Renormalization-group methods for critical dynamics: I. Recursion relations and effects of energy conservation

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(Received 25 February 1974)

The renormalization-group method for studying critical phenomena is generalized to a class of dynamical systems—the time-dependent Ginzburg-Landau models. The effects of conservation laws on the critical dynamics are investigated through the study of models with different conservation properties for the energy and the space integral of the order parameter. Dynamic critical exponents near four dimensions ($d \approx 4$) are obtained from recursion relations, analogous to those of Wilson and Fisher. The physical significance of the time-dependent Ginzburg-Landau models is explored and the applicability of the results to experiments on the NMR linewidth of FeF_2 is discussed.

I. INTRODUCTION

The renormalization-group approach to static critical phenomena developed by Wilson and coworkers[1-4] has provided a new mathematical formalism for calculating critical exponents approximately, and for answering certain detailed questions about critical behavior. For example, in certain limits it was possible to verify the hypothesis of universality, which states that the critical exponents do not depend on the interaction strength, but do depend on such parameters as the spatial dimensionality d and the order-parameter dimensionality n. Furthermore, the scaling hypothesis has been verified explicitly to several nontrivial orders in the expansion parameters. In a recent work,[5] the present authors generalized the renormalization-group approach to dynamic properties, and calculated the dynamic scaling exponent z to order ϵ^2 for two particular cases of a time-dependent Ginzburg-Landau model. The results of that analysis provided a partial confirmation of the extension of the scaling hypothesis to dynamic critical phenomena,[6,7] and of a form of universality for dynamic properties. In one of the two cases, the calculated dynamic critical exponents turned out to disagree with the values expected on the basis of mode-mode coupling theories,[8,9] and the "conventional" (Van Hove) theory.[10] Additional confirmation of scaling behavior was provided subsequently by Suzuki and Igarashi,[11] who also generalized the model of Ref. 5 to a system with long-range forces.[12]

In the present paper, the previously considered[5] time-dependent Ginzburg-Landau model is extended to study the effect of energy conservation on critical behavior.[13,14] The methods we have used to investigate dynamic critical phenomena correspond closely to those which have been developed for the static case, namely, approximate recursion relations,[2,12] diagrammatic ϵ and $1/n$ expansions,[1,3,15] and a formal analysis of the renormalization group to arbitrary order in ϵ.[4] The principal results of these investigations, together with a detailed description of the models, and of the recursion relations, will be given below. We shall also discuss the relationship of the models to some real physical systems, and the relevance of our predictions to possible experimental measurements. We shall defer to a second paper,[16] however, a detailed description of the diagrammatic ϵ and $1/n$ expansions, and the formal renormalization-group analysis. In both the present paper and Ref. 16, our analysis is confined to $T \geq T_c$, with zero external field.

The time-dependent Ginzburg-Landau models are continuum generalizations of the kinetic Ising models, first introduced by Glauber.[17] In the kinetic Ising models, the spin system is supposed to interact with a "bath," which causes spins to flip in a stochastic manner. The spin-flip probabilities in turn are chosen consistent with the principle of detailed balance, so that the system tends to the equilibrium ensemble of the Ising model, at some specified temperature T. Consistent with this requirement, there are still many possibilities. One may choose the transition probabilities so that the order parameter ψ of the system is conserved, or in a manner that it is not conserved. Similarly, one may define the model so that energy E is conserved, and therefore there is a slow long-wavelength mode associated

with thermal diffusion, or one may define the model so that energy is not conserved. The latter case corresponds physically to an infinite heat bath at each lattice site, or to a bath with an infinite thermal conductivity. We designate the four cases by the following: *case A—ψ not conserved, E not conserved; case B—ψ conserved, E not conserved; case C—ψ not conserved, E conserved; case D—ψ conserved, E conserved.* As mentioned above, analyses based on the mode-mode coupling approach[8,9] have suggested that the conventional (Van Hove) theory[10] should be correct in these models.

The influence of energy conservation in the time-dependent Ginzburg-Landau models arises from the fact that the relaxation rate of the energy diffusion mode

$$\omega_E(k) = \lambda_E k^2/C \ , \tag{1.1}$$

is very slow at long wavelengths (λ_E is the thermal conductivity and C is the specific heat). If this mode couples to the order parameter ψ, it might influence the critical slowing down of ψ and modify its dynamic critical exponent. The important role of conservation laws in determining dynamic critical behavior in various systems has been emphasized particularly by Kawasaki[8] and by Kadanoff and Swift.[18]

In the earlier work[5] it was found that when the order parameter ψ was itself conserved (case B), then its transport coefficient λ_ψ remained finite at T_c, in accordance with the conventional (Van Hove) theory.[10] When the order parameter was not conserved (case A), the kinetic coefficient was found to go to zero at T_c, thus making the relaxation frequency *slower* than the conventional theory would predict. It turns out that in case D, the imposition of energy conservation does not cause any change, relative to case B, in the critical behavior of the order parameter (at or above T_c). As will be shown in detail below, however, the imposition of energy conservation in case C causes significant modification relative to case A. First, there is a portion of the d-n plane, for $d<4$, where the exponents z_ψ and z_E for the relaxation rates of the order parameter and the energy density are equal to each other, and *greater* than the value of z_ψ in case A. Thus, in this region, the order-parameter relaxation rate is slower than in case A, and even further from the conventional prediction. Second, there is another region for $d<4$ where energy conservation does not have any effect on the critical behavior of ψ. Third, there may be an anomalous region where energy conservation modifies the exponent for spin relaxation, but it appears that $z_\psi < z_E$ and the simple dynamic scaling picture breaks down. (An alternate interpretation of the behavior in this region is that dynamic scaling holds, but it is the ϵ expansion which breaks down). Finally, in case C, as in the other three cases, there is Gaussian behavior for $d>4$, where the energy and the order parameter behave as non-interacting fields, and trivially satisfy the conventional theory. The results of our analysis are described in detail in Sec. V and illustrated in Figs. 3–5.

In Sec. II the time-dependent Ginzburg-Landau models are defined, and the four cases distinguished, with different conservation properties for the order parameter ψ and the energy E. In Sec. III the existing theories of dynamic critical behavior for these models are reviewed (scaling and conventional theories), and the diagrammatic perturbation expansion for response functions is briefly introduced. Sec. IV discusses the lowest-order recursion relations, and Sec. V contains a summary of results on the time-dependent Ginzburg-Landau models. In Sec. VI the relationship between these models and more general physical systems is explored, and the question of universality of dynamic critical behavior is briefly discussed. Section VII contains some comments on the experimental observability of the results of the present work.

II. DYNAMICAL MODELS

Let us consider the Ginzburg-Landual-Wilson[1,4] model in which the equilibrium distribution of the order paramter ψ is given by

$$P^{\mathrm{I}}[\psi] = Z^{-1}\exp(-T^{-1}\mathcal{H}^{\mathrm{I}}[\psi]), \tag{2.1}$$

with

$$T^{-1}\mathcal{H}^{\mathrm{I}} = \int d^dx\left[\tfrac{1}{2}r_0\psi^2(x) + u_0\psi^4(x) + \tfrac{1}{2}|\nabla\psi(x)|^2\right] + T^{-1}F_0(T) \ , \tag{2.2}$$

$$\psi^2 = \sum_{\alpha=1}^{n}\psi_\alpha^2 \ ,$$

$$\psi^4 = (\psi^2)^2 \ , \tag{2.3}$$

$$|\nabla\psi|^2 = \sum_{\alpha=1}^{n}|\nabla\psi_\alpha|^2 \ ,$$

$$Z \equiv \int d\{\psi\}\exp(-T^{-1}\mathcal{H}^{\mathrm{I}}[\psi]), \tag{2.4}$$

where F_0, r_0, and u_0 are regular functions of the temperature T (which we measure in units of energy). For simplicity we shall choose u_0 to be independent of T, and r_0 to be linear in T^{-1}:

$$r_0 = r_0^{(0)} - r_0^{(1)}T^{-1} \ . \tag{2.5}$$

The field $\psi(x)$ is understood to have variations with

wave vectors less than a cutoff Λ, and Eq. (2.4) contains a functional integral over all such configurations of ψ. From the partition function (2.4) we may calculate the energy E and the specific heat C in the usual way, and we find

$$E = T^2 \frac{\partial}{\partial T} \ln Z = -T^2 \frac{d}{dT}\left(\frac{F_0}{T}\right) + r_0^{(1)} \frac{\partial}{\partial r_0} \ln Z , \tag{2.6}$$

$$E = -T^2 \frac{d}{dT}\left(\frac{F_0}{T}\right) - \frac{\Omega}{2} r_0^{(1)} \langle \psi^2 \rangle \equiv \Omega [c_1(T) - \tfrac{1}{2} r_0^{(1)} \langle \psi^2 \rangle] , \tag{2.7}$$

$$\Omega C = \frac{\partial E}{\partial T} = -TF_0'' + (r_0^{(1)})^2 T^{-2} \frac{\partial^2}{\partial r_0^2} \ln Z , \tag{2.8}$$

where the primes on F_0 denote differentiation with respect to T, Ω is the volume of the system, and the angular bracket is a functional average with weight $P^{\mathrm{I}}[\psi]$. It is clear from Eq. (2.7) that we may associate an energy density $\epsilon_\psi(x)$ with the spin system, defined by

$$\epsilon_\psi(x) \equiv c_1 - \tfrac{1}{2} r_0^{(1)} \psi^2(x) . \tag{2.9}$$

One may readily check that the singular part of the specific heat is proportional to the $k = 0$ component of the energy-energy correlation function

$$C + TF_0''/\Omega = T^{-2} \int d^d x [\langle \epsilon_\psi(x) \epsilon_\psi(0) \rangle - \langle \epsilon_\psi \rangle^2] . \tag{2.10}$$

In order to introduce dynamics into this model we write down an equation of motion for the field ψ, in the form

$$\frac{\partial \psi_\alpha(x,t)}{\partial t} = -\frac{\Gamma_0}{T} \frac{\delta \mathcal{H}^{\mathrm{I}}}{\delta \psi_\alpha} + \Gamma_0 h_\alpha(x,t) + \eta_\alpha(x,t), \tag{2.11}$$

where Γ_0 is a constant which sets the time scale, and $h_\alpha(x,t)$ is a space- and time-varying external field. The function $\eta_\alpha(x,t)$ is a statistically defined Langevin noise source with mean zero and correlation function

$$\langle \eta_\alpha(x,t) \eta_{\alpha'}(x',t') \rangle = 2\Gamma_0 \delta(x-x') \delta(t-t') \delta_{\alpha\alpha'} , \tag{2.12}$$

where the bracket denotes an average over the fluctuations of the Langevin force $\eta_\alpha(x,t)$. As is well known,[19] the dynamical model defined by Eqs. (2.11) and (2.12) relaxes to an equilibrium state described by the static probability distribution (2.1).

The low-frequency properties of the above model (which we denote as model I) depend crucially on the form of the quantity Γ_0. Two cases may be distinguished

Case A: Order parameter not conserved. If Γ_0 is a constant, then the time derivative of $\int d^d x \psi(x,t)$ is in general finite, and at any $T > T_c$, a fluctuation in ψ will relax to zero at a finite rate.

Case B: Order parameter conserved. If we let $\Gamma_0 = -\lambda_0 \nabla^2$, then $\int d^d x \psi(x,t)$ is independent of time, and we say that ψ is a conserved quantity. It also follows that any disturbance of ψ from its equilibrium value will relax very slowly if the wavelength of the disturbance is large.

Let us now discuss the time dependence of the "energy density" ϵ_ψ [Eq. (2.9)]. If we apply the equation of motion (2.11), it is easy to see that for *both* cases A and B, the variable ϵ_ψ is *not* conserved, and a long-wavelength disturbance in the energy may relax at a finite rate. In fact, these models represent situations in which the order parameter (e.g., the spin) can exchange energy with a reservoir that has either an infinite specific heat or an infinite thermal conductivity.

A model which satisfies energy conservation is most easily constructed by introducing the energy density $\epsilon(x)$ as a *separate* field, coupled to the order parameter $\psi(x)$. The equilibrium distribution for this model, which we denote as model II, is given by a joint probability density

$$P^{\mathrm{II}}[\psi, \epsilon] = Z^{-1} \exp(\mathcal{S}[\psi, \epsilon] - E/T), \tag{2.13}$$

where

$$Z = \int d\{\epsilon\}\, d\{\psi\} \exp(\mathcal{S} - E/T) , \tag{2.14}$$

E is the total energy, and the integral in Eq. (2.14) is over all functions $\psi(x)$ and $\epsilon(x)$ whose variations have wave vectors less than cutoffs Λ and Λ_E, respectively. From a microscopic point of view, $\exp(\mathcal{S}[\psi, \epsilon])$ is the phase-space volume associated with given values of the functions $\psi(x)$ and $\epsilon(x)$, and $\mathcal{S}$ is the corresponding entropy functional. We shall choose the explicit simple form

$$\mathcal{S}[\psi, \epsilon] = -\int d^d x [\tfrac{1}{2} \tilde{r}_0 \psi^2(x) + \tilde{u}_0 \psi^4(x) + \tfrac{1}{2} |\nabla \psi(x)|^2 + \gamma_0 T_0^{-1} \psi^2(x) \tilde{\epsilon}(x) + \tfrac{1}{2} C_0^{-1} \tilde{\epsilon}^2(x)] + S_0 + T_0^{-1} E , \tag{2.15}$$

$$\tilde{\epsilon}(x) \equiv \epsilon(x) - \epsilon_0 , \tag{2.16}$$

where $\tilde{r}_0$, $\tilde{u}_0$, γ_0, T_0, C_0, S_0, and ϵ_0 are parameters of the model. The physical significance of these parameters is as follows: $\tilde{r}_0$ and $\tilde{u}_0$ are analogous to r_0 and u_0 in model I [Eq. (2.2)]; γ_0 is a coupling constant between long-wavelength fluctuations of $\psi(x)$ and $\epsilon(x)$, and T_0, S_0, $E_0 \equiv \Omega \epsilon_0$, and $C_0 T_0^{-2}$ are approximate values of the transition temperature,

entropy, energy, and specific heat of the system, respectively, associated with the short-wavelength components ($q > \Lambda, \Lambda_E$).

Since the entropy functional is quadratic in the energy density, we may integrate Eq. (2.14) over $\tilde{\epsilon}$ to obtain a probability distribution for ψ alone. Defining

$$P^{\mathrm{II}}[\psi] \equiv Z^{-1} \exp(-T_0^{-1}\mathcal{F}[\psi]) , \tag{2.17}$$

$$Z \equiv \int d\{\psi\} \exp(-T_0^{-1}\mathcal{F}[\psi]) , \tag{2.18}$$

we find that $T_0^{-1}\mathcal{F}[\psi]$ has the same form as $T^{-1}\mathcal{H}^{\mathrm{I}}[\psi]$, Eq. (2.2), with parameters[20]

$$r_0 = \tilde{r}_0 - 2\gamma_0 C_0 \delta\beta_0 , \tag{2.19}$$

$$u_0 = \tilde{u}_0 - \tfrac{1}{2}\gamma_0^2 C_0 , \tag{2.20}$$

$$T^{-1}F_0 = (\text{const}) + E_0\delta\beta_0 - \tfrac{1}{2}\Omega C_0 T_0 (\delta\beta_0)^2 , \tag{2.21}$$

where

$$\delta\beta_0 \equiv T^{-1} - T_0^{-1} . \tag{2.22}$$

From Eq. (2.14) we may calculate the total heat capacity of the system,

$$\Omega C = \left(\frac{1}{T_0}\right)^2 \left(\Omega C_0 + 4(\gamma_0 C_0)^2 \frac{\partial^2 \ln Z}{\partial r_0^2}\right) , \tag{2.23}$$

which has the same singular part as in model I, Eq. (2.8). In fact, Eq. (2.23) may be considered as a justification of the calculation of the specific heat in model I by taking a second derivative with respect to r_0 [Eq. (2.8)]. Note also that the conditional expectation value of $\epsilon(x)$ for a specific configuration of the field ψ, may be found from Eq. (2.13) to be

$$\langle \epsilon(x) \rangle |_{\{\psi(x)\}} = \epsilon_\psi(x) , \tag{2.24}$$

where $\epsilon_\psi(x)$ is defined by Eq. (2.9).

The simplest equations of motion which obey energy conservation and are consistent with the equilibrium distribution $P^{\mathrm{II}}[\psi, \epsilon]$, are

$$\frac{\partial \psi_\alpha}{\partial t}(x,t) = \Gamma_0 \left(\frac{\delta \mathcal{S}}{\delta \psi_\alpha} + h_\alpha(x,t)\right) + \eta_\alpha(x,t) , \tag{2.25}$$

$$\frac{\partial \tilde{\epsilon}}{\partial t}(x,t) = -\lambda_0^E \nabla^2 \left(\frac{\delta \mathcal{S}}{\delta \tilde{\epsilon}} - \delta\beta(x,t)\right) + \zeta(x,t), \tag{2.26}$$

$$\langle \zeta \rangle = 0;$$
$$\langle \zeta(x,t)\zeta(x',t') \rangle = -2\lambda_0^E \nabla^2 \delta(x-x')\delta(t-t'), \tag{2.27}$$

where the correlations of η_α are still given by Eq. (2.12), $\delta\beta(\vec{x}, t)$ is a space and time varying external (temperature) field analogous to $h_\alpha(x,t)$, and λ_0^E/T_0^2 is the thermal conductivity arising from the short-wavelength components. From Eq. (2.26) it is clear that the energy is a *conserved* quantity, independent of the conservation properties of $\psi(x,t)$. We again distinguish two cases: *Case C—order parameter not conserved, energy conserved,* where Γ_0 is a constant; and *case D:—order parameter conserved, energy conserved,* where $\Gamma_0 = -\lambda_0 \nabla^2$.

The various cases considered above also have analogs in the discrete kinetic Ising models. In particular, Kawasaki[21] has considered models in which spin is conserved (case B), while Kadanoff and Swift[22] have discussed a model in which both spin and energy are conserved (case D). The relevance of our models to a wider class of physical systems, and in particular the question of universality of dynamic critical phenomena, will be discussed in Sec. VI.

III. RESPONSE FUNCTIONS AND DIAGRAMS

A. Response functions and characteristic frequency

Given a physical quantity Q and a field h_Q which couples linearly to it, we define the linear-response function $\chi_Q(k,\omega)$ by the relation

$$\langle Q(k,\omega) \rangle_{h_Q} = \chi_Q(k,\omega) h_Q(k,\omega) , \tag{3.1}$$

where Fourier transforms in space and time are defined by

$$h_Q(x,t) = \int \frac{d^d k}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(k\cdot x - \omega t)} h_Q(k,\omega), \tag{3.2}$$

etc., and the system is assumed to start from thermal equilibrium at time $t = -\infty$. The angular brackets in Eq. (3.1) denote the expectation value in the presence of the field h_Q. In Eq. (3.1), the field h_Q is assumed to be infinitesimal. We may also define the correlation function

$$C_Q(x,t) = \langle Q(x,t)Q(0,0) \rangle - \langle Q(x,t) \rangle \langle Q(0,0) \rangle \tag{3.3}$$

in an ensemble with $h_Q = 0$. The Fourier transform $C_Q(k,\omega)$, and the partial Fourier transform $\tilde{C}_Q(k,t)$ satisfy the well-known classical relations

$$C_Q(k,\omega) = (T/\omega)\mathrm{Im}\chi_Q(k,\omega) , \tag{3.4}$$

$$\tilde{C}_Q(k, t=0) = T\chi_Q(k,\omega=0) \equiv T\chi_Q(k) . \tag{3.5}$$

The last equation follows from Eq. (3.4) and the Kramers-Kronig relation for $\chi_Q(k,\omega)$.

It is convenient for the subsequent discussion to define the *characteristic frequency* $\omega_Q(k)$ as

$$\omega_Q(k) = \Gamma_Q(k)/\chi_Q(k) , \tag{3.6}$$

where the "kinetic coefficient" $\Gamma_Q(k)$ is defined by

$$\frac{1}{\Gamma_Q(k)} = \frac{\partial \chi_Q^{-1}(k,\omega)}{\partial(-i\omega)}\bigg|_{\omega=0} . \tag{3.7}$$

If the variable Q is conserved by the dynamics, then the kinetic coefficient has the form

$$\Gamma_Q(k) = \lambda_Q k^2 , \tag{3.8}$$

for $k \to 0$, at any fixed $T \neq T_c$, and λ_Q is known as a "transport coefficient." From Eqs. (3.4)–(3.7) it is easy to show that

$$\frac{1}{\omega_Q(k)} = \frac{1}{\chi_Q(k)} \frac{\partial \chi_Q(k,\omega)}{\partial(i\omega)}\bigg|_{\omega=0} , \tag{3.9a}$$

$$\frac{1}{\omega_Q(k)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \omega^{-1} C_Q(k,\omega) \Big/ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} C_Q(k,\omega). \tag{3.9b}$$

In addition, one may define the *initial decay rate* $\nu_Q(k)$, in terms of the short-time (high-frequency) behavior, as

$$\nu_Q(k) = \left[\frac{-1}{\tilde{C}_Q(k,t)} \frac{\partial \tilde{C}_Q(k,t)}{\partial t} \right]_{t=0} , \tag{3.10}$$

which may be shown to be equal to

$$\nu_Q(k) = \chi_Q^{-1}(k) \lim_{\omega\to\infty} [-i\omega \chi_Q(k,\omega)] . \tag{3.11}$$

When the dynamics of the system is purely relaxational, i.e., the Liouville operator is Hermitian, as is the case in the models we consider here, then a rigorous inequality may be proved,[21,22] namely,

$$\omega_Q(k) \leq \nu_Q(k). \tag{3.12}$$

The characteristic frequency $\omega_Q(k)$ will be used in formulating the dynamic scaling hypothesis for quantities Q whose static response $\chi_Q(k=0)$ is divergent at T_c. In cases where χ_Q is finite but its derivatives diverge, it can be useful to define a characteristic frequency for the "singular part" of Q, in analogy with Eq. (3.9a) as

$$\frac{1}{\omega_Q^{\rm sing}(k)} = \frac{1}{\chi_Q^{\rm sing}(k)} \frac{\partial \chi(k,\omega)}{\partial(i\omega)}\bigg|_{\omega=0} , \tag{3.13}$$

where $\chi_Q^{\rm sing}(k)$ is the "singular part" of the static response.

B. Dynamic scaling and conventional theory

According to the dynamic scaling hypothesis,[6,7] the response function $\chi_Q(k,\omega)$ at T_c has the form

$$\chi_Q(k,\omega) = \chi_Q(k) f_Q[\omega/\omega_Q(k)] , \tag{3.14}$$

$$\omega_Q(k) \sim (\text{const}) k^{z_Q} , \tag{3.15}$$

for $k \to 0$, $\omega \to 0$, where f is a smooth function of its argument satisfying the condition $f(0) = 1$. For temperatures near T_c, the response function has a form similar to (3.14), but with a characteristic dependence on the ratio k/κ, where κ is the inverse correlation length. Specifically, the static response function and the characteristic frequency[23] are hypothesized to behave as

$$\chi_Q(k) = k^{x_Q} g_Q(k/\kappa) \tag{3.16}$$

and

$$\omega_Q(k) = k^{z_Q} \Omega_Q(k/\kappa) . \tag{3.17}$$

Moreover, the shape of the function f_Q in Eq. (3.14) may also depend on k/κ. When the dynamic scaling hypothesis is applied to the order parameter for the transition, it is referred to as "restricted scaling," as opposed to "extended scaling," which assumes that the scaling form applies to other variables as well.[7]

The conventional or Van Hove theory[10] makes the assumption that the kinetic and transport coefficients remain finite near the critical point. In our present models, this implies that the characteristic frequency is proprotional to the initial decay rate ν_Q, since the quantity $-dC_Q(k,t)/dt\,|_{t=0}$ on the right-hand side of Eq. (3.10) can be shown to have no divergent temperature dependence (when Q is ϵ or ψ):

$$\nu_Q(k) \propto [\omega_Q(k)]^{\rm conv} . \tag{3.18}$$

Thus the inequality (3.12) implies that

$$\omega_Q(k) \leq [\omega_Q(k)]^{\rm conv} , \tag{3.19}$$

from which a rigorous exponent inequality follows:

$$z_Q \geq [z_Q]^{\rm conv} . \tag{3.20}$$

C. Perturbation theory for model II

Let us specialize the discussion to the model of Eqs. (2.25) and (2.26), where the variable Q is either the order parameter ψ or the energy density ϵ. The conjugate fields are h and $-\delta\beta$, and averages are taken with respect to the probability distribution $P[\psi, \epsilon; t]$ which is assumed to be equal to the equilibrium distribution $P^{\rm II}[\psi, \epsilon]$ of Eq. (2.13) at $t = -\infty$. The response functions are

$$G(k,\omega) = \chi_\psi(k,\omega) = \langle \psi_\alpha(k,\omega) \rangle_{h,\,\delta\beta} [h_\alpha(k,\omega)]^{-1} , \tag{3.21}$$

$$D(k,\omega) = \chi_E(k,\omega) = -\langle \epsilon(k,\omega) \rangle_{h,\,\delta\beta} [\delta\beta(k,\omega)]^{-1} , \tag{3.22}$$

where the averages are taken with respect to the probability $P[\psi, \epsilon; t]$ in the presence of the infinitesimal fields h and $\delta\beta$. [Note: In Eq. (3.22) we have used the notation χ_E rather than χ_ϵ to avoid possible confusion with the parameter $\epsilon = 4 - d$. We shall similarly use the notation ω_E, z_E, λ_E, etc.]

We wish to develop a diagrammatic formalism for calculating the response functions for model

II as power series in the "interaction vertices" γ_0 and u_0 of Eqs. (2.15) and (2.20).[24] The formalism appropriate for model I is then easily obtained by setting γ_0 equal to zero, or, as we shall see, by setting $\lambda_0^E = \infty$.

In the noninteracting case ($\gamma_0 = u_0 = 0$) we may solve the linear equations of motion [(2.25) and (2.26)] to find

$$G_0(k, \omega) = (-i\omega/\Gamma_0 + k^2 + r_0)^{-1} , \tag{3.23}$$

$$D_0(k, \omega) = (-i\omega/\lambda_0^E k^2 + C_0^{-1})^{-1} . \tag{3.24}$$

The response functions for the interacting case are then obtained by drawing diagrams using G_0 and D_0 as propagators, u_0 as a vertex[24] joining four G_0 lines, and γ_0 as a vertex joining two G_0's and one D_0. In addition, the effect of the Langevin noise sources η_α and ζ is included by means of noise vertices, which are inserted in the propagators, as explained in a similar case by Tucker and Halperin.[25] A detailed derivation of this perturbation theory and precise rules for calculating the contribution of each diagram will be given in Ref. 16. We shall simply give an illustrative example here.

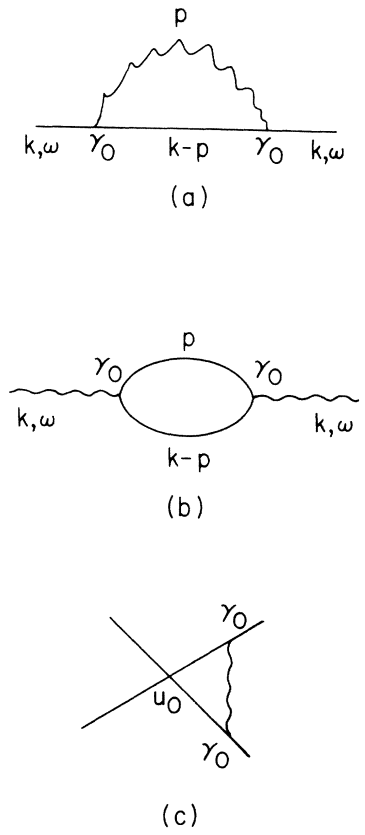

FIG. 1. Typical low-order diagrams in the perturbation expansion for the dynamical equations (2.25)–(2.27). Solid lines denote order-parameter propagators, and wavy lines energy propagators. (a) Contribution to the self-energy $\Sigma(k,\omega)$, of order ϵ. (b) Contribution to $\Pi(k,\omega)$. (c) Contribution to the dynamic four-point vertex. This diagram does not contribute to the static ($\omega = o$) vertex, since it is already included in going from $\tilde{u}_0$ [Eq. (2.15)] to u_0 [Eq. (2.20)].

Let us write

$$G^{-1}(k, \omega) \equiv G_0^{-1}(k, \omega) + \Sigma(k, \omega) , \tag{3.25}$$

$$D^{-1}(k, \omega) \equiv D_0^{-1}(k, \omega) + \Pi(k, \omega) . \tag{3.26}$$

For the lowest-order recursion relations obtained in the present paper, only the diagrams depicted in Figs. 1(a) and 1(b) contribute frequency-dependent terms. These terms are

$$\Sigma_a(k, \omega) = -\gamma_0^2 \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(p^2 + r_0) C_0} \right) \times \left(\frac{\Gamma_0 (p^2 + r_0) + (\lambda_0^E / C_0)(p - k)^2}{-i\omega + \Gamma_0 (p^2 + r_0) + (\lambda_0^E / C_0)(p - k)^2} \right) \tag{3.27}$$

$$\Pi_b(k, \omega) = -\tfrac{1}{2} n \gamma_0^2 \int \frac{d^d p}{(2\pi)^d} \times \frac{1}{(p^2 + r_0)[(p - k)^2 + r_0]} \times \left(\frac{p^2 + r_0 + (p - k)^2 + r_0}{(-i\omega/\Gamma_0) + p^2 + r_0 + (p - k)^2 + r_0} \right) . \tag{3.28}$$

In the noninteracting case described by Eqs. (3.23) and (3.24), the characteristic frequencies and kinetic coefficients for ψ and ϵ are easily identified, in all four cases (A–D) considered in Sec. II, namely,

$$\text{case A: } \Gamma_\psi(k) = \Gamma_0; \tag{3.29a}$$

$$\text{case B: } \Gamma_\psi(k) = \lambda_0 k^2; \tag{3.29b}$$

$$\text{case C: } \Gamma_\psi(k) = \Gamma_0, \ \Gamma_E(k) = \lambda_0^E k^2; \tag{3.29c}$$

$$\text{case D: } \Gamma_\psi(k) = \lambda_0 k^2, \ \Gamma_E(k) = \lambda_0^E k^2; \tag{3.29d}$$

where Γ_0, λ_0, and λ_0^E are independent of k.

In the presence of interactions we expect that the static response functions (χ_ψ and χ_E) and the kinetic coefficients (Γ_ψ and Γ_E) will depend on the vertices u_0 and γ_0 in a complicated way. If the kinetic coefficients were to remain finite at the critical point, then the conventional theory[10] would apply. For fixed $T > T_c$, and for $k \to 0$, conventional theory then predicts

$$\omega_\psi(k) \propto \kappa^{2-\eta} k^2 , \ \psi \text{ conserved (cases B and D);} \tag{3.30}$$

$$\omega_\psi(k) \propto \kappa^{2-\eta} , \ \psi \text{ not conserved (cases A and C);} \tag{3.31}$$

$$\omega_E(k) \propto \kappa^{\tilde{\alpha}/\nu} k^2, \quad E \text{ conserved (cases C and D);} \tag{3.32}$$

$$\omega_E(k) \propto \kappa^{\tilde{\alpha}/\nu}. \quad E \text{ not conserved (cases A and B).} \tag{3.33}$$

Here ν is the exponent of the inverse correlation length κ, and $\tilde{\alpha}$ is the exponent of the temperature dependence of the *largest* term in the specific heat. The exponent $\tilde{\alpha}$ is related to the usual exponent α by

$$\tilde{\alpha} = \alpha \quad \text{if } \alpha > 0, \tag{3.34a}$$

$$\tilde{\alpha} = 0 \quad \text{if } \alpha < 0. \tag{3.34b}$$

Thus in the limit $T \to T_c$ we have

$$C = a_1 |T - T_c|^{-\alpha} + a_2 \sim (\text{const})\, \kappa^{-\tilde{\alpha}/\nu}. \tag{3.35}$$

Equation (3.32) follows from the assumption that λ_E remains finite at T_c, together with the energy diffusion equation

$$\omega_E(k) = \lambda_E k^2 / C, \tag{3.36}$$

which corresponds to Eqs. (3.6) and (3.8) with $\chi_E(k=0) \propto C$.

The conventional result for $\omega_\psi(k)$ is consistent with (restricted) dynamic scaling if

$$z_\psi = 2 - \eta, \quad \text{cases A and C;} \tag{3.37}$$

$$z_\psi = 4 - \eta, \quad \text{cases B and D.} \tag{3.38}$$

For the energy, the conventional theory is consistent with extended dynamic scaling for $T \geq T_c$ if

$$z_E = \tilde{\alpha}/\nu, \quad \text{cases A and B;} \tag{3.39}$$

$$z_E = 2 + \tilde{\alpha}/\nu, \quad \text{cases C and D.} \tag{3.40}$$

IV. RECURSION RELATIONS

A. Static properties

We wish to find the fixed point of the "Hamiltonian" (2.2) or "entropy" (2.15) under the renormalization-group transformation R_b defined by

$$R_b = R_b^s R_b^i, \tag{4.1}$$

where R_b^i is an integration over intermediate wave vectors in the domain[20]

$$b^{-1}\Lambda < p < \Lambda, \tag{4.2}$$

and R_b^s is a change of scale

$$x \to x' = x/b, \tag{4.3a}$$

$$\psi \to \psi' = b^a \psi, \tag{4.3b}$$

$$\vec{\epsilon} \to \vec{\epsilon}\,' = b^{a_E} \vec{\epsilon}. \tag{4.3c}$$

Here b is a constant greater than unity, and a and a_E are constants to be specified below. The action of R_b^i is to "dress" the propagators and vertices in a manner suggested by Fig. 2, where all intermediate wave vectors are in the shell $b^{-1}\Lambda < p < \Lambda$. In fact, the diagrams specifically enumerated in Fig. 2 contain all terms necessary to determine the exponents to order ϵ. Furthermore, it is correct to this order to neglect the dependence of the diagrams on the incoming momenta, and we may set all of these momenta equal to zero. It then follows that under the total action of R_b, the entropy functional (2.15) takes the form

$$\begin{aligned}(\mathcal{S} - E/T)' = -\int d^d x' \{ &\tfrac{1}{2}\tilde{r}_1 |\psi'(x')|^2 + \tilde{u}_1 |\psi'(x')|^4 \\ &+ \tfrac{1}{2} |\nabla\psi'(x')|^2 + \gamma_1 |\psi'(x')|^2 \vec{\epsilon}\,'(x') \\ &+ \tfrac{1}{2} C_1^{-1} |\vec{\epsilon}\,'(x')|^2 \\ &+ \delta\beta_1 \vec{\epsilon}\,'(x') \} + \text{const}, \end{aligned} \tag{4.4}$$

where the quantities $\tilde{r}_1$, $\tilde{u}_1$, γ_1, C_1, and $\delta\beta_1$ may be expressed in terms of the original parameters $\tilde{r}_0$, $\tilde{u}_0$, γ_0, C_0, and $\delta\beta_0$. We assume, moreover, that the constant a of Eq. (4.3b) has been chosen so that the coefficient of $|\nabla\psi|^2$ in (4.4) remains unchanged. The procedure is analogous to the one employed by Wilson and Fisher[2] and justified in Ref. 4 for the Hamiltonian (2.2), except that expansions are made in both u_0 and γ_0, and the bare static energy propagator is equal to C_0.[24] The recursion relations for the quantities $\tilde{r}_l$, $\tilde{u}_l$,

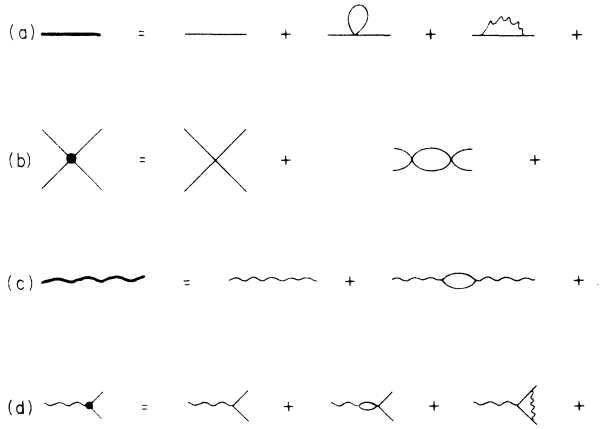

FIG. 2. Schematic representation of the diagrams necessary to obtain the recursion relations for case C to lowest order in ϵ. Intermediate lines are integrated over the range $b^{-1}\Lambda \leq p \leq \Lambda$ and $-\infty < \omega' < \infty$ for the wave vector and the frequency, respectively. (a) yields the recursion relation for r_l and Γ_l; (b) yields u_l; (c) yields C_l and λ_l^E; and (d) yields a recursion relation for γ_l.

γ_l, C_l, and $\delta\beta_l$ may be reduced to equations for the four quantities r_l, u_l, γ_l, and C_l. To lowest order these are

$$r_{l+1} = b^{d-2a}\{r_l + 2(n+2)Bu_l \times [\Lambda^2(1-b^{-2}) - 2r_l \ln b]\}, \quad (4.5)$$

$$u_{l+1} = b^{d-4a}[u_l - 4(n+8)u_l^2 B \ln b], \quad (4.6)$$

$$C_{l+1}^{-1} = b^{d-2a_E}[C_l^{-1} - 2n\gamma_l^2 B \ln b], \quad (4.7)$$

$$\gamma_{l+1} = b^{2-2a-a_E}[\gamma_l - 4(n+2)u_l\gamma_l B \ln b - 2n\gamma_l^3 C_l], \quad (4.8)$$

where

$$B \ln b \equiv \int_{\Lambda/b}^{\Lambda} \left(\frac{d^4p}{(2\pi)^4}\right) \frac{1}{p^4} = \frac{\ln b}{8\pi^2}. \quad (4.9)$$

Let us find the fixed point of the transformation (4.5)–(4.9). First we recall that at T_c we have

$$\langle \psi(k)\psi(-k')\rangle \propto k^{-2+\eta}\delta(k-k'), \quad (4.10)$$

$$\langle \tilde{\epsilon}(k)\tilde{\epsilon}(-k')\rangle \propto k^{-\tilde{\alpha}/\nu}\delta(k-k'), \quad (4.11)$$

where $\tilde{\alpha}$ is defined in Eq. (3.34). Equation (4.10) follows from the definition of the exponent η and Eq. (4.11) results from the usual identification of the specific heat with static energy correlations, combined with static scaling. The assumption that δ_l approaches a fixed point at T_c is consistent with Eqs. (4.10) and (4.11) if and only if

$$a = \tfrac{1}{2}(d-2+\eta), \quad (4.12)$$

$$a_E = \tfrac{1}{2}(d - \tilde{\alpha}/\nu). \quad (4.13)$$

The fixed point of the recursion relations (4.5-9) may be found for d close to 4, i.e.,

$$0 < \epsilon = 4 - d \ll 1. \quad (4.14)$$

One finds $\eta = 0$ to first order ϵ, and the fixed point of Eq. (4.6) is

$$u^* = \epsilon/4B(n+8) + \mathcal{O}(\epsilon^2). \quad (4.15)$$

The equation for C_l [Eq. (4.7)] yields

$$\gamma^{*2}C^* = \tilde{\alpha}/2n\nu B, \quad (4.16)$$

and Eq. (4.8) gives either

$$\gamma^* = \tilde{\alpha} = 0 \quad (4.17)$$

or

$$\tilde{\alpha}/\nu = (4-n)\epsilon/(n+8) + \mathcal{O}(\epsilon^2), \quad \gamma^* \neq 0. \quad (4.18)$$

Equation (4.18) for α agrees with the value obtained using the relation

$$d\nu = 2 - \alpha, \quad (4.19)$$

and the ϵ expansion of ν, calculated by Wilson,[1] directly for model I. It is easy to verify that the solution with $\gamma^* \neq 0$ is stable if and only if $\alpha > 0$, i.e., if $n < 4$. In the opposite case, the specific heat does not diverge at T_c ($\alpha < 0$), and the order parameter and energy density are uncoupled at the fixed point.

B. Dynamics

We shall next generalize[12] the approximate recursion formulas to the dynamical equations (2.11), (2.25), and (2.26). Let us extend the transformation R_b^s so that frequencies are rescaled by

$$\omega \to \omega' = b^z\omega. \quad (4.20)$$

We have not found it necessary to introduce a special cutoff for the frequency integrals in our perturbation theory, and we merely generalize R_b^i so that one integrates intermediate frequencies from $-\infty$ to $+\infty$ for all lines with $b^{-1}\Lambda < p < \Lambda$. We identify the renormalized value of the coefficient Γ_l^{-1} with the value of $\partial G_l^{-1}(k,\omega)/\partial(-i\omega)$ in the limit $\omega \to 0$ and $k \to 0$. The recursion relation for Γ_l^{-1} implied by Fig. 2(a) is then given in case C by

$$\Gamma_{l+1}^{-1} = b^{d-2a-z}(\Gamma_l^{-1} + 4\gamma_l^2 C_l F_l), \quad (4.21)$$

$$F_l \equiv \int_{\Lambda/b}^{\Lambda} \left(\frac{d^dp}{(2\pi)^d}\right) \times \frac{1}{(p^2+r_l)[\Gamma_l(p^2+r_l) + (\lambda_l^E/C_l)p^2]}. \quad (4.22)$$

For case A, on the other hand, there is no frequency-dependent contribution of order ϵ to the self-energy, and we have simply

$$\Gamma_{l+1}^{-1} = b^{d-2a-z}\Gamma_l^{-1}. \quad (4.23)$$

[Note that this result may be obtained from (4.21) either by setting $\gamma_l = 0$, or by setting $\lambda_l^E = \infty$, as expected.] The dynamic scaling exponent z_ψ may be identified with the parameter z, when the latter has been chosen so that Γ_l approaches a finite nonzero value as $l \to \infty$. It follows from Eq. (4.23) that for case A, a fixed point with finite Γ occurs if and only if

$$z = d - 2a = 2. \quad (4.24)$$

Thus the conventional theory is valid, to order ϵ, for case A.

Returning to case C, we find that the recursion relation for λ_l^E has no contribution from R_b^i, since the self-energy contributions in Fig. 2(c) have no term which behaves as ω/k^2 as ω and k go to zero [cf. Eq. (3.28)]. Thus we have

$$1/\lambda_{l+1}^E = b^{d+2-2a_E-z}/\lambda_l^E. \quad (4.25)$$

Inserting the fixed point values of $\gamma_l^2 C_l$ and r_l in (4.21), and using (4.7), (4.25), (4.22), and (4.9)

we find, to lowest order,

$$F_l = B\ln b[\Gamma_l(1+\mu_l)]^{-1}\,, \tag{4.26}$$

$$\mu_{l+1} = \mu_l\{1-[\tilde{\alpha}/\nu - (2\tilde{\alpha}/n\nu)(1+\mu_l)^{-1}]\ln b\} \tag{4.27}$$

where

$$\mu_l \equiv \lambda_l^B/\Gamma_l C_l \tag{4.28}$$

and

$$\Gamma_{l+1}^{-1} = \Gamma_l^{-1}\{1+[2-z+(2\tilde{\alpha}/n\nu)(1+\mu_l)^{-1}]\ln b\}\,. \tag{4.29}$$

Equation (4.27) has three possible fixed points, for $n<4$:

$$\mu^* = \infty\,, \tag{4.30}$$

$$\mu^* = 0\,, \tag{4.31}$$

and

$$\mu^* = (2/n) - 1\,. \tag{4.32}$$

In fact, we see that (4.32) is the stable solution for $n<2$, while (4.31) is stable for $2<n<4$. If we choose z such that Γ_l goes to a finite nonzero value as $l\to\infty$, then $z=z_\psi$, and according to (4.29),

$$z_\psi = 2+\tilde{\alpha}/\nu, \text{ for } n<2 \tag{4.33}$$

$$2<z_\psi = 2+2\tilde{\alpha}/n\nu<2+\tilde{\alpha}/\nu, \text{ for } 2<n<4. \tag{4.34}$$

For $n>4$, $\tilde{\alpha}=0$, and Eq. (4.29) implies that $z=2$ to order ϵ, regardless of the value of μ^*. According to Eq. (4.27), in fact, μ_l may take on any value one pleases as $l\to\infty$. When terms of order ϵ^2 are considered, however, one finds[16] that $\mu^*=\infty$ is the stable fixed point, and we have

$$z_\psi = 2+\mathcal{O}(\epsilon^2), \text{ for } n>4\,. \tag{4.35}$$

V. RESULTS OF THE FULL RENORMALIZATION-GROUP ANALYSIS–A SUMMARY

In the present section we wish to summarize the results of our renormalization-group analysis[16] of the time-dependent Ginzburg-Landau models, and compare them with the predictions of dynamic scaling[6,7] and the conventional theory.[10] Our discussion will be confined to $d<4$, since the conventional theory holds in a trivial fashion for $d>4$, i.e., $\eta=0$ and $z=2$ for cases A and C, $z=4$ for cases B and D. (Henceforth we shall drop the subscript ψ when referring to the dynamic critical exponent for the order parameter z_ψ.)

A. Case A

Case A was studied previously[5] using expansion techniques to order ϵ^2 and $1/n$. The results are indicated in Fig. 3, where the exponent z for the order-parameter relaxation frequency is shown as a function of ϵ and n. Let us write

$$z = 2+c\eta\,, \tag{5.1}$$

so that $c=-1$ corresponds to the conventional theory [Eq. (3.37)]. We have found, in contrast, that $c=0.7261$ as $d\to 4^-$, for any $n<\infty$. Furthermore, for $n\to\infty$ and $2<d<4$, c approaches a known function of d, whose value is $\frac{1}{2}$ at $d=3$, 0.7261 at $d=4$, and 0 as $d\to 2$. Results of high-temperature-series expansions and Monte Carlo calculations for the two-dimensional kinetic Ising model indicate that $c\approx 0$ for $d=2$, $n=1$ as well.[26,27] The above results suggest that $c\gtrsim 0$ throughout the region $2\leq d<4$, $n<\infty$, for case A, in violation of the conventional theory. Note, however, that η is of order ϵ^2, so that the conventional theory is correct to order ϵ in this case, in agreement with the recursion relation analysis of Sec. IV.

The scaling function f in Eq. (3.14) has been calculated to lowest nontrivial order in $1/n$ or ϵ, and the dynamic scaling behavior has been confirmed in detail to those orders, both at T_c,[5] and for $T>T_c$.[11]

General renormalization-group arguments[16] indicate that in the region where the specific heat diverges, the nonconserved energy ϵ relaxes with the same characteristic exponent as the order parameter ($z_E = 2+c\eta$) to all orders in ϵ. For exam-

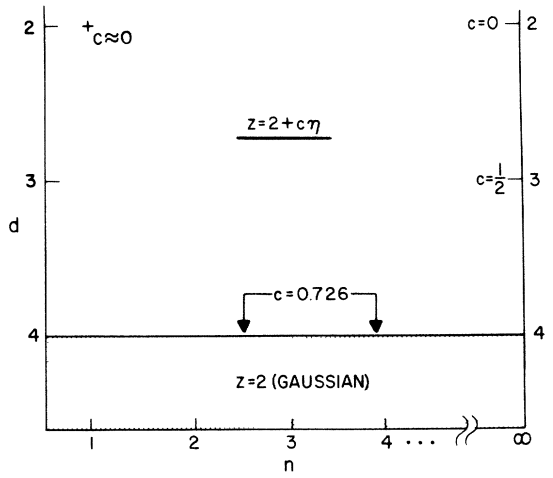

FIG. 3. Dynamic critical exponent z for the time-dependent Ginzburg-Landau model with neither energy nor order parameter conserved (case A). Ordinate and abscissa are lattice dimensionality d and order parameter dimensionality n. Values of z at boundaries come from expansions in $(4-d)$ or n^{-1}, value at $d=2$, $n=1$ comes from high-temperature-series expansion for the two-dimensional kinetic Ising model (Ref. 26).

ple, at T_c we have

$$\omega_E(k) \propto k^{2+c\eta} \tag{5.2}$$

when $\alpha > 0$.

In the region of the d-n plane where the specific heat does not diverge ($\alpha < 0$), we find that the *singular* part of the energy correlation function relaxes with the exponent of the order parameter, i.e.,

$$\omega_E^{\text{sing}} \propto k^{2+c\eta}. \tag{5.3}$$

The energy relaxation rate has been calculated explicitly only to first order in ϵ,[16] or zeroth order in $1/n$ (spherical model limit),[28] but is in agreement with the renormalization-group arguments to these orders. High-temperature-series expansions by Yahata[26] for the two-dimensional kinetic Ising model are also consistent with the result $z = z_E$.

Note that these results are already very far from what one would predict on the basis of the conventional theory applied to the nonconserved energy field: $z_E^{\text{conv}} = \tilde{\alpha}/\nu$ [Eq. (3.39)].

B. Case C

According to the recursion relation analysis of Sec. IV, the dynamic critical exponent for case C, to first order in ϵ, obeys different formulas in three regions of n [cf. Eqs. (4.33)–(4.35)]:

$$\text{Region I } (n > 4): \quad z = 2; \tag{5.4a}$$

$$\text{Region II } (n < 2): \quad z = 2 + \tilde{\alpha}/\nu; \tag{5.4b}$$

$$\text{Region III } (2 < n < 4): \quad 2 < z < 2 + \tilde{\alpha}/\nu. \tag{5.4c}$$

A more detailed renormalization-group analysis,[16] to all orders in ϵ, suggests that these definitions be generalized to read

$$\text{Region I:} \quad z = 2 + c\eta; \tag{5.5a}$$

$$\text{Region II:} \quad z = 2 + \tilde{\alpha}/\nu; \tag{5.5b}$$

$$\text{Region III:} \quad 2 + c\eta < z < 2 + \tilde{\alpha}/\nu. \tag{5.5c}$$

(We use the notation $2 + c\eta$ to represent the value of the exponent for the relaxation rate of ψ in case A, where energy is not conserved.) From Eq. (5.5a) it is seen that in region I, energy conservation has no effect on the value of z. In region II, on the other hand, the characteristic rates of order parameter and energy relaxation are equal to the conventional relaxation rate for the *energy*. This rate is slower than the relaxation rate of the order parameter in case A, and still farther from the prediction of the conventional theory for the order parameter. In region III, the order parameter relaxes more slowly than in case A, but more rapidly than the energy, which continues to obey

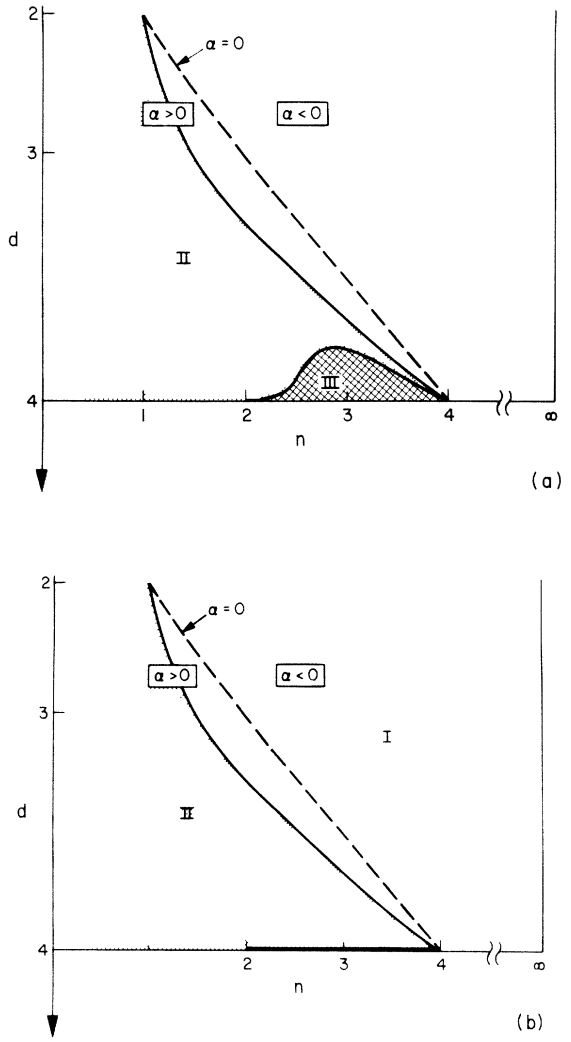

FIG. 4. Possible "phase diagram" for the dynamic critical behavior in case C, where energy is conserved but the order parameter is not. In region I energy conservation does not affect the dynamic critical exponent for the order parameter, which has the value $z = 2 + c\eta$, found in case A (Fig. 3). In region II the energy and order parameter have the same critical exponent $z = z_E = 2 + \alpha/\nu$, which is larger than the value $z = 2 + c\eta$ for case A. In region III, z obeys the inequality $2 + c\eta < z < 2 + \alpha/\nu$, and dynamic scaling does not hold. The dotted line $\alpha = 0$ separates the region where the specific heat diverges ($\alpha > 0$), from the region where it is finite at T_c ($\alpha < 0$). The boundary between regions I and II is determined by the relation $c\eta = \alpha/\nu$. We have assumed $c > 0$ for $2 < d < 4$, and $c = 0$ for $d = 2$, so that the dotted line ($\alpha = 0$) lies in the interior of region I for $2 < d < 4$, but meets the boundary at $d = 2$ and $d = 4$. The existence of region III, as indicated in Fig. (a), is suggested by the recursion relation analysis of Sec. IV. An alternate phase diagram for case C, in which region III is absent, is indicated in Fig (b). If this latter phase diagram is correct, then the approximate recursion relations do not give the exponent z correctly to first order in $\epsilon = 4 - d$, for $2 < n < 4$.

the conventional theory. Region III is expected to have a number of anomalous properties, including a breakdown of dynamic scaling even in its "restricted" sense.[7]

In order to understand the significance of the three regions discussed above, it is important to determine their boundaries in the d-n plane, away from the line $d=4$. As discussed in Ref. 16, there is a boundary between regions I and II given by the relation

$$2+c\eta = 2+\alpha/\nu, \tag{5.6}$$

(assuming $c \geq 0$). This curve, is close to, but not identical with the curve $\alpha = 0$ (cf. Fig. 4).

The boundaries, and indeed the very existence, of region III are more problematical. From the analysis of the full renormalization group, to be presented in Ref. 16, it is not possible to determine the boundaries of region III unambiguously because the corrections of order ϵ^2 to the recursion relations appear to be divergent for $2 < n < 4$. One possibility is that region III extends to finite ϵ, and possesses boundaries as shown in Fig. 4(a). (Note that the curve separating regions III and II has zero slope near $n=2$, $d=4$, and there is a "tail" of region II coming down between region I and region III near $n=4$, $d=4$.) Another possibility, however, is that the recursion relations of Sec. IV do not give the exponent z correctly even to first order in ϵ, for $2 < n < 4$. In that case, region III would not exist for finite ϵ, and the phase diagram would be the one shown in Fig. 4(b), where the area $2 < d < 4$, is simply divided into two regions. This possibility was not considered in earlier presentations[13,14] of the results of this work, but it is consistent with our calculations.

It should be noted that in the case where region III has a finite area [Fig. 4(a)], its "upper" boundary is not determined by our reasoning, and the shape depicted in Fig. 4(a) merely represents a plausible phase diagram. It has the property that the point $d=3$, $n=1$ (i.e., the three-dimensional Ising-like model) belongs to region II. Another phase diagram, which is equally consistent with our information, but perhaps aesthetically less attractive, is shown in Fig. 5(a), where there are two disjoint pieces of region II, and region III may even extend far enough to include the three-dimensional kinetic Ising model. Finally, we remind the reader that in the problem of critical dynamics, just as in the simpler question of the static critical behavior, it is possible in principle to have an arbitrarily complicated phase diagram, such that the points of greatest physical interest (i.e., points along the line $d=3$) lie within various interior regions where the critical behavior cannot be divined by any analytic continuation from $\epsilon \to 0$ or $n \to \infty$. One of the more pessimistic pictures of this situation is shown in Fig. 5(b).

We shall end this subsection by making some remarks on the behavior of the energy relaxation in case C. First, let us note that a diagrammatic analysis[16] implies that for any fixed $T > T_c$, in the limit $k \to 0$, the exact transport coefficient λ_E will be precisely equal to the bare transport coefficient λ_0^E. This would seem to imply a scaling exponent for the energy equal to $z_E = 2+\tilde{\alpha}/\nu$, even in region I. In fact, however, it turns out that in region I, the energy response function does not have the scaling form, with a unique frequency scale. In particular, at T_c the relaxation rate for the energy is the same in case C as in case A, and

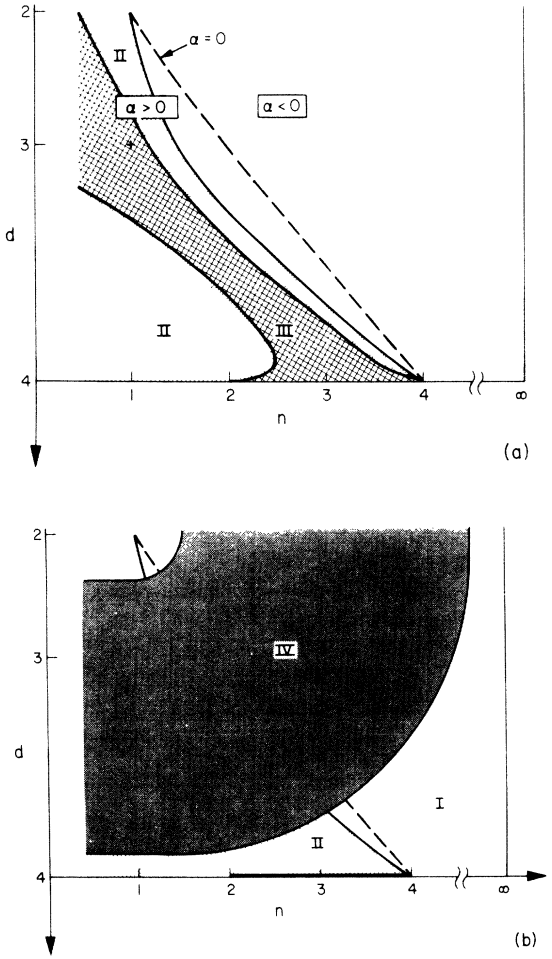

FIG. 5. (a) An alternate form of the phase diagram similar to Fig. 4(a), but where the anomalous region (III), extends up to the three-dimensional Ising-like model. (b) A schematic diagram showing a situation in which most the the d-n plane, and in particular the line $d=3$, is occupied by an unknown region (IV) whose properties cannot be inferred from expansions near $d=4$ or $n \to \infty$.

the energy scales with the exponents given in (5.2) or (5.3), and not with the exponent $2+\tilde{\alpha}/\nu$.

From a heuristic point of view we may now characterize the phase diagram for case C as follows. In that part of the phase diagram (region I) where the characteristic relaxation rate of the energy, at T_c, is already slower in case A than the value $\lambda_0^E k^2/C(k)$, the imposition of energy conservation does not affect the critical behavior of either the energy relaxation or the order-parameter relaxation. Where the relaxation rate $k^{2+c\eta}$ is faster than $\lambda_0^E k^2/C(k)$, the imposition of energy conservation must reduce the energy relaxation to a rate $\simeq \lambda_0^E k^2/C(k) \sim k^{2+\alpha/\nu}$ [cf. the inequality (3.12)], and the nonlinear coupling between the energy and order-parameter relaxation modes will tend to reduce the relaxation rate of the order parameter as well. If the coupling between the modes is strong enough, in some sense, the order-parameter relaxation rate will be pulled down until it has the same exponent as the energy (region II). If the coupling is not sufficiently strong, the order-parameter relaxation may lie somewhere between $k^{2+\alpha/\nu}$ and $k^{2+c\eta}$ (region III).

C. Cases B and D

In these cases the order parameter ψ is conserved, and the recursion relations do not lead to any renormalization of the transport coefficient λ_ψ, in analogy to the result for λ_E in case C. Thus the conventional theory holds for the order parameter, and we have $z=4-\eta$ [Eq. (3.38)]. Furthermore, at least in case B, we find[16] that there are no terms in the complete perturbation expansion of $\Sigma(k,\omega)$ which behave as $i\,\omega/k^2$ when ω and k go to zero, for fixed $T>T_c$, so that in the long-wavelength limit the exact transport coefficient λ_ψ is equal to its bare value λ_0. For case B we have also calculated $\Sigma(k,\omega)$ explicitly *at* T_c to order ϵ^2, and have confirmed that the conventional theory applies to that order.[5]

VI. APPLICABILITY OF MODELS AND UNIVERSALITY OF DYNAMIC CRITICAL BEHAVIOR

In this section we wish to elucidate the connection between the models we have studied and microscopically defined systems; in particular, we shall attempt to identify those properties of a real system which affect dynamic critical behavior.

As our first example, let us consider a collection of Ising spins on a lattice, each one of which is coupled to a reservoir consisting of a set of oscillators. The total system obeys reversible equations of motion, which may be transformed into *irreversible* equations for the spin system alone, by a trace over the reservoir variables. In this way it is possible to "derive" Glauber's kinetic Ising model[17] from a microscopic starting point. Depending on the form of the spin-reservoir coupling in the original Hamiltonian, the model may or may not obey spin conservation or energy conservation (i.e., conservation of energy in the spin system alone). In order to obtain the models of the present paper from the Ising models, one merely takes spatial averages over regions of linear dimension Λ^{-1}, just as one passes from an Ising model to the Ginzburg-Landau-Wilson model[1,4] in the static case. When properly carried out, this procedure must preserve the conservation laws.

As a second example, consider a model described by the classical *Hamiltonian:*

$$\mathcal{H}=\int d^dx\,(r_0\psi^2+u_0\psi^4+|\nabla\psi|^2+\Pi^2), \tag{6.1}$$

where $\psi(x)$ is a scalar displacement field and $\Pi(x)$ is the conjugate momentum field, with $r_0<0$ and wave vectors restricted to be less than the cutoff Λ. This model may be used to represent the anharmonic phonon gas in a solid undergoing a structural phase transition. Far from the transition this system can be considered to be a set of weakly damped modes ("dressed phonons") with finite frequencies. As the temperature approaches T_c, the frequencies of one group of modes (the "soft modes") go to zero,[29] and the modes become overdamped. [For situations represented by (6.1), in which ψ is not "conserved," these soft modes may either be acoustic phonons at a point in the Brillouin zone away from the zone center, or they may be optic phonons anywhere in the zone.] In principle, one could imagine applying the renormalization group directly to the diagrammatic series generated by the Hamiltonian (6.1). Once the long-wavelength modes are overdamped, however, the renormalized propagator for ψ will look precisely like (3.23), in the limit of long wavelengths and low frequencies. The renormalized four-phonon vertex will not be regular in frequency, however, but will have a singularity near zero frequency for small values of the momentum transfer, which is a consequence of the energy conservation in this model. The introduction of the energy density as an auxiliary field in model C, enables one to keep track separately of the singular and nonsingular parts of the four-phonon vertex, in a simple approximate fashion. Thus, we expect that the time-dependent critical behavior of the anharmonic phonon system described by (6.1) should be the same as that of model C above, for $n=1$. It must be noted, however, that the temperature region over which this equivalence holds, i.e., where one may neglect

the real part or propagating component of the phonon frequencies, may be quite small.

Turning now to purely magnetic systems, consider a three-dimensional anisotropic Heisenberg ferromagnet or antiferromagnet, in which the coupling constants for the three components of the spin are all different. In this system there is no conservation law other than conservation of energy, and the only low-frequency mode at long wavelengths, for $T \neq T_c$, is the thermal diffusion mode. Since the spin has a single easy direction, the order parameter is characterized by $n = 1$, and the static exponents are expected to be the same as those of the Ising model. Similarly we expect that the dynamic critical properties should be the same as those of the model with energy conservation, i.e., of case C with $n = 1$.

In the uniaxial Heisenberg ferromagnet, the couplings of the x and y components of the spin are equal, but smaller than the coupling of the z component. The order parameter is the z component of the spin, which is now conserved by the Hamiltonian. We expect that the dynamic critical behavior should be the same as that of case D.

In the uniaxial antiferromagnet, the order parameter is the z component of the staggered magnetization which is not conserved. In addition to the energy, there is now a second conserved quantity, the z component of the total magnetization. However, under repeated application of the renormalization group, the (nonlinear) coupling of the order parameter and the energy to long-wavelength fluctuations of the z component of the magnetization becomes vanishingly small, and we do not expect this variable to affect the critical dynamics. Thus the uniaxial antiferromagnet should have the same critical behavior as case C above.

Considering more general systems undergoing phase transitions, we can immediately identify a number of Hamiltonians for which the phenomenological models discussed in this paper are *not* appropriate. Examples include systems whose hydrodynamics lead to low-frequency propagating modes,[7] such as a superfluid, or an isotropic magnetic system obeying ordinary spin dynamics and not coupled to a reservoir. The phenomenological equations appropriate to an isotropic Heisenberg ferromagnet, for example, would not be those in models B or D, but rather

$$\frac{\partial \vec{s}}{\partial t} = \gamma_0 \vec{s} \times \frac{\delta \mathcal{H}}{\delta \vec{s}} , \tag{6.2}$$

with $\mathcal{H}$ given by Eq. (2.2), with $n = 3$.

It is clear from the preceding discussion that a class of systems showing the same static critical behavior may be divided into several classes of dynamic behavior. For example, the four cases treated in this paper all correspond to systems with the same static properties, for given n and d. Moreover, these four cases by no means exhaust the possible types of dynamic behavior, since the classical isotropic Heisenberg ferromagnet and antiferromagnet also correspond to the same static Hamiltonian.

As in the static case, a complete classification of all universality classes of dynamic critical behavior has not been achieved, but it is apparent that the important features are conservation laws, and the symmetry of the coupling of the order parameter to other slowly varying physical quantities. These features determine the low-lying modes of the system, which in turn govern the dynamic critical behavior.

VII. EXPERIMENTAL CONSEQUENCES

Generally speaking, the time-dependent Ginzburg-Landau models considered in the present paper are rather well approximated by the conventional theory, since the deviations of the critical exponents from their conventional values are of order α or η, which are numerically small in all known cases. Thus, in order to observe these deviations, which constitute the main result of our work, extremely accurate measurements would have to be made. It was argued above that there exist systems in nature whose critical dynamics should in principle be well approximated by the time-dependent Ginzburg-Landau models. Interesting examples are uniaxial antiferromagnets such as FeF_2, and perhaps also binary alloys. A straightforward method of obtaining the critical relaxation frequency $\omega_\psi(k)$ is by inelastic neutron diffraction. The results on FeF_2[30] certainly support the conclusions of the present work, that the exponent z is close to 2, but they are not able to distinguish between the conventional result and the various cases calculated above. A more promising method of measuring z is via the NMR linewidth $\Delta \sim \kappa^{-x/\nu}$, which according to restricted dynamic scaling[7] has an exponent

$$x/\nu = z - d + 2 - \eta . \tag{7.1}$$

The temperature dependence of Δ has been measured quite accurately by Gottlieb and Heller,[31] who found

$$x^{\text{expt}} = 0.67 \pm 0.02 . \tag{7.2}$$

A comparison with the different possible theoretical predictions is rendered difficult by uncertainties in the values of the static exponents for either FeF_2 or the ideal three-dimensional Ising

TABLE I. Exponent x for the NMR linewidth.

	$x=\nu(z-1-\eta)$	$\alpha=0.125$ $\nu=0.625$ $\eta=0$	$\alpha=0.08$ $\nu=0.64$ $\eta=0.047$	$\alpha=0.125$ $\nu=0.64$ $\eta=0.047$
Conventional $z^{\text{conv}}=2-\eta$	$x^{\text{conv}}=\nu(1-2\eta)$	0.625	0.58	0.58
Case A $z^{A}=2+c\eta$ $=2+\frac{1}{2}\eta$	$x^{A}=\nu(1-\frac{1}{2}\eta)$	0.625	0.625	0.625
Case C $z^{C}=2+\alpha/\nu$	$x^{C}=\nu+\alpha-\nu\eta$	0.75	0.69	0.735
Expt. (Ref. 31)		$x^{\text{expt}}=0.67\pm0.02$		

model. In particular, the presently accepted "best values" for the latter do not satisfy static scaling,[32] which is certainly necessary for the renormalization-group ideas to be valid. Nevertheless, we may make different choices of α, ν, and η, and compare the predictions for z in the conventional theory, and in cases A and C treated above. The corresponding values of x [Eq. (7.1)] are listed in Table I for three sets of plausible values of α, ν, and η; the first two sets satisfy static scaling, and the last one represents the best values from series estimates.[32] It is seen that in all cases the conventional result is too small, and cases A and C agree somewhat better with the experiment.

It is difficult to decide *a priori* whether case A or case C is more applicable to the experimental situation, since in a real crystal the phonon thermal conductivity may be much larger than that of the isolated spin system. In the experimentally obtainable range of temperatures, the extrapolated thermal relaxation rate given by $\lambda^{E}\kappa^{2}/\rho C_{p}$ may be much faster than the characteristic rate of relaxation of the order parameter. Thus one has not attained the critical region described by case C, in which the two relaxation rates are comparable. The exchange of energy between the lattice and spins may be sufficiently slow so that one cannot describe the dynamics as in case A, in which the system freely interacts with a heat bath of infinite thermal conductivity. At the same time, energy exchange with the lattice may be too fast to consider the spin system as thermally isolated from the lattice, so that case C could be applied to the spin system alone. It is therefore likely that the observable dynamics might be intermediate between case A and case C, as seems to be suggested by the experimentally measured exponent (7.2). However, in the absence of a more detailed analysis of the spin-phonon system it is not possible to draw firm conclusions from the NMR linewidth, other than to say that the deviations from the conventional theory are in the same direction as our theory.

A method of obtaining similar information to the NMR linewidth is via perturbed-angular-correlation measurements.[33] At present, however, experiments have only been performed on Ni, which is Heisenberg-like. The results there are rather far from the scaling prediction,[7] and qualitatively different from the neutron scattering results.[34]

In closing let us mention the binary alloy Ni_3Mn, whose time-dependent behavior has recently been investigated by neutron scattering.[35] The quoted exponent ($z\nu=1.04\pm0.09$) is rather far from the expected value $z\nu\approx1.3$ for an Ising-like system. The reason for this difference is not presently understood.

Note added in proof. The recursion relation approach has recently been applied to a number of models in which the master equation is non-Hermitian, and where one or more transport coefficients are found to diverge at the critical point. These include simplified models for the binary fluid separation and gas liquid critical points, the isotropic Heisenberg antiferromagnet, the XY model, and the superfluid transition in helium. Results are given in B. I. Halperin, P. C. Hohenberg, and E. Siggia (unpublished).

*Alfred P. Sloan Foundation Fellow. Supported in part by the National Science Foundation under Grant GP-38627X, and U. S. AFOSR Contract F44620-70-C-0028.

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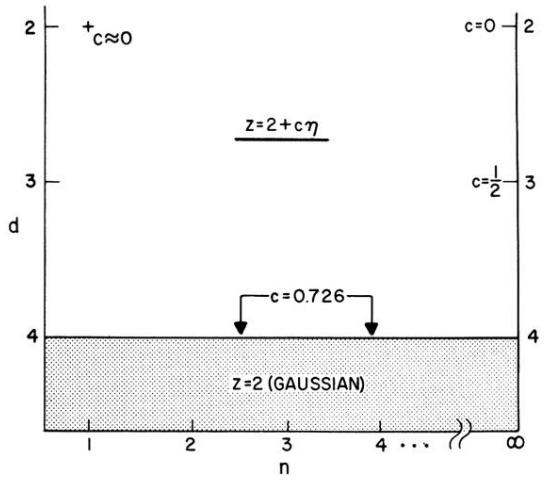
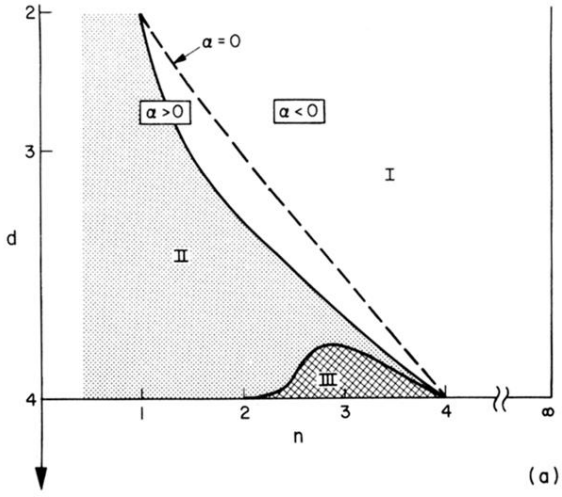


FIG. 3. Dynamic critical exponent z for the time-dependent Ginzburg-Landau model with neither energy nor order parameter conserved (case A). Ordinate and abscissa are lattice dimensionality d and order parameter dimensionality n. Values of z at boundaries come from expansions in $(4-d)$ or n^{-1}, value at $d=2$, $n=1$ comes from high-temperature-series expansion for the two-dimensional kinetic Ising model (Ref. 26).



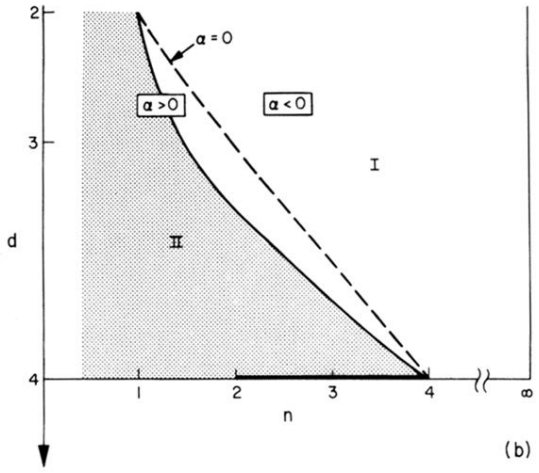


FIG 4. Possible "phase diagram" for the dynamic critical behavior in case C, where energy is conserved but the order parameter is not. In region I energy conservation does not affect the dynamic critical exponent for the order parameter, which has the value $z = 2 + c\eta$, found in case A (Fig. 3). In region II the energy and order parameter have the same critical exponent $z = z_E = 2 + \alpha/\nu$, which is larger than the value $z = 2 + c\eta$ for case A. In region III, z obeys the inequality $2 + c\eta < z < 2 + \alpha/\nu$, and dynamic scaling does not hold. The dotted line $\alpha = 0$ separates the region where the specific heat diverges ($\alpha > 0$), from the region where it is finite at T_c ($\alpha < 0$). The boundary between regions I and II is determined by the relation $c\eta = \alpha/\nu$. We have assumed $c > 0$ for $2 < d < 4$, and $c = 0$ for $d = 2$, so that the dotted line ($\alpha = 0$) lies in the interior of region I for $2 < d < 4$, but meets the boundary at $d = 2$ and $d = 4$. The existence of region III, as indicated in Fig. (a), is suggested by the recursion relation analysis of Sec. IV. An alternate phase diagram for case C, in which region III is absent, is indicated in Fig (b). If this latter phase diagram is correct, then the approximate recursion relations do not give the exponent z correctly to first order in $\epsilon = 4 - d$, for $2 < n < 4$.

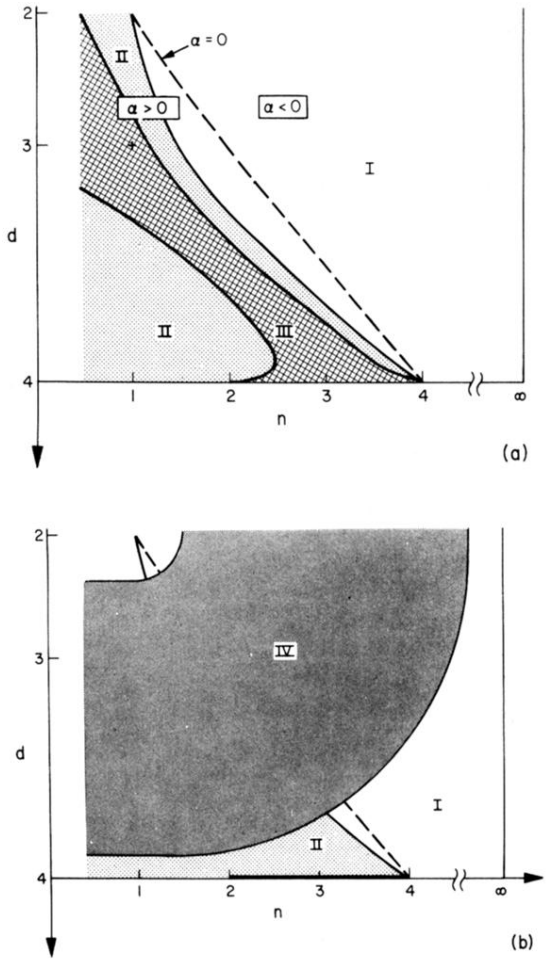


FIG. 5. (a) An alternate form of the phase diagram similar to Fig. 4(a), but where the anomalous region (III), extends up to the three-dimensional Ising-like model. (b) A schematic diagram showing a situation in which most the the d-n plane, and in particular the line $d=3$, is occupied by an unknown region (IV) whose properties cannot be inferred from expansions near $d=4$ or $n\to\infty$.