^\alpha_{\vec{q}}$ with wave vectors in the range $b^{-1} < |\vec{q}| < 1$ are integrated out of the partition function (2.4). The wave-vector space is rescaled by a factor b, and the spin variables are rescaled through the relation

$$\vec{\sigma}_{\vec{q}} \to \zeta \sigma'_{b\vec{q}} \ . \qquad (2.7)$$

By repeated application of this transformation, we generate a sequence of effective Hamiltonians of the form (2.5), but with the parameters (r_0, g_0, h_0, u_0) replaced by new parameters (r_l, g_l, h_l, u_l), related to the starting values by a set of recursion relations.

We may remark that in performing this transformation we also generate terms not contained in the starting Hamiltonian (2.5)—for example, a six-spin term. Within the framework of the recursion relations, such "irrelevant" variables may not, in general, be neglected, since they may have fixed-point values of order ϵ^2.[2] Following Aharony,[9,17] however, we adopt a modified recursion relations approach, close in spirit to the Feynman graph method, in which we choose a large-wave-vector cutoff factor b and ignore all irrelevant variables. This procedure has been examined extensively elsewhere,[20] and has been shown to yield the correct exponents.

The recursion relations for the parameters (r_l, g_l, h_l, u_l) are obtained in the usual way, with the aid of a perturbation expansion of the partition function (2.4). The terms contributing to the equation for r_l are represented graphically in Fig. 1 (we ignore mass renormalization terms, since we require only the linearized r equation, and do not need r^* explicitly[20]). They yield

$$r_{l+1} = \zeta_l^2 b^{-d}\Big(r_l + 4(d+2)u_l \int_{\vec{q}}^{>} G_l^{\alpha\alpha}(\vec{q}) - 32u_l^2 \sum_{\beta,\delta}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} [G_l^{\alpha\alpha}(\vec{q})G_l^{\beta\delta}(\vec{q}') \times G_l^{\beta\delta}(\vec{q}'') + 2G_l^{\alpha\delta}(\vec{q})G_l^{\alpha\beta}(\vec{q}')G_l^{\beta\delta}(\vec{q}'')] \times \delta(\vec{q}+\vec{q}'+\vec{q}'') + O(u_l^3)\Big) \ . \qquad (2.8)$$

Here we have adopted the notation of I, in which

$$\int_{\vec{q}}^{>} \text{ stands for } \frac{1}{(2\pi)^d}\int_{1/b<|\vec{q}|<1} d^dq \ ,$$

while the propagator for the system is given by

$$G_l^{\alpha\beta}(\vec{q}) = \frac{1}{r_l + q^2}\left(\delta_{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2}\right)$$

$$+\frac{1}{r_l+g_l+(1-h_l)q^2}\,\frac{q^\alpha q^\beta}{q^2}\,. \tag{2.9}$$

The scale factor ζ_l is determined by imposing the usual condition[2,7] that the coefficient of q^2 in $\overline{\mathcal{H}}_{l+1}$ should remain equal to unity. As pointed out in Ref. 11, one may also choose ζ_l so that the coefficient of $q^\alpha q^\beta/q^2$, which is the term that grows most rapidly under iteration of the recursion relations, will remain constant. This results in a clearer picture of the dipolar fixed-point Hamiltonian, but makes little practical difference to the calculation. We therefore choose to adopt the usual definition, namely,

$$\zeta_l^2 = b^{d+2-\eta_l}\,, \tag{2.10}$$

where the exponent η_l is determined, as in I, from an examination of the q-dependent contributions of the graph shown in Fig. 1(b), and is hence of order u_l^2.

As argued in I, the recursion relation for g_{l+1} is easily derived once one notes that, because of the restricted range of q-space integration involved in the evaluation of the perturbation expansion of the partition function, all the terms in this expansion are analytic in q, and thus make no contribution to g_l. We then simply have

$$g_{l+1} = \zeta_l^2 b^{-d} g_l = b^{2-\eta_l} g_l\,. \tag{2.11}$$

We shall see in Sec. III that the distinctive behavior of the parameter g_l (in the case of ferromagnets[9]) under the renormalization group transformation obviates the necessity of a detailed examination of the recursion relation for h_{l+1}; it is sufficient to note that h remains finite.[7] We are then left with the task of deriving the equation for u_{l+1}. We find a relation of the form

$$u_{l+1} = \zeta_l^4 b^{-3d} \sum_{i=1}^{4} X^{(i)}(u_l, g_l, h_l, r_l) + O(u_l^4)\,, \tag{2.12}$$

where the $X^{(i)}$ are defined by the coefficients of the four-spin terms in the four diagrams shown in Fig. 2. The detailed expressions, which are rather complex, are given in Appendix A.

As discussed in I, the form of critical behavior of the system described by Eq. (2.5) depends primarily upon the initial value g_0. For $g_0 = 0$ (and $h_0 = 0$) the propagator (2.9) reduces to that for the isotropic short-range problem, and the recursion relations (2.8) and (2.12) assume the form given in Ref. 17. The corresponding fixed points are the Gaussian and Heisenberg fixed points, with the usual critical exponents.[5,17] For $g_0 \neq 0$, Eq. (2.11) ensures that, since η_l in Eq. (2.10) is of order u_l^2, g_l will grow large so that the system will eventually cross over (from Gaussian-like or Heisenberg-like behavior) to a characteristic dipolar behavior. The cross-over exponent[21] ϕ_D is simply the exponent γ appropriate to the fixed point from which the cross over occurs.[7,12] It should be emphasized here that the exponent ν_G, related to the g instability, is exactly equal to $1/(2-\eta)$, to all orders in the diagrammatic expansion, and therefore $\phi_D = \nu/\nu_G = \nu(2-\eta) = \gamma$ is also exact to all orders.

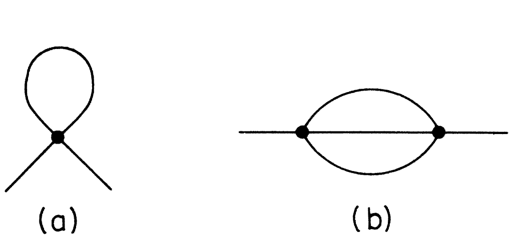

FIG. 1. Set of graphs for the recursion relation for r_l (or $r_{\alpha,l}$).

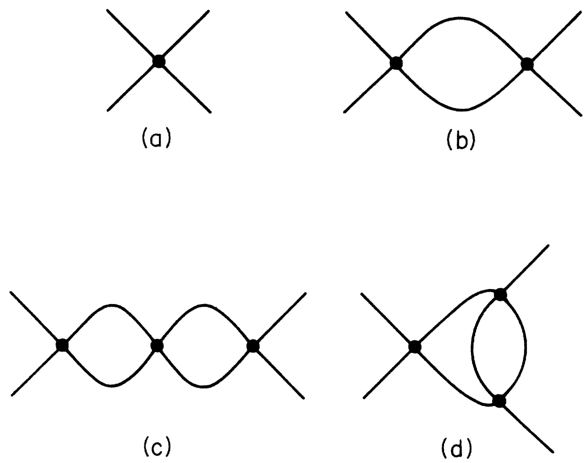

FIG. 2. Set of graphs for the recursion relation for u_l (and v_l).

III. ISOTROPIC DIPOLAR BEHAVIOR

With the divergence of g_l for large l, the propagator (2.9) assumes the simpler form

$$G_l^{\alpha\beta}(\vec{q}) \simeq [1/(r_l+q^2)](\delta_{\alpha\beta} - q^\alpha q^\beta/q^2)\,, \tag{3.1}$$

so that, as remarked above, the detailed behavior of the parameter h_l becomes unimportant. (It remains important, however, for *anti*ferromagnets, when $g_0 = 0$.[9]) In this limit, the integrals implicit in the recursion relation (2.12), and given explicitly in Appendix A, may be evaluated. As usual in ϵ expansions,[2,4] we assume that $\epsilon = 4-d$ is small, and that u_l and r_l are of order ϵ. To obtain results to order ϵ^2, we may thus calculate the integrals which multiply u_l^3 for $n = d = 4$. This calculation is outlined in Appendix B. The integrals which multiply u_l^2 were calculated in I (e.g., Eqs. (31) and (44) of I). The resulting recursion relation for u_{l+1} is

$$u_{l+1}=b^{\epsilon-2\eta_l}\{u_l-4K_du_l^2[7+d-12/d+12/d(d+2)]\times(b^\epsilon-1)/\epsilon+K_4^2u_l^3(533\tfrac{7}{9}+1156\ln b)\ln b+O(u_l^4)\}\,, \quad (3.2)$$

where

$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d)\,, \quad (3.3)$$

while, from Refs. 8 and 9, we have

$$\eta_l=\tfrac{80}{3}K_4^2u_l^2+O(u_l^3)\,. \quad (3.4)$$

[See especially the discussion following Eq. (59) in Ref. 9.]

The relation (3.2) leads to a dipolar Gaussian fixed point $u^*=0$, and to an isotropic dipolar nontrivial fixed point

$$K_du^*=(\tfrac{1}{34}\epsilon)(1+\tfrac{6211}{10404}\epsilon)+O(\epsilon^3)\,. \quad (3.5)$$

To determine the critical exponent ν, we must linearize Eq. (2.8) about the corresponding nontrivial fixed-point value for r, namely, r^*. Again, the order-u terms were calculated in I, and the formulas of Appendix B must be used to obtain the four-dimensional integrals related to the order-u^2 terms. The linearized second-order terms are

$$32K_d^2u_l^2\sum_{\beta\delta}\int_{\vec q}^{>}\frac{1}{q^2}[G_l^{\alpha\alpha}(\vec q)J_2^{\beta\delta\beta\delta}(\vec q)+2G_l^{\beta\delta}(\vec q)J_2^{\alpha\alpha\beta\delta}(\vec q)+2G_l^{\beta\delta}(\vec q)J_2^{\alpha\delta\alpha\beta}(\vec q)+4G_l^{\alpha\delta}(\vec q)J_2^{\alpha\beta\beta\delta}(\vec q)]\Delta r_l\,, \quad (3.6)$$

where $J_2^{\alpha\beta\gamma\delta}(\vec q)$ is defined in Eq. (A6). On substituting Eq. (B18) for $J_2^{\alpha\beta\gamma\delta}(\vec q)$, and performing the integrations, the linearized r equation finally becomes

$$\Delta r_{l+1}\equiv r_{l+1}-r^*\simeq b^{2-\eta^*}\Delta r_l\times[1-4K_du^*(d+2)(1-d^{-1})(b^\epsilon-1)/\epsilon+126K_d^2u^{*2}\ln b]\,, \quad (3.7)$$

where we have retained only those terms which are proportional to $\ln b$,[22] and where[8,9]

$$\eta^*=\tfrac{80}{3}K_4^2u^{*2}=\tfrac{20}{867}\epsilon^2+O(\epsilon^3)\,. \quad (3.8)$$

Using the relation[2,7]

$$\Delta r_{l+1}=b^{1/\nu}\Delta r_l\,, \quad (3.9)$$

we find from Eqs. (3.7) and (3.9)

$$\nu=\tfrac{1}{2}[1+\tfrac{9}{34}\epsilon+\tfrac{7013}{58956}\epsilon^2+O(\epsilon^3)]\,. \quad (3.10)$$

A discussion of this result will be presented in Sec. V, where we will also tabulate the values of all the other critical exponents, which follow (within scaling theory) directly from Eqs. (3.8) and (3.10).

We now turn to discuss the q dependence of the correlation function. As explained at length in Sec. VII of I, the correlation function may be written

$$\Gamma^{\alpha\beta}(\vec q)=b^{l(2-\eta)}\Gamma_l^{\alpha\beta}(b^l\vec q)=b^{l(2-\eta)}[G_l^{\alpha\beta}(b^l\vec q)+\text{“diagrams”}]\,. \quad (3.11)$$

The general structure of the terms which result from the diagrams is of the form

$$\sum_{\gamma\delta}G_l^{\alpha\gamma}(b^l\vec q)G_l^{\delta\beta}(-b^l\vec q)A^{\gamma\delta}(b^l\vec q)\,, \quad (3.12)$$

where $A^{\gamma\delta}(\vec q)$ is a sum of integrals over internal lines of the appropriate diagrams. By rotational invariance, $A^{\gamma\delta}(\vec q)$ must be of the form

$$A^{\gamma\delta}(\vec q)=a(q)\delta_{\gamma\delta}+b(q)q^\gamma q^\delta\,. \quad (3.13)$$

Explicit examples for $A^{\gamma\delta}(q)$ to order u_l were given in I. The order u^2 terms may also be checked explicitly, using the formulas in Appendix B. Substituting Eq. (3.13) into Eq. (3.12), and using the projection property of $G^{\alpha\gamma}$, we find

$$\text{“diagrams”}=\frac{a(b^lq)}{(r_l+b^{2l}q^2)^2}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right), \quad (3.14)$$

and hence

$$\Gamma^{\alpha\beta}(\vec q)=\frac{b^{l(2-\eta)}}{r_l+b^{2l}q^2-a(b^lq)}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right). \quad (3.15)$$

Since we choose b^l to be equal to ξ, the correlation length, this finally gives

$$\Gamma^{\alpha\beta}(\vec q)=(C/t^\gamma)\hat D(\xi^2q^2)(\delta_{\alpha\beta}-q^\alpha q^\beta/q^2)\,, \quad (3.16)$$

where $\gamma=(2-\eta)\nu$, and where $\hat D$ is a scaling function, which may be determined as in Ref. 23. As discussed in I,

$$1/\hat D(x^2)=1+x^2+O(\epsilon^2x^4) \quad (3.17)$$

for small x. It was shown in II that

$$\hat D(x^2)\sim 1/x^{2-\eta} \quad (3.18)$$

for large x.

The susceptibiltiy was discussed in Sec. VII of I, and in II. The result,

$$\chi^{\alpha\beta}\propto\delta_{\alpha\beta}/(C^{-1}t^\gamma+g_0\mathfrak{D}^{\alpha\alpha})\,, \quad (3.19)$$

where $\mathfrak{D}^{\alpha\alpha}$ is a demagnetization sum, remains true to all orders in ϵ.

IV. CORRECTIONS TO SCALING: CROSSOVERS

The Hamiltonian (2.5) is idealized; most real systems do not have the high isotropy, both in space and in spin space, which Eq. (2.5) has. The possible symmetry-breaking terms which may appear in a Hamiltonian for magnetic systems are reviewed in Ref. 12. Most of the conclusions in this reference are independent of the expansion in ϵ (e.g., regarding the crossovers due to long-range interactions or to changed dimensions, which depend only on the exponents γ and ν). We shall concentrate here on those additional interactions which do lead to exponents that have independent ϵ expansions. In particular, we shall discuss the leading correction to scaling, and the instabilities due to spin anisotropy[10,18,19] and to cubic anisotropy.[7,17,18]

The general structure of corrections to scaling has been discussed by Wegner.[24] If the Hamiltonian can be written in the form

$$\overline{\mathcal{H}} = \overline{\mathcal{H}}^* + \sum_i \mu_i O_i , \tag{4.1}$$

where $\overline{\mathcal{H}}^*$ is the fixed-point Hamiltonain, and O_i are eigenoperators of the linearized recursion relations near the fixed point, so that after l iterations $\mu_i \to \mu_i b^{\lambda_i l}$, the free energy per unit volume can be written in the scaling form

$$F(t, \{\mu_i\}) = b^{-dl} F(t b^{\lambda l}, \{\mu_i b^{\lambda_i l}\}) , \tag{4.2}$$

where $t = (T - T_c)/T_c$. Substituting $b^l = \xi \sim t^{-\nu}$ and using $\lambda = 1/\nu$ [see Eq. (3.9)], we then find

$$F(t, \{\mu_i\}) = t^{2-\alpha} f(\{\mu_i t^{-\nu\lambda_i}\}) . \tag{4.3}$$

The leading variable μ_i is the field conjugate to the order parameter. This leads to the usual scaling form of F. The additional variables lead to corrections to this scaling form near the appropriate fixed point. If one or more of these variables is relevant (i.e., $\lambda_i > 0$, for some i), then the system eventually crosses over to a new type of behavior, and the exponent $\phi = \nu\lambda_i$ is called a "crossover exponent."[18,19,21] To find these exponents, we thus have to consider the linearized recursion relations for the additional parameters near the fixed point.

Consider first the parameter u_0. In the discussion of the Sec. III we have assumed that $u_0 = u_{0c} = u^*$,[25] so that we did not allow for a transient variation in u. If we linearize the recursion relation for u [Eq. (3.2)] near u^*, we find

$$\Delta u_{l+1} = b^{\epsilon - 2\eta^*} [\Delta u_l - 8K_d \ln b (1 + (\tfrac{1}{2}\epsilon)\ln b) \times (\tfrac{17}{2} - \tfrac{37}{24}\epsilon) u^* \Delta u_l + K_4^2 u^{*2} \ln b (\tfrac{4804}{3} + 3468 \ln b) \Delta u_l + \cdots] - 2u^* \ln b \Delta \eta_l , \tag{4.4}$$

with $\Delta\eta_l$ coming from the linearization of Eq. (3.4). This can readily be written in the form $\Delta u_{l+1} = b^{\lambda_u} \Delta u_l$, with

$$\lambda_u = -\epsilon + \tfrac{1081}{2601}\epsilon^2 + O(\epsilon^3) . \tag{4.5}$$

Therefore, u is irrelevant; however, the exponent associated with it is quite small, and hence its effect may be felt even quite close to T_c. Expanding f in Eq. (4.3) about $u = u_{0c}$, and ignoring all other variables, we find

$$F(t) \approx t^{2-\alpha} + a(u_0 - u_{0c}) t^{2-\alpha+\omega} + \cdots , \tag{4.6}$$

where

$$\omega = -\nu\lambda_u = \tfrac{1}{2}\epsilon (1 - \tfrac{785}{5202}\epsilon) + O(\epsilon^3) . \tag{4.7}$$

We now turn to discuss two relevant operators. First, we replace the exchange interaction (2.2) by an anisotropic one,

$$\mathcal{H}_{SR} = -\frac{1}{2} \sum_{\vec{R},\vec{\delta}} \sum_{\alpha} J_\alpha S^\alpha_{\vec{R}} S^\alpha_{\vec{R}+\vec{\delta}} . \tag{4.8}$$

In the dipolar limit, $g \to \infty$, the anisotropic dipolar propagator becomes[10]

$$G^{\alpha\beta}(\vec{q}) = \frac{1}{r_\alpha + q^2} \left(\delta_{\alpha\beta} - \frac{1}{Q^2} \frac{q^\alpha q^\beta}{r_\beta + q^2} \right) , \tag{4.9}$$

with

$$Q^2 = \sum_\gamma (q^\gamma)^2 \big/ (r_\gamma + q^2) . \tag{4.10}$$

Here, r_α is the parameter associated with the exchange parameter J_α. A simple generalization of Eq. (2.8) now gives

$$r_{\alpha,l+1} = b^{2-\eta_l} \Bigg[r_{\alpha,l} + 4\left(\sum_\gamma u^l_{\alpha\gamma} \int_{\vec{q}}^{>} G_l^{\gamma\gamma}(\vec{q}) + 2u^l_{\alpha\alpha} \int_{\vec{q}}^{>} G^{\alpha\alpha}(\vec{q}) \right) - 32 \sum_{\gamma\delta} u^l_{\alpha\gamma} u^l_{\alpha\delta} \int_{\vec{q}}^{>} \int_{\vec{q}'}^{>} [G_l^{\alpha\alpha}(\vec{q}) G_l^{\gamma\delta}(\vec{q}') G_l^{\gamma\delta}(\vec{q}+\vec{q}') + 2G_l^{\alpha\gamma}(\vec{q}) G_l^{\gamma\delta}(\vec{q}') G_l^{\delta\alpha}(\vec{q}+\vec{q}')] + O(u^3) \Bigg] , \tag{4.11}$$

where $u^l_{\alpha\beta}$ now replaces u_l as the coefficient of $\sigma^\alpha\sigma^\alpha\sigma^\beta\sigma^\beta$ in Eq. (2.5). The second-order integrals are complicated for the propagator (4.9). They are quite difficult, even near the anisotropic dipolar fixed point found in Ref. 10. We therefore limit ourselves to the behavior near the isotropic dipolar fixed point, at which $r_\alpha \equiv r^*$ and $u_{\alpha\beta} \equiv u^*$. Linearizing Eq. (4.11) about this fixed point gives

$$\Delta r_{\alpha,l+1} = b^{2-\eta^*} \Bigg[\Delta r_\alpha + 4u^* \left(\sum_\gamma \int_{\vec{q}}^{>} \Delta G^{\gamma\gamma}(\vec{q}) + 2 \int_{\vec{q}}^{>} \Delta G^{\alpha\alpha}(\vec{q}) \right) - 32 K_d u^{*2} \int_{\vec{q}}^{>} \sum_{\gamma\delta} [\Delta G^{\alpha\alpha}(\vec{q}) J_2^{\gamma\delta\gamma\delta}(\vec{q}) + 2\Delta G^{\gamma\delta}(\vec{q}) J_2^{\alpha\alpha\gamma\delta}(\vec{q}) + 2\Delta G^{\gamma\delta}(\vec{q}) J_2^{\alpha\gamma\alpha\delta}(\vec{q}) + 4\Delta G^{\alpha\gamma}(\vec{q}) J_2^{\gamma\delta\delta\alpha}(\vec{q})] + O(u^{*3}) \Bigg] , \tag{4.12}$$

where $J_2^{\alpha\beta\gamma\delta}(\vec{q})$ is defined in Eq. (A6) and is given explicitly by Eq. (B18), while, from Eq. (4.9),

$$\Delta G^{\alpha\beta}(\vec{q}) = \frac{-\Delta r_{\alpha,l}\delta_{\alpha\beta} + (\Delta r_{\alpha,l} + \Delta r_{\beta,l})q^\alpha q^\beta/q^2 - \sum_\epsilon \Delta r_{\epsilon,l}(q^\epsilon)^2 q^\alpha q^\beta/q^4}{q^4} . \qquad (4.13)$$

The integrals in Eq. (4.12) can now be calculated explicitly, and Eq. (4.12) becomes

$$\Delta r_{\alpha,l+1} = b^{2-\eta^*}\Big[1 - 4K_d u^* \ln b\Big((\tfrac{7}{6} - \tfrac{13}{72}\epsilon)\Delta r_{\alpha,l} + (\tfrac{5}{6} - \tfrac{1}{36}\epsilon)\sum_\gamma \Delta r_{\gamma,l}\Big) + K_d^2 u^{*2}\ln b\Big(\frac{1402}{27}\Delta r_{\alpha,l} + \frac{500}{27}\sum_\gamma \Delta r_{\gamma,l}\Big) + O(\epsilon^3)\Big] , \qquad (4.14)$$

where we have again ignored the $(\ln b)^2$ terms. Equation (4.14) has two eigenvalues. The first eigenvalue is simply $b^{1/\nu}$, with the eigenvector $\Delta r_\alpha \equiv \Delta r$. The second eigenvalue is $(d-1)$-fold degenerate, with the eigenvectors satisfying $\sum_\gamma \Delta r_\gamma \equiv 0$, and is b^{λ_1}, with

$$\lambda_1 = 1 - \tfrac{7}{102}\epsilon - \tfrac{5153}{265302}\epsilon^2 + O(\epsilon^3) . \qquad (4.15)$$

Thus, the crossover exponent is

$$\phi = \nu\lambda_1 = 1 + \tfrac{10}{51}\epsilon + \tfrac{10793}{132651}\epsilon^2 + O(\epsilon^3) . \qquad (4.16)$$

We now turn to the question of instabilities associated with cubic perturbations of the Hamiltonian (2.5). As discussed in I, such perturbing terms arise when the Fourier transform of the dipolar interactions is calculated over a cubic lattice. The perturbation is characterized by two parameters, namely, f, the coefficient of

$$\sum_\alpha \int (q^\alpha)^2 \sigma_{\vec{q}}^\alpha \sigma_{-\vec{q}}^\alpha ,$$

and v, the coefficient of the four-spin term

$$\sum_\alpha \int\int\int \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}'}^\alpha \sigma_{\vec{q}''}^\alpha \sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\alpha .$$

A full analysis of the effects of these parameters is complicated by the fact that they generate one another under the renormalization-group transformation, so that such an analysis must use the complicated propagator [Eq. (23) of I] appropriate for nonzero f. Since the basic instability seems to be that associated with the parameter v, we shall restrict ourselves here to an examination of the behavior of this parameter (with $f=0$) close to the isotropic dipolar fixed point. We thus require the recursion relation for v, linearized about this fixed point; this requires consideration of all the configurations associated with the graphs in Fig. 2, in which one vertex $u\sigma^\alpha\sigma^\alpha\sigma^\beta\sigma^\beta$ is replaced by $v\sigma^\alpha\sigma^\alpha\sigma^\alpha\sigma^\alpha$. Collecting all such tems and using the integrals evaluated in Appendix B, one finds the result $\Delta v_{l+1} = b^{\lambda v}\Delta v_l$, with

$$\lambda_v = \tfrac{3}{17}\epsilon + \tfrac{5878}{44217}\epsilon^2 + O(\epsilon^3) \qquad (4.17)$$

and a crossover exponent

$$\phi_v = \tfrac{3}{34}\epsilon + \tfrac{15887}{176868}\epsilon^2 + O(\epsilon^3) . \qquad (4.18)$$

The instability with respect to v predicted by the first-order analysis of I is therefore confirmed in second order, and the system must eventually cross over and exhibit a cubic dipolar behavior, the nature of which must still be elucidated. We shall discuss this further in Sec. V.

V. DISCUSSION

Having calculated the two exponents η and ν to second order in ϵ, we can now use scaling relations[26] to obtain all the other critical exponents. In fact we have actually demonstrated one of these relations, namely, $\gamma = (2-\eta)\nu$, in Eq. (3.16). We shall report such checks of other relations in future work. The results for all the exponents, including the crossover and leading correction exponents, are summarized in Table I.

We first note that at $\epsilon = 1$, the exponents η, ν, γ, α_s, and ϕ are still farther away from their classical values than the corresponding short-range exponents. The changes in β and in δ are so small, however, that the direction of the changes is rather meaningless. The changes in 2ν and in γ are of the order of 0.01, and even these are beyond the accuracy of most present experiments. We may mention again, in this context, the apparent experimental discrepancy in the values of γ for EuO, when measured statically[27] or by neutron scattering.[28] However, there are several remarks to be made. First, the temperature range of both experiments is about $0.01 < t < 0.5$. It is not clear that the leading corrections to scaling [see, e.g., the relation Eq. (4.6)] or merely analytic terms in t are really negligible in this region. The relative magnitude of these corrections could be quite different in the two experiments, since in the neutron scattering case the coefficients are functions of the wave-vector transfer q. Second, the value of g_0 for EuO is approximately 0.017.[6] Taking the crossover exponent as $\phi_D = \gamma \simeq 1.37$, the crossover temperature to dipolar behavior is of the order of $t^\times \simeq g_0^{1/\gamma} = 0.05$. This is right in the middle of the experimental range! Thus, the "constant" amplitude of the susceptibility should actually be replaced by a scaling function $C(g/t^\gamma)$, and this might well affect both experiments. (Again, in the neutron scattering case, we have a function of both

TABLE I. Critical exponents to order ϵ^2 ($n = d = 4 - \epsilon$).

Exponent	Classical (Gaussian)	Isotropic short range ϵ expansion	Isotropic short range $\epsilon = 1$	Isotropic dipolar ϵ expansion	Isotropic dipolar $\epsilon = 1$
η	0	$\frac{1}{48}\epsilon^2$	0.0208	$\frac{20}{867}\epsilon^2$	0.0231
2ν	1	$1 + \frac{1}{4}\epsilon + \frac{1}{8}\epsilon^2$	1.375	$1 + \frac{9}{34}\epsilon + \frac{7013}{58956}\epsilon^2$	1.384
γ	1	$1 + \frac{1}{4}\epsilon + \frac{11}{96}\epsilon^2$	1.365	$1 + \frac{9}{34}\epsilon + \frac{2111}{19652}\epsilon^2$	1.372
α_s	0	$-\frac{1}{8}\epsilon^2$	−0.125	$-\frac{1}{34}\epsilon - \frac{6223}{58956}\epsilon^2$	−0.135
δ	3	$3 + \epsilon + \frac{11}{24}\epsilon^2$	4.458	$3 + \epsilon + \frac{787}{1734}\epsilon^2$	4.454
β	$\frac{1}{2}$	$\frac{1}{2} - \frac{1}{8}\epsilon + \frac{1}{192}\epsilon^2$	0.380	$\frac{1}{2} - \frac{2}{17}\epsilon - \frac{55}{58956}\epsilon^2$	0.381
ϕ	1	$1 + \frac{1}{6}\epsilon + \frac{11}{144}\epsilon^2$	1.243	$1 + \frac{10}{51}\epsilon + \frac{10793}{132651}\epsilon^2$	1.277
ϕ_v	$\frac{1}{2}\epsilon$	$\frac{1}{24}\epsilon^2$	0.0417	$\frac{3}{34}\epsilon + \frac{15887}{176868}\epsilon^2$	0.178
ω	$-\frac{1}{2}\epsilon$	$\frac{1}{2}\epsilon - \frac{7}{48}\epsilon^2$	0.354	$\frac{1}{2}\epsilon - \frac{785}{10404}\epsilon^2$	0.425

g/t^{γ} and of ξq.) Unfortunately, we have no quantitative estimates of such scaling functions as yet. At present, the only way to eliminate their effects is to work in the temperature range $t \ll t^{\times}$! Finally, the values of ξq used in the experiments of Ref. 28 are in the range $0.2 < \xi q < 5$. For this range, it is again not clear that one is justified in using the Ornstein-Zernike form for the scaling function $\hat{D}(\xi q)$ in Eq. (3.15), as is done in the analysis of Ref. 28. The deviations from the Ornstein-Zernike form could be quite large.[23] Again, no theoretical calculation of $\hat{D}(\xi q)$ for the dipolar case has yet been carried out, so that we are unable to estimate its effects quantitatively.

The values of the specific-heat exponent α_s are very interesting. As we mentioned in Sec. I, the experimental value for EuO seems to be $\alpha_s \simeq -0.04 \pm 0.02$.[13,14] The present theoretical value to second order is, at $\epsilon = 1$, $\alpha_s \simeq -0.135$. It is possible, of course, that the asymptotic character of the ϵ expansion is such that the first-order result for the dipolar case gives better estimates for α_s than the second-order expression. However, experience with the short-range exchange systems suggests that the latter is likely to be better.

It is interesting to note that the correction and crossover exponents ω, ϕ, and ϕ_v all change quite significantly. In general, these exponents are difficult to compare with experiments, since we have insufficient theoretical knowledge about the scaling functions in which they appear [see Eq. (4.3)]. Still, the exponent ω has recently been estimated experimentally for superfluid helium and for the Heisenberg antiferromagnet $RbMnF_3$.[29] The relatively large difference between its values in the dipolar and in the nondipolar cases gives at least some hope for an experimental test of the theoretical prediction made here. Moreover, an inclusion of a correction term in the analysis of the specific-heat data may lead to new estimates of α_s for EuO which may be in better agreement with our new theoretical estimates.

Turning now to the question of instabilities associated with cubic perturbations, it is of interest to note that the value of the crossover exponent ϕ_v is increased by the second-order term to one that is significantly larger than that of its short-range counterpart. Thus, the effects of such perturbations may become evident experimentally somewhat further away from the critical point than estimated in I. They should certainly be more easily observed in dipolar systems than in systems (such as the antiferromagnet[9]) where the dipolar perturbation is absent ($g_0 = 0$) so that the asymptotic critical exponents are those of the short-range model.

However, in contrast to the short-range system, where a cubic fixed point is known to exist,[17] the nature of the cubic dipolar behavior, to which the dipolar system must eventually cross over, is not known. Should a cubic dipolar fixed point exist, it must, of necessity, have a nonzero value of f^*, so analysis of this problem is not easy! It is possible, however, as noted in I, that no such fixed point exists, and that v and (probably) f grow to precipitate a first-order phase transition.[30] This possibility is in line with the suggestion[31] that critical fluctuations may accentuate the strong anisotropy observed in the soft-mode dispersion relations of cubic ferroelectrics such as $BaTiO_3$,[32] in which the (ferroelectric) phase transitions tend to be of first order.

Finally, we remark that the value of the crossover exponent ϕ describing the effects resulting from anisotropy in the exchange forces suggests that, again, a crossover to a smaller effective value of n[10] should occur at a larger value of $T - T_c$ than in the equivalent short-range problem. In the case of uniaxial anisotropy, this may be particularly important, because of the strong contrast in the asymptotic dipolar behavior in that case.[11]

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APPENDIX A: RECURSION RELATION FOR u_{l+1}

The recursion relation for u_{l+1} is determined by the coefficients $X^{(i)}$ introduced in Eq. (2.12), which are themselves defined in terms of the diagrams of Fig. 2. The contribution of Fig. 2(a) is trivially $X^{(1)}=u_l$. $X^{(2)}$ is determined by identifying the coefficient of the four-spin term in the graph of Fig. 2(b), which gives

$$-4K_d u_l^2 \sum_{\alpha\beta\gamma\delta}\left(\sum_{\mu\nu} J_1^{\mu\nu\mu\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}+4\sum_{\mu} J_1^{\mu\gamma\mu\delta}\delta_{\alpha\beta}+4J_1^{\alpha\gamma\beta\delta}\right)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\beta}\sigma_{\vec{q}''}^{\gamma}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^{\delta}\,, \tag{A1}$$

where

$$J_1^{\alpha\gamma\beta\delta}=K_d^{-1}\int_{\vec{q}}^{>} G_l^{\alpha\gamma}(\vec{q})G_l^{\beta\delta}(-\vec{q})\,. \tag{A2}$$

Similarly, $X^{(3)}$ is determined from the contribution made by the graph of Fig. 2(c), namely

$$16K_d^2u_l^3\sum_{\alpha\beta\gamma\delta}\left\{\left[\left(\sum_{\mu\nu}J_1^{\mu\nu\mu\nu}\right)^2+2\sum_{\substack{\mu\nu\\ \kappa\rho}}J_1^{\mu\kappa\mu\rho}J_1^{\nu\kappa\nu\rho}\right]\delta_{\alpha\beta}\delta_{\gamma\delta}+\left(4\sum_{\mu\nu}J_1^{\mu\nu\mu\nu}\sum_{\rho}J_1^{\gamma\rho\delta\rho}\right.\right.$$
$$\left.\left.+8\sum_{\mu\nu}J_1^{\mu\nu\mu\rho}J_1^{\gamma\rho\delta\nu}\right)\delta_{\alpha\beta}+\left(4\sum_{\mu}J_1^{\alpha\mu\beta\mu}\sum_{\nu}J_1^{\gamma\nu\delta\nu}+8\sum_{\mu\nu}J_1^{\alpha\mu\beta\nu}J_1^{\gamma\mu\delta\nu}\right)\right\}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\beta}\sigma_{\vec{q}''}^{\gamma}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^{\delta}\,. \tag{A3}$$

Finally, the contribution of Fig. 2(d), which determines $X^{(4)}$, can be written as

$$64K_d^2u_l^3\sum_{\alpha\beta\gamma\delta}\left(\sum_{\mu\nu\rho}(I^{\mu\gamma\mu\delta\nu\rho\nu\rho}+2I^{\mu\nu\mu\rho\nu\rho\delta\gamma}+4I^{\mu\nu\mu\gamma\nu\rho\delta\rho}+2I^{\mu\nu\mu\rho\delta\rho\gamma\nu})\delta_{\alpha\beta}\right.$$
$$\left.+2\sum_{\mu\nu}(I^{\beta\gamma\alpha\delta\nu\mu\nu\mu}+2I^{\beta\nu\alpha\mu\nu\mu\delta\gamma}+4I^{\beta\nu\alpha\gamma\nu\mu\delta\mu}+2I^{\alpha\nu\beta\mu\delta\mu\gamma\nu})\right)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\beta}\sigma_{\vec{q}''}^{\gamma}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^{\delta}\,, \tag{A4}$$

where

$$I^{\alpha\beta\gamma\delta\epsilon\eta\mu\nu}=K_d^{-1}\int_{\vec{q}}^{>} G_l^{\alpha\beta}(\vec{q})G_l^{\gamma\delta}(-\vec{q})J_2^{\epsilon\eta\mu\nu}(\vec{q})\,, \tag{A5}$$

with

$$J_2^{\epsilon\eta\mu\nu}(\vec{q})=K_d^{-1}\int_{\vec{p}}^{>} G_l^{\epsilon\eta}(\vec{q}+\vec{p})G_l^{\mu\nu}(-\vec{p})\,. \tag{A6}$$

The various integrals can now be calculated using the appropriate propagators. Near the isotropic Heisenberg fixed point, one must use $G_l^{\alpha\beta}(\vec{q})=\delta_{\alpha\beta}/(r_l+q^2)$, whereas near the dipolar isotropic fixed point, one must use Eq. (3.1). In practice, we shall replace r_l in the propagator by zero and ignore mass-renormalization graphs.[17,20,22]

Near the Heisenberg fixed point ($g_l=0$), this reproduces the recursion relations of Ref. 17. For $g_l\to\infty$, the integral $J_1^{\alpha\gamma\beta\delta}(\vec{q})$ was given in Eq. (44) of I, and is

$$J_1^{\alpha\gamma\beta\delta}(\vec{q})=\{\delta_{\alpha\gamma}\delta_{\beta\delta}(1-2/d)+(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta})[d(d+2)]^{-1}\}(b^{\epsilon}-1)/\epsilon\,. \tag{A7}$$

The integral $J_2^{\epsilon\eta\mu\nu}(\vec{q})$ is evaluated, for these conditions, at $d=4$, in Appendix B. The result, to order q^2, is given in Eq. (B18). Substituting in Eq. (A5) then gives a general expression for $I^{\alpha\beta\gamma\delta\epsilon\eta\mu\nu}$. In practice, we need only sums over such expressions, and this leads to some simplification. For example, the first term in (A4) involves

$$\sum_{\mu\nu\rho}I^{\mu\gamma\mu\delta\nu\rho\nu\rho}=K_d^{-1}\int_{\vec{q}}^{>}\sum_{\mu}G_l^{\mu\gamma}(\vec{q})G_l^{\mu\delta}(\vec{q})\sum_{\nu\rho}J_2^{\nu\rho\nu\rho}(\vec{q})$$
$$=K_d^{-1}\int_{\vec{q}}^{>}\frac{1}{q^4}\left(\delta_{\gamma\delta}-\frac{q^\gamma q^\delta}{q^2}\right)(-3\ln q+\tfrac{3}{4})$$
$$=\tfrac{9}{16}\ln b(1+2\ln b)\delta_{\gamma\delta}\,. \tag{A8}$$

A similar calculation of the other sums in (A4) finally leads to Eq. (3.2).

APPENDIX B: CALCULATION OF INTEGRAL $J_2^{\alpha\beta\gamma\delta}(\vec{q})$

Using the definition (A6), and ignoring the mass terms in the propagators [putting $r_l=0$ in Eq. (3.1)[17,20,22]], $J_2^{\alpha\beta\gamma\delta}(\vec{q})$ for the dipolar propagator may be written

$$J_2^{\alpha\beta\gamma\delta}(\vec{q})=J_3(\vec{q})\delta_{\alpha\beta}\delta_{\gamma\delta}-J_4^{\alpha\beta}(\vec{q})\delta_{\gamma\delta}-J_4^{\gamma\delta}(\vec{q})\delta_{\alpha\beta}+J_5^{\alpha\beta\gamma\delta}(\vec{q})\,, \tag{B1}$$

where

$$J_3(\vec{q})=K_d^{-1}\int_{\vec{p}}^{>}\frac{1}{p^2(\vec{p}+\vec{q})^2}\,, \tag{B2}$$

$$J_4^{\alpha\beta}(\vec{q})=K_d^{-1}\int_{\vec{p}}^{>}\frac{p^\alpha p^\beta}{p^4}\frac{1}{(\vec{p}+\vec{q})^2}\,, \tag{B3}$$

and

$$J_5^{\alpha\beta\gamma\delta}(\vec{q})=K_d^{-1}\int_{\vec{p}}^{>}\frac{(p^\alpha+q^\alpha)(p^\beta+q^\beta)p^\gamma p^\delta}{p^4(\vec{p}+\vec{q})^4}\,. \tag{B4}$$

The first two integrals, J_3 and $J_4^{\alpha\beta}$ were calculated in Ref. 9 with a sharp cutoff and a range $b^{-1} < |\vec{p}| < 1$, and in Ref. 8, with a smooth cutoff and a range $0 < |\vec{p}| < \infty$. It turns out that the leading terms, $\ln q$ and $q^\alpha q^\beta / q^2$ are the same in the two cases. The constant terms can be obtained as described in Ref. 9, following Eq. (31). The results are, for $d = 4$,

$$J_3(\vec{q}) = -\ln q + \tfrac{1}{2} + O(q^2) \tag{B5}$$

and

$$J_4^{\alpha\beta}(\vec{q}) = \tfrac{1}{4}[-(\ln q - \tfrac{1}{4})\delta_{\alpha\beta} + q^\alpha q^\beta / q^2 + O(q^2)] \;. \tag{B6}$$

In these results, we have also ignored terms of order $1/b^2 q^2$, since these will finally lead to terms of the form $b^{-2}\ln b$, and we assume $b \gg 1$.

To calculate $J_5^{\alpha\beta\gamma\delta}(\vec{q})$, we can adopt several methods. One method is to write the most general form allowed by rotational invariance, namely,

$$\begin{aligned} J_5^{\alpha\beta\gamma\delta}(\vec{q}) = {} & A(q)\delta_{\alpha\beta}\delta_{\gamma\delta} + B(q)\,(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \\ & + C(q)\,(q^\alpha q^\beta \delta_{\gamma\delta} + q^\gamma q^\delta \delta_{\alpha\beta})/q^2 \\ & + D(q)\,(q^\alpha q^\gamma \delta_{\beta\delta} + q^\alpha q^\delta \delta_{\beta\gamma} + q^\beta q^\gamma \delta_{\alpha\delta} \\ & + q^\beta q^\delta \delta_{\alpha\gamma})/q^2 + E(q) q^\alpha q^\beta q^\gamma q^\delta / q^4 \;, \end{aligned} \tag{B7}$$

and then calculate explicitly J_5^{1111}, J_5^{2211}, J_5^{1221}, J_5^{2222}, and $\sum_\gamma J_5^{\alpha\beta\gamma\gamma} = J_4^{\alpha\beta}$, where $\vec{q}$ is in the direction of the 1 axis, using formulas (33) and (34) of Ref. 9. These yield the functions A, B, C, D, and E. Since this approach has already been used in Ref. 9, we choose to present here an alternative method, leading to the same results.

We start with the substitution

$$\frac{1}{p^2} \to \int_{1/b^2}^{1} \frac{dt}{(p^2+t)^2} \tag{B8}$$

and extend the p integration to cover all space. We can expect that this method will enable us to pick up the correct q-dependent terms (to the order required). The substitution (B8) will, however, change the terms of order unity. These will be determined separately, using any of the integrals mentioned following Eq. (B7). We thus have

$$J_5^{\alpha\beta\gamma\delta}(\vec{q}) = K_d^{-1}\int_{\vec{p}} \int_{1/b^2}^{1} \frac{dt}{(p^2+t)^2} \times \int_{1/b^2}^{1} \frac{dt'}{(p^2+t')^2} \frac{p^\alpha p^\beta (p^\gamma + q^\gamma)(p^\delta + q^\delta)}{(\vec{p}+\vec{q})^4} \;. \tag{B9}$$

Using the identity

$$\frac{1}{a^2 b^2 c^2} = 120 \int_0^1 dx \int_0^{1-x} dy \frac{xy(1-x-y)}{[a + (b-a)x + (c-a)y]^6} \tag{B10}$$

and changing variables according to

$$\vec{k} = \vec{p} + y\vec{q} \;, \tag{B11}$$

we have

$$J_5^{\alpha\beta\gamma\delta} = 120 K_d^{-1} \times \int_{\vec{k}} \int_{1/b^2}^{1} dt \int_{1/b^2}^{1} dt' \int_0^1 dx \int_0^{1-x} dy\, xy(1-x-y) \left\{ \frac{(k^\alpha - yq^\alpha)(k^\beta - yq^\beta)[k^\gamma - (y-1)q^\gamma][k^\delta - (y-1)q^\delta]}{(k^2+z)} \right\} \;, \tag{B12}$$

where

$$z = t + x(t'-t) + y(q^2 - t) - y^2 q^2 \;. \tag{B13}$$

The integrals over $\vec{k}$, t, and t' are now straightforward. Using the identities of Appendix B of I, we find

$$\begin{aligned} J_5^{\alpha\beta\gamma\delta}(\vec{q}) = \int_0^1 dx \int_0^{1-x} dy \Big(& -\tfrac{1}{8} y \ln z' (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta}) + \frac{1}{4z'} y^3 q^\alpha q^\beta \delta_{\gamma\delta} + \frac{1}{4z'} y(y-1)^2 q^\gamma q^\delta \delta_{\alpha\beta} \\ & + \frac{1}{4z'} y^2 (y-1)\,(q^\alpha q^\gamma \delta_{\beta\delta} + q^\alpha q^\delta \delta_{\beta\gamma} + q^\beta q^\delta \delta_{\alpha\gamma} + q^\beta q^\gamma \delta_{\alpha\delta}) + \frac{1}{2z'^2} y^3 (y-1)^2 q^\alpha q^\beta q^\gamma q^\delta \Big), \end{aligned} \tag{B14}$$

where $z' = (1-y)(b^{-2} + yq^2)$, and where we have dropped those terms arising from the upper limits in the t and t' integrations, since they do not make significant contributions to the final result. Finally, performing the integrations over the Feynman parameters we obtain from Eq. (B14)

$$\begin{aligned} J_5^{\alpha\beta\gamma\delta}(\vec{q}) = {} & -\tfrac{1}{24}\ln q(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + (1/12q^2)\,(q^\alpha q^\beta \delta_{\gamma\delta} + q^\gamma q^\delta \delta_{\alpha\beta}) - (1/24q^2) \\ & \times (q^\alpha q^\delta \delta_{\beta\gamma} + q^\alpha q^\gamma \delta_{\beta\delta} + q^\beta q^\gamma \delta_{\alpha\delta} + q^\beta q^\delta \delta_{\alpha\gamma}) + (1/12q^4) q^\alpha q^\beta q^\gamma q^\delta + \text{const.} + O(q^2, 1/b^2 q^2) \;. \end{aligned} \tag{B15}$$

The constant is determined from the identity

$$\sum_\gamma J_5^{\alpha\beta\gamma\gamma}(\vec{q}) = J_4^{\alpha\beta}(\vec{q}) \tag{B16}$$

to be

$$(\text{const.}) \to -\tfrac{1}{288}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \quad . \qquad \text{(B17)}$$

Combining Eqs. (B1), (B5), (B6), (B15), and (B17), we finally find

$$\begin{aligned} J_2^{\alpha\beta\gamma\delta}(\vec{q}) = & -\tfrac{1}{24}\ln q(13\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - (1/6q^2)\,(q^\alpha q^\beta \delta_{\gamma\delta} + q^\gamma q^\delta \delta_{\alpha\beta}) \\ & - (1/24q^2)\,(q^\alpha q^\gamma \delta_{\beta\delta} + q^\alpha q^\delta \delta_{\beta\gamma} + q^\beta q^\gamma \delta_{\alpha\delta} + q^\beta q^\delta \delta_{\alpha\gamma}) \\ & + (1/12q^4) q^\alpha q^\beta q^\gamma q^\delta + \tfrac{1}{288}(107\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + O(q^2, 1/b^2q^2) \quad . \end{aligned} \qquad \text{(B18)}$$

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