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Ornstein-Zernike expression for correlation functions near a tricritical point*†

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We calculate the Fourier transform of the spin-spin correlation function of a metamagnet (a simple-cubic array of Ising spins 1/2 with ferromagnetic planes coupled antiferromagnetically) in the Ornstein-Zernike (OZ) approximation. We evaluate this expression near λ points and at the tricritical point. The divergence of the staggered susceptibility is found to have the usual OZ form near λ and tricritical points. The uniform susceptibility diverges as the tricritical point is approached from the ordered phase, but the nature of the divergence differs radically from the usual OZ form. We also present and compare the correlation functions of the Blume-Emery-Griffiths (BEG) model with those of the metamagnet to find a complete isomorphism between them.

I. INTRODUCTION

Mean-field calculations have provided a useful qualitative understanding of thermodynamic properties of systems with tricritical points,[1–5] and also a guide for the understanding of more ambitious approximations.[6–10] In this paper we extend the mean-field calculations to the evaluation of correlation functions in the Ornstein-Zernike (OZ) approximation. In particular, we study two models which have tricritical points: the Ising spin-$\frac{1}{2}$ metamagnet, and a spin-1 Ising model in a crystalline field.[2] We calculate the Fourier transform of the spin-spin correlation function of the metamagnet in the OZ approximation, and present the results of a similar calculation for the spin-1 model in the Appendix. In Sec. II we introduce our notation for the metamagnet, and consider general features of the thermodynamics which constrain by fluctuation theorems the behavior of the correlation functions. In Sec. III we calculate the correlation function in terms of the equilibrium sublattice magnetizations of the metamagnet. In Sec. IV we develop the mean-field thermodynamics of the metamagnet in a form useful for the evaluation of the OZ correlation function. We then present expressions for the correlation function near lines of second-order phase transitions, and near the tricritical point. In Sec. V we summarize the formulas we have developed. We also discuss the various exponents associated with the correlation function, the application of the Ginsburg criterion for the metamagnet, and Blume-Emery-Griffiths (BEG) models and the OZ value for the specific heat. The Appendix is devoted to the correlation functions of the BEG model and a comparison of these correlation functions with those of the metamagnet.

For convenience we summarize our results. The general form for the Fourier transform of the correlation function $\langle S_{p\vec{q}} S_{-p\,-\vec{q}} \rangle$ [see Eqs. (5) and (2) for definitions] in the OZ approximation is presented in Eq. (23). This expression is shown to diverge in the $(p, \vec{q}) = (\pi, \vec{0})$ direction at λ and tricritical points, and also in the $(0, \vec{0})$ direction near the tricritical point in the ordered phase. Equation (23) is then specialized in the vicinity of λ and tricritical points. Figures 3 and 4 are a pictorial summary of the cases we have considered. The correlation function in the $(\pi, \vec{0})$ direction is found to have the usual OZ form near λ and tricritical points. The correlation function in the $(0, \vec{0})$ direction near the tricritical point differs radically from the usual OZ form. While the scaling form Eq. (49) indicates the assignment of $\eta_u = 1$ for the divergence in the $(0, \vec{0})$ direction near the tricritical point, the limits $(p, \vec{q}) \to (0, \vec{0})$ and $(T, H) \to (T_t, H_t)$ are not interchangeable. The fluctuations in the sublattice magnetization and total magnetization are shown in Eq. (52) to be decoupled, allowing the successful application of the Ginsburg criterion for the metamagnet near the tricritical point. The OZ expression for the specific heat gives $\alpha = \frac{1}{2}$ for λ and tricritical points. Finally, a complete isomorphism is established between the correlation functions of the BEG and metamagnet models in the OZ approximation.

II. MODEL

The model studied in this calculation is a simple-cubic Ising metamagnet of N spins $\frac{1}{2}$. With the coordinate axes parallel to the axes of the unit cell, the spins within each plane parallel to the xy plane are ferromagnetically coupled, whereas spins in adjacent xy planes are antiferromagnetically coupled. The cylindrical coordinates used, and the ordered phase in zero magnetic field, are displayed in Fig. 1. The Hamiltonian of this system is

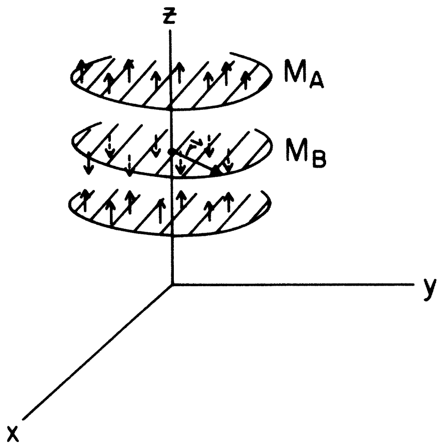

FIG. 1. Displayed are the cylindrical coordinates $(z\vec{r})$ used for the metamagnet and ground state at $H=0$. M_A and M_B are the sublattice magnetizations.

$$\mathcal{H} = -\frac{1}{2}\sum_{\substack{z\vec{r}\\ z'\vec{r}'}} \mathcal{J}(z\vec{r}, z'\vec{r}')S_{z\vec{r}}\, S_{z'\vec{r}'} - H\sum_{z\vec{r}} S_{z\vec{r}} - H_S \sum_{z\vec{r}} (-1)^z S_{z\vec{r}} \;, \tag{1}$$

$$\mathcal{J}(z\vec{r}, z'\vec{r}') = \begin{cases} J(\vec{r}-\vec{r}') & \text{if } z = z', \quad J>0 \\ \mathcal{J}(\vec{r}-\vec{r}') & \text{if } |z-z'| = 1, \; \mathcal{J}<0 \\ 0 & \text{otherwise .} \end{cases}$$

Because of the anisotropy of this system, we introduce separate Fourier transforms in the z and $\vec{r}$ direction, e.g.,

$$S_{p\vec{r}} = \frac{1}{N^{1/3}}\sum_z e^{-ipz} S_{z\vec{r}} \;,$$

$$S_{z\vec{q}} = \frac{1}{N^{2/3}}\sum_{\vec{r}} e^{-i\vec{q}\cdot\vec{r}} S_{z\vec{r}} \;, \tag{2}$$

$$S_{p\vec{q}} = \frac{1}{N}\sum_{z\vec{r}} e^{-ipz-i\vec{q}\cdot\vec{r}} S_{z\vec{r}} \;.$$

This combination of antiferromagnetic and ferromagnetic couplings gives a ground state at $H=0$ which is ferromagnetically ordered in each xy plane, with the spin direction alternating from plane to plane. At $H=0$, as the temperature is increased, the magnetizations on the alternate layers, M_A and M_B, will decrease monotonically and vanish at T_N, where the system undergoes a second-order phase transition. At $T=0$, as the magnetic field is increased, the magnetic field will eventually be large enough so that it is energetically favorable for the spins on the sublattice which point in a direction opposite to H to flip in the direction of H. This situation corresponds to a first-order phase transition. While not apparent, the first-order transition persists (for a range of values of $J(\vec{r}-\vec{r}')$ and $\mathcal{J}(\vec{r}-\vec{r}')$ if we fix the temperature to be nonzero, and raise the magnetic field. If this is true, then somewhere in the H-T phase diagram the transition changes from first to second order. The point at which this occurs is the tricritical point (T_t, H_t). This state of affairs is represented by the phase diagram of Fig. 2. Four paths of interest are indicated: (1) a path through the second-order line from below; (2) through the second-order line from above; (3) through the tricritical point from above; and (4) through the tricritical point from below. The critical indices will be different for each of these paths, and this will be reflected in the behavior of the spin-spin correlation function.

Two quantities of thermodynamic interest for this system are the order parameter ϵ, which is the difference between the magnetizations of the two sublattices, and M, which is the sum of the two sublattice magnetizations;

$$\epsilon = M_A - M_B \;, \quad M = M_A + M_B \;. \tag{3}$$

In the H-T plane the ordered phase $(\epsilon \neq 0)$ is separated from the paramagnetic phase $(\epsilon = 0)$ by a λ line $(T_t < T \le T_N)$ of second-order phase transitions, terminating at the tricritical point, where a first-order line originates. Along a (1), (2), (3), or (4) path, $(\partial\epsilon/\partial H_S)_{H_S=0}$ diverges strongly. Along a (1) or (2) path, $\partial M/\partial H$ is thought to diverge weakly, i.e., like a specific heat. This is borne out by the mean-field thermodynamics, which shows that $\partial M/\partial H$ has a jump discontinuity. At the tricritical point along a (4) path, $\partial M/\partial H$ diverges, although along a (3) path mean-field gives the result that $\partial M/\partial H$ is finite.

The correlation function, with $\langle\cdots\rangle$ representing a thermal average with respect to the Hamiltonian of Eq. (1), is denoted by

$$f(z,\vec{r}) = \frac{1}{N}\sum_{z'\vec{r}'} \langle S_{z'\vec{r}'}\, S_{z+z',\vec{r}+\vec{r}'}\rangle \;, \tag{4}$$

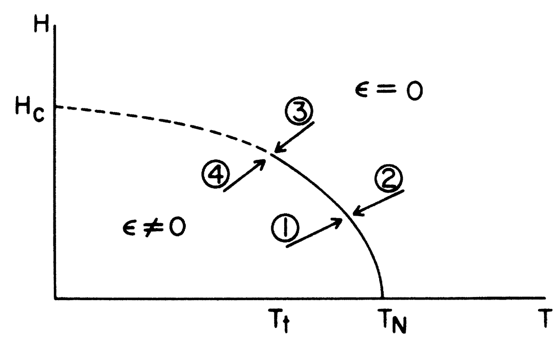

FIG. 2. Phase diagram of the metamagnet. Dotted lines are lines of first-order phase transitions and solid lines are lines of critical points. The paths (1) and (2) approach λ points, and the paths (3) and (4) approach the tricritical point.

and the Fourier transform of f is denoted by

$$\tilde{f}(p, \vec{q}) = \langle S_{p\vec{q}} S_{-p\,-\vec{q}} \rangle \; . \tag{5}$$

By using the well-known connection between susceptibility and correlation functions, we have $\partial M/\partial H \propto \tilde{f}(0, \vec{0})$, while $\partial\epsilon/\partial H_S \propto \tilde{f}(\pi, 0)$. Therefore, in an OZ calculation of $\tilde{f}$, we expect that $\tilde{f}(\pi, \vec{0})$ should diverge and $\tilde{f}(0, \vec{0})$ should be finite along a (1) or (2) path. Along a (4) path we expect that both $\tilde{f}(\pi, \vec{0})$ and $\tilde{f}(0, \vec{0})$ should diverge. Hence, along a (1) or (2) path, we expect only minor modifications in the usual OZ form for the correlation function. Along a (4) path the correlation function $\tilde{f}$ will have to diverge in the $(p, \vec{q}) = (\pi, \vec{0})$ and $(0, \vec{0})$ directions. Thus the simple single-pole divergence of the OZ correlation function will change its analytic form to show critical fluctuations of two types at the tricritical point.

III. CALCULATION

To calculate the correlation function we add to (1), with $H_S = 0$, an oscillating magnetic field to form the new Hamiltonian

$$\mathcal{H}' = \mathcal{H} - h \sum_{z\vec{r}} e^{ip'z + i\vec{q}'\cdot\vec{r}} S_{z\vec{r}} \equiv \mathcal{H} - hNS_{-p',\,-\vec{q}'} \; . \tag{6}$$

Then with $\langle\cdots\rangle_{\mathcal{H}}$ and $\langle\cdots\rangle_{\mathcal{H}'}$ denoting thermal averages with density matrices proportional to $e^{-\beta\mathcal{H}}$ and $e^{-\beta\mathcal{H}'}$, respectively, it is easily seen that

$$\left.\frac{\partial\langle S_{z\vec{r}}\rangle_{\mathcal{H}'}}{\partial h}\right|_{h\to 0} = \beta N(\langle S_{-p',\,-\vec{q}'}\, S_{z\vec{r}}\rangle_{\mathcal{H}} - \langle S_{-p',\,-\vec{q}'}\rangle_{\mathcal{H}} \langle S_{z\vec{r}}\rangle_{\mathcal{H}}) \; . \tag{7}$$

Equation (7) is exact. In order to obtain an approximate expression for the correlation function on the right-hand side, we evaluate $\langle S_{z\vec{r}}\rangle_{\mathcal{H}}$ in the mean-field approximation. Linearization in h of the resulting expression then yields the derivative on the left-hand side of (7), and hence the correlation function.

We derive the mean-field result by using the variational principle[11]

$$F(\beta, H, h) \leq \phi(\beta, H, h) = \mathrm{Tr}\rho\mathcal{H}' + \frac{1}{\beta}\mathrm{Tr}(\rho\ln\rho) \; . \tag{8}$$

$F(\beta, H, h)$ is the true free energy of the system with Hamiltonian $\mathcal{H}'$, and ρ is any density matrix, with an equality of $\rho \propto e^{-\beta\mathcal{H}'}$.

Mean-field theory follows from a single-spin approximation for ρ,

$$\rho = \prod_{z\vec{r}} \rho_{z\vec{r}} \; , \quad \rho_{z\vec{r}} = \tfrac{1}{2}(1 + M_{z\vec{r}} S_{z\vec{r}}) \; . \tag{9}$$

Performing the traces with this ρ, and using Eqs. (6) and (9), we find

$$\phi(\beta, H, h, M_{z\vec{r}}) = -\frac{1}{2}\sum_{\substack{z\vec{r}\\ z'\vec{r}'}} \mathcal{J}(z\vec{r}, z'\vec{r}') M_{z\vec{r}} M_{z'\vec{r}'} - H\sum M_{z\vec{r}} - h\sum e^{i(p'z + \vec{q}'\cdot\vec{r})} M_{z\vec{r}}$$
$$+ \frac{1}{\beta}\sum\left[\left(\frac{1+M_{z\vec{r}}}{2}\right)\ln\left(\frac{1+M_{z\vec{r}}}{2}\right) + \left(\frac{1-M_{z\vec{r}}}{2}\right)\ln\left(\frac{1-M_{z\vec{r}}}{2}\right)\right] . \tag{10}$$

To find the lowest upper bound to F, we minimize ϕ with respect to $M_{z\vec{r}}$:

$$\frac{\partial\phi}{\partial M_{z\vec{r}}} = -\sum_{z'\vec{r}'} \mathcal{J}(z\vec{r}, z'\vec{r}') M_{z'\vec{r}'} - H - he^{ip'z + i\vec{q}'\cdot\vec{r}} + \frac{1}{2\beta}\ln\frac{1+M_{z\vec{r}}}{1-M_{z\vec{r}}} = 0 \; . \tag{11}$$

The $M_{z\vec{r}}$ are functions of h but, since we are interested in the linear term, we expand in a series in h,

$$M_{z\vec{r}} = M^{o}_{z\vec{r}} + h\delta M_{z\vec{r}} + \cdots \; , \tag{12}$$

where from Eq. (7)

$$\delta M_{z\vec{r}} = \beta N(\langle S_{-p',\,-\vec{q}'} S_{z\vec{r}}\rangle - \langle S_{-p',\,-\vec{q}'}\rangle \langle S_{z\vec{r}}\rangle) \; . \tag{13}$$

Substituting Eq. (12) in Eq. (11) and equating factors of orders zero and one in h,

$$\frac{1}{2\beta}\ln\frac{1+M^{o}_{z\vec{r}}}{1-M^{o}_{z\vec{r}}} = \sum_{z'\vec{r}'} \mathcal{J}(z\vec{r}, z'\vec{r}') M^{o}_{z'\vec{r}'} + H \; , \tag{14}$$

$$\frac{\delta M_{z\vec{r}}}{\beta[1-(M^{o}_{z\vec{r}})^2]} = \sum_{z'\vec{r}'} \mathcal{J}(z\vec{r}, z'\vec{r}')\delta M_{z'\vec{r}'} + e^{ip'z + i\vec{q}'\cdot\vec{r}} \; . \tag{15}$$

Equation (14) is the usual self-consistent relation for the magnetization. When there is more than one solution, we choose the one which gives the smaller ϕ. Equation (15) then yields a solution for $\delta M_{z\vec{r}}$ and hence for the correlation function.

To solve Eq. (15), we note that $M^{o}_{z\vec{r}}$ is independent of $\vec{r}$, and depends only on z, taking on the values M^{o}_{A} or M^{o}_{B} depending on whether z is even or odd, respectively. Equation (15) becomes

$$\frac{\delta M_{z\vec{r}}}{\beta[1-(M^{o}_{z})^2]} = \sum_{\vec{r}'} J(\vec{r}-\vec{r}')\delta M_{z\vec{r}'} + \sum_{\vec{r}'} \mathcal{J}(\vec{r}-\vec{r}')\delta M_{z+1,\vec{r}'} + \sum_{\vec{r}'} \mathcal{J}(\vec{r}-\vec{r}')\delta M_{z-1,\vec{r}'} + e^{ip'z + i\vec{q}'\cdot\vec{r}} \; . \tag{16}$$

Taking a Fourier transform in $\vec{r}$, we find

$$\frac{\delta M_{z\vec{q}}}{\beta[1-(M^{o}_{z})^2]} = J(\vec{q})\delta M_{z,\vec{q}} + \mathcal{J}(\vec{q})\delta M_{z-1,\vec{q}} + \mathcal{J}(\vec{q})\delta M_{z+1,\vec{q}} + \delta_{\vec{q}\vec{q}'} e^{ip'z} \; , \tag{17}$$

$$J(\vec{q}) = \sum_{\vec{r}} e^{-i\vec{q}\cdot\vec{r}} J(\vec{r}) \; ,$$

$$\mathcal{J}(\vec{q}) = \sum_{\vec{r}} e^{-i\vec{q}\cdot\vec{r}} \mathcal{J}(\vec{r}) \ . \tag{18}$$

Now we define even $E_{p\vec{q}}$ and odd $O_{p\vec{q}}$ transforms in the z direction:

$$E_{p\vec{q}} = \frac{1}{N^{1/3}} \sum_{z\ \text{even}} e^{-ipz} \delta M_{z\vec{q}} \ , \tag{19}$$

$$O_{p\vec{q}} = \frac{1}{N^{1/3}} \sum_{z\ \text{odd}} e^{-ipz} \delta M_{z\vec{q}} \ . \tag{20}$$

Clearly, the Fourier transform $\delta M_{p\vec{q}} = E_{p\vec{q}} + O_{p\vec{q}}$. Summing Eq. (17) over z even or z odd,

$$\frac{E_{p\vec{q}}}{\beta[1-(M_A^o)^2]} = J(\vec{q}) E_{p\vec{q}} + 2\mathcal{J}(\vec{q}) O_{p\vec{q}} \cos p + \tfrac{1}{2}\delta_{\vec{q}\vec{q}'}\delta_{pp'} \ , \tag{21}$$

$$\frac{O_{p\vec{q}}}{\beta[1-(M_B^o)^2]} = J(\vec{q}) O_{p\vec{q}} + 2\mathcal{J}(\vec{q}) E_{p\vec{q}} \cos p + \tfrac{1}{2}\delta_{\vec{q}\vec{q}'}\delta_{pp'} \ . \tag{22}$$

Solving for $E_{p\vec{q}}$ and $O_{p\vec{q}}$ and using Eq. (7) we have, finally,

$$C(p,\vec{q}) \equiv 2\beta(\langle S_{-p\,-\vec{q}} S_{p\vec{q}}\rangle - \langle S_{p\vec{q}}\rangle \langle S_{-p\,-\vec{q}}\rangle)$$
$$= \frac{\left[\left(\frac{1}{\beta}\frac{1}{1-M_A^{o\,2}} - J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p\right) + \left(\frac{1}{\beta}\frac{1}{1-M_B^{o\,2}} - J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p\right)\right]}{\left[\left(\frac{1}{\beta}\frac{1}{1-M_A^{o\,2}} - J(\vec{q})\right)\left(\frac{1}{\beta}\frac{1}{1-M_B^{o\,2}} - J(\vec{q})\right) - 4\mathcal{J}^2(\vec{q})\cos^2 p\right]} \ . \tag{23}$$

As an immediate check, and to gain some insight into Eq. (23), suppose $\mathcal{J}(\vec{q}) > 0$, so that the system is a ferromagnet. In this case $M_A^o = M_B^o = \mathfrak{M}$, so that the denominator of Eq. (23) factors, and

$$C(p,\vec{q}) = 2\Big/\left(\frac{1}{\beta}\frac{1}{1-\mathfrak{M}^2} - J(\vec{q}) - 2\mathcal{J}(\vec{q})\cos p\right) \ . \tag{24}$$

$J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p$ is just the Fourier transform of the total potential. Furthermore, since the Curie temperature is given by $KT_c = J(0) + 2\mathcal{J}(0)$, and $\mathfrak{M} \sim 0$ from below, the denominator of Eq. (24) vanishes in the forward direction at T_c.

Note that in comparison with Eq. (24), Eq. (23) may allow a double-pole structure in Fourier space, with the numerator of (23) cancelling out the second pole except for one point in the (H, T) plane. While (23) may stand on its own as an interpolation scheme for experimental data, it is useful to use Eq. (14) to develop the mean-field thermodynamics, find M_A^o and M_B^o, and use these results in (23).

IV. THERMODYNAMICS

The solutions to Eq. (17) have been studied by Bideaux, Carrara, and Vivet[4]; for completeness we shall rederive some of their results by methods similar to those used by Blume, Emery, and Griffiths.[12] As an alternative to solving Eq. (14) directly, it is convenient to do a Taylor series expansion on Eq. (10) (with $h=0$) in the order parameter ϵ;

$$\phi(\beta, H, h=0\,;\,\epsilon, M) = \phi_0 + \epsilon^2\phi_1 + \epsilon^4\phi_2 + \epsilon^6\phi_3 + \cdots \ . \tag{25}$$

After a straightforward but tedious calculation, one finds

$$\phi_0 = (-\tfrac{1}{8}J_0 - \tfrac{1}{4}\mathcal{J}_0)M_0^2 - \tfrac{1}{2}HM_0$$
$$+ \frac{1}{4\beta}M_0 \ln\frac{2+M_0}{2-M_0} + \frac{1}{2\beta}\ln(4-M_0^2) \ , \tag{26}$$

$$\phi_1 = -\tfrac{1}{8}J_0 + \tfrac{1}{4}\mathcal{J}_0 + \frac{1}{2\beta}\frac{1}{4-M_0^2} \ , \tag{27}$$

$$\phi_2 = \left(\frac{1}{2\beta(4-M_0^2)^3}\right) \times \frac{(3M_0^2-4) + (4+3M_0^2)\beta(\tfrac{1}{4}J_0 + \tfrac{1}{2}\mathcal{J}_0)(4-M_0^2)}{6\{[(\tfrac{1}{4}J_0 + \tfrac{1}{2}\mathcal{J}_0)\beta(4-M_0^2)] - 1\}} \ , \tag{28}$$

where $M_0 = M_A^o + M_B^o$ is determined by

$$\frac{1}{4\beta}\ln\frac{2+M_0}{2-M_0} = (\tfrac{1}{4}J_0 + \tfrac{1}{2}\mathcal{J}_0)M_0 + \tfrac{1}{2}H \ ,$$
$$J_0 = \sum_{\vec{r}} J(\vec{r}) \ , \quad \mathcal{J}_0 = \sum_{\vec{r}} \mathcal{J}(\vec{r}) \ , \tag{29}$$

and for future reference

$$M = M_0 + M_1\epsilon^2 + M_2\epsilon^4 + \cdots \ , \tag{30}$$

$$\frac{1}{\beta}\frac{M_1}{4-M_0^2} = M_1(\tfrac{1}{4}J_0 + \tfrac{1}{2}\mathcal{J}_0) - \frac{1}{\beta}\frac{M_0}{(4-M_0^2)^2} \ . \tag{31}$$

One can then conclude on the basis of Eq. (25), by an argument of Landau,[12] that if

(i) $\phi_1 = 0$, $\phi_2 > 0$,

we have a critical (λ) point; if

(ii) $\phi_1 = 0$, $\phi_2 = 0$,

we have a tricritical point; and if

(iii) $4\phi_1\phi_3 = \phi_2^2$,

we have a first order point. The phase diagram of Fig. 2 was constructed in this way. For this particular model, $\phi_1 = \phi_2 = 0$ at a temperature T_t:

$$T_t/T_N = 1 + \tfrac{2}{3}\mathcal{J}_0/J_0 \ , \tag{32}$$

when $-\frac{3}{2} < \mathcal{J}_0/J_0 < 0$.

The order parameter which we shall need for the calculation of the correlation function is given, in be ordered phase, by[5]

$$\epsilon^2(T,H) = -\frac{\phi_2}{3\phi_3} + \left[\left(\frac{\phi_2}{3\phi_3}\right)^2 - \frac{\phi_1}{3\phi_3}\right]^{1/2} . \quad (33)$$

$\epsilon(T,H)$ vanishes as $T \to T_\lambda$ and $H \to H_\lambda$ in the case of a (1) path. We introduce the exponent β by

$$\epsilon \sim (T - T_\lambda)^\beta \text{ or } \epsilon \sim [H - H(T_\lambda)]^\beta \quad (34)$$

as $T \to T_\lambda$, $H \to H_\lambda$. Similarly, near the tricritical point along a (4) path

$$\epsilon \sim (T - T_t)^{\beta_t} \text{ or } \epsilon \sim (H - H_t)^{\beta_t} . \quad (35)$$

The essential point to recognize is that for a (1) path, ($\phi_1 \to 0$, $\phi_2 > 0$) and $\beta = \frac{1}{2}$, while for a (4) path, $\beta_t = \frac{1}{4}$. These results are found by expansion of Eq. (33), and by noting from Eq. (27) that ϕ_1 will vanish linearly with its distance from a λ or tricritical point, and that ϕ_2 will vanish linearly with its distance from the tricritical point; the exception to this general rule occurs if a λ or tricritical point is approached asymptotically close to the λ line. In this case, ϕ_1 would vanish quadratically. We shall not consider approaches to critical points in this "weak" direction here.[13] With this thermodynamic information, we turn to the evaluation of Eq. (23) in some special cases.

A. Correlation function along a (2) path

In this case we are in the paramagnetic phase ($\epsilon = 0$), so that $M_A^o = M_B^o = \frac{1}{2} M^o$, and the same cancellations that occurred in deriving Eq. (24) occur here. Equation (23) becomes

$$C(p,\vec{q}) = 2\bigg/\left(\frac{1}{\beta}\,\frac{1}{1-(\frac{1}{2}M_0)^2} - J(\vec{q}) - 2\mathcal{J}(\vec{q})\cos p\right) . \quad (36)$$

Using Eq. (27), we can rewrite this as

$$C(p,\vec{q}) = \frac{2}{8\phi_1 - [J(\vec{q}) - J(0) + 2\mathcal{J}(0) + 2\mathcal{J}(\vec{q})\cos p]} . \quad (37)$$

$C(\pi,\vec{0})$ will diverge along a (2) path when $\phi_1 = 0$, i.e., when a λ point is approached. Note that as we approach a λ point along a (2) path, $C(0,\vec{0})$ approaches the value $-1/2\mathcal{J}(0)$. Expanding Eq. (37) for $\vec{q} \sim \vec{0}$ and $\pi - p \sim 0$, retaining the quadratic terms,

$$C(\pi - p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{2}{8\phi_1 + q^2[\mathcal{J}''(0) - \frac{1}{2}J''(0)] + p^2[-\mathcal{J}(0)]} ,$$

where

$$q^2 = |\vec{q}|^2 , \quad J''(0) = \frac{d^2}{dq^2} J(\vec{q})\bigg|_{q=0} , \text{ etc.}$$

The quantity $q^2[\mathcal{J}''(0) - \frac{1}{2}J''(0)] + p^2[-\mathcal{J}(0)]$ frequently occurs in our expressions for correlation functions. We define

$$|p,\vec{q}|^2 = q^2[\mathcal{J}''(0) - \tfrac{1}{2}J''(0)] + p^2[-\mathcal{J}(0)] ;$$

then

$$C(\pi - p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to 0}}{\sim} \frac{2}{8\phi_1 + |p,\vec{q}|^2} . \quad (38)$$

Expanding Eq. (37) for $\vec{q} \to 0$ and $p \to 0$ gives

$$C(p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to 0}}{\sim} \frac{2}{8\phi_1 + |p,\vec{q}|^2 - 4\mathcal{J}(0)} . \quad (39)$$

Equation (38) has the usual OZ form. With ξ the correlation length, the exponent ν defined by

$$\xi \sim (T - T_\lambda)^{-\nu} ,$$

and the exponent η defined by

$$C(\pi - p,\vec{q}) \sim 1/|p,\vec{q}|^{2-\eta} ,$$

where $T = T_\lambda$, $H = H(T_\lambda)$, and $|p,\vec{q}| \to 0$, we find $\nu = \frac{1}{2}$ and $\eta = 0$ for the divergence of $C(\pi,\vec{0})$ along a (2) path.

B. Correlation function along a (3) path

This case is identical to that in Sec. IV A. As far as the OZ expression for the correlation function is concerned, the tricritical point is not distinguished from a λ point in the disordered phase.

C. Correlation function in the ordered phase

In the ordered phase, $\epsilon \neq 0$, and Eq. (23) no longer factors. It is convenient first to specialize (23) when $\epsilon \sim 0$, and then consider the behavior of (23) along (1) and (4) paths.

Expressing M_A^o and M_B^o in terms of M and ϵ, and using Eqs. (26)–(31) in Eq. (23), we find to $O(\epsilon^4)$

$$C(p,\vec{q}) = \frac{2[4/\beta(4-M_0^2) - J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p] + 2\delta\epsilon^2 + O(\epsilon^4)}{[4/\beta(4-M_0^2) - J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p][4/\beta(4-M_0^2) - J(\vec{q}) - 2\mathcal{J}(\vec{q})\cos p] + \epsilon^2 f(p,\vec{q}) + O(\epsilon^4)} , \quad (40)$$

where

$$f(p,\vec{q}) = \delta[J(0) - J(\vec{q}) + 2\mathcal{J}(0) - 2\mathcal{J}(\vec{q})\cos p]$$
$$+ \delta[4/\beta(4-M_0^2) - J(\vec{q}) + 2\mathcal{J}(\vec{q})\cos p] + 48\phi_2[4/\beta(4-M_0^2) - J(0) - 2\mathcal{J}(0)] \quad (41)$$

and

$$\delta = 48\phi_2 - 16M_1M_0/\beta(4-M_0^2)^2 \,. \tag{42}$$

The behavior of $f(p,\vec{q})$ can be summarized as follows:

$$f(0,\vec{0}) = \begin{cases} -48\times 4\phi_2\mathcal{I}(0) & (\lambda \text{ point}) \\ 0 & (\text{tricritical point}) \end{cases}$$

$$f(\pi,\vec{0}) = \begin{cases} -48\times 4\phi_2\mathcal{I}(0) & (\lambda \text{ point}) \\ 0 & (\text{tricritical point}) \,. \end{cases}$$

Also, by expansion of Eq. (33) to leading order,

$$\epsilon^2(T,H) \sim -\phi_1/2\phi_2 \tag{43}$$

along a (1) path near a λ point, and

$$\epsilon^2(T,H) \sim (-\phi_1/3\phi_3)^{1/2} \tag{44}$$

along a (4) path near the tricritical point. Now we specialize Eq. (40) for (1) and (4) paths.

D. Correlation function along a (1) path

Consider first $C(\pi,\vec{0})$. The numerator of (40) is finite, whereas the denominator of (40) is vanishing We can write (40) in the limit $\pi - p \to 0$, $\vec{q}\to\vec{0}$, $\epsilon\to 0$, with ϵ^2/p^2 and ϵ^2/q^2 finite as

$$C(\pi-p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{2}{-16\phi_1 + |p,\vec{q}|^2} \,. \tag{45}$$

In contrasting the behavior of $C(\pi-p,\vec{q})$ along (1) and (2) paths [Eqs. (45) and (38)], we see that the amplitude of the $C(\pi,\vec{0})$ divergence is different. This change of amplitude is a usual feature of the OZ correlation function above and below the critical temperature.

The behavior of $C(0,\vec{0})$ is a bit more subtle. Both the numerator and denominator are vanishing, but the ratio is finite. In the limit $p\to 0$, $\vec{q}\to\vec{0}$, $\epsilon^2\to 0$, with ϵ^2/p^2 and ϵ^2/q^2 fixed, we find

$$C(p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{2(8\phi_1 + |p,\vec{q}|^2) - \delta\phi_1/\phi_2}{-4\mathcal{I}(0)(8\phi_1 + |p,\vec{q}|^2) + 2\times 48\phi_1\mathcal{I}(0)} \,. \tag{46}$$

If we set $H=0$, $T\to T_N^-$, then $M_0=0$, and from Eq. (42)

$$\delta = 48\phi_2 \,.$$

Using this in Eq. (46),

$$C(0,\vec{0}) \underset{T\to T_N}{\sim} = -\frac{1}{2\mathcal{I}(0)} \,.$$

Since from Sec. IV A, Eq. (39), $C(0,\vec{0}) = -1/2\mathcal{I}(0)$ for a (2) path at a λ point, $C(0,\vec{0})$ is continuous at $T=T_N$, $H=0$. Along other (2) paths, $H\neq 0$ and (46) no longer reduces to $-1/2\mathcal{I}(0)$. This jump discontinuity of $C(0,\vec{0})$ upon crossing a λ line grows as ϕ^{-1} as the tricritical point is approached. At the tricritical point the discontinuity becomes infinite. On the other hand, if we consider $|p,\vec{q}|^2\neq 0$ and $T=T_\lambda$, $H=H_\lambda$, $C(p,\vec{q}) = -1/2\mathcal{I}(0)$ as $|p,\vec{q}|^2\to 0$, and there is no discontinuity.

E. Correlation function along a (4) path

$C(\pi,\vec{0})$, from Eq. (40), have a finite numerator and vanishing denominator on a (4) path as the tricritical point is approached. For $(p,\vec{q}) = (\pi,\vec{0})$, the order of the denominator terms are as follows: The product term is of order ϕ_1, $\epsilon^2 f(\pi,\vec{0})$ is of order $(-\phi_1)^{1/2}\phi_2$, and $O(\epsilon^4)$ is of order ϕ_1. To leading order, therefore, the denominator vanishes as ϕ_1. If $f(\pi,\vec{0})$ did not vanish at the tricritical point, the divergence of $C(\pi,\vec{0})$ would be of order $(-\phi_1)^{-1/2}$, which would not lead to the proper susceptibility. For $\pi-p\to 0$, $\vec{q}\to\vec{0}$ and ϵ^4/p^2, ϵ^4/q^2 fixed,

$$C(\pi-p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{2}{-32\phi_1 + |p,\vec{q}|^2} \,. \tag{47}$$

This is in the usual OZ form, and we note $\nu_t = \frac{1}{2}$, $\eta_t = 0$. $C(0,\vec{0})$ from Eq. (40) has a numerator which vanishes as $(-\phi_1)^{1/2}$ to leading order, while the denominator vanishes as ϕ_1. $C(0,\vec{0})$ therefore diverges along a (4) path. For $\vec{q}\to\vec{0}$, $p\to 0$, $\epsilon\to 0$, retaining terms of order q^2, p^2, and ϵ^4 but neglecting terms of order $q^2\epsilon^2$, $p^2\epsilon^2$, we find

$$C(p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{2(8\phi_1 + |p,\vec{q}|^2) + [32/\beta(4-M_0^2)]M_1M_0(-\phi_1/3\phi_3)^{1/2}}{-4\mathcal{I}(0)(A\phi_1 + |p,\vec{q}|^2)} \,. \tag{48}$$

A is a constant determined from the nonordering susceptibility. If we consider $|p,\vec{q}|^2\neq 0$, $T=T_t$, $H=H_t$, then $C(p,\vec{q}) = 1/-2\mathcal{I}(0)$. Therefore, if $|p,\vec{q}|^2$ is small but not zero, $C(p,\vec{q})$ is finite and continuous across $T=T_t$, $H=H_t$ in going from a (3) to a (4) path. This behavior is not peculiar to the tricritical point; Eq. (46) shows this same continuity behavior at a λ point.

It is helpful to consider the scaling limit in Eq. (48): ϵ^4/q^2 and ϵ^4/p^2 fixed; $|p,\vec{q}|^2\to 0$, $\epsilon\to 0$. In this limit

$$C(p,\vec{q}) \underset{\substack{p\to 0\\ \vec{q}\to\vec{0}}}{\sim} \frac{1}{|p,\vec{q}|}\left[\frac{32}{\beta(4-M_0^2)}\,\frac{M_1M_0}{|p,\vec{q}|}\left(\frac{-\phi_1}{3\phi_3}\right)^{1/2} \Big/ -4\mathcal{I}(0)\left(1+\frac{A\phi_1}{|p,\vec{q}|^2}\right)\right] \equiv |p,\vec{q}|^{\eta-2}\mathcal{F}\left(\frac{\phi_1}{|p,\vec{q}|^2}\right) \,. \tag{49}$$

Associating an exponent ν_u and η_u with (49), $\nu_u = \frac{1}{2}$ and $\eta_u = 1$.[14]

F. Correlation function along (4) path near the tricritical point for arbitrary $(p, \vec{q})$

The last case we consider is Eq. (40) near the tricritical point along a (4) path but $(p, \vec{q})$, not $(\pi, \vec{0})$ or $(0, \vec{0})$. In this case, the behavior of $C(p, \vec{q})$ is dominated by the fact that $f(p, \vec{q})$ is not small. First we define $K(p, \vec{q})$, the Fourier transform of the potential:

$$K(p, \vec{q}) = J(\vec{q}) + 2\mathcal{J}(\vec{q}) \cos p \,. \tag{50}$$

Expanding Eq. (40) to order ϵ^2, we find

$$C(p, \vec{q}) \underset{\epsilon \to 0}{\sim} \frac{2[K(\pi, \vec{0}) - K(\pi - p, \vec{q})] + 2\alpha\epsilon^2 + O(\epsilon^4)}{[K(\pi, \vec{0}) - K(\pi - p, \vec{q})][K(\pi, \vec{0}) - K(p, \vec{q})] + 2\alpha\epsilon^2[K(\frac{1}{2}\pi, \vec{0}) - K(\frac{1}{2}\pi, \vec{q})]}\,, \tag{51}$$

where

$$\alpha = -16 M_1 M_0 / \beta (4 - M_0^2)^2 \,.$$

Equation (51) in addition to (38), (39), (47), and (48), serves as the OZ expression for the correlation function near a tricritical point.

V. DISCUSSION

The critical behavior of a metamagnet is represented by the behavior of the approximate OZ correlation function. At λ points, the correlation function diverges in the $(\pi, \vec{0})$ direction, whereas at tricritical points it diverges additionally in the $(0, \vec{0})$ direction along a (4) path. Equation (23) interpolates these regimes by having a two-pole single-zero structure, in distinction to the single-pole structure of the OZ expression for a system with only λ points.

Equation (23) is the basic result, all other expressions being limiting cases of (23). Figures (3) and (4) catalog the limits we have taken. The numbers in parentheses correspond to the appropriate expression in the text for $C(p, \vec{q})$. Additionally (57) is an interpolation formula for $C(p, \vec{q})$ for $(p, \vec{q}) \neq (0, \vec{0})$ or $(\pi, \vec{0})$ near the tricritical point from below.

We have noted that the expression for that $C(p, \vec{q})$ for $p \sim \pi$ $\vec{q} \sim \vec{0}$ is of the usual OZ form near critical points or tricritical points. Along a (4) path $C(p, \vec{q})$ shows critical fluctuations in the forward direction $(|p, \vec{q}|^2 \sim 0)$. The expression for these critical fluctuations [Eqs. (48) and (49)] is, however, not the usual OZ form. In the scaling limit (49) we find an $\eta_u = 1$. Additionally, the limits $|p, \vec{q}|^2 \to 0$, $\epsilon \to 0$ may not be interchanged on a (4) path. If $\epsilon = 0$ and $|p, \vec{q}|^2 \to 0$, then $C(p, \vec{q})$ is finite and continuous. If $|p, \vec{q}|^2 = 0$ and $\epsilon \to 0$, then $C(p, \vec{q})$ diverges as $(-\phi_1)^{-1/2}$.

The recent predictions that tricritical exponents have mean-field values in three dimensions[10] have been made plausible by an appeal to the Ginsburg criterion.[5] [It is pointed out in Ref. 5 that the Ginsburg criterion, in addition to being satisfied along a (4) path, is also valid along a path in the ordered phase asymptotically close to the first-order line terminating at the tricritical point.] The Ginsburg criterion involves the comparison of the magnitude of the order parameter with the magnitude of fluctuations over a correlation volume. If the magnitude of the order parameter is large compared with fluctuations of the order parameter, it is argued that mean-field theory is consistent. The tricritical point is characterized by critical fluctuations of two order parameters, whereas the Ginsburg criterion is usually envisaged to treat one order parameter. In the case of the BEG model (see Appendix) this is no problem, since fluctuations in the two order parameters are expressed in terms of two distinct correlation functions. In the BEG model, one merely applies the Ginsburg criterion twice. One finds in this case that the ordering and nonordering, fluctuations separately satisfy the Ginsburg criterion. On the other hand, in the case of the metamagnet there is the concern that the relatively fast vanishing of the magnetization $(\beta_u = \frac{1}{2})$ may be coupled with the strong divergence of ordering fluctuations. This does not happen, because the critical fluctuations in the magnetization are effectively decoupled from those in ϵ. To see this, we define the local quantities

$$\epsilon(z, \vec{r}) = S_{z+1, \vec{r}} - S_{z\vec{r}} \,, \quad M(z, \vec{r}) = S_{z+1, \vec{r}} + S_{z\vec{r}} \,.$$

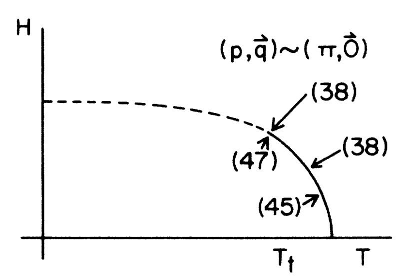

FIG. 3. Superimposed on the phase diagram of the metamagnet are the paths for which limiting expressions for $C(p, \vec{q})$ for $(p, \vec{q}) \sim (\pi, \vec{0})$ have been given. The numbers in parentheses refer to relevant equations in the text.

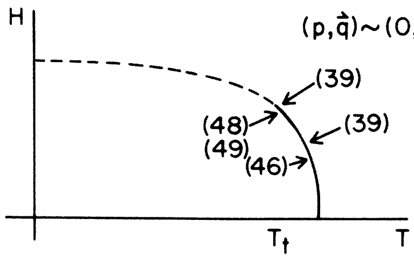

FIG. 4. Superimposed on the phase diagram of the metamagnet are the paths for which limiting expressions for $C(p, \vec{q})$ for $(p, \vec{q}) \sim (0, \vec{0})$ have been given. The numbers in parentheses refer to relevant equations in the text.

One can then express objects $\langle \epsilon(z,\vec{r})\epsilon(z',\vec{r}')\rangle$ in terms of the $\langle S_{z\vec{r}} S_{z'\vec{r}'}\rangle$. Doing this, and taking a Fourier transform, we find

$$\langle \epsilon(p,\vec{q})\epsilon(-p,-\vec{q})\rangle = (2-2\cos p)\langle S_{p\vec{q}} S_{-p\,-\vec{q}}\rangle \,,$$
$$\langle M(p,\vec{q})M(-p,-\vec{q})\rangle = (2+2\cos p)\langle S_{p\vec{q}} S_{-p\,-\vec{q}}\rangle \,. \qquad (52)$$

This shows in an explicit fashion that the fluctuations $\langle S_{\pi\vec{0}} S_{-\pi\vec{0}}\rangle$ are effectively absent from critical fluctuations in the magnetization. This allows the separate invocation of the Ginsburg criterion for M and ϵ, and the proof of the plausibility argument proceeds as usual.

Mean-field theory predicts that specific heat diverges as $(-\phi_1)^{-1/2}$ along a (4) path, whereas it is finite along a (3) path. This asymmetry of the specific heat is presumably a consequence of the neglect of short-range order in mean-field theory. Since the energy may be expressed as

$$\mathcal{E} = -\sum_{p,\vec{q}} K(p,\vec{q})\langle S_{p\vec{q}} S_{-p\,-\vec{q}}\rangle - HM$$

differentiation of $\mathcal{E}$ with respect to T yields the specific heat. Using our results, we find that the specific heat diverges along a (3) or (2) path as $(-\phi_1)^{-1/2}$. This restores the symmetry of the specific heat along a (3) and (4) path, but as a result of the inconsistency of the calculation,[15] the specific heat now diverges as $(-\phi_1)^{-1/2}$ at λ points.

In the Appendix we develop the correlation functions for the BEG model. Both the metamagnet and BEG models have tricritical points. In fact, within mean field, the thermodynamics of the two models are identical. Since in the metamagnet two critical fluctuations are present in the same correlation function, whereas in the BEG model two separate correlation functions diverge, it is natural to ask if the BEG model and metamagnet differ in the behavior of their correlation functions. We show in the Appendix that there is a complete isomorphism between the correlation functions of the metamagnet with those of the BEG model.

VI. CONCLUSION

We have presented the Ornstein-Zernike expression for the Fourier transform of the correlation function of the metamagnet and the correlation function of the BEG model. That the correlation function has a single divergence at λ points and two divergences at the tricritical point, is a fact constrained by the thermodynamics. Universality along the λ line implies that the usual OZ form should obtain for the ordering fluctuation at λ points. In this context we mention that even along a path through the tricritical point, the ordering fluctuations assume the usual OZ form. An interesting aspect of this calculation is the unusual expression for the nonordering fluctuations at the tricritical point. We have noted that $\eta_u = 1$, and that the limits $(T,H)\to(T_t,H_t)$ and $|p,\vec{q}|^2\to 0$ are not interchangeable for these nonordering fluctuations.

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APPENDIX

We present in this Appendix the correlation functions of a spin-1 model with crystalline field, the BEG model.[2] The Hamiltonian of this model is

$$\mathcal{H} = -\frac{1}{2}\sum_{\vec{r},\vec{r}'} \mathcal{J}(\vec{r}-\vec{r}')S_{\vec{r}}S_{\vec{r}'} + \Delta\sum_{\vec{r}} S_{\vec{r}}^2 \,, \qquad (A1)$$

where the spins are on a simple-cubic lattice, and the spin $S_{\vec{r}}$, at lattice site $\vec{r}$, takes on the values ± 1 or 0. The two order parameters of this model are

$$M = \langle S_{\vec{r}}\rangle$$

and

$$Q = \langle (S_{\vec{r}})^2\rangle \,.$$

Three correlation functions are of interest in this model:

$$C_1(\vec{r}) = \beta[\langle S_{\vec{r}'} S_{\vec{r}+\vec{r}'}\rangle - M^2] \,, \qquad (A2)$$

$$C_2(\vec{r}) = \beta[\langle (S_{\vec{r}'})^2 (S_{\vec{r}+\vec{r}'})^2\rangle - Q^2] \,, \qquad (A3)$$

$$C_3(\vec{r}) = \beta[\langle (S_{\vec{r}'})^2 (S_{\vec{r}+\vec{r}'})\rangle - MQ] \,. \qquad (A4)$$

The Fourier-transform variable $\vec{q}$ [to be distinguished from the $(p,\vec{q})$ transform variables of the metamagnet] is a three-component vector and, for example,

$$C_1(\vec{q}) = \sum_{\vec{r}} e^{-i\vec{q}\cdot\vec{r}} C_1(\vec{r}) \,.$$

Utilizing the same approximations as were used for the metamagnet, we find the Fourier transform of Eqs. (A2)–(A4),

$$C_1(\vec{q}) = \frac{\beta(Q - M^2)}{1 - \beta Q \mathcal{J}(\vec{q}) + \beta \mathcal{J}(\vec{q}) M^2} , \tag{A5}$$

$$C_2(\vec{q}) = \frac{\beta(1 - Q)\{Q[1 - \beta \mathcal{J}(\vec{q}) Q] + \beta \mathcal{J}(\vec{q}) M^2\}}{1 - \beta Q \mathcal{J}(\vec{q}) + \beta \mathcal{J}(\vec{q}) M^2} , \tag{A6}$$

$$C_3(\vec{q}) = \frac{\beta M(1 - Q)}{1 - \beta Q \mathcal{J}(\vec{q}) + \beta \mathcal{J}(\vec{q}) M^2} . \tag{A7}$$

The thermodynamic information needed to evaluate these expressions is: $M = 0$ in the disordered phase and $M \to 0$ as a λ or tricritical point is approached from the ordered phase; $\beta \mathcal{J} Q = 1$ at a λ point and, additionally, $Q = \frac{1}{3}$ at the tricritical point; $Q = M \coth \beta \mathcal{J} M = Q_0 + M^2 Q_1 + \cdots$ in the ordered phase; and $\mathcal{J} = \sum_{\vec{r}} \mathcal{J}(\vec{r})$.

A. Behavior of $C_1(\vec{q})$

Along a (2) or (3) path we find

$$C_1(\vec{q}) = Q_0 / [1 - \beta \mathcal{J}(\vec{q}) Q_0] ,$$

which clearly diverges in the forward ($\vec{q} = 0$) direction at a λ or tricritical point. Along a (1) or (4) path to order M^4

$$C_1(\vec{q}) = \frac{\beta[1/\beta\mathcal{J} + M^2(\frac{1}{3}\beta\mathcal{J} - 1)]}{1 - \mathcal{J}(\vec{q})/\mathcal{J} + \beta \mathcal{J}(\vec{q})(1 - \frac{1}{3}\beta\mathcal{J}) M^2 + O(M^4)} . \tag{A8}$$

The reader may readily convince himself that the behavior of $C_1(\vec{q})$ is identical to the behavior of $C(p, \vec{q})$ of the metamagnet for $(p, \vec{q}) \sim (\pi, \vec{0})$.

B. Behavior of $C_2(\vec{q})$

Along a (2) or (3) path

$$C_2(\vec{q}) = \beta(1 - Q_0) Q_0$$

which is finite. Along a (1) or (4) path

$$C_2(\vec{q}) = \frac{\beta(1 - Q_0)[Q_0(1 - \mathcal{J}(\vec{q})/\mathcal{J}) + \beta \mathcal{J}(\vec{q}) M^2 (1 - \frac{1}{3}\beta\mathcal{J} Q_0)]}{1 - \mathcal{J}(\vec{q})/\mathcal{J} + \beta \mathcal{J}(\vec{q})(1 - \frac{1}{3}\beta\mathcal{J}) M^2 + O(M^4)} . \tag{A9}$$

Along a (1) path for $\vec{q} = 0$, the numerator and denominator are of order M^2 so $C_2(\vec{q})$ is finite. For a (4) path, the M^2 term in the denominator of Eq. (A9) vanishes, and (A9) diverges. Comparison with Eq. (48) leads to the identification of $C_2(\vec{q})$ with $C(p, \vec{q})$ for $(p, \vec{q}) \sim (0, \vec{0})$. Equation (A6) may be evaluated in the QT plane along the superfluid branch of the first-order line near the tricritical point. The result is

$$C_2(\vec{q}) = \tfrac{2}{9} \beta \frac{aq^2 + 15(1 - T/T_t)}{aq^2 + \frac{1}{4} 15(1 - T/T_t)^2} ,$$

where $\mathcal{J}(\vec{q}) = \mathcal{J}(1 - aq^2 + \cdots)$.

C. Behavior of $C_3(\vec{q})$

This correlation function measures a crossed susceptibility, the response of the ordering density to the nonordering field, or the nonordering density to the ordering field. From Eq. (A7), $C_3(\vec{q}) = 0$ for a (2) or (3) path. Along a (1) or (4) path,

$$C_3(\vec{q}) = \frac{\beta M(1 - Q)}{1 - \mathcal{J}(\vec{q})/\mathcal{J} + \beta \mathcal{J}(\vec{q})(1 - \frac{1}{3}\beta\mathcal{J}) M^2 + O(M^4)} . \tag{A10}$$

If we take $\vec{q} = 0$ along a (1) path, C_3 will diverge as M^{-1}, i.e., with an exponent of $\frac{1}{2}$. Taking $\vec{q}$ small, we can write (A10) as

$$C_3(\vec{q}) \underset{\vec{q} \to 0}{\sim} \frac{\beta M(1 - Q)}{[-\mathcal{J}''(0)/2\mathcal{J}](q^2) + (1 - \frac{1}{3}\beta\mathcal{J})(M^2/Q_0) + O(q^2 M^2, M^4)} . \tag{A11}$$

In the scaling limit $\vec{q} \to \vec{0}$, $M \to 0$, with M^2/q^2 fixed, Eq. (A11) gives us $\eta = 1$. This result is for a λ point.

Near the tricritical point, the M^2 term in the denominator of Eq. (A10) is of order M^4, so that for a (4) path

$$C_3(\vec{q}) \underset{\vec{q} \to 0}{\sim} \frac{\beta M(1 - Q)}{-[\mathcal{J}''(0)/2\mathcal{J}] q^2 + O(M^4)} . \tag{A12}$$

$C_3(\vec{0})$ diverges at the tricritical point as M^{-3}, i.e., with an exponent $\frac{3}{4}$. Taking the scaling limit on Eq. (A12) (M^4/q^2 fixed) yields $\eta = \frac{1}{2}$.

A crossed correlation function can also be defined in the metamagnet. We write

$$D(p, \vec{q}) = E_{p\vec{q}} - O_{p\vec{q}} ,$$

where $E_{p\vec{q}}$ and $O_{p\vec{q}}$ are determined from Eqs. (21) and (22). $D(p, \vec{q})$ measures for the metamagnet the response of ϵ to a magnetic field, or the response of M to a staggered field—the analog of $C_3(\vec{q})$. One readily finds

$$D(p, \vec{q}) = \left(\frac{1}{\beta} \frac{1}{1 - (M_A^0)^2} - \frac{1}{\beta} \frac{1}{1 - (M_B^0)^2}\right) \Big/ \left[\left(\frac{1}{\beta} \frac{1}{1 - (M_A^0)^2} - \mathcal{J}(\vec{q})\right)\left(\frac{1}{\beta} \frac{1}{1 - (M_B^0)^2} - \mathcal{J}(\vec{q})\right) - 4\mathcal{J}^2(\vec{q}) \cos^2 p\right] . \tag{A13}$$

This expression can be analyzed as was $C(p,\vec{q})$. The behavior of both $D(p\sim 0,\vec{q}\sim\vec{0})$ and $D(p\sim\pi, q\sim\vec{0})$ is found to be identical with that of $C_3(q\sim\vec{0})$.

The isomorphism between the correlation functions of the BEG model and metamagnet is

$$C_1(\vec{q}\sim\vec{0})\Longleftrightarrow C(p\sim\pi,\vec{q}\sim\vec{0})\ ,$$
$$C_2(\vec{q}\sim 0)\Longleftrightarrow C(p\sim 0,\vec{q}\sim\vec{0})\ ,$$
$$C_3(\vec{q}\sim\vec{0})\Longleftrightarrow\begin{cases}D(p\sim 0,\vec{q}\sim\vec{0})\\ D(p\sim\pi,\vec{q}\sim\vec{0})\ .\end{cases}$$

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