Longitudinal susceptibility and the question of zero magnons in low-temperature magnetic insulators*†

T. L. Reinecke
Department of Physics, Brown University, Providence, Rhode Island 02912
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The longitudinal susceptibility of the low-temperature Heisenberg ferromagnet is treated in random-phase approximation (RPA) in order to discuss whether a zero magnon (spin-density mode in the collisionless regime) can exist. In the unrenormalized RPA, this mode is found at large wave vectors in systems which have an exchange anisotropy which enhances the repulsion between spin-density fluctuations. It is shown, however, that (with the possible exception of certain two-dimensional systems) spin-wave damping suppresses this mode. A similar approach is applied to the low-temperature Heisenberg antiferromagnet. For systems with non-negligible anisotropy, it is shown that, with the possible exception of certain two-dimensional systems, spin-wave damping suppresses the zero-magnon mode. An argument is then given which suggests that the mode does not exist in the isotropic antiferromagnet either.

I. INTRODUCTION

The longitudinal susceptibilities of low-temperature Heisenberg-like magnetic insulators in the collisionless frequency regime and their possible excitations, zero magnons, are discussed here. These susceptibilities in the hydrodynamic frequency regime and their second-magnon resonances have been investigated in detail.[1-4] Rather less work has been done on them in the collisionless regime. Recently, however, Natoli and Ranninger[5] and Izuyama and Saitoh[6] have suggested that a zero-magnon mode supported by the mechanical repulsion between spin-wave densities exists there.

The susceptibility of the ferromagnet with and without exchange anisotropy is obtained in a random-phase approximation (RPA) and is evaluated for small and large wave vectors. The results are used to discuss the characteristics of this system which are of importance in the zero-magnon question; they are found to include exchange anisotropy, spin-wave interactions, spin-wave damping, wave-vector magnitude and direction, dimensionality, and spin magnitude. In the unrenormalized RPA this mode is found at large wave vector in systems with an exchange anisotropy which enhances the repulsion between spin-density fluctuations. Except possibly in certain two-dimensional systems, however, spin-wave damping suppresses the mode.

This work shows that previous results for the longitudinal susceptibility of the isotropic low-temperature ferromagnet[2,5,7] cannot be used to treat the zero-magnon problem in that system because of their inadequate treatment of the spin-wave interactions. This same conclusion has been reached by Harris,[8] who first obtained the correct susceptibility of the low-temperature isotropic ferromagnet. The present work uses this same susceptibility as a starting point for the discussion of the zero-magnon question in the isotropic case. The analysis is here extended to other systems (principally those with anisotropy), and a fairly complete discussion of the possible existence of a zero-magnon mode is given.

Finally, the zero-magnon question is the low-temperature antiferromagnet is discussed by an approach similar to that used for the ferromagnet. The denominator of the components of the matrix longitudinal susceptibility is obtained for systems with non-negligible anisotropy using a random phase approximation. Its analysis shows that, with the possible exception of certain two-dimensional systems, a well defined mode does not exist owing to the effects of spin-wave damping. An argument is then given which suggests that such a mode does not exist in the isotropic antiferromagnet either.

This paper is organized as follows. Sections II–V deal with the ferromagnet. In Sec. II the Hamiltonian is presented, and the effects of exchange anisotropy on spin-wave interactions are discussed. In Sec. III the RPA susceptibility is obtained. The susceptibility is evaluated in Sec. IV, and its zero-magnon resonance is found. In Sec. V the suppression of this mode by spin-wave damping is demonstrated. In Sec. VI the antiferromagnet is discussed.

II. HAMILTONIAN FOR THE LOW-T FERROMAGNET

The Heisenberg ferromagnetic system will be represented by the Hamiltonian

$$H = -h\sum_i S_i^z - \tfrac{1}{2}J\sum_{i,\vec{\delta}} \vec{S}_i \cdot \vec{S}_{i+\vec{\delta}} - \tfrac{1}{2}\sigma J \sum_{i,\vec{\delta}} S_i^z S_{i+\vec{\delta}}^z, \tag{2.1}$$

where the label i refers to the sites on a simple cubic lattice of N spins S and of lattice constant a, and $\vec{\delta}$ are the vectors to the nearest neighbors of i. The spins interact by nearest-neighbor coupling with a coupling constant J. The term involving σ is a uniaxial exchange anisotropy.

In his work on the low-T thermodynamics of the

Heisenberg ferromagnet, Dyson[9] showed that the spin Hamiltonian could be replaced by an equivalent boson Hamiltonian when quantities of order $e^{-\beta J}$ are neglected, where $\beta=(k_B T)^{-1}$ and k_B is Boltzmann's constant. In the resulting treatment, the spin waves interact weakly at low T, and therefore the contributions of the spin-wave interactions can be calculated perturbatively in temperature. Silberglitt and Harris[10] have discussed the validity of using this boson Hamiltonian in the calculation of dynamic quantities.

It might be mentioned here that the other common low-T boson Hamiltonian for the ferromagnet, viz., that of Holstein and Primakoff,[11] does not correctly represent the weakness of the spin-wave interactions at low T. Thus it cannot be used to determine the effects of those interactions perturbatively without a further resummation (usually in $1/S$). The use of the Holstein-Primakoff Hamiltonian has led other authors to incorrect conclusions for the longitudinal susceptibility and the zero-magnon question.

A simple substitution for the spin operators equivalent to that of Dyson is provided by the Dyson-Maleev[9,12] operators:

$$S_i^+=(2S)^{1/2}[a_i+(1/2S)a_i^\dagger a_i a_i],\quad S_i^-=(2S)^{1/2}a_i^\dagger,$$
$$S_i^z=S-a_i^\dagger a_i, \tag{2.2}$$

where $a_i^\dagger$, a_i are boson operators. Their Fourier lattice transforms are

$$a_{\vec{k}}^\dagger=\frac{1}{\sqrt{N}}\sum_j e^{-i\vec{k}\cdot\vec{r}_j}a_j^\dagger,\quad a_{\vec{k}}=\frac{1}{\sqrt{N}}\sum_j e^{i\vec{k}\cdot\vec{r}_j}a_j.$$

Making this substitution, the Hamiltonian (2.1) becomes

$$H^{\rm DM}=(\text{const.})+\sum_q \omega_q a_q^\dagger a_q + \frac{1}{N}\sum_{K,q,q'}V^{\rm DM}(K,q,q')a_{K+q}^\dagger a_{q'}^\dagger a_q a_{K+q'}, \tag{2.3}$$

where

$$\omega_q=h+(1+\sigma)JzS-JzS\gamma_q,$$
$$V^{\rm DM}(K,q,q')=\tfrac{1}{4}Jz[\gamma_{K+q}+\gamma_{q'}-(1+\sigma)\gamma_K-(1+\sigma)\gamma_{q-q'}], \tag{2.4}$$

and as usual $\gamma_{\vec{q}}=(1/z)\sum_{\vec{\delta}}e^{i\vec{q}\cdot\vec{\delta}}$, with z the number of nearest neighbors (six for the sc lattice).

For $\sigma=0$, if either of the incident wave vectors, q or $K+q'$, goes to zero, then $V^{\rm DM}$ vanishes. Because only long-wavelength spin waves are thermally excited at low T, this is the basis for treating this low-T Hamiltonian as a weakly interacting system. For $\sigma\neq 0$, on the other hand, $V^{\rm DM}$ does not go to zero for an incident wave vector going to zero, and this effect on the spin-wave interactions will have important results in the zero-magnon problem. Many magnetic insulators have non-negligible exchange anisotropy (see, e.g., Elliott and Thorpe[13]); the term in σ will be used as a simple and reasonably realistic way in which to include such effects.

The effect of σ on spin-wave interactions is easily seen. The process of interest in the dynamic susceptibility involves the system of thermal magnons being acted upon by a disturbance of wave vector $\vec{K}$. A subsequent spin-wave interaction involving the propagation of this disturbance can be roughly described at low T by taking q and q' small in the interaction part of $H^{\rm DM}$, which gives for that part

$$H_I^{\rm DM}\simeq\frac{Jz}{4N}\sum_{K,q,q'}[(-\sigma)(1+\gamma_K)+T_K(q,q')]a_{K+q}^\dagger a_{q'}^\dagger a_q a_{K+q'}. \tag{2.5}$$

Here $T_K(q,q')$ is a term dependent on $\vec{K}$ and proportional to powers of the small q and q'. The first term in (2.5) dominates the interaction for at least $\sigma\gg(k_BT/JS)$, and it will be shown later that it is the more important interaction contribution in the zero-magnon problem for all nonzero σ.

From

$$S_K^z=\frac{1}{\sqrt{N}}\sum_q a_{K+q}^\dagger a_q$$

for $K\neq 0$, Eq. (2.5) represents the interaction of spin-density fluctuations. For $\sigma<0$ the first term is repulsive; in direct space its interaction potential falls off in a range of about the lattice spacing. The term involving $T_K(q,q')$ also gives an interaction between spin-density fluctuations, but its nature cannot be seen easily because of the q, q' dependence.

There now arises the question of whether these repulsive interactions will support a mechanical spin-density oscillation similar to that of zero sound in a Fermi liquid or the plasma mode of the electron gas. This question will be investigated for the ferromagnet in Secs. III–V. The results of Secs. III and IV for the longitudinal susceptibility will verify the existence of a repulsive interaction for $\sigma<0$ and will give results for $\sigma=0$.

In discussing the zero-magnon question it will occasionally be convenient to formally evaluate the susceptibility for the choice of parameters which maximizes the possibility of the mode occurring. These results will then be used to demonstrate the nonexistence of the mode for these parameters and will therefore demonstrate its nonexistence more generally. These parameters will be shown to be obtained by choosing $|\sigma|$ non-negligible ($\sigma<0$), and taking

$$\omega_0=[h-(-\sigma)6JS]\ll k_BT \tag{2.6}$$

(the magnetization is taken to remain along $\hat{z}$ and $\omega_0 \gtrsim 0$ throughout).[14] This choice of parameters will not be made in obtaining the susceptibilities, and its use in the zero-magnon problem will be specifically discussed.

III. LONGITUDINAL SUSCEPTIBILITY OF THE LOW-T FERROMAGNET IN RPA

In the presence of a locally varying field perturbation, the spin (spin-wave) densities of this system move to screen out that field. This self-consistent response in spin density will be described in the collisionless regime by the dynamic longitudinal spin susceptibility in the random phase approximation. The validity of the RPA in this low-density system has not been rigorously established. Its use in the isotropic case is based on the physical idea that the interaction of some spin wave with thermal spin waves is small at low T and that it is correctly represented by the Dyson-Maleev interaction in the first Born approximation. This approach to spin-wave scattering in the isotropic ferromagnet has been shown to give a physically correct representation of that interaction in the calculation of both the free energy[9] and the spin-wave damping.[15] The discussion of Sec. II shows that for anisotropic exchange, the spin-wave interactions are enhanced. The use of the RPA in the anisotropic case (σ will be taken to be negative hereafter) also requires that the spin-wave interactions be small; hence σ will be taken to be small in the following. In fact it will be shown that the effect of the exchange anisotropy dominates the spin-wave interactions in the zero-magnon problem for all nonzero σ. If the anisotropy and hence the spin-wave interactions are fairly large, a more complete description of the multiple scattering processes between pairs of spin waves will probably become necessary. The effects of bound states upon the spin-wave scattering states in the RPA will be ignored because they have been shown to have little effect on the spectral densities of the spin waves.[10]

The RPA is a good description only in the collisionless regime, i.e., for frequencies ω such that $\omega \gg \bar{\Gamma}$, where $\bar{\Gamma}$ is some upper limit on the spin-wave relaxation frequencies of importance (see, e.g., Kadanoff and Baym[16]). In the hydrodynamic regime a more complete theory is necessary in order to preserve conservation properties.[17]

The susceptibility is given in terms of the longitudinal thermodynamic spin Green's function:

$$\langle T S_K^z(\tau) S_{-K}^z(0)\rangle . \tag{3.1}$$

The usual thermodynamic perturbation formalism will be used here (see, e.g., Abrikosov *et al.*[18]). The dynamic part of (3.1) is given by

$$\frac{1}{N^2}\sum_{q,q'} G(K,\ \tau;\ q,\ q') = \frac{1}{N}\sum_{q,q'}\langle T a_{K+q}^\dagger(\tau)\, a_q(\tau)\, a_{q'-K}^\dagger(0)\, a_{q'}(0)\rangle , \tag{3.2}$$

which gives this spin correlation function by the spin-wave-density correlation function.

The RPA for G can be obtained diagrammatically as in Fig. 1. G is given in terms of $G^{(0)}$, the unperturbed two-particle spin-wave Green's function. The spin-wave Green's function is $g(q,\ \tau) = \langle T a_q^\dagger(\tau)\, a_q(0)\rangle$, and has the unperturbed form

$$g^{(0)}(q,\ i\omega_n) = (i\omega_n - \omega_q)^{-1} . \tag{3.3}$$

The frequency components of boson Green's functions are given by the form

$$g(q,\ \tau) = \frac{1}{\beta}\sum_n g(q,\ i\omega_n) e^{-i\omega_n \tau}, \tag{3.4}$$

where $\omega_n = (2\pi n/\beta)$, n any integer. $G^{(0)}$ becomes

$$G^{(0)}(K,\ i\omega_n;\ q) = \left(\frac{f(\omega_q) - f(\omega_{q+K})}{i\omega_n - [\omega_{q+K} - \omega_q]}\right), \tag{3.5}$$

where $f(x) = (e^{\beta x} - 1)^{-1}$. Because V^{DM} is frequency independent, G can be written in terms of one frequency; the RPA Dyson equation for G is then

$$\frac{1}{N}\sum_{q'} G(K;\ i\omega_n;\ q,\ q') = G^{(0)}(K,\ i\omega_n;\ q) + G^{(0)}(K,\ i\omega_n;\ q)\frac{1}{N}\sum_p V^{DM}(K,\ p,\ q) \times \frac{1}{N}\sum_{q'} G(K,\ i\omega_n;\ p,\ q') . \tag{3.6}$$

The solution of this integral equation in momentum is somewhat complicated because of the momentum dependences of V^{DM}. It can, however, be separated, and it has been solved for a number of

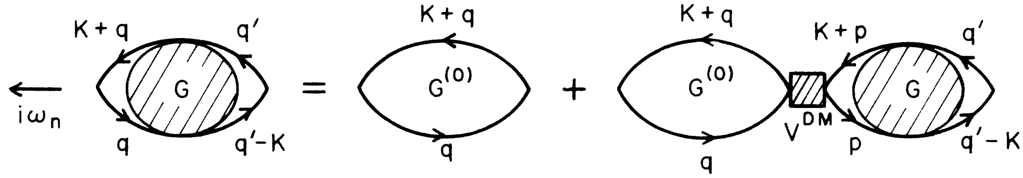

FIG. 1. Dyson equation for RPA longitudinal Green's function.

interesting $\vec{K}$. The discussion of the zero-magnon mode will be by several illustrative calculations, and the longitudinal susceptibility will now be obtained for these cases. The simplest cases will be given first. For definiteness, the susceptibility (obtained by $i\omega_n \to \omega + i\epsilon$, ϵ small) will be discussed for $\omega > 0$.[19]

A. G when σ dominates V^{DM}

The approximation $V^{DM} \cong \frac{1}{4}(-\sigma)[J(0)+J(K)]$ is valid when $|\sigma|$ is sufficiently large that the first term in (2.5) dominates the spin-wave interaction. Its region of validity in the zero-magnon problem will be examined in Sec. IV. Equation (3.6) then gives

$$\frac{1}{N^2}\sum_{q,q'} G(K,\omega;q,q') = \frac{(1/N)\sum_q\{[f(\omega_q)-f(\omega_{q+K})]/[\omega+i\epsilon-(\omega_{q+K}-\omega_q)]\}}{1-\frac{1}{4}(-\sigma)[J(0)+J(K)](1/N)\sum_q\{[f(\omega_q)-f(\omega_{q+K})]/[\omega+i\epsilon-(\omega_{q+K}-\omega_q)]\}} . \tag{3.7}$$

The denominator of (3.7) is suggestive of a similar one for the Fermi liquid in the RPA. It will be shown in Sec. IV that the momentum sum in the denominator has a positive real part; therefore, the suggestion in Sec. II that a negative σ is most likely to give a resonance in the self-consistent response of the spin-density fluctuations is borne out.

B. G for $\sigma = 0$

The dependence of $V^{DM}(K,q,q')$ on q and q' must be included for $\sigma=0$. Factorizing V^{DM}, (3.6) becomes

$$\frac{1}{N}\sum_{\vec{q}'} G(\vec{K},\omega;\vec{q},\vec{q}') = G^{(0)}(\vec{K},\omega;\vec{q}) + G^{(0)}(\vec{K},\omega;\vec{q})\,\tfrac{1}{4}J\sum_{\vec{\delta}}(e^{i\vec{K}\cdot\vec{\delta}}-e^{-i\vec{q}\cdot\vec{\delta}}) \times \frac{1}{N}\sum_{\vec{p}}(e^{i\vec{p}\cdot\vec{\delta}}-1)\frac{1}{N}\sum_{\vec{q}'}G(\vec{K},\omega;\vec{p},\vec{q}') . \tag{3.8}$$

By iterating (3.6), G is seen to consist of a series of terms involving $G^{(0)}$'s connected by V^{DM}'s:

$$\frac{1}{N}\sum_{q'} G(K,i\omega_n;q,q') = G^{(0)}(K,i\omega_n;q) + G^{(0)}(K,i\omega_n;q)\frac{1}{N}\sum_p V^{DM}(K,p,q) \times G^{(0)}(K,i\omega_n;p)+\cdots . \tag{3.9}$$

It will be shown later that for $\omega>0$, $G^{(0)}(K,\omega;q)$ is non-negligible only for $qa \lesssim (k_BT/JS)^{1/2}$ (see Sec. IV). Thus in order to obtain the low-T contributions to the series (3.9), all of the $V^{DM}(K,p,q)$ must be expanded for both small p and q.

Expanding the interaction term in (3.8) up to terms quadratic in p and q and their products and deleting terms which vanish for a lattice of inversion symmetry, (3.8) becomes

$$\frac{1}{N}\sum_{\vec{q}'} G(\vec{K},\omega;\vec{q},\vec{q}') = G^{(0)}(\vec{K},\omega;\vec{q}) + G^{(0)}(\vec{K},\omega;\vec{q})\frac{J}{4N}\sum_{\vec{p}\,\vec{\delta}}\{[\cos(\vec{K}\cdot\vec{\delta})-1+\tfrac{1}{2}(\vec{q}\cdot\vec{\delta})^2+\cdots][-\tfrac{1}{2}(\vec{p}\cdot\vec{\delta})^2+\cdots] - [\sin(\vec{K}\cdot\vec{\delta})+(\vec{q}\cdot\vec{\delta})+\cdots][(\vec{p}\cdot\vec{\delta})+\cdots]\}\,\frac{1}{N}\sum_{\vec{q}'}G(\vec{K},\omega;\vec{p},\vec{q}') . \tag{3.10}$$

Equation (3.10) can be solved especially easily for two interesting $\vec{K}$ values near the Brillouin-zone boundary by exploiting the q dependence of the approximations that will be used for $G^{(0)}$ in these cases.

For $\vec{K}=(k,k,k)$ and $\sin(ka)\ll 1$ (ka near π), the approximation for $G^{(0)}(K,\omega;q)$ depends on q through q^2 only [see (4.7)]. Then (3.10) yields

$$\frac{1}{N^2}\sum_{q,q'} G(K,\omega;q,q') = \frac{(1/N)\sum_q\{[f(\omega_q)-f(\omega_{q+K})]/[\omega+i\epsilon-(\omega_{q+K}-\omega_q)]\}}{1-\frac{1}{4}J[1-\cos(ka)](1/N)\sum_q(qa)^2\{[f(\omega_q)-f(\omega_{q+K})]/[\omega+i\epsilon-(\omega_{q+K}-\omega_q)]\}} . \tag{3.11}$$

For $\vec{K}=(k,0,0)$ and $\sin(ka)\ll 1$ (ka near π), the approximation for $G^{(0)}(K,\omega;q)$ depends on q through only q_x^2 [see (4.13)]. Then

$$\frac{1}{N^2}\sum_{q,q'} G(K,\ \omega;\ q,\ q') = \frac{(1/N)\sum_q \{[f(\omega_q) - f(\omega_{q+K})]/[\omega + i\epsilon - (\omega_{q+K} - \omega_q)]\}}{1 - \frac{1}{4}J[1-\cos(ka)](1/N)\sum_q (q_x a)^2 \{[f(\omega_q) - f(\omega_{q+K})]/[\omega + i\epsilon - (\omega_{q+K} - \omega_q)]\}}\ . \tag{3.12}$$

The q^2 and q_x^2 present in the sums in these two denominators arise from $V^{\rm DM}$ for $\sigma = 0$ and represent the weakness of the long-wavelength spin-wave interactions.

When lowest-order terms in both σ and q^2, p^2 are retained in $V^{\rm DM}(K,\ p,\ q)$, the denominator of G for large $\vec{\rm K} = (k,\ k,\ k)$, $\sin(ka) \ll 1$, becomes

$$1 - \tfrac{1}{4}(-\sigma)[J(0) + J(K)]\frac{1}{N}\sum_q\left(\frac{f(\omega_q) - f(\omega_{q+K})}{\omega + i\epsilon - [\omega_{q+K} - \omega_q]}\right) + \tfrac{1}{4}J[\cos(ka) - 1]\frac{1}{N}\sum_q (qa)^2\left(\frac{f(\omega_q) - f(\omega_{q+K})}{\omega + i\epsilon - [\omega_{q+K} - \omega_q]}\right) \tag{3.13}$$

and a similar result for $\vec{\rm K} = (k,\ 0,\ 0)$.

The results obtained here for $\sigma = 0$ are not in agreement with those of Natoli and Ranninger,[5] of Liu,[7] and of Reiter[2] in the collisionless regime because those authors have not properly treated the average field due to the spin-wave interactions at low T. These results for $\sigma = 0$, however, are in essential agreement with those of Harris.[8]

Equation (3.6) can be solved for other $\vec{\rm K}$, but for $\vec{\rm K}$ away from the zone boundary, the resulting susceptibility has a more complicated denominator and is less straightforward to discuss. These results will not be written down here.

IV. ANALYSIS OF LOW-T FERROMAGNETIC SUSCEPTIBILITY

The momentum sums in the denominators of the susceptibilities will now be carried out for several $\vec{\rm K}$ of interest. In particular, the possibility of a zero-magnon mode will be examined.

A. $Ka \ll (k_B T/JS)^{1/2}$

The case of small K is most similar to zero sound in a Fermi liquid (see, e.g., Fetter and Walecka[20]). Consider first systems in which the σ term dominates the spin-wave interactions. Expanding the susceptibility in (3.7) for small K gives for its denominator, $D(K,\ \omega)$:

$$D(K,\ \omega) = 1 - \tfrac{1}{2}J(0)(-\sigma) \times \frac{1}{N}\sum_q\left[\left(-\frac{\partial f(\omega_q)}{\partial\omega_q}\right)\left(\frac{\partial\omega_{\vec{q}}}{\partial\vec{q}}\cdot\vec{\rm K}\right)\Big/\left(\omega + i\epsilon - \frac{\partial\omega_{\vec{q}}}{\partial\vec{q}}\cdot\vec{\rm K}\right)\right]. \tag{4.1}$$

Non-negligible contributions to the momentum sum in D come from $qa \lesssim (k_B T/JS)^{1/2}$, for which the spin-wave energy can be expanded

$$\omega_q \cong \omega_0 + JSa^2q^2, \quad \omega_0 = h + 6JS\sigma\ . \tag{4.2}$$

In the several examples of this section, the small-q values of $f(\omega_q)$ in the momentum sums will play the most important role, and the following approximations, which give correctly both the small- and large-q behaviors of $f(\omega_q)$, will be used:

$$f(\omega_q) \to \frac{e^{-\beta\omega_q}}{\beta\omega_q} \cong \frac{e^{-\beta JSa^2q^2}}{\beta JSa^2q^2} \quad \text{for } \omega_0 = 0\,, \tag{4.3a}$$

$$f(\omega_q) \to \frac{e^{-\beta(\omega_0 + JSa^2q^2)}}{(1 - e^{-\beta\omega_0}) + \beta JSa^2q^2} \quad \text{for } \omega_0 \neq 0\,. \tag{4.3b}$$

Using (4.3b) and performing the angular integral and other manipulations, D becomes

$$D(K,\ \omega) = 1 - \left(\frac{J(0)(-\sigma)}{8\pi^2 JS}\right)\left(\frac{e^{-\beta\omega_0}}{(\beta JS)^{1/2}}\right) \times \int_0^\infty \frac{t^2 e^{-t^2}\,dt}{(B - t^2)(A + t^2)}\,, \tag{4.4}$$

where

$$A = (1 - e^{-\beta\omega_0})\,, \quad B = \frac{\beta}{4JSa^2}\left(\frac{\omega + i\epsilon}{K}\right)^2.$$

Performing the integrals in (4.4) (taking care with the signs of the imaginary part of B and the real part of A), gives for D

$$D(K,\ \omega) = 1 - \left(\frac{J(0)(-\sigma)}{8\pi^2 JS}\right)\left(\frac{e^{-\beta\omega_0}}{(\beta JS)^{1/2}}\right) \times\left\{\frac{1}{B + A}\right\}\{-e^{-B}[\tfrac{1}{2}\pi i\sqrt{B}\,][1 + {\rm erf}(i\sqrt{B}\,)] - (\tfrac{1}{2}\pi e^{A}\sqrt{A}\,)[1 - {\rm erf}(\sqrt{A}\,)]\}\,, \tag{4.5}$$

where

$${\rm erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt\,.$$

Because of the small factor $(k_B T/JS)^{1/2}e^{-\beta\omega_0}$ preceding the integral in (4.4), a resonance of the zero-magnon kind can occur only if the integral becomes large for some B. The best chance for this to occur would be for small $A(\omega_0 \ll k_B T)$. in which case the integral appears to diverge for small B. Taking A to be negligible in (4.5), however, gives that the maximum value of the real part of the last two curly brackets is a constant ($\sqrt{\pi}$) and that for small B their imaginary part diverges as $B^{-1/2}$. Therefore, even for this optimum choice of values of A and B, this denominator does not give a resonance, and for small B the imaginary part of the denominator is dominant.

The physics of this case can be understood by comparing it with zero sound in a Fermi liquid. These results for the real and imaginary parts of D can also be obtained by replacing the denominator in the q sum of D in (4.4) by a principal part and a δ function and proceeding as above. The T dependence in front of the real integral [c.f. (4.5)] represents the fact that any resulting mode would be proportional to the density of particles in the system [c.f. Fetter and Walecka, Ref. 19, Eq. (16.10)], which for these boson spin waves is small in $(k_B T/JS)$. The small ω which is therefore necessary to make the real part of the integral large means that the δ function in the imaginary part of D will be satisfied throughout a large part of the range of q in the sum. Thus, the Bose character of the spin waves dominates the behavior of the longitudinal susceptibility for small K.

A similar analysis of a solution of (3.6) for $\sigma = 0$ and small K shows that a zero-magnon mode does not exist in that case either.

B. Large $\vec{K}$

i. For $\vec{K} = (k, k, k)$ *and* $\sin(ka) \ll 1$ *(ka near* π*).* First, the $\sigma = 0$ form of D will be examined. D is given by the denominator of (3.11). When the origin of summation of the term involving $f(\omega_{q+K})$ in the q sum in D is shifted by $-K$, its denominator contains $(\omega + \omega_{K+q})$. For $\omega > 0$, this term gives a contribution small in $(k_B T/JS)$, and it will be discarded. For a simple-cubic lattice with nearest-neighbor coupling

$$\omega_{\vec{q}+\vec{K}} = \omega_K + 2JS \sum_{i+} [1 - \cos(q_i a)] \cos(K_i a) + 2JS \sum_{i+} [\sin(q_i a)] \sin(K_i a) \qquad (4.6)$$

for all $\vec{q}$ and $\vec{K}$. Using (4.3a), expanding (4.6) for small q, replacing the sum by an integral and taking its upper limit to infinity, D becomes

$$D(K, \omega) = 1 - \left(\frac{J(0) - J(K)}{2\omega_K}\right)\left(\frac{k_B T}{JS}\right)^{3/2} \times \int_0^\infty \frac{t^2 e^{-t^2}\, dt}{[(\omega + i\epsilon - \omega_K)/\omega_K](6\beta JS) + t^2} . \qquad (4.7)$$

For $\omega > \omega_K$, the imaginary part of this integral is zero, and the maximum value of the real part as a function of $(\omega - \omega_K)$ is $(\frac{1}{2}\sqrt{\pi})$. Because of the $(k_B T/SJ)^{3/2}$ preceding the integral, there can be no resonance. Harris[8] has reached a similar conclusion for this case. For $\omega < \omega_K$ both real and imaginary parts of the integral are nonzero, but both are bounded by constants of order unity. It is the presence of q^2 in the momentum sum of (3.11) representing the weakness of the spin-wave interactions for $\sigma = 0$ that precludes a zero-magnon mode in this case. Thus the spin-wave interactions in the isotropic low-T ferromagnet are not strong enough to support this mechanical mode.

The above analysis shows that the last term in (3.13) is always small in $(k_B T/JS)$ and cannot contribute to a zero-magnon mode. Therefore, for nonzero σ it remains only to consider the first two terms in (3.13), which is equivalent to the approximation (3.7), and gives

$$D(K, \omega) = 1 - \tfrac{1}{4}(-\sigma)[J(0) + J(K)]\frac{1}{N} \times \sum_q \left(\frac{f(\omega_q) - f(\omega_{q+K})}{\omega + i\epsilon - [\omega_{q+K} - \omega_q]}\right) . \qquad (4.8)$$

Once again, for $\omega > 0$, only the first term in the sum is non-negligible. ω_K will be written $\omega_K = \omega_0 + W_K$. The integral is treated in much the same manner as in the previous cases and becomes

$$D(K, \omega) = 1 - \left(\frac{(-\sigma)3[J(0) + J(K)]}{(2\pi)^2 W_K}\right)\left(\frac{e^{-\beta\omega_0}}{(\beta JS)^{1/2}}\right) \times \int_0^\infty \frac{t^2 e^{-t^2}\, dt}{(A + t^2)(C + t^2)} , \qquad (4.9)$$

where

$$A = 1 - e^{-\beta\omega_0}, \quad C = (6\beta JS)\left(\frac{\omega - W_K + i\epsilon}{W_K}\right).$$

Integrating, this becomes

$$D(K, \omega) = 1 - \left(\frac{3\pi(-\sigma)[J(0) + J(K)]}{2(2\pi)^2 W_K}\right) \times \left(\frac{e^{-\beta\omega_0}}{(\beta JS)^{1/2}}\right)\left(\frac{1}{C - A}\right) \times \{(\sqrt{C})e^{C}[1 - \mathrm{erf}(\sqrt{C})] - \sqrt{A}\, e^{A}[1 - \mathrm{erf}(\sqrt{A})]\} . \qquad (4.10)$$

The presence of the small factor $(k_B T/SJ)^{1/2} e^{-\beta\omega_0}$ in the second term of D in (4.9) requires that the integral become large for there to be a resonance, and the best chance for this to occur is for $A(\omega_0)$ and C to be small. Taking $\omega > W_K$, the imaginary part of D is zero. Taking A to be negligible ($\omega_0 \ll k_B T$) and expanding (4.10) for small C gives

$$D(K, \omega) = 1 - \left(\frac{3(-\sigma)[J(0) + J(K)]}{8\pi W_K}\right) \times \left(\frac{k_B T}{JS}\right)^{1/2}\left(\frac{1}{\sqrt{C}}\right). \qquad (4.11)$$

The frequency of the resonance in the susceptibility is given by equating $D(K, \omega) = \mathrm{Re}D(K, \omega) = 0$, which gives

$$\left(\frac{\omega - W_K}{W_K}\right) = \left(\frac{3}{128\pi^2}\right)\left(\frac{J(0) + J(K)}{W_K}\right)^2\left(\frac{k_B T}{JS}\right)^2 \sigma^2 . \qquad (4.12)$$

Thus the random-phase approximation for the longitudinal susceptibility gives this undamped mode above the W_K band at the zone boundary.

Since the last term in (3.13) is always negligible, (4.12) gives the resonance in the RPA susceptibility for all nonzero σ. That is, for the zero-magnon question, $V^{\rm DM}$ is dominated by the anisotropy term for all nonzero values of σ.

In Sec. V the effect of including spin-wave damping in the intermediate states of the RPA will be obtained. These effects tend to suppress the zero-magnon mode. In order to complete the discussion of the behavior of the RPA longitudinal susceptibility, however, the results of calculations performed for other values of $\vec{K}$ will be summarized here.

ii. Other large $\vec{K}$. The discussion of the momentum sums above shows how important the $\vec{q}$ behavior of the summand is in producing a divergence of the sum and thus a resonance. Equation (4.6) shows that for various geometries and magnitudes of $\vec{K}$, the denominator of the summand has different $\vec{q}$ behaviors. For example, for $\vec{K}$ neither small nor on the zone boundary, there are terms in q_i as well as q_i^2 in the denominator. Also for $\sigma = 0$ the numerators of the summands acquire factors of q_i. An analysis of the often complicated sums for various $\vec{K}$ has been carried out, and in all cases for $\sigma = 0$ there are no divergences for any ω. For $\sigma \neq 0$ the modes from the resonances of the susceptibility have very nearly the same T and σ dependences as in (4.12); all of these modes in three dimensions are suppressed in the same way as that to be shown in Sec. V.

As a simple and useful example of the effect of a different geometry of $\vec{K}$, consider another $\vec{K}$ near the zone boundary: $\vec{K} = (k, 0, 0)$, $\sin(ka) \ll 1$. Equations (3.7) and (3.12) give the forms of D, and from (4.6) the denominators of the summands now involve only q_x^2. Using the same approximations used before, the q_y and q_z integrals give to the q sums a factor of

$$\int_0^\infty \int_0^\infty \frac{e^{-\beta JSa^2(q_y^2+q_z^2)}\, dq_y\, dq_z}{\beta JSa^2(q_x^2+q_y^2+q_z^2)} = \frac{2\pi e^{\beta JSa^2 q_x^2}}{\beta JSa^2}\, \mathrm{Ei}(\beta JSa^2 q_x^2)\,, \qquad (4.13)$$

where

$$\mathrm{Ei}(z) = \int_z^\infty \frac{e^{-t}\, dt}{t}\,.$$

The weak divergence of the Ei function for small q_x has no appreciable effect on the q_x integrals, and the remaining integrals in q_x have the same form as those in q^2 for $\vec{K} = (k, k, k)$ [see (4.7) and (4.9)]. The momentum sum for $\sigma = 0$ has no divergence for any ω, and for $\sigma \neq 0$ there is a mode which is the same as that given in (4.12) except for minor numerical differences; it should be noted that the coefficient $[J(0) + J(K)]$ in (4.12) which is small near the zone boundary in (1, 1, 1) is not small in (1, 0, 0).

V. SPIN-WAVE DAMPING IN INTERMEDIATE STATES OF LOW-T FERROMAGNETIC SUSCEPTIBILITY

Although the mode at the zone boundary in (4.12) is a high-frequency mode, the effect involved, i.e., the "gap" $(\omega - W_K)$ between it and the band W_K, is small, and therefore it may be important to include the "thermal blurring" effects of the spin-wave damping in the intermediate states of the RPA. These effects are seen to be important by noting that the damping of a large-wave-vector spin wave in the first Born approximation in a system with non-negligible σ is proportional to $(k_B T/JS)^{3/2}$, whereas the gap $(\omega - W_K)$ is proportional to $(k_B T/JS)^2$.

Expressions for the temperature and anisotropy dependence of the low-temperature spin-wave damping Γ_k will be required. Γ_k will be calculated by using the first Born approximation to the spin-wave scattering involved. Tahir-Kheli and ter Haar[21] first obtained this approximation for the damping in the isotopic Heisenberg ferromagnet. Harris[15] used a full T-matrix calculation to show that in the isotropic system essentially the same result (at least for $\beta\omega_k \ll 1$) is obtained by summing all orders in the Born series. Therefore it will be assumed that the first Born approximation is an adequate description of Γ_k in systems with the small values of σ being discussed here. This gives

$$\begin{aligned}\Gamma_k = \frac{\pi}{2N^2} \sum_{q,q'} [J(q) + J(k) - (1+\sigma) J(q-q') \\ - (1+\sigma) J(k-q')]^2 \{[1 + f(\omega_{q'}) + f(\omega_{k+q-q'})] \\ \times f(\omega_q) - f(\omega_{q'}) f(\omega_{k+q-q'})\} \\ \times \delta(\omega_k - \omega_{q'} - \omega_{k+q-q'} + \omega_q)\,. \qquad (5.1)\end{aligned}$$

The spin-wave damping is included in the RPA susceptibility by renormalizing the spin-wave propagators in $G^{(0)}$, which gives (see e.g., Baym and Sessler[21])

$$G^{(0)}(K, i\omega_n; q) = \frac{1}{\pi^2\beta} \sum_m \int_{-\infty}^\infty \int_{-\infty}^\infty \rho(q+K, \omega_1)\rho(q, \omega_2) \times \frac{d\omega_1\, d\omega_2}{(i\lambda_m + i\omega_n - \omega_1)(i\lambda_m - \omega_2)}\,, \qquad (5.2)$$

where ρ is the spectral density for the renormalized spin-wave Green's function:

$$\rho(q, \omega) = -\lim_{\epsilon\to 0} \mathrm{Im}\, g(q, \omega + i\epsilon)\,.$$

Performing the frequency sum,

$$\frac{1}{N}\sum_q G^{(0)}(K, i\omega_n; q) = \frac{1}{\pi^2 N} \sum_q \int_{-\infty}^\infty \int_{-\infty}^\infty \rho(q+K, \omega_1)\rho(q, \omega_2) \times \left(\frac{f(\omega_2) - f(\omega_1)}{i\omega_n - \omega_1 + \omega_2}\right) d\omega_1\, d\omega_2\,. \qquad (5.3)$$

At low T each $\rho(K, \omega)$ is a sharply peaked function of ω about the renormalized spin-wave energy with a width given by the spin-wave damping. The renormalizations of the real spin-wave energy will be dropped, and the combined widths, Γ_q and Γ_{K+q}, will be approximated by Γ_K for large K and small q. Then

$$\frac{1}{N}\sum_q G^{(0)}(K,\ \omega;\ q) = \frac{1}{N}\sum_q\left(\frac{f(\omega_q)-f(\omega_{q+K})}{\omega-[\omega_{q+K}-\omega_q]+i\Gamma_K}\right). \quad (5.4)$$

Using this form of $G^{(0)}$ in the analysis of Sec. IV B i for D with nonzero σ and for large $\vec{K}=(k,k,k)$, $\sin(ka)\ll 1$, and taking A to be negligible, the analog of (4.9) becomes

$$D(K,\ \omega)=1-\left(\frac{(-\sigma)3[J(0)+J(K)]}{(2\pi)^2 W_K}\right)\left(\frac{k_BT}{JS}\right)^{1/2} \times\int_0^\infty \frac{e^{-t^2}\,dt}{[(6\beta JS)(\omega-W_K+i\Gamma_K)/W_K]+t^2}. \quad (5.5)$$

This gives

$$D(K,\ \omega)=1-\left(\frac{3(-\sigma)[J(0)+J(K)]}{(2\pi)^2 W_K}\right)\left(\frac{k_BT}{JS}\right)^{1/2} \times\left[e^{\zeta^2}\left(\frac{\pi}{2\zeta}\right)[1-\mathrm{erf}(\zeta)]\right], \quad (5.6)$$

where the magnitude of ζ is

$$|\zeta|=\{(6\beta JS/W_K)^2[(\omega-W_K)^2+\Gamma_K^2]\}^{1/4},$$

and its argument is $\frac{1}{2}\theta$, where

$$\theta=\arctan\left(\frac{\Gamma_K}{\omega-W_K}\right).$$

Once again the maximum value of the real part of the integral is obtained by taking ζ small. Expanding (5.6) for small ζ gives

$$\mathrm{Re}D(K,\ \omega)=1-\left(\frac{3(-\sigma)[J(0)+J(K)]}{(2\pi)^2 W_K}\right)\left(\frac{k_BT}{JS}\right)^{1/2} \times\frac{(\frac{1}{2}\pi)\cos\frac{1}{2}\theta}{\{(6\beta JS/W_K)^2[(\omega-W_K)^2+\Gamma_K^2]\}^{1/4}}. \quad (5.7)$$

Γ_K from (5.1) for large K and $\sigma\gtrsim(k_BT/JS)^{1/2}$ (taking $\omega_0\ll k_BT$) is

$$\Gamma_K\propto\sigma^2\frac{J}{S}\left(\frac{k_BT}{JS}\right)^{3/2}, \quad (5.8)$$

with a constant of proportionality of about the same order as for $(\omega-W_K)/W_K$ in (4.12). Using this expression for Γ_K and considering the powers of all the parameters σ, S^{-1}, and (k_BT/JS), the maximum value of the second term on the right-hand side of (5.7) [obtained for $(\omega-W_K)=0$] is proportional to $(k_BT/JS)^{1/4}$. For $\sigma\lesssim(k_BT/JS)^{1/2}$ different expressions for the spin-wave damping must be used, and in these cases the second term on the right-hand side of (5.7) is even smaller.

Strictly interpreted this result shows that the mode given by (4.12) is not well defined owing to the effects of spin-wave damping. Clearly, however, it should not be claimed that the maximum size found here for the second term of ReD in (5.7) is precise. This calculation shows the important role played by the spin-wave damping is suppressing the zero-magnon mode in this system; furthermore, because the assumption of a negligible ω_0 was used to maximize the possibility of the mode appearing in (4.12), this calculation makes it appear very unlikely that such a mode could be well defined in a realistic system.

In a final example it will be pointed out that, like anisotropy, dimensionality plays a qualitatively different role in determining the "gap" for a possible zero-magnon mode than it does in the spin-wave damping. Consider a two-dimensional ferromagnetic system. In this case the system orders only for some (small) over-all anisotropy field. Treating the momentum sums as usual, the denominator D for large $\vec{K}=(k,\ k)$, $\sin(ka)\ll 1$, and σ nonzero [cf. (4.9)] becomes

$$D(K,\ \omega)=1-\left(\frac{2(-\sigma)[J(0)+J(K)]e^{-\beta\omega_0}}{2\pi W_K}\right) \times\int_0^\infty\frac{te^{-t^2}\,dt}{(A+t^2)(C+t^2)}, \quad (5.9)$$

where

$$A=1-e^{-\beta\omega_0},\quad C=4\beta JS\,\frac{\omega-W_K}{W_K}.$$

Performing the integral, taking A to be negligible, and expanding for small C gives the following unrenormalized zero-magnon mode at the zone boundary in two dimensions:

$$\left(\frac{\omega-W_K}{W_K}\right)\propto\sigma\left(\frac{k_BT}{JS}\right)\frac{1}{W_K}. \quad (5.10)$$

The size of Γ_K for large K and $\sigma\gtrsim(k_BT/JS)^{1/2}$ in two dimensions (taking $\omega_0\ll k_BT$), however, is

$$\Gamma_K\propto\sigma\frac{\sigma}{S}\left(\frac{k_BT}{JS}\right). \quad (5.11)$$

Thus Γ_K is smaller than the gap $(\omega-W_K)$ by a factor (σ/S), which could be quite small. If D in (5.9) were renormalized with Γ_K as in (5.5), a sharp resonance for the zero magnon would occur when (σ/S) is sufficiently small.

Because of the assumptions used in obtaining this mode in two dimensions, this result can only be used to suggest that further investigation of the two-dimensional case may be worthwhile. In particular, it must be determined whether in a realistic system it is possible to have non-negligible σ, the magnetization aligned along $\hat{z}$, and also $\omega_0 \ll k_B T$.

V. QUESTION OF THE ZERO MAGNON IN THE LOW-T ANTIFERROMAGNET

The zero-magnon question in the antiferromagnet will be treated in a manner similar to that used for the ferromagnet. The role of the weak spin-wave interactions at low T in the isotropic antiferromagnet is considerably more subtle than in the ferromagnet,[22] and that case will not be treated in detail. It will, however, be shown that the mode is not expected to exist in three-dimensional systems with anisotropy, and an argument will be given that suggests that it does not exist in isotropic systems either.

The formalism developed by Harris *et al.*[22] for the low-T antiferromagnet will be employed here; their expressions and notation will be taken over directly and referred to as HKHH. Because the calculations for the antiferromagnetic longitudinal susceptibility are rather lengthy only an outline of them will be given. The required results will then be given and discussed.

In the two-sublattice antiferromagnet, the longitudinal spin-spin Green's function [cf. (3.1)] becomes a 2×2 matrix because of the two spin sublattices. Each component of this matrix is expressed in terms of 16 of the total of 64 two-particle Green's functions involving the α_k and β_k spin-wave operators with coefficients given by the Bogoliubov parameters m_k, l_k (which relate spin operators to spin-wave operators).

The random-phase approximation will be used for systems with zero or small anisotropy. In the RPA each of the α, β two-particle Green's functions is given in terms of four of these Green's functions in an integral equation in momentum. A typical example is

$$\frac{1}{N}\sum_{q'}\langle T\beta_{-(K+q)}(\tau)\alpha_q(\tau)\alpha^\dagger_{q'-K}(0)\alpha_{q'}(0)\rangle_{i\omega_n} = l_q l_{q+K}\left[(k_B T)\sum_{\lambda_m} G^{(0)}_{\beta\beta}(K+q,\ -i\omega_n - i\lambda_m)G^{(0)}_{\alpha\alpha}(q,\ -i\lambda_m)\right]$$
$$\times\left(\frac{-H_E}{4SN^2}\right)\sum_{p,q'} l_p l_{p+K}\{\langle T\alpha^\dagger_{K+p}(\tau)\,\alpha_p(\tau)\alpha^\dagger_{q'-K}(0)\alpha_{q'}(0)\rangle_{i\omega_n}\Psi^{(3)}(q,\ p,\ K)$$
$$+\langle T\beta_{-(K+p)}(\tau)\alpha_p(\tau)\alpha^\dagger_{q'-K}(0)\alpha_{q'}(0)\rangle_{i\omega_n}\Psi^{(4)}(q,\ p,\ K)+\langle T\beta_{-(K+p)}(\tau)\beta^\dagger_{-p}(\tau)\alpha^\dagger_{q'-K}(0)\alpha_{q'}(0)\rangle_{i\omega_n}\Psi^{(6)}(q,\ p,\ K)$$
$$+\langle T\alpha^\dagger_{p+K}(\tau)\beta^\dagger_{-p}(\tau)\alpha^\dagger_{q'-K}(0)\alpha_{q'}(0)\rangle_{i\omega_n}\Psi^{(7)}(q,\ p,\ K)\}\ , \tag{6.1a}$$

where

$$\begin{aligned}
\Psi^{(3)}(q,\ p,\ K) &= 2[\Phi^{(3)}_{q,K+p,p,K+q}+\Phi^{(3)}_{K+p,q;p,K+q}]\ ,\\
\Psi^{(4)}(q,\ p,\ K) &= 4[\Phi^{(4)}_{q,K+p,p,K+q}]\ ,\\
\Psi^{(6)}(q,\ p,\ K) &= 2[\Phi^{(6)}_{q,K+p,p,K+q}+\Phi^{(6)}_{q,K+p,K+q,p}]\ ,\\
\Psi^{(7)}(q,\ p,\ K) &= 4[\Phi^{(7)}_{q,K+p,K+q,p}+\Phi^{(7)}_{K+p,q,K+q,p}+\Phi^{(7)}_{q,K+p,p,K+q}+\Phi^{(7)}_{K+p,q,p,K+q}]\ .
\end{aligned} \tag{6.1b}$$

The $\Phi^{(i)}$ come from the vertices of the HKHH spin-wave Hamiltonian [HKHH (2.17b)]. The $i\omega_n$ indicates the frequency component, and $G^{(0)}_{\mu\mu}$ are the unperturbed one-particle spin-wave Green's functions [HKHH (2.23)]. The two-particle Green's functions are given by 16 sets of four coupled integral equations. They are difficult to solve in general because of the dependence of $\Phi^{(i)}$ on p and q.

A. Denominator of the longitudinal susceptibility for anisotropic systems

To treat the antiferromagnet, we begin by considering systems with anisotropy, which is given in HKHH by H_A.[23] (Unlike HKHH, we retain the sublattice field h, which enters only the spin-wave energy E_q.) For $\omega > 0$, unperturbed two-particle Green's functions,

$$\sum_{\lambda_m}[G^{(0)}_{\mu\mu}(K+q,\ -i\lambda_m - i\omega_n)G^{(0)}_{\nu\nu}(q,\ -i\lambda_m)]\ ,$$

have non-negligible contributions for $q \lesssim (k_B T/JZS)$ and $|q+K| \lesssim (k_B T/JZS)$. When the $\Psi^{(i)}$ in the interaction terms of the RPA equations are evaluated for these small wave vectors [cf. Eq. (3.9) and the following discussion for the ferromagnet], they become constants (dependent on H_A, E_0, and K) plus terms proportional to the small wave vectors. As discussed in Sec. IV B i for the ferromagnet, the terms proportional to the small wave vectors have less likelihood of giving a zero-magnon resonance, and they will be dropped. Then the α, β two-particle Green's functions and thus the longitudinal-susceptibility matrix components are solved for. The denominator of each component consists of unity plus a sum of the following terms plus all of their products (up to five different factors in a term):

$$\sum_q (l_q l_{q+K})^2 \left(\frac{f(E_q) - f(E_{q+K})}{\omega + i\epsilon - E_{q+K} + E_q} \right),$$

$$\sum_q (l_q l_{q+K})^2 \left(\frac{-1 - [f(E_q) - f(E_{q+K})]}{\omega + i\epsilon + E_{q+K} + E_q} \right),$$

$$\sum_q (l_q l_{q+K})^2 \left(\frac{1 - [f(E_q) - f(E_{q+K})]}{\omega + i\epsilon - E_{q+K} - E_q} \right), \qquad (6.2)$$

$$\sum_q (l_q l_{q+K})^2 \left(\frac{1 - [f(E_{q+K}) - f(E_q)]}{\omega + i\epsilon - E_{q+K} - E_q} \right),$$

$$\sum_q (l_q l_{q+K})^2 \left(\frac{f(E_q) - f(E_{q+K})}{\omega + i\epsilon + E_{q+K} - E_q} \right).$$

The coefficients of these terms arise from the K-dependent constants $\Psi^{(i)}$. The forms in (6.2) will now be examined for a possible zero-magnon resonance.

B. Question of a zero magnon at large K in anisotropic systems

As in the ferromagnetic case, the momentum sums will be carried out for a formal choice of parameters (given by taking $E_0 \ll k_B T$) which maximizes the chance of a mode occurring; the mode will be shown not to exist for this choice, and thus it will not exist more generally.

In a lattice with inversion symmetry, for small k (with E_0 negligible), $E_k \propto |k|$, and $l_k \cong (1/\sqrt{E_k}) \propto (1/\sqrt{k})$. For large K the most divergent sum containing either $f(E_q)$ or $f(E_{q+K})$ in (6.2) is, in three dimensions, proportional to (taking E_0 negligible)

$$\left(\frac{k_B T}{JSz} \right) \int_0^\infty \frac{dx\, e^{-x}}{[\beta(\omega - E_K) - x]}. \qquad (6.3)$$

The integral in (6.3), which must become large in order to make (6.3) of order unity, gives a contribution which goes like $\ln\beta(\omega - E_K)$ for small $\beta(\omega - E_K)$. Such a divergence however is so weak that the inclusion of spin-wave damping in the RPA would certainly blur it out.

In two dimensions for large K, the integral corresponding to that in (6.3) has an additional factor of q^{-1}, and therefore the term corresponding to (6.3) has a leading contribution proportional to $[\beta(\omega - E_K)]^{-1}$ for small $[\beta(\omega - E_K)]$. Equating this term to unity, gives a resonance $(\omega - E_K) \propto (JS)(k_B T/JSz)$. A calculation of the spin-wave damping for this large K and non-negligible anisotropy in the low-T antiferromagnet has not been performed, but the results of HKHH suggest that Γ_K would be smaller in temperature than $(k_B T/JS)$. This argument therefore suggests that further consideration of the zero-magnon question in the two-dimensional antiferromagnet may be worthwhile.

The behavior of the sums in (6.2) which do not involve some Bose distribution must also be considered, but the comments made about them cannot be as conclusive as those made above. Such sums must be taken over the entire Brillouin zone and can therefore give contributions as large as order unity. On the other hand, the integrals corresponding to these sums in both two and three dimensions appear to never be more than logarithmically divergent, and when spin-wave damping is included in the intermediate states, the results are fairly smooth functions of ω; such terms do not appear to give a sharp resonance which would correspond to a zero-magnon mode.

C. Question of a zero-magnon mode at small K in anisotropic systems

The behavior of the momentum sums for small K in the antiferromagnet is quite different from that in the ferromagnet. For $\omega > 0$, the first and last sums in (6.2) are most likely to give a mode. In order to maximize the possibility of a mode occurring (by maximizing the power of q in the denominator of the integrand) the K dependence of l_{q+K} will be dropped. Taking $Ka \ll (k_B T/JZS)$ and $E_q = cq$ for small q, the denominator of these integrands become

$$\omega \pm (E_{q+K} - E_q) \cong \omega \pm \frac{\partial E_{\vec{q}}}{\partial \vec{q}} \cdot \vec{K} = \omega \pm cK \cos\theta_K, \qquad (6.4)$$

where θ_K is the angle between $\vec{K}$ and $\vec{q}$. When the angular integral is done, the most divergent contribution to these sums in three dimensions is

$$-\left(\frac{\omega}{cK} \right) \left(\frac{k_B T}{JSz} \right) \int_0^\infty e^{-\beta c q} \left(\frac{dq}{q^2} \right) \ln\left[\left| \frac{\omega}{cK} - 1 \right| \right]. \qquad (6.5)$$

For $\omega \sim cK$, equating this term to unity gives a resonance which is similar to zero sound in a Fermi liquid. This logarithmic divergence is weak, however, and it is likely that it is blurred out by the dispersion in c from the q^4 term in $J(q)$ and also by the thermal spin-wave damping. Indeed, if the q^{-2} is absent from the integral in (6.5) (which would occur if there were a non-negligible gap in E_q or if K in l_{q+K} could not be ignored), then the dispersion in c alone entirely obscures the resonance. Taken together, the above effects make it seem unlikely that (6.5) will give a well-defined resonance in realistic systems. The small-K contributions from other than the first and last sums in (6.2) are similar to (6.5) except that the argument of the logarithm contains additional q-dependent terms and thus the q integrals are even less divergent that in (6.5).

D. Isotropic Heisenberg antiferromagnet

The set of coupled integral equations for the α, β two-particle Green's functions in the RPA, of which (6.1) is an example, holds with or without anisotropy. Once again at low T, the $\Psi^{(i)}$ are expanded for the appropriate small wave vectors. In the isotropic system, however, the $\Psi^{(i)}$ become proportional to one or both of the small wave vectors. Using these expansions of $\Psi^{(i)}$, the RPA equations can be solved algebraically for the two-particle Green's functions, and from them the spin-spin Green's functions are formed. Because of the complexity of this calculation, it has not yet been fully carried out. The form of the results can, however, be seen: The denominators of the components of the spin-spin Green's-function matrix will consist of unity plus a sum of products of various momentum sums. These momentum sums have the same forms as those in (6.2) except that many of them will have an additional power of q or of its components in their numerators (cf., ferromagnetic case in Sec. III). Such momentum sums will diverge even less strongly than their counterparts for systems with anisotropy. Therefore a zero-magnon mode is even less likely to exist in the isotropic antiferromagnet than in the anisotropic one.

A reservation should be made concerning this discussion of the isotropic antiferromagnet: Because the calculation of the susceptibility matrix has not been fully carried out, it remains possible that some unexpected cancellation or combination of terms might occur in the momentum sums in the denominators of this matrix which could change the behavior of those sums from that stated above. This, however, seems quite unlikely.

VII. CONCLUSION

The question of the existence of a zero-magnon mode in the low-T Heisenberg ferromagnet was treated by obtaining and evaluating the RPA longitudinal susceptibility in systems with and without exchange anisotropy for large and small wave vector. The zero-magnon mode, which was found in the RPA susceptibility at large wave vectors in systems with an exchange anisotropy which enhances the repulsion between spin-wave densities, was shown to be suppressed by spin-wave damping (except possibly in certain two-dimensional systems).

The analysis was extended to the low-temperature antiferromagnet with exchange anisotropy, for which it was shown that a well defined mode is not expected except possibly in certain two-dimensional systems. An argument was given which suggests that the zero-magnon mode is not expected in the isotropic antiferromagnet either.

The author has also examined the question of the existence of a zero-magnon mode in magnetic insulators at higher temperatures throughout their ordered regimes (outside of the critical region), where the spin-density fluctuations are greater. It has been found that a well-defined mode does not exist in the three-dimensional ferromagnet or antiferromagnet at these temperatures. These results will be presented in a separate publication.

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