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Approximate method for ferromagnetic resonance in metals*

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We present a useful new approximate calculation of ferromagnetic resonance in isotropic metal plates. The key step in this calculation is a transformation, previously applied to magnetoelastic insulators, which permits the separation of the resonant and nonresonant senses of polarization. As a result, we are able to reduce the dispersion relation resulting from simultaneous solution of the Landau-Lifshitz equation and of Maxwell's equations from a quartic, for arbitrary directions of the magnetization, to a quadratic. Assuming a simple (but arbitrary) spin-pinning boundary condition on the surface rf magnetization, the surface impedance Z can then be obtained from a linear equation. Near resonance, at X or K band, we find that Z agrees with values from the exact calculation to within a fraction of 1% for all magnetization directions. The resonant frequency and linewidth both agree with the exact calculation to better than 1 Oe.

I. INTRODUCTION

The problem of ferromagnetic resonance (FMR) in metals has been studied extensively, both theoretically and experimentally since the pioneering work of Ament and Rado[1] in 1955. However, the complexity of calculations for general cases has meant that theoretical studies have largely been confined to parallel[1–4] or perpendicular[5] resonance, or have involved approximations which are not necessarily justified. It is only relatively recently that a method had been presented for the general solution of the problem of resonance in an infinite isotropic metallic plate, in which the dc magnetic field is at an arbitrary direction with respect to the plate normal.[6] The anisotropic plate has been discussed even more recently.[7,8] While this method has been used with success for some problems,[9,10] it has not been widely applied, owing to the fact that the dispersion relation is quartic in k^2, and that the boundary conditions involve a quadratic whose coefficients are determinants.[6] This has meant that a given problem involves a fairly lengthy computer calculation. Very recently, Kobayashi *et al.*[11] have reported an approximate method for a related problem, the magnetoelastic insulating plate. For this problem, even in the magnetostatic limit, the dispersion relation is quintic in k^2. It was found that by a simple transformation one can reduce the solution of the part with physical interest to a quadratic, or at worse, a cubic. This approximation has been applied both to resonance[12] and to phonon generation.[13] We shall demonstrate here that by the same transformation the problem of FMR in metals can be considerably simplified, making it easily applicable. We treat only the isotropic case here, although the extension to the anisotropic problem is not very difficult, and will be reported elsewhere.[14]

II. DISPERSION RELATIONS

Macroscopic analysis of FMR in metals[1–8,15] consists of the simultaneous solution of Maxwell's equations and the Landau-Lifshitz equation. Assuming that the dielectric terms in the electric field are negligible compared to the conduction terms, the former may be written

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{H}+4\pi\vec{M}), \qquad (1)$$

$$\nabla\times\vec{H} = (4\pi\sigma/c)\vec{E}, \qquad (2)$$

$$\nabla\cdot(\vec{H}+4\pi\vec{M}) = 0, \qquad (3)$$

while the latter is written

$$\frac{1}{\gamma}\frac{\partial\vec{M}}{\partial t} = \vec{M}\times\left[\vec{H}+\frac{2A}{|M|^2}\nabla^2\vec{M}-\frac{\lambda}{\gamma|M|^2}(\vec{M}\times\vec{H})\right]. \qquad (4)$$

Equations (1) and (2) may be combined to eliminate E:

$$\nabla\times\nabla\times\vec{H} = -\frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t}(\vec{H}+4\pi\vec{M}). \qquad (5)$$

We now assume that the sample is a slab, infinite in the x and z directions, and that the static magnetization $\vec{M}_0$ lies in the y-z plane. $\vec{H}_0$, the sum of the static applied and demagnetizing fields, lies parallel to $\vec{M}_0$. Then we may write

$$\vec{H} = \vec{H}_0+\vec{h}\,e^{i(\omega t-ky)}, \qquad (6a)$$

$$\vec{M} = \vec{M}_0+\vec{m}\,e^{i(\omega t-ky)}. \qquad (6b)$$

The form of Eq. (4) assures that $\vec{m}$ is normal to

$\vec{M}_0$, so that only two components of $\vec{m}$ are independent. We now substitute Eqs. (6) into Eqs. (3)–(5), and make a small-signal approximation, so that we may neglect terms second order and higher in h and m. We find that we may eliminate h, by applying the relation

$$m_\alpha = -Qh_\alpha, \quad \alpha = x, z \tag{7}$$

where

$$Q = (1/4\pi)(1 - \tfrac{1}{2} i\,\delta^2 k^2),$$

and

$$\delta^2 = c^2/2\pi\sigma\omega .$$

Neglecting the exchange field in the damping term[16] we find[15]

$$-(a+i\Omega)m_x + (c\sin\theta + b\cos\theta\cot\theta)m_y = 0, \tag{8a}$$

$$b\sin\theta\, m_x + (d\sin^2\theta + a\cos^2\theta + i\Omega)m_y = 0 . \tag{8b}$$

In Eqs. (8), θ is the angle between M_0 and the y axis, and

$$a = \frac{\lambda}{\gamma}\left(\frac{H_0}{M_0} + \frac{1}{Q}\right), \qquad b = H_0 + \frac{M_0}{Q} + \frac{2A}{M_0}k^2,$$

$$c = H_0 + 4\pi M_0 + \frac{2A}{M_0}k^2, \quad d = \frac{\lambda}{\gamma}\left(\frac{H_0}{M_0} + 4\pi\right), \quad \Omega = \omega/\gamma.$$

For a solution to Eqs. (8) to exist, the determinant of the coefficients of m must be zero. This yields the quartic secular equation[6] which is the starting point of the previous work. At $\theta = 0$, this equation factors into two quadratics,[5] and at $\theta = 90°$, it factors into a cubic and a linear equation.[1–3] But for other angles it does not factor exactly. However, it is possible to find an approximate factorization which is extremely accurate in almost all cases. Following Kobayashi *et al.*[11] we define

$$m_y = -m_\theta \sin\theta, \quad m_x = m_\varphi . \tag{9}$$

Then,

$$\mathcal{G}\begin{pmatrix} m_\theta \\ m_\varphi \end{pmatrix} = 0, \tag{10}$$

where

$$\mathcal{G} = \begin{pmatrix} c\sin^2\theta + b\cos^2\theta & a + i\Omega \\ -(d\sin^2\theta + a\cos^2\theta + i\Omega) & b \end{pmatrix}. \tag{11}$$

We wish to find an approximate diagonalization of G. In the absence of conductivity, G reduces to

$$\mathcal{G} = \begin{pmatrix} H_0 + (2A/M_0)k^2 + 4\pi M_0\sin^2\theta & i\Omega \\ -i\Omega & H_0 + (2A/M_0)k^2 \end{pmatrix}. \tag{12}$$

$\mathcal{G}$ can be exactly diagonalized. In fact this is done by the same transformation

$$\overline{\mathcal{G}} = U^\dagger \mathcal{G} U, \tag{13}$$

previously used by Kobayashi and co-workers[11–13] to diagonalize the uniform precession mode. That is, U is given by

$$U_{11} = |\Omega|/(2\Omega_0^2 + 4\pi M_0\sin^2\theta\,\Omega_0)^{1/2},$$

$$U_{12} = i|\Omega|/(2\Omega_0^2 - 4\pi M_0\sin^2\theta\,\Omega_0)^{1/2},$$

$$U_{21} = \frac{-i(2\pi M_0\sin^2\theta + \Omega_0)}{(2\Omega_0^2 + 4\pi M_0\sin^2\theta\,\Omega_0)^{1/2}}, \tag{14}$$

$$U_{22} = \frac{(2\pi M_0\sin^2\theta - \Omega_0)}{(2\Omega_0^2 - 4\pi M_0\sin^2\theta\,\Omega_0)^{1/2}},$$

$$\Omega_0 = [(2\pi M_0\sin^2\theta)^2 + \Omega^2]^{1/2} .$$

Now let us find

$$G' = 2\Omega_0 \overline{G}, \tag{15}$$

where

$$\overline{G} = U^\dagger G U . \tag{16}$$

Then,

$$G'_{11} = \left\{2\left(H_0 + \frac{2A}{M_0}k^2\right) + M_0\left[4\pi\sin^2\theta + \frac{1}{Q}(1+\cos^2\theta)\right] - 2\Omega_0\right\}\Omega_0 + \frac{2\pi(M_0\sin^2\theta)^2}{Q} + \frac{i\lambda}{\gamma}\Omega\left[\frac{2H_0}{M_0} + 4\pi\sin^2\theta + \frac{1}{Q}(1+\cos^2\theta)\right], \tag{17a}$$

$$G'_{22} = \left\{2\left(H_0 + \frac{2A}{M_0}k^2\right) + M_0\left[4\pi\sin^2\theta + \frac{1}{Q}(1+\cos^2\theta)\right]\right\}\Omega_0 + \left(4\pi - \frac{1}{Q}\right)2\pi(M_0\sin^2\theta)^2 + 2\Omega^2 - \frac{i\lambda}{\gamma}\Omega\left[\frac{2H_0}{M_0} + 4\pi\sin^2\theta + \frac{1}{Q}(1+\cos^2\theta)\right], \tag{17b}$$

$$G'_{12} = \left\{\frac{iM\Omega}{Q} + \frac{\lambda}{\gamma}\left[4\pi H + \left(4\pi - \frac{1}{Q}\right)\Omega_0 + 2\pi M_0\left(4\pi\sin^2\theta + \frac{1+\cos^2\theta}{Q}\right)\right]\right\}\sin^2\theta, \tag{17c}$$

$$G'_{21} = -\left\{\frac{iM\Omega}{Q} + \frac{\lambda}{\gamma}\left[4\pi H + \left(\frac{1}{Q} - 4\pi\right)\Omega_0 + 2\pi M_0\left(4\pi\sin^2\theta + \frac{1+\cos^2\theta}{Q}\right)\right]\right\}\sin^2\theta . \tag{17d}$$

As before, the dispersion relations are given by $\det G' = 0$. If G'_{12} and G'_{21} are negligible, the two branches with the resonant sense of polarization, that is, the "uniform precession" and propagating spin-wave branches are given by $G'_{11} = 0$, or

$$\frac{iA\Omega_0\delta^2}{M_0}k^4-\left[\frac{2A\Omega_0}{M_0}+(H_0+2\pi M_0\sin^2\theta)\left(\frac{\lambda\Omega}{\gamma M_0}-i\Omega_0\right)\frac{\delta^2}{2}+\frac{i\Omega_0^2\delta^2}{2}\right]k^2-(H_0+4\pi M_0)\left(\Omega_0+\frac{i\lambda\Omega}{\gamma M}\right)+\Omega^2=0\ . \tag{18}$$

The two branches with a nonresonant sense of polarization, that is, the branches corresponding to ordinary electromagnetic propagation and to the rapidly decaying spin wave, are given by $G'_{22}=0$, or

$$\frac{iA\Omega_0\delta^2}{M_0}k^4+\left[(H_0+2\pi M_0\sin^2\theta)\left(\frac{\lambda\Omega}{\gamma M_0}+i\Omega_0\right)\frac{\delta^2}{2}-\frac{2A\Omega_0}{M_0}+\frac{i\Omega_0^2\delta^2}{2}\right]k^2-(H_0+4\pi M_0)\left(\Omega_0-\frac{i\lambda\Omega}{\gamma M}\right)-\Omega^2=0\ . \tag{19}$$

It is obvious that for $\theta=0$ (perpendicular resonance), $G'_{12}=G'_{21}\equiv 0$. That is, the factorization is exact, and we obtain the well-known results for this case.[5] For other angles the factorization is always approximate. However, the off-diagonal terms are typically of the order of 10^{-2} compared to the diagonal terms, for materials such as nickel or permalloy, so that they produce a term in the determinant which is 10^4 smaller than that due to the diagonal terms. The only case in which the approximation fails is near antiresonance[2] (FMAR), at which the wave vector of the "uniform-precession" branch becomes extremely small. Thus, for this branch, $1/Q$ is no longer small, but approaches 4π. Even in this case, the approximation works for angles up to about 20° since $\sin^2\theta$ keeps the off-diagonal terms small for these angles. At FMR, since the k for the uniform-precession branch becomes relatively large, the approximation works even better than usual. Comparing with the exact calculation[6] for Ni near resonance at K- and X-band frequencies, we find that the approximate calculation gives the dispersion relation for the branches with resonant polarization to better than 1%. That is, it can be applied with confidence to such problems.

III. CHARACTERISTIC POLARIZATION AND BOUNDARY CONDITIONS

There are two pairs of roots to Eq. (18), corresponding to the almost uniform precession and the resonant spin wave. They both have approximately the polarization μ_1, given by

$$\begin{pmatrix} m_\theta \\ m_\varphi \end{pmatrix}=U\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}. \tag{20}$$

The nonresonant spin wave and normal electromagnetic propagation branch, the solutions of Eq. (19), have approximately the polarization μ_2. These two, μ_1 and μ_2 are approximately the characteristic polarizations[6,15] for which Z, the surface impedance, is a constant. That is,[1,6]

$$Z=\frac{4\pi\sigma}{c}\frac{e_{0z}}{h_{0x}}=-\frac{4\pi\sigma}{c}\frac{e_{0x}}{h_{0z}}\ . \tag{21}$$

In Eq. (21) e_0 and h_0 are the electric and magnetic field components at the surface. It is easy to show[6,15] that for orthogonal components at any angle to the x and z axes, the same relationship holds. The characteristic polarizations are transmitted and reflected without change of polarization[6,15] so that any arbitrary polarization, which does change, can be studied by decomposing it into μ_1 and μ_2.

Following Kobayashi *et al.*,[11] we may assume that μ_2 is negligibly excited. Then

$$m_\theta=U_{11}\mu_1,\quad m_\varphi=U_{21}\mu_1\ . \tag{22}$$

That is, the resonant polarization is determined [see Eq. (14)] by M, Ω, and θ, but is virtually independent of conductivity, damping, magnetoelastic coupling, etc.

Now, Eqs. (21) and (22) imply that it is sufficient to satisfy three boundary conditions at each surface: continuity of h_x, continuity of e_z, and a pinning condition on μ_1. Conditions on h_z and e_x are redundant, and a condition on μ_2 is unnecessary as $\mu_2\equiv 0$ will automatically be satisfied. Thus, we have half the boundary conditions necessary for the exact calculation. These conditions are to be satisfied by

$$\mu_1=\left(\sum_{i=1,2}A_i\cos k_iy+\sum_{i=1,2}B_i\sin k_iy\right)e^{i\omega t}\ , \tag{23}$$

where k_1, k_2 are the solutions of Eq. (18) with positive real part, and A_i and B_i are to be determined. We lose no physical interest by assuming symmetrical surfaces and symmetrical excitation, so that (omitting the time dependence)

$$\mu_1=C_1\cos k_1y+C_2\cos k_2y\ . \tag{24}$$

We assume that the plate surfaces are at $y=\pm d$. Then the continuity of h is satisfied by applying Eq. (7), to yield

$$\frac{C_1}{Q_1}\cos k_1d+\frac{C_2}{Q_2}\cos k_2d=-\frac{h_{0x}}{U_{21}}\ . \tag{25}$$

Inside the medium, by applying Eq. (2), we find[1,4]

$$e_{iz}=\frac{ic}{4\pi\sigma}k_ih_{ix}=\frac{c}{4\pi\sigma}\left(\frac{C_1}{Q_1}k_1\sin k_1y+\frac{C_2}{Q_2}k_2\sin k_2y\right)\ . \tag{26}$$

Outside, Eq. (21) is satisfied at $y=\pm d$. Then the continuity of e is given by

$$\frac{C_1}{Q_1}k_1\sin k_1d+\frac{C_2}{Q_2}k_2\sin k_2d=-\frac{Zh_{0x}}{U_{21}}\ . \tag{27}$$

For the condition on μ, we assume a simple pinning condition, that is, if μ is pinned

$$C_1 \cos k_1 d + C_2 \cos k_2 d = 0 \ . \tag{28}$$

If μ is free

$$C_1 k_1 \sin k_1 d + C_2 k_2 \sin k_2 d = 0 \ . \tag{29}$$

For intermediate pinning

$$a(C_1 \cos k_1 d + C_2 \cos k_2 d) + b(C_1 k_1 \sin k_1 d + C_2 k_2 \sin k_2 d) = 0 \ . \tag{30}$$

Here, the ratio of b to a represents the effective pinning strength. This ratio is assumed known.

It is relatively simple to extend the calculation to more complicated boundary conditions, such as surface anisotropy.

Simultaneous solution of Eqs. (25), (27), and one of (28)–(30) permits us to obtain the C's, and Z. For the spin-pinned case, we find

$$Z_p = \frac{k_1 Q_2 \sin k_1 d \cos k_2 d - k_2 Q_1 \sin k_2 d \cos k_1 d}{(Q_2 - Q_1) \cos k_1 d \cos k_2 d} \ . \tag{31}$$

For the spin-unpinned case

$$Z_u = \frac{(Q_1 - Q_2) k_1 k_2 \sin k_1 d \sin k_2 d}{k_1 Q_1 \sin k_1 d \cos k_2 d - k_2 Q_2 \sin k_2 d \cos k_1 d} \ . \tag{32}$$

In the general case

$$Z = \frac{[(Q_2 - Q_1) b\, k_1 k_2 \sin k_1 d \sin k_2 d + a(Q_2 k_1 \sin k_1 d \cos k_2 d - Q_1 k_2 \sin k_2 d \cos k_1 d)]}{[(Q_2 - Q_1) a \cos k_1 d \cos k_2 d + b(Q_2 k_2 \sin k_2 d \cos k_1 d - Q_1 k_1 \sin k_1 d \cos k_1 d)]} \ . \tag{33}$$

For the situations described above, in which the approximate dispersion relation is accurate, we find that both Z_u and Z_p agree with their values from the exact calculation to within a fraction of 1%, for all angles. They are, of course, exact at $\theta = 0$.

IV. POWER ABSORBED, RESONANT FREQUENCY, AND LINEWIDTH

We may calculate the power absorbed from[4]

$$P = \mathrm{Re}(Z) |h_0|^2 \ . \tag{34}$$

We may also calculate the change in the Q of the cavity[17,18]

$$\Delta(1/Q) = -c \,\mathrm{Im} \int m h_0^* \, dv \ . \tag{35}$$

This requires using Eqs. (25), (27), and the pinning condition to obtain C_1 and C_2. We find that for pinned boundary conditions

$$\Delta\left(\frac{1}{Q}\right) \propto \left(1 + \frac{|U_{11}|^2}{|U_{21}|^2}\right) \cos^2\theta \ \mathrm{Im}\left[\frac{Q_1 Q_2}{Q_1 - Q_2} \frac{k_1 \sin k_2 d \cos k_1 d - k_2 \sin k_1 d \cos k_2 d}{k_1 k_2 \cos k_1 d \cos k_2 d}\right] \ . \tag{36}$$

For the unpinned case

$$\Delta\left(\frac{1}{Q}\right) \propto \left(1 + \frac{|U_{11}|^2 \cos^2\theta}{|U_{21}|^2}\right) \mathrm{Im}\left[\frac{Q_1 Q_2 (k_1^2 - k_2^2) \sin k_1 d \sin k_2 d}{k_1 k_2 (k_1 Q_1 \sin k_1 d \cos k_2 d - k_2 Q_2 \sin k_2 d \cos k_1 d)}\right] \ . \tag{37}$$

Near resonance, a plot of P vs field or frequency, or of $\Delta(1/Q)$ vs H or ω yields very similar lines, and essentially the same resonant frequency and line shape.[15] For the conditions mentioned above the approximate values of resonant field and linewidth agree with that obtained from the exact calculation to within less than 1 Oe out of several thousand for the field, and out of several hundred for the width, for both pinned and unpinned boundaries, for all angles. For a (100) plate, including anisotropy,[14] the results are as good as these for the ⟨001⟩ plane, and only slightly inferior (3 Oe) for the ⟨011⟩ plane.

V. CONCLUSIONS

In order to perform FMR calculations for a metal plate, one solves Eq. (18) for the resonant k's. To find the power absorbed, the k values are substituted into Eq. (31) for pinned surfaces, or into Eq. (32) for unpinned surfaces. To find the change of Q of the cavity, one uses Eq. (36) or (37). The results of (31), (32), (36), or (37) as a function of H or ω yield the desired comparison with experimental results. Equation (18) is a quadratic in k^2 whereas Eq. (1) of Ref. 6 is quartic in k^2. The others are relatively simple, merely involving algebraic substitution, in contrast to Eq. (3) of Ref. 6, which is a quadratic whose coefficients are 4×4 determinants. Thus, the approximate method results in a much simplified calculation, with no essential loss in accuracy, except near FMAR, when the "uniform-precession" root is not accurately described by Eq. (18). For this case, it is possible to develop new approximations,[19] but these do not seem to be sufficiently advantageous, and it appears that one should apply the exact method to this case. But we believe that for all other problems, the approximate method is advantageous.

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