Thus, when the difference frequency ω_d is small compared to those of the fundamental signals, and the conductivity relaxation frequency ω_c is chosen to maximize the attenuation of the fundamental ($\omega_c \approx \omega_1$), the detected difference-frequency signal amplitude at ω_d will be comparable to that of the dc acoustoelectric voltage, and both will be much greater than the second-harmonic and sum-frequency amplitudes. The sequence of photographs presented in Fig. 3 illustrates the inverse frequency dependence of the difference frequency and, further, confirms that for small ω_d this detected voltage is comparable to its dc counterpart. This is to be expected since, in the limit as $\omega_d \rightarrow 0$, the difference-frequency acoustoelectric voltage becomes the dc acoustoelectric voltage.

These results describing the direct-voltage measurement of nonlinear acoustic-wave interactions in piezoelectric semiconductors may prove significant, since they indicate that an ultrasonic amplifier employed under gain conditions is capable of amplifying a modulated acoustic wave, detecting the modulation, filtering out the carrier and harmonics of the original signal, and converting the modulation portion of the acoustic wave directly into a large-amplitude voltage. This detected modulation signal is, of course, directly analogous to the aforementioned difference-frequency voltage.

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Carrier Lifetimes in Semiconductors with Two Interacting or Two Independent Recombination Levels

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Carrier recombination under (a) arbitrary steady-state and (b) small-signal near-equilibrium transient conditions has been studied theoretically for a two-interacting-level (ITL) model and a two-independent-level (IDL) model. Analytic solutions for carrier lifetimes have been obtained and manipulated into a form which facilitates comparison between the two models, as well as comparison between the steady-state and transient lifetimes as predicted by each model. It is shown that under small-signal steady-state and transient conditions the two interacting levels may be treated, with little loss of accuracy, as two independent levels, provided we describe the effective flaw density at each level by interacting-level equilibrium statistics. However, under appropriate conditions, the use of either ITL or IDL equilibrium statistics leads to essentially the same lifetimes; the ITL model is then indistinguishable from the IDL model. A comparison of the steady-state and transient lifetimes, whether of two interacting or two independent levels, shows that in certain circumstances the transient lifetime can exceed the sum of the steady-state electron and hole lifetimes, a possibility which does not exist if only one level is present. As a numerical example, the lifetimes in gold-doped silicon have been calculated and compared. Some possible applications of this work are proposed.

I. INTRODUCTION

THE recombination statistics for holes and electrons through a set of single-level flaws have been treated extensively in the literature.[1–6] In the classic treatment by Shockley and Read,[1] steady-state solutions were obtained for the lifetime of electrons and holes. The extension to the small-signal[7] transient situation was given by Sandiford[3] and Wertheim[4] and recently by Sah[5] who, in applying the equivalent-circuit approach to single-level flaws, also examined the transient case of small signals superimposed on arbitrary steady-state conditions. As pointed out by Normura and Blakemore,[6] a complete analytic solution is not possible for transient decay involving signal levels and flaw densities of arbitrary magnitude; but some numerical calculations, with analytic approximations in various ranges, have been given by these authors.

An obvious extension to a set of single-level flaws is the case of two or more sets of single-level flaws acting in concert. Steady-state solutions for arbitrary flaw densities in a two-independent-level (IDL) model were obtained by Okada[8] and Kalashnikov,[9] while the small-signal transient solution was given by Wertheim[4] for n-type material with the restriction that the total density of flaws is less than that of the free carriers. The IDL case is reducible to the trapping model of Hornbeck

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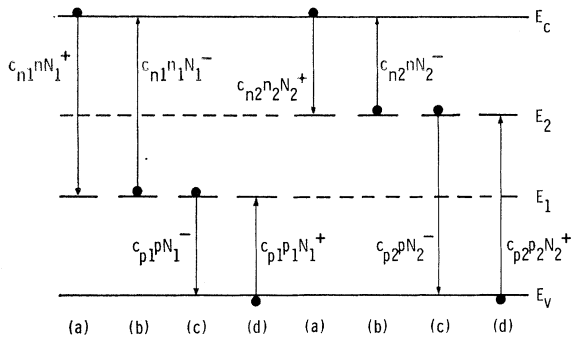

FIG. 1. Transitions between the bands and the flaws in a IDL model are (a) electron capture, (b) electron emission, (c) hole capture, and (d) hole emission.

and Haynes,[10] if the capture asymmetry at one of the levels is very large.[4,11,12]

In ascribing a single energy level to the flaws, the implicit assumption is that only two charge states are effective for each flaw. However, as is well known, impurities which are most effective as recombination centers in silicon and germanium are multiply charged, with more than one energy level. The equilibrium statistics for this case are formally different from those for the same number of independent levels, since the occupancy of the different levels is now interdependent.[13] The nonequilibrium statistics for multiply charged flaws have been considered in detail by Sah and Shockley[14] under steady-state conditions. Explicit solution for the small-signal steady-state carrier lifetimes for divalent centers has been given by Kalashnikov and Tissen[15] and by Laff and Fan.[16] More recently, Sah[17] has developed a small-signal equivalent circuit model for the general interacting-level time-dependent case. The effect of coupling between the levels could not be expressed in terms of a convenient circuit element, however, and it was necessary to resort to the use of a negative capacitance to represent this effect. Both steady-state and transient solutions were considered by Sah, but explicit solutions for the transient time constants were not given.

Although the statistical theory of the two interacting-level (ITL) model is formally different from that of the IDL model, it has often been supposed that the two cases are indistinguishable if their energy levels are more than a few kT's apart. Sah and Shockley[14] have shown this to be the case in terms of the steady-state recombination rate. However, it should be pointed out that any conclusion regarding the recombination rate does not necessarily apply to the lifetimes even in the steady state, since when trapping effects occur, the electron and hole lifetimes cannot be determined from the recombination rate alone.

The purpose of this paper is to obtain, from a ITL model and a IDL model, analytic solutions for carrier lifetimes under (a) arbitrary steady-state and (b) small-signal transient conditions, and to use these solutions as the basis for comparison between the two models, as well as comparison between the steady-state lifetimes and transient lifetime, as predicted by each model. Both the steady-state and transient solutions are valid for arbitrary flaw densities in both n- and p-type material. The transient solution for the IDL case is therefore an extension of Wertheim's result, while that for the ITL case is, to our knowledge, not available elsewhere.

II. THEORY

We assume that the semiconductor is homogeneous and nondegenerate. Moreover, we exclude any consideration of the effects due to excited states and spin degeneracy, so that the energy associated with each recombination level is an effective energy.[1,17] With little loss of generality, our treatment will be confined to a system of two, rather than an arbitrary number of recombination levels, since no more than two levels are generally effective at any one time. Although some of the results obtained are, as pointed out in Sec. I, already available in the literature, the derivation of these and other new results will be given in order to show the basic similarities and differences of the IDL and ITL models. The results to be presented have been manipulated into a form which facilitates comparison between the two models.

We shall begin with the IDL case which is algebraically simpler than the ITL case.

A. IDL Model

1. Rate Equations

Suppose that there are two sets of centers lying at energy E_1 and E_2, respectively, in the band gap of a homogeneous semiconductor. Their densities will be, for the moment, distinguished by N_1 and N_2, although later for comparison with the ITL case they will be made equal. Each flaw of type $j(j=1, 2)$ is capable of capturing one electron at an average rate c_{nj} when vacant, and one hole at a rate c_{pj} when occupied by an electron. The number of filled and empty centers are denoted by N_j^- and N_j^+, respectively, and these are related by the totality condition

$$N_j^-+N_j^+=N_j. \tag{1}$$

The four transition processes which occur between each set of flaws and the valence and conduction bands are shown in Fig. 1 where we have used the parameters

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n_j and p_j to denote the equilibrium carrier densities when the Fermi level passes through the flaws lying at E_j. For these two quantities, the following relationships apply:

$$\begin{aligned} n_j &= n_0 p_0/p_j = n_0 N_{j0}^{+}/N_{j0}^{-} \\ &= n_0 \exp[(E_j - F)/kT] \\ &= N_c \exp[(E_j - E_c)/kT], \end{aligned} \tag{2}$$

where we have used the subscript 0 to denote equilibrium quantities, F is the Fermi level, and N_c is the effective density of states at the conduction band edge E_c.

The rate equations describing the interaction of the flaws with the conduction and valence bands are

$$\begin{aligned} dn/dt &= -\sum_j R_{nj} + G, \\ dp/dt &= -\sum_j R_{pj} + G, \\ dN_j^{-}/dt &= R_{nj} - R_{pj}, \quad (j = 1, 2) \end{aligned} \tag{3}$$

where

$$\begin{aligned} R_{nj} &= c_{nj}(nN_j^{+} - n_j N_j^{-}), \\ R_{pj} &= c_{pj}(pN_j^{-} - p_j N_j^{+}), \end{aligned} \tag{4}$$

and G is the generation rate of electron-hole pairs due to external excitation. We assume that charge neutrality conditions apply so that, in terms of deviations from equilibrium densities, we have

$$\Delta p - \Delta n - \sum \Delta N_j^{-} = 0. \tag{5}$$

The above system of Eqs. (2)–(4) is valid for arbitrary values of excess carrier concentrations. For small deviations from thermal equilibrium condition such that $\Delta n \ll n_0$ and $\Delta p \ll p_0$, the equations are linear, and may by written in operator form as follows:

$$\begin{bmatrix} D+a_{11} & a_{12} & a_{13} \\ a_{21} & D+a_{22} & a_{23} \\ a_{31} & a_{32} & D+a_{33} \end{bmatrix} \times \begin{bmatrix} \Delta n \\ \Delta p \\ \Delta N_2^{-} \end{bmatrix} = \begin{bmatrix} g \\ g \\ 0 \end{bmatrix}, \tag{6}$$

where g is the small-signal component of the excitation function, D is the operator d/dt, and the a_{ij}'s are functions of the equilibrium statistics and capture coefficients, as given by Eqs. (A1) in the Appendix.

2. Steady-State Lifetimes

In the steady state, the time derivatives in Eqs. (3) disappear. By definition, the lifetimes for electrons τ_n and for holes τ_p are given by

$$\tau_n = \Delta n / \sum_j R_{nj}, \tag{7}$$

$$\tau_p = \Delta p / \sum_j R_{pj}. \tag{8}$$

From Eqs. (3) then, along with Eq. (5), we obtain the following expressions valid for arbitrary signal levels:

$$\tau_p/\tau_n = \Delta p/\Delta n = (1+\sum_j \mu_{nj})/(1+\sum_j \mu_{pj}), \quad (j=1, 2) \tag{9}$$

$$\tau_n^{-1} = [p_0 + \Delta p + (\tau_p/\tau_n) n_0] \sum_j (c_{nj} c_{pj} N_j / H_j), \tag{10}$$

where

$$\mu_{nj} = c_{nj} N_{j0}^{+}/H_j, \tag{11}$$

$$\mu_{pj} = c_{pj} N_{j0}^{-}/H_j, \tag{12}$$

$$H_j = c_{nj}(n_0 + \Delta n + n_j) + c_{pj}(p_0 + \Delta p + p_j). \tag{13}$$

Although not required in the above solutions, the following relationship is useful for subsequent comparison with the ITL model:

$$N_j^{+}/N_j^{-} = (c_{nj} n_j + c_{pj} p)/(c_{nj} n + c_{pj} p_j). \tag{14}$$

In general, Eqs. (9)–(13) can be solved only by directly or indirectly (through G) specifying either Δp or Δn, but not both, since these two quantities are implicitly related by Eq. (9), which contains H, as a function of Δn and Δp. To find Δn explicitly in terms of Δp or vice versa would involve the solution of a cubic equation. The resulting solution is by no means compact and will not be given here.

However, a convenient explicit relationship between Δp and Δn exists for signal levels where $\Delta p \ll (p_0 + p_j)$ and $\Delta n \ll (n_0 + n_j)$. Then,

$$\tau_n^{-1} = (\sum \tau_{jd}^{-1})/(1+\sum \mu_{pj}^{0}), \tag{15}$$

$$\tau_p^{-1} = (\sum \tau_{jd}^{-1})/(1+\sum \mu_{nj}^{0}), \quad (j=1, 2) \tag{16}$$

where the superscript 0 is used to designate equilibrium values,

$$\tau_{jd}^{-1} = c_{nj} c_{pj} N_j \left(n_0 + p_0 + \sum_{j=1}^{2} \frac{N_{j0}^{-} N_{j0}^{+}}{N_j} \right) \Big/ H_{j0}, \tag{17}$$

and

$$H_{j0} = c_{nj}(n_0 + n_j) + c_{pj}(p_0 + p_j). \tag{18}$$

3. Transient Lifetimes

In a study of transient decay such as the photoconductivity decay, the main concern is usually with the time constants of the system which are given by the negative inverse of the roots of the characteristic equation for the system matrix [Eq. (6)]

$$D^3 + \alpha_2 D^2 + \alpha_1 D + \alpha_0 = 0, \tag{19}$$

where

$$\begin{aligned} \alpha_0 &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{32} + a_{13}a_{21}a_{32}) \\ &\quad - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}), \\ \alpha_1 &= (a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33}) \\ &\quad - (a_{12}a_{21} + a_{23}a_{32} + a_{13}a_{31}), \\ \alpha_2 &= a_{11} + a_{22} + a_{33}. \end{aligned} \tag{20}$$

A general solution for the time constants cannot be obtained in any convenient analytical form unless cer-

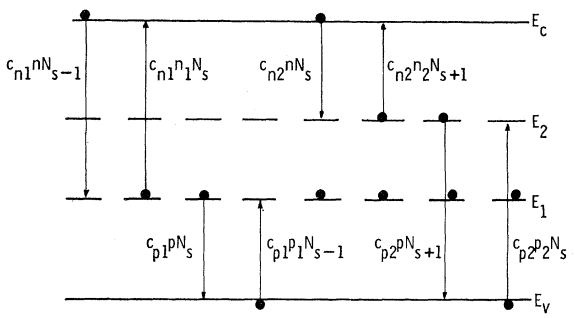

FIG. 2. Transitions between the bands and the flaws in a ITL model are (a) electron capture, (b) electron emission, (c) hole capture, and (d) hole emission.

tain simplifications are made. In most practical situations of interest, either one or two of the time constants are dominant. The measurement of photoconductivity decay would normally yield the longer time constants since these control the final transient decay.

When two of the constants are longer than the third, they are given approximately by

$$\tau_{a,b}=(\alpha_1/2\alpha_0)\pm[(\alpha_1/2\alpha_0)^2-(\alpha_2/\alpha_0)]^{1/2}. \quad (21)$$

If only one time constant is dominant, it is then given approximately by

$$\tau=\alpha_1/\alpha_0. \quad (22)$$

The solution for the case of a single dominant time constant is often the most useful, and this is found from Eqs. (22), (20), and (A1) to be

$$\tau_d{}^{-1}=(\sum_j \tau_{jd}{}^{-1})/(1+\sum_j \gamma_{jd}), \quad (j=1, 2) \quad (23)$$

where

$$\gamma_{jd}=\mu_{nj}{}^0(1+\mu_{pi}{}^0)+\mu_{pj}{}^0+(\tau_j H_{i0})^{-1} \quad (i\neq j;\ i, j=1, 2) \quad (24)$$

and

$$\tau_j{}^{-1}=c_{nj}c_{pj}N_j(n_0+p_0+N_{j0}{}^-N_{j0}{}^+/N_j)/H_{j0} \quad (j=1, 2). \quad (25)$$

When the total density of flaws is smaller than that of free carriers, i.e., $N_1+N_2\ll n_0+p_0$, some simplification results:

$$\tau_d{}^{-1}=(\sum \tau_j{}^{-1})/(1+\sum \gamma_{jd}), \quad (26)$$

where now

$$\tau_j=\frac{c_{nj}(n_0+n_j)+c_{pj}(p_0+p_j)}{c_{nj}c_{pj}N_j(n_0+p_0)}. \quad (27)$$

Equation (27) is simply the Shockley-Read lifetime for the individual levels.

For p-type material, further simplification is possible,

$$\gamma_{jd}=\mu_{nj}{}^0[1+(c_{pj}(p_0+p_j)/H_{i0})], \quad (i\neq j;\ i, j=1, 2) \quad (28)$$

while for n-type material,

$$\gamma_{jd}=\mu_{pj}{}^0[1+(c_{nj}(n_0+n_j)/H_{i0})], \quad (i\neq j;\ i, j=1, 2). \quad (29)$$

The result for n-type material under similar conditions has been given by Wertheim,[4] but in error. In his lifetime expression, the $\tau_j{}^{-1}$ terms do not appear on their own as in Eq. (26), but are each multiplied by a factor $(1+\mu_{pj}{}^0)$.

B. ITL Model

1. Rate Equations

We consider a set of flaws with three possible charge states, $(s-1)$, s, and $(s+1)$ of electronic charges. For amphoteric impurities such as gold in silicon, $s=0$, while for divalent donors and acceptors, $s=-1$ and $+1$, respectively. We assume that the energies associated with the transitions, E_1 for $(s-1)\rightleftarrows s$ and E_2 for $s\rightleftarrows(s+1)$, along with their capture coefficients, are identical to those for the IDL case. The flaw densities in the three charge states are denoted by N_{s-1}, N_s, and N_{s+1}, and these are related by the totality condition

$$N_{s-1}+N_s+N_{s+1}=N_f, \quad (30)$$

where N_f is the total flaw density.

The interaction of the flaws with the valence and conduction bands is shown in Fig. 2. The parameters n_1, p_1, n_2, and p_2 are defined by analogy to single-level flaws as follows[14]:

$$\begin{aligned} n_1&=n_0p_0/p_1=n_0N_{s-1}{}^0/N_s{}^0\\ &=n_0\cdot\exp[(E_1-F)/kT]\\ &=N_c\cdot\exp[(E_1-E_c)/kT], \end{aligned} \quad (31)$$

$$\begin{aligned} n_2&=n_0p_0/p_2=n_0N_s{}^0/N_{s+1}{}^0\\ &=n_0\cdot\exp[(E_2-F)/kT]\\ &=N_c\cdot\exp[(E_2-E_c)/kT], \end{aligned} \quad (32)$$

where the superscript 0 denotes equilibrium conditions.

The above equations also give the relationship of the equilibrium flaw densities in the various charge states to the Fermi level. From this equation and the totality condition, Eq. (30), the number of flaws in each charge state can be determined for a given value of Fermi level.

From Fig. 2, the rate equations are given by

$$\begin{aligned} dn/dt&=\sum_j R_{nj}+G,\\ dp/dt&=\sum_j R_{pj}+G, \quad (j=1, 2)\\ dN_{s+1}/dt&=R_{n2}-R_{p2},\\ dN_s/dt&=R_{n1}-R_{p1}-dN_{s+1}/dt, \end{aligned} \quad (33)$$

where

$$\begin{aligned} R_{n1}&=c_{n1}(nN_{s-1}-n_1N_s),\\ R_{p1}&=c_{p1}(pN_s-p_1N_{s-1}),\\ R_{n2}&=c_{n2}(nN_s-n_2N_{s+1}),\\ R_{p2}&=c_{p2}(pN_{s+1}-p_2N_s). \end{aligned}$$

The charge neutrality condition expressed in terms of deviations from equilibrium condition is now given by

$$\Delta p-\Delta n-\Delta N_s-\Delta N_{s+1}=0. \quad (34)$$

Comparison of the rate equations, (33), with those of the IDL model, Eqs. (3), shows that their main difference lies in the appearance of the last term in Eqs. (33). This term expresses the fact that the creation of a flaw in the $s+1$ charge state corresponds to the annihilation of a flaw in the s charge state. The inclusion of this term indicates that the flaws in the s charge state, being common to both the upper and lower levels, participate in all eight transitions as shown in Fig. 2, instead of four in the case of flaws in other charge conditions.

The small-signal expansion of Eqs. (33) leads to a 3×3 matrix equation in the variables, Δn, Δp, and ΔN_{s+1}, of the same form as Eq. (6) for the IDL case. The matrix elements are different, however, as given by Eqs. (A2) in Appendix.

2. Steady-State Lifetimes

Following the same procedure as in the IDL case, we obtain the following expressions for the steady-state lifetimes of electrons and holes, valid for arbitrary signal levels:

$$\frac{\tau_p}{\tau_n}=\frac{\Delta p}{\Delta n}=\frac{1-\beta_1\beta_2+(1+\beta_2)\mu_{n1}+(1+\beta_1)\mu_{n2}}{1-\beta_1\beta_2+(1+\beta_2)\mu_{p1}+(1+\beta_1)\mu_{p2}}, \quad (35)$$

$$\tau_n^{-1}=[p_0+p+(\tau_p/\tau_n)n_0][c_{n1}c_{p1}(N_{s-1}+N_s)/H_1+c_{n2}c_{p2}(N_s+N_{s+1})/H_2], \quad (36)$$

where $H_{1,2}$ are as given in the IDL case, but

$$\mu_{n1}=c_{n1}N_{s-1}{}^0/H_1, \quad \mu_{n2}=c_{n2}N_s{}^0/H_2, \quad (37)$$

$$\mu_{p1}=c_{p1}N_s{}^0/H_1, \quad \mu_{p2}=c_{p2}N_{s+1}{}^0/H_2, \quad (38)$$

$$N_{s-1}/N_s=(c_{n1}n_1+c_{p1}p)/(c_{n1}n+c_{p1}p_1), \quad (39)$$

$$N_s/N_{s+1}=(c_{n2}n_2+c_{p2}p)/(c_{n2}n+c_{p2}p_2), \quad (40)$$

$$\beta_1=N_{s-1}/(N_s+N_{s-1}), \quad (41)$$

$$\beta_2=N_{s+1}/(N_s+N_{s+1}). \quad (42)$$

For near-equilibrium conditions, the following simplifications apply:

$$\tau_n^{-1}=(\tau_{1t}^{-1}+\tau_{2t}^{-1})/[1-\beta_{10}\beta_{20}+(1+\beta_{20})\mu_{p1}{}^0+(1+\beta_{10})\mu_{p2}{}^0], \quad (43)$$

$$\tau_p^{-1}=(\tau_{1t}^{-1}+\tau_{2t}^{-1})/[1-\beta_{10}\beta_{20}+(1+\beta_{20})\mu_{n1}{}^0+(1+\beta_{10})\mu_{n2}{}^0], \quad (44)$$

where

$$\tau_{1t}^{-1}=c_{n1}c_{p1}(N_s{}^0+N_{s-1}{}^0)\lambda/H_{10}, \quad (45)$$

$$\tau_{2t}^{-1}=c_{n2}c_{p2}(N_s{}^0+N_{s+1}{}^0)\lambda/H_{20}, \quad (46)$$

$$\lambda=(1-\beta_{10}\beta_{20})(n_0+p_0)+(1+\beta_{20})N_s{}^0N_{s-1}{}^0/(N_s{}^0+N_{s-1}{}^0)+(1+\beta_{10}\beta_{20})N_s{}^0N_{s+1}{}^0/(N_{s+1}{}^0+N_s{}^0), \quad (47)$$

$$\beta_{10}=N_{s-1}{}^0/(N_s{}^0+N_{s-1}{}^0)=\{1+\exp[(F-E_1)/kT]\}^{-1}, \quad (48)$$

$$\beta_{20}=N_{s+1}{}^0/(N_s{}^0+N_{s+1}{}^0)=\{1+\exp[(E_2-F)/kT]\}^{-1}. \quad (49)$$

In a study of divalent donors, Laff and Fan[16] have given the solutions for the electron and hole lifetimes. Their result for τ_p/τ_n is in agreement with ours, but their expression for τ_n is incorrect, since it can be shown that when the Fermi level is well below level 1, their τ_n tends to infinity.[18]

3. Transient Lifetimes

The time constants for the ITL case are found in the same manner as the IDL case. The result for the case of a single dominant time constant is

$$\tau_t^{-1}=(\tau_{1t}^{-1}+\tau_{2t}^{-1})/(1-\beta_{10}\beta_{20}+\gamma_{1t}+\gamma_{2t}), \quad (50)$$

where

$$\gamma_{1t}=\mu_{n1}{}^0[1+\mu_{p2}{}^0+(c_{n2}n_0/H_{20})]+\mu_{p1}{}^0[1+(c_{p2}p_2/H_{20})]+(\tau_1H_{20})^{-1}, \quad (51)$$

$$\gamma_{2t}=\mu_{n2}{}^0[1+\mu_{p1}{}^0+(c_{n1}n_1/H_{10})]+\mu_{p2}{}^0[1+(c_{p1}p_0/H_{10})]+(\tau_2H_{10})^{-1}, \quad (52)$$

$$\tau_1^{-1}=c_{n1}c_{p1}(N_s{}^0+N_{s-1}{}^0)[n_0+p_0+N_s{}^0N_{s-1}{}^0/(N_s{}^0+N_{s-1}{}^0)]/H_{10}, \quad (53)$$

$$\tau_2^{-1}=c_{n2}c_{p2}(N_s{}^0+N_{s+1}{}^0)[n_0+p_0+N_s{}^0N_{s+1}{}^0/(N_s{}^0+N_{s+1}{}^0)]/H_{20}. \quad (54)$$

For flaw densities which are small compared to the free carrier densities $N_f\ll n_0+p_0$, τ_t reduces to

$$\tau_t^{-1}=(1-\beta_{10}\beta_{20})(\tau_1^{-1}+\tau_2^{-1})/(1-\beta_{10}\beta_{20}+\gamma_{1t}+\gamma_{2t}), \quad (55)$$

where

$$\tau_1=\frac{c_{n1}(n_0+n_1)+c_{p1}(p_0+p_1)}{c_{n1}c_{p1}(N_s{}^0+N_{s-1}{}^0)(n_0+p_0)}, \quad (56)$$

$$\tau_2=\frac{c_{n2}(n_0+n_2)+c_{p2}(p_0+p_2)}{c_{n2}c_{p2}(N_s{}^0+N_{s+1}{}^0)(n_0+p_0)}. \quad (57)$$

τ_1 and τ_2 are the Shockley-Read lifetimes for effective flaw densities of $(N_s{}^0+N_{s-1}{}^0)$ and $(N_s{}^0+N_{s+1}{}^0)$ at the lower and upper levels, respectively.

[18] Using Laff and Fan's notation, their Eq. (22), when corrected, should read

$$\tau_n=\left[N\left(1+\frac{n}{p}\frac{\tau_p}{\tau_n}\right)\right]^{-1}\left((1-F_1)\frac{r_{C1}}{B_1}+F_1\frac{r_{C2}}{B_2}\right)^{-1}\left(\frac{1-F_2(1-F_1)}{1-F_2}\right).$$

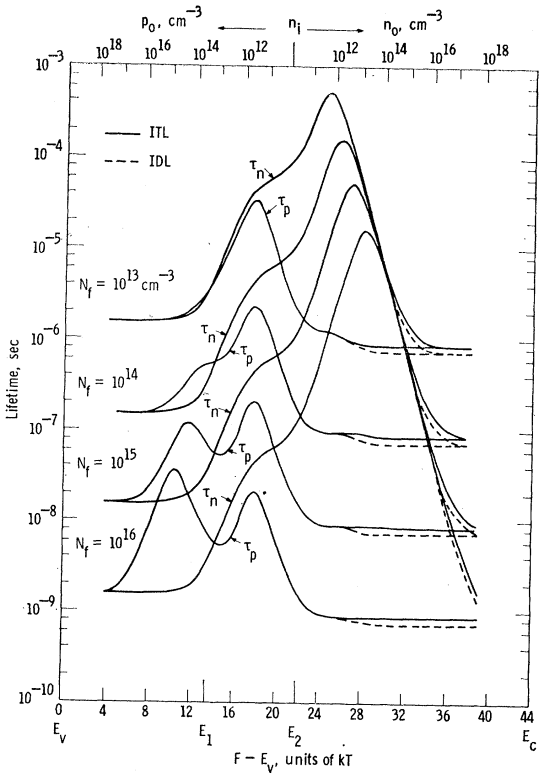

FIG. 3. Dependence of small-signal steady-state electron and hole lifetimes on Fermi level in gold-doped silicon, based on an ITL model and an IDL model.

For p-type material, in particular,

$$\gamma_{1t}=\mu_{n1}^0\{1+H_{20}^{-1}[c_{p1}(p_0+p_1)+c_{n2}n_0]\}, \quad (58)$$

$$\gamma_{2t}=\mu_{n2}^0\{1+H_{10}^{-1}[c_{p2}(p_0+p_2)+c_{n1}n_1]\}, \quad (59)$$

while for n-type material

$$\gamma_{1t}=\mu_{p1}^0\{1+H_{20}^{-1}[c_{n1}(n_0+n_1)+c_{p2}p_2]\}, \quad (60)$$

$$\gamma_{2t}=\mu_{p2}^0\{1+H_{10}^{-1}[c_{n2}(n_0+n_2)+c_{p1}p_0]\}. \quad (61)$$

III. DISCUSSION OF RESULTS

A. Comparison of IDL and ITL Lifetimes

Comparison of the solutions for the IDL and ITL steady-state lifetimes, as given, respectively, by Eqs. (9), (10), (35), and (36), shows that they are identical in form, aside from the appearance of the additional factors β_1 and β_2 in the ITL solutions. In the absence of β_1 and β_2, the two interacting levels would behave exactly as two independent levels with a total effective flaw density of $(N_{s-1}+N_s)$ and of (N_s+N_{s+1}) at the lower and the upper level, as pointed out by Sah and Shockley[14] in their study of steady-state recombination rate.

The presence of β_1 and β_2 is clearly the result of the dynamic interaction of the two levels, but, as we shall now show, their effect is small in most cases. From their defining equations (41) and (42), we observe that β_1 and β_2 are simply the fractions of empty and filled states at the lower and upper levels, respectively, and obviously neither quantity can exceed unity. For arbitrary signal levels, a determination of β_1 and β_2 would require a solution for Δn and Δp from Eq. (35), and, as mentioned earlier, this goes beyond the scope of the present paper. However, in the limit of very large signal levels such that $n\simeq p\gg N_f$, we have μ_{nj}, $\mu_{pj}\ll 1$ in Eq. (35), so that the presence of β_1 and β_2 is immaterial.

For near-equilibrium conditions, Eqs. (48) and (49) show that β_{10} and β_{20} are governed by the position of the flaw levels in relation to the Fermi level. The product $\beta_{10}\beta_{20}$ depends specifically on the energy separation, and even for energy separation as small as $1kT$, it is less than 0.15. A survey of multiply charged impurities in silicon and germanium indicates that the interacting levels are seldom less than a few kT apart, so that for most practical purposes, $\beta_{10}\beta_{20}$ is negligible.

In view of the inequality, $\beta_{10}\beta_{20}<0.15$, if $\beta_{10}\cong 1$, $\beta_{20}<0.15$, and vice versa, so that, at most, only one of these factors is likely to matter in any given circumstance. If $\beta_{10}\cong 1$, the corresponding μ_{n2}^0 or μ_{p2}^0 in Eqs. (43) and (44) will be multiplied by, at worst, a factor of 2 instead of 1. The latter two quantities are, moreover, associated with level 2, and if the Fermi level is such that $\beta_{10}\cong 1$, they are not likely to be important anyway. These considerations suggest that the presence of β_{10} and β_{20} does not have a significant effect on the steady-state lifetimes.

Turning now to the transient lifetimes as given by Eqs. (23)–(25) for the IDL and Eqs. (50)–(54) for the ITL model, we note that the same factors β_{10} and β_{20} recur in the ITL solution. The foregoing remarks for the steady-state lifetimes apply here as well in comparing the two models. There are, however, additional "coupling" terms in the ITL solution which did not appear in the steady-state lifetimes, but these terms are again always less than unity, and their effect is of the same order as β_{10} and β_{20}.

The above discussion indicates that at least for small-signal conditions, we may regard the two levels in the ITL model as essentially independent, or, to be strictly accurate, as quasi-independent. The reason that the two levels are not truly independent is that we still have to describe the effective flaw density at each level by interacting-level equilibrium statistics, rather than by independent-level statistics. To illustrate some of the consequences of this distinction between the quasi-IDL and the real IDL models, consider a simple situation where the Fermi level is, say, well above E_2. Under such conditions, $N_{s+1}^0\gg N_s^0\gg N_{s-1}^0$, and $N_{s+1}^0\cong N_f$, i.e., the lower level in the quasi-IDL model practically ceases to exist. If we now set $N_1=N_2=N_f$, we find that since $N_{10}^-\gg N_{10}^+$ and $N_{20}^-\gg N_{20}^+$, we have $N_{10}^-\cong N_1=N_f$ and $N_{20}^-\cong N_2=N_f$. It is apparent that in such circumstances, while the effect of the upper level on lifetime would be the same in both the quasi-IDL and

the real IDL models, this is not true of the lower level.

We have examined the conditions under which the difference between the IDL model and the ITL or quasi-IDL model is minimized, and found that for them to yield essentially the same lifetime under steady-state conditions, we should have

$$\exp[(E_2-E_1)/kT]\gg(c_{n1}/c_{n2}),\quad (c_{p2}/c_{p1});$$
$$(c_{n1}/c_{n2})\gg 1;\quad (c_{p2}/c_{p1})\gg 1.$$

For transient conditions, we require, in addition, that the flaw density not be too large, $N_f\ll[(c_{p2}/c_{n2})p_1+(c_{n1}/c_{p1})n_2]$. Depending on the actual location of E_1 and E_2 in the band gap, some of these conditions can be relaxed. We note that the first inequality involving (E_2-E_1) is easily satisfied for energy separation of more than a few kT's. The explanation for this condition is rather lengthy and will not be given. However, the origin and significance of the remaining inequalities may be seen from the discussion below which deals with a specific numerical example.

B. Application to Gold in Silicon

We shall now apply the ITL small-signal steady-state and transient solutions to gold in silicon, an amphoteric impurity with two well-separated energy levels, and compare these solutions with those obtained from an IDL model, subject, of course, to the condition that $N_1=N_2=N_f$. From the data of Fairfield and Gokhale,[19] the energy levels and capture coefficients of the gold atom at $T=300°$K are as follows:

For the donor level,

$$E_1-E_v=0.35\text{ eV},$$
$$c_{n1}=6.3\times10^{-8}\text{ cm}^{-3}\text{ sec}^{-1},$$
$$c_{p1}=2.4\times10^{-8}\text{ cm}^{-3}\text{ sec}^{-1}.$$

For the acceptor level,

$$E_2-E_v=0.57\text{ eV},$$
$$c_{n2}=1.65\times10^{-9}\text{ cm}^{-3}\text{ sec}^{-1},$$
$$c_{p2}=1.15\times10^{-7}\text{ cm}^{-3}\text{ sec}^{-1}.$$

Other numerical constants used in the computation are[20] $n_i=1.4\times10^{10}$ cm^{-3}, $m_e=1.18m_0$, and $m_h=0.81m_0$, where n_i is the intrinsic carrier density, m_0 is the free-electron mass, m_e is the effective electron, and m_h is the effective hole mass.

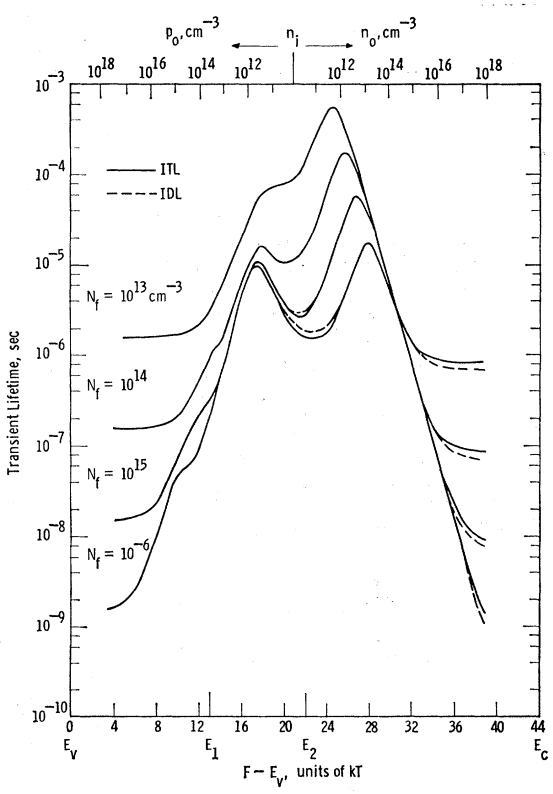

FIG. 4. Dependence of small-signal transient lifetimes on Fermi level in gold-doped silicon, based on an ITL model and an IDL model.

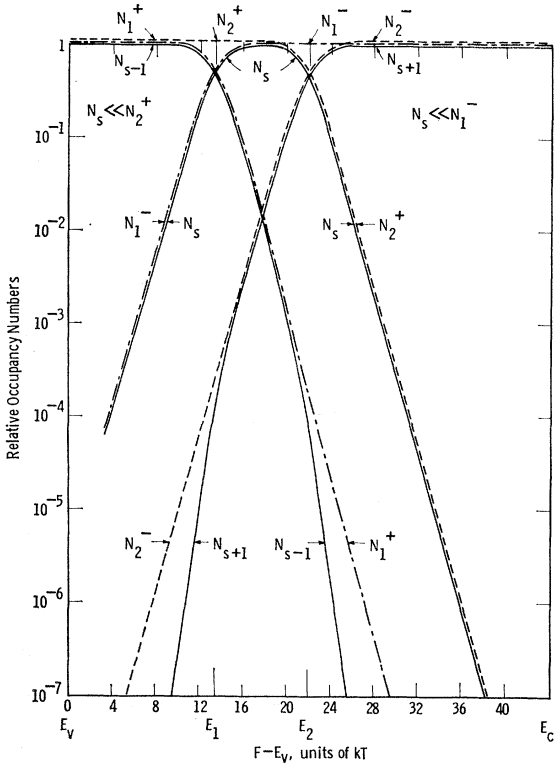

FIG. 5. Variation with Fermi level of the occupancy numbers at the two flaw levels in gold-doped silicon, based on an ITL and an IDL model. For simplicity, the subscript or superscript 0, affixed to the flaw densities to denote equilibrium conditions, has been omitted.

[19] J. M. Fairfield and B. V. Gokhale, Solid-State Electron. **8**, 685 (1965).

[20] H. D. Barber, Solid-State Electron. **10**, 1039 (1967).

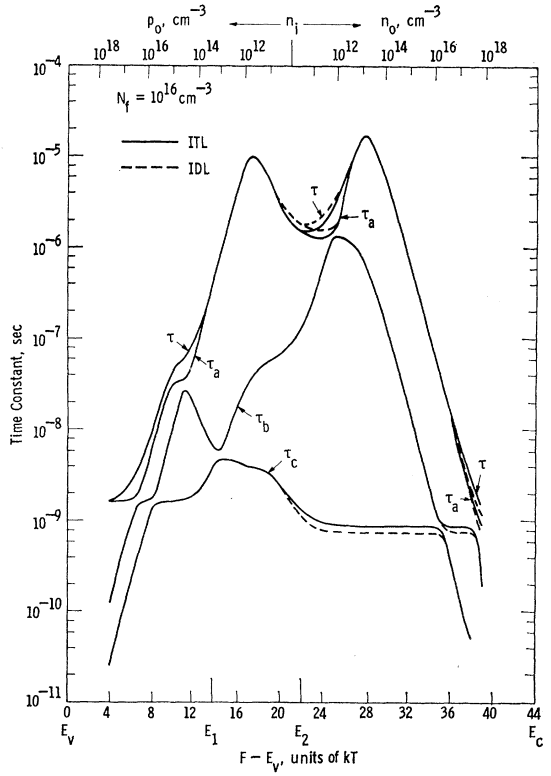

FIG. 6. Comparison of transient lifetime τ, based on the approximation of a single dominant time constant, and the exact solutions for the three time constants, τ_a, τ_b, and τ_c.

The recombination parameters clearly satisfy two of the conditions required for the ITL and IDL models to yield the same steady-state lifetimes, $\exp[(E_2-E_1)/kT]\gg(c_{n1}/c_{n2})$, (c_{p2}/c_{p1}); $(c_{n1}/c_{n2})\gg 1$; however, the condition $(c_{p2}/c_{p1})\gg 1$ does not strictly apply. Also, since $(c_{p2}/c_{n2})p_1\gg(c_{n1}/c_{p1})n_2$, we must have $N_f<(c_{p2}/c_{n2})p_1$, or a flaw density of less than 10^{15} cm^{-3}, for the ITL and IDL transient lifetimes to be identical.

The results of the calculations are shown in Figs. 3–5. As we see from Fig. 3, the ITL and IDL steady-state lifetimes are in excellent agreement, except for a consistent, but small, difference in extrinsic n-type material for $F-E_v>25kT$. The difference, which amounts to about 20% regardless of the flaw densities, is due to the fact that the condition $(c_{p2}/c_{p1})\gg 1$ does not strictly apply, and this difference can be readily estimated by restricting ourselves to the flat portion of the τ_p curve. As may be seen from Fig. 5, in this region the lower level in the ITL model practically ceases to exist; $N_{20}^-\cong N_{s+1}^0$, $\beta_{10}\ll 1$, $\beta_{20}\cong 1$. Moreover, in both the ITL and IDL lifetime expressions μ_{n1}^0, $\mu_{n2}^0\ll 1$, and $\mu_{p1}^0\ll 1$, so that (τ_p/τ_n) is identical in both models. However, $\tau_p(\mathrm{IDL})=(c_{p1}N_1+c_{p2}N_2)^{-1}$, as compared to $\tau_p(\mathrm{ITL})=(c_{p2}N_{s+1}^0)^{-1}=(c_{p2}N_f)^{-1}$. The extent to which these two lifetimes differ depends therefore on the ratio $(c_{p1}/c_{p2})=0.2$ which accounts for the 20% difference noted above.

The above discussion also serves to explain why in extrinsic p-type material the presence of the upper level in the IDL model does not result in any significant difference between the IDL and ITL lifetimes. Here, the effect of the upper level is governed by the ratio (c_{n2}/c_{n1}) and is negligible if $(c_{n2}/c_{n1})\gg 1$. In the present example this ratio is 0.025.

In this connection, it is worth mentioning that for divalent donors the consideration of the Coulombic interaction between the flaw and the carrier leads to the inequality $c_{n1}>c_{n2}>c_{p2}>c_{p1}$, while for divalent acceptors $c_{p2}>c_{p1}>c_{n1}>c_{n2}$. As an example of a divalent acceptor, nickel in germanium gives $(c_{p1}/c_{p2})\leq 0.35$ and $(c_{n2}/c_{n1})=0.12$[21]; both ratios are quite small.

As shown in Fig. 4, the agreement between the ITL and IDL transient lifetimes is again satisfactory. The discrepancy in the extrinsic n-type region occurs for the same reasons mentioned above. There is, however, in addition a small difference in the near intrinsic region which begins to develop for gold concentrations greater than 10^{15} atoms cm^{-3} when $N_f>(c_{p2}/c_{n2})p_1$. The origin of this difference can be traced to the product term $\mu_{p1}^0\mu_{n2}^0$ which occurs in the denominator of Eqs. (23) and (50) through γ_{2d} and γ_{2t}, respectively. Figure 5 shows that in this region $N_s^0=N_{20}^+$, but $N_s^0<N_{10}^-$; hence, $\mu_{n2}^0(\mathrm{IDL})=\mu_{n2}^0(\mathrm{ITL})$, but $\mu_{p1}^0(\mathrm{ITL})=c_{p1}N_s^0/H_{10}<\mu_{p1}^0(\mathrm{IDL})=c_{p1}N_{10}^-/H_{10}$. It is clear, however, that since $\mu_{p1}^0\mu_{n2}^0$ involves the square of the flaw density, its contribution to lifetime rapidly decreases at the lower flaw densities, and the difference between the ITL and IDL lifetime vanishes.

The transient lifetimes considered thus far have been obtained from Eq. (22) based on the assumption of a single dominant time constant. It is of interest to compare these solutions with the exact solutions obtained by numerically solving the cubic equation (19). The results for $N_f=10^{16}$ cm^{-3}, given in Fig. 6 for both the ITL and IDL models, are typical and show that the use of Eq. (22) is a reasonable approximation to the largest time constant over nearly the whole range of Fermi levels considered. We have also used Eq. (21) to give a better approximation than Eq. (22) and obtained results practically indistinguishable from the exact solutions. These are not shown in the figure.

C. Comparison of Steady-State and Transient Lifetimes

As originally pointed out by Sandiford[3] for a single level of flaws, the transient lifetimes are not necessarily identical to the steady-state electron and hole lifetimes when the flaw density is sufficiently large. The divergence of the transient lifetimes from the steady-state lifetimes in a two-level situation, as shown in Fig. 7 for

[21] F. M. Klaassen, J. Blok, and H. C. Booy, Physica **27**, 48 (1961).

$N_f = 10^{13}$ cm^{-3} and 10^{16} cm^{-3}, is therefore not surprising. We shall presently show, however, that there are a number of interesting new features in a two-level system, which do not occur in a single-level system. In view of the close similarity in the lifetimes calculated from the IDL and ITL models, only one of these models need be considered, and we shall use, as the basis of our comparison of steady-state and transient lifetimes, the much simpler expressions for the IDL model.

We begin by considering the simplest situation where the flaw density is sufficiently small that no trapping effects occur. Under such conditions, $\sum \mu_{nj}{}^0 \ll 1$, $\sum \mu_{pj}{}^0 \ll 1$, and $\sum \gamma_{jd} \ll 1$, and hence $\tau_p = \tau_n = \tau_d$. This type of behavior is observed in Fig. 7 at either end of the plots (particularly in the case of $N_f = 10^{13}$ cm^{-3}), where the majority carrier density n_0 or $p_0 \gg N_f$. It is interesting to observe from Eqs. (26) and (27) that τ_{1d} and τ_{2d} are now the Shockley-Read lifetimes due to the individual levels and that they add in parallel. The presence of the other level therefore tends to reduce the over-all lifetime.

The situation is much more complicated when trapping effects become important. In the steady state, trapping is deemed to occur when $\tau_p \neq \tau_n$ and arises because $\sum \mu_{nj}{}^0$ and $\sum \mu_{pj}{}^0 \gg 1$, while in the transient decay, trapping is evidenced by an increase in the lifetime beyond the value $(\tau_{1d}{}^{-1} + \tau_{2d}{}^{-1})$, because $\sum \gamma_{jd} \gg 1$. As shown in Fig. 7, trapping behavior can be observed over the major portions of the curves. We note that for $N_f = 10^{13}$ cm^{-3}, where $\tau_n \gg \tau_p$, $\tau_d = \tau_n$. However, as the flaw density is increased, an unusual effect comes into play, as shown in the plots for $N_f = 10^{16}$ cm^{-3}. Thus, while it remains true that where $\tau_n \gg \tau_p$ for $F - E_v > 25kT$, $\tau_d = \tau_n$, there exists a region $15kT < (F - E_v) < 25kT$, where $\tau_n > \tau_p$, but where τ_d now actually exceeds $(\tau_n + \tau_p)$. This effect is peculiar to the two-level system, and as we shall show below, does not occur in a single-level system.

In order to understand the relationship between the transient and steady-state lifetimes, we combine Eq. (23) with Eqs. (15) and (16) to give

$$\tau_d/\tau_p = (1+\sum \gamma_{jd})/(1+\sum \mu_{nj}{}^0),$$
$$\tau_d/\tau_n = (1+\sum \gamma_{jd})/(1+\sum \mu_{pj}{}^0).$$

It is clear, therefore, that if $\tau_n \gg \tau_p$, implying $\sum \mu_{pj}{}^0 \gg \sum \mu_{nj}{}^0$, and if furthermore, $\sum \gamma_{jd} \cong \sum \mu_{pj}{}^0$, $\tau_d = \tau_n$. This accounts for the equality between the transient lifetime and the longer of the two steady-state lifetimes in Fig. 7. However, it is equally apparent that if the presence of terms other than $\sum(\mu_{nj}{}^0 + \mu_{pj}{}^0)$ is important in $\sum \gamma_{jd}$, we may have a situation where $\sum(\mu_{nj}{}^0 + \mu_{pj}{}^0) \gg 1$, but more importantly, $\sum \gamma_{jd} > \sum(\mu_{nj}{}^0 + \mu_{pj}{}^0)$, so that $\tau_d > (\tau_n + \tau_p)$, as observed above in connection with the plots for $N_f = 10^{16}$ cm^{-3}.

Consider now the situation where a single level of flaws exists. The lifetime expressions reduce then to the well-known results

$$\tau_p = (1+\mu_{n1}{}^0)\tau_1,$$
$$\tau_n = (1+\mu_{p1}{}^0)\tau_1,$$
$$\tau_d = (1+\mu_{n1}{}^0+\mu_{p1}{}^0)\tau_1,$$

and, therefore, the inequality must apply:

$$\tau_d < (\tau_n + \tau_p).$$

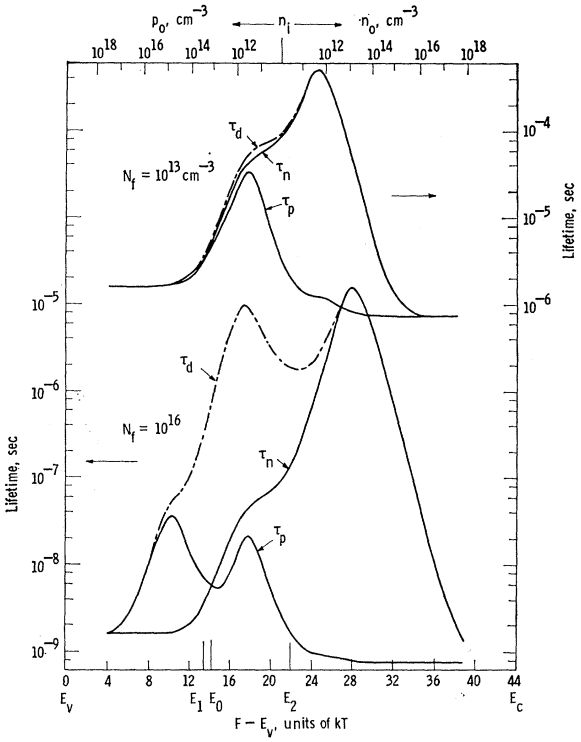

FIG. 7. Comparison of small-signal, steady-state, and transient lifetimes in gold-doped silicon.

From the above discussion, we conclude that while the transient lifetime can exceed the sum of the steady-state lifetimes in a two-level system, this is not true of a one-level system.[22] This result has an interesting application, since it can be used to distinguish experimentally a single-level system from a two-level system.

Included in the region where $\tau_d > (\tau_n + \tau_p)$ for $N_f = 10^{16}$ cm^{-3} in Fig. 7, there is a position at E_0 where $\tau_p = \tau_n$. It may be shown that here $\mu_{n2}{}^0$, $\mu_{p2}{}^0 \ll 1$, and although $\mu_{n1}{}^0$, $\mu_{p1}{}^0 > 1$ (which normally implies trapping), $\mu_{n1}{}^0 = \mu_{p1}{}^0$, so that in the steady state the electron and hole lifetimes are identical. This unique situation was first pointed out by Kalashnikov[9] for a single level of flaws and later elaborated on by Blakemore[11] who also showed that in this circumstance, the transient lifetime is indistinguishable from the steady-state lifetime.

[22] Both the one-level and two-level transient lifetime expressions under consideration were based on the assumption of a single dominant time constant. The latter restriction can, however, be removed without invalidating our conclusion.

This is not the case, however, in the present two-level system, as shown by the considerable difference which exists between the transient and steady-state lifetimes in Fig. 7. This difference is attributed to the presence of the term $(\tau_2 H_{10})^{-1}$ in γ_{2d}, which now has a dominant effect. A conclusion from this result is that the transient lifetime is much more sensitive to the presence of the second level than the steady-state lifetimes.

IV. CONCLUSIONS

We have shown that under small-signal, steady-state, and transient conditions, two interacting flaw levels may be treated as two independent levels, so long as we describe the effective flaw density at each level by interacting-level equilibrium statistics. However, under certain conditions, the use of either interacting-level or independent-level equilibrium statistics leads to essentially the same lifetime; the ITL model is then indistinguishable from the IDL model. Some of the possible applications of this result are: (a) simplification in the small-signal equivalent circuit proposed by Sah[16] for the ITL model, since the negative-capacitance element used to express the dynamic coupling effect between the two levels may be removed with little loss of accuracy. The resultant network in the case of a homogeneous semiconductor is simply a parallel combination of two "tee" networks each representing one of the levels, as may be obtained by a simple extension of Shockley's one-level equivalent circuit model[23]; (b) applicability in lifetime analysis of the much simpler IDL lifetime expressions as a good first approximation to any two-level system, interacting or otherwise.

A comparison of the steady-state and transient lifetimes has shown that the latter is more sensitive to the presence of a second level. Under certain circumstances, the transient lifetime can even be larger than the sum of the steady-state electron and hole lifetimes, in contrast to a single-level system where this possiblity does not exist. This implies that whenever the condition $\tau_d > (\tau_n + \tau_p)$ is observed experimentally, a model involving at least more than one level must be invoked. In this connection, it is interesting to note that in lifetime analysis, the presence of a two-level system is usually deduced from the temperature dependence of either the transient or steady-state lifetimes. This procedure is subject to uncertainties, since it has often been found that the temperature dependence can be equally well explained in terms of a single level with temperature-dependent capture cross sections. The present work suggests that if circumstances allow the inequality $\tau_d > (\tau_n + \tau_p)$ to be observed, a comparison of the transient and steady-state lifetimes should resolve this ambiguity.

[23] W. Shockley, Proc. IRE **46**, 973 (1958).

APPENDIX: RATE EQUATION MATRIX OF IDL AND ITL MODELS

The elements of the rate equation matrix for both the IDL and ITL models have previously been given[24] in a slightly different notation. They are repeated here for easy reference.

For the IDL model,

$$\begin{aligned}
a_{11} &= c_{n1}(n_0+n_1+N_{10}^{+})+c_{n2}N_{20}^{+},\\
a_{12} &= -c_{n1}(n_0+n_1),\\
a_{13} &= c_{n1}(n_0+n_1)-c_{n2}(n_0+n_2),\\
a_{21} &= -c_{p1}(p_0+p_1),\\
a_{22} &= c_{p1}(p_0+p_1+N_{10}^{-})+c_{p2}N_{20}^{-},\\
a_{23} &= -c_{p1}(p_0+p_1)+c_{p2}(p_0+p_2),\\
a_{31} &= -c_{n2}N_{20}^{+},\\
a_{32} &= c_{p2}N_{20}^{-},\\
a_{33} &= c_{n2}(n_0+n_2)+c_{p2}(p_0+p_2).
\end{aligned} \tag{A1}$$

For the ITL model,

$$\begin{aligned}
a_{11} &= c_{n1}(n_0+n_1+N_{s-1}^{0})+c_{n2}(N_s^{0}-n_0),\\
a_{12} &= -c_{n1}(n_0+n_1)+c_{n2}n_0,\\
a_{13} &= c_{n1}(n_0+2n_1)-c_{n2}(2n_0+n_2),\\
a_{21} &= -c_{p1}(p_0+p_1)+c_{p2}p_2,\\
a_{22} &= c_{p1}(p_0+p_1+N_s^{0})+c_{p2}(N_{s+1}^{0}-p_2),\\
a_{23} &= -c_{p1}(2p_0+p_1)+c_{p2}(p_0+2p_2),\\
a_{31} &= -c_{n2}(N_s^{0}-n_0)+c_{p2}p_2,\\
a_{32} &= -c_{n2}n_0+c_{p2}(N_{s+1}^{0}-p_2),\\
a_{33} &= c_{n2}(2n_0+n_2)+c_{p2}(p_0+2p_2).
\end{aligned} \tag{A2}$$

[24] J. E. L. Hollis, S. C. Choo, and E. L. Heasell, J. Appl. Phys. **38**, 1626 (1967).