Microwave, Flux Flow, and Fluctuation Resistance of Dirty Type-II Superconductors*

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We have reexamined the recent theories by Caroli and Maki of the dynamic response of dirty superconductors and find them inconsistent and incomplete. Additional contributions, which were not included by them, are calculated, and new expressions are presented for the microwave and flux-flow resistance. In particular, we find that the use of the time-dependent Ginzburg-Landau equation for the order parameter, coupled with the static Ginzburg-Landau equation for the current, is not generally adequate to calculate the leading superconducting contributions to the resistance. These results imply that the theory of Aslamazov and Larkin for the extra conductivity due to fluctuations above the transition temperature is also incomplete and that, in general, additional contributions of the type suggested by Maki must be included.

I. INTRODUCTION

RECENTLY, a very interesting series of articles has appeared by Caroli and Maki in which the microwave impedance[1,2] and the flux-flow resistance[3] of dirty type-II superconductors near the upper critical field H_{c2} were calculated from the microscopic theory of superconductivity.[4] The impedance was calculated in the gauge in which the small time-varying field was expressed entirely in terms of a spatially constant, time-dependent vector potential $A(t)=A_\omega e^{-i\omega t}$. The impedance was found to depend on the relative orientation of the static magnetic field H and the rf electric field $E(t)=-\partial A(t)/\partial t$, as had been found earlier experimentally.[5] The flux-flow resistance was calculated for a static field E, expressed entirely as a static scalar potential $\varphi=Ex$, perpendicular to H.

In principle, the results of these calculations should be gauge-invariant and give the same result for the same physical limit of a static field $E\perp H$, which may be obtained from the impedance calculation in the limit $\omega\rightarrow 0$. Taking this limit, we immediately see that the impedance calculation is incorrect, since a prediction of infinite conductivity is obtained contradicting the experimentally established result that a finite resistance results from the motion of the lines of flux.[6] [The corresponding calculations of Caroli and Maki[3,7] in the pure limit are also inconsistent. The leading corrections to the normal-state resistance are given as proportional to $(H_{c2}-H)^{1/2}$ in the microwave case and to $(H_{c2}-H)$ in the flux-flow case. However, at least they obtained a finite conductivity in the $\omega\rightarrow 0$ limit, i.e., no Meisner effect for electric fields perpendicular to the flux lines.]

In Sec. II we reexamine Caroli and Maki's impedance calculation and find important contributions which they ignored which cancel their infinite contributions to the conductivity. The finite value remaining is, however, not the same as they obtained for the flux-flow resistivity, except at zero temperature. As has recently been pointed out by Gor'kov and Éliashberg (GE),[8] the time-dependent response of a superconductor is rather singular. Aided by their analysis, we find that additional contributions arise when the scalar potential is given a time dependence $\varphi e^{-i\omega t}$, and the limit $\omega\rightarrow 0$ is taken. When they are added to the results of Caroli and Maki, we obtain the same resistivity as we got from the impedance calculation, thus arriving at a consistent, gauge-invariant answer. The initial slope of the resistivity near $H_{c2}(T)$ is found to be twice as steep as predicted by Caroli and Maki near the critical temperature, gradually reducing to their value as zero temperature is approached.

Having obtained a consistent theory of the response of a superconductor to electric fields, we turn our attention in Sec. III to the paraconductivity σ' arising from fluctuations in a superconductor above its transition temperature T_c. We find two dominant contributions in the region not too close to the transition. The first has been calculated by Aslamazov and Larkin (AL),[9] and for films gives a "Curie-Weiss" behavior $R_s/R_n=(1+\tau_0/\tau)^{-1}$ for the ratio of the resistance R_s to the constant normal-state resistance R_n, as a function of the normalized difference of the temperature from T_c, where $\tau=(T-T_c)/T_c$. The second contribution, which we call anomalous, is of the type proposed by Maki[10] in the three-dimensional case, which for two dimensions is logarithmically divergent $\sigma' R_n=(2\tau_0/\tau)$

* Work performed under the auspices of the U. S. Atomic Energy Commission. A summary was presented earlier: R. S. Thompson, Bull. Am. Phys. Soc. 14, 128 (1969).
[1] K. Maki, Phys. Rev. **141**, 331 (1966).
[2] C. Caroli and K. Maki, Phys. Rev. **159**, 306 (1967).
[3] C. Caroli and K. Maki, Phys. Rev. **164**, 591 (1967).
[4] The earlier work of A. Schmid [Phys. Kondensierten Materie **5**, 302 (1966)] obtained the same flux-flow resistance as Ref. 3 in the limit $T\rightarrow T_c$.
[5] M. Cardona and B. Rosenblum, Phys. Letters **8**, 308 (1964); B. Rosenblum and M. Cardona, *ibid.* **9**, 220 (1964).
[6] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **139**, A1163 (1965); A. R. Strnad, C. F. Hempstead, and Y. B. Kim, Phys. Rev. Letters **13**, 794 (1964).
[7] C. Caroli and K. Maki, Phys. Rev. **159**, 316 (1967).
[8] L. P. Gor'kov and G. M. Éliashberg, Zh. Eksperim. i Teor. Fiz. **54**, 612 (1968) [English transl.: Soviet Phys.—JETP **27**, 328 (1968)].
[9] L. G. Aslamazov and A. I. Larkin, Phys. Letters **26A**, 238 (1968); Fiz. Tverd. Tela **10**, 1104 (1968) [English transl.: Soviet Phys.—Solid State **10**, 875 (1968)].
[10] K. Maki, Progr. Theoret. Phys. (Kyoto) **39**, 897 (1968); **40**, 193 (1968).

$\times \ln\tau/\tau_c$, where τ_c is a cutoff proportional to the shift in T_c resulting from an assumed pair-breaking interaction.

From a theoretical point of view, this divergence may not be too surprising, since there are several arguments which for other reasons say that fluctuations should be logarithmically divergent in two-dimensional systems.[11,12] Experimentally, relatively sharp transitions have been observed,[13,14] with widths in good agreement with the prediction of AL. One does not know how much pair-breaking interaction was present in these films, but it could conceivably have been large enough to make the anomalous contributions insignificant. The usual estimates of the shift of T_c due to electron-phonon or electron-electron scattering, T^3/(Debye temperature)^2, and T^2/Fermi temperature, appear to be too small to account for the experimental results, but we do not know how much pair breaking arises from the imperfect film structure, looking, for example, like paramagnetic impurities. On the other hand, it could be that τ_c was small and the condition $\tau_0/\tau \ll \tau_c$ we derive for the validity of the expansion of the anomalous contributions was violated, so that nonlinear effects became important, which damped out the anomalous contributions to σ'. Recently, Masker and Parks[15] have observed transitions in less dirty aluminum films far from the transitions and found values of τ_0 about ten times that predicted by AL. The influence of an additional pair-breaking interaction, such as the application of a magnetic field parallel to the film, on such films to see if the transitions then became narrower would be a particularly interesting study.

II. MICROWAVE IMPEDANCE AND FLUX-FLOW RESISTANCE

We will first turn our attention to the calculation of the anisotropic electromagnetic conductivity for dirty, bulk, type-II superconductors given in Sec. IV of the paper by Caroli and Maki.[2] As they noted, one must consider two different geometrical arrangements. The parallel orientation of the rf electric field E to the static magnetic field $H[\approx H_{c2}(t)]$ may be calculated more simply and is considered first. The response function $Q_{\parallel}(\omega)$ relating the current j to the rf vector potential A_ω by $j=-QA$ was originally calculated by Maki,[1] although he did not then realize its validity was restricted only to the parallel case. In addition to the normal-state response $Q_n=-i\omega\sigma$, the first additional term Q' in an expansion in powers of the order parameter squared is evaluated. The sum over imaginary frequencies which must be computed to obtain Q' is given by Eq. (14) of Ref. 1, and the diagrams considered are illustrated in his Fig. 1.

Following GE,[8] we find it useful to divide the sum into two parts, called regular and anomalous. The regular part contains products of only retarded or only advanced Green's functions $[\omega'(\omega'+\omega_0)>0]$;

$$Q'_{\parallel r}=\frac{\sigma\langle|\Delta|^2\rangle}{2\pi T}\left\{\frac{1}{2}\left[\psi'\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)+\psi'(\tfrac{1}{2}+\rho)\right]+\frac{2\pi T}{-i\omega}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]\right\}. \quad (1)$$

The notation is identical to that of Ref. 2: $\langle|\Delta|^2\rangle$ is the spatial average of the order parameter squared, T is the temperature, ψ is the digamma function, ψ' is its derivative, $\rho=\epsilon_0/4\pi T$, $\epsilon_0=2eDH$, where D is the diffusion constant which equals $\frac{1}{3}$ of the product of the Fermi velocity v and the electron mean free path l. For low frequencies, $\omega\ll\pi T_{c0}$ (T_{c0} is the critical temperature for $H=0$) and Eq. (1) may be expanded in powers of $\omega/\pi T_{c0}$. The results may be expressed in terms of the magnetization M of the sample:

$$\mathrm{Re}Q'_{\parallel r}=-4eM,\qquad \mathrm{Im}Q'_{\parallel r}=4eM\frac{\omega}{8\pi T_{c0}}\frac{2\psi''(\frac{1}{2}+\rho)}{t\psi'(\frac{1}{2}+\rho)}. \quad (2)$$

ψ'' is the second derivative of ψ, and $t=T/T_{c0}$. The magnetization is given by

$$-4eM=\frac{\sigma}{\pi T}\psi'(\tfrac{1}{2}+\rho)\langle|\Delta|^2\rangle=\frac{e}{\pi}\frac{H_{c2}(t)-H}{1.16[2\kappa_2^2(t)-1]+n}, \quad (3)$$

using $\kappa_2(t)$ as calculated by Caroli *et al.*,[16] which reduces to the Ginzburg-Landau κ at T_{c0}. The quantity n is the demagnetization coefficient.[17]

The anomalous contribution to Q' contains products of retarded and advanced Green's functions $[\omega'(\omega'+\omega_0)<0]$;

$$Q'_{\parallel a}=\frac{\sigma\langle|\Delta|^2\rangle}{2\pi T}\left\{\frac{1}{2}\left[\psi'\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)-\psi'(\tfrac{1}{2}+\rho)\right]+\frac{2\pi T}{-i\omega+\epsilon_0}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]\right\}. \quad (4)$$

The sum $Q_{\parallel}=Q_n+Q'_{\parallel r}+Q'_{\parallel a}$ is the same as Eq. (16) of Ref. 1 and Eq. (29) of Ref. 2. It may be noted that we have included in $Q'_{\parallel a}$ two additional diagrams, which we illustrate in Fig. 1, to obtain this result. Maki also must have included them in his calculation but without explicitly showing them. Again for small frequencies

[11] T. M. Rice, Phys. Rev. **140**, A1889 (1965); J. Math. Phys. **8**, 1581 (1967).
[12] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).
[13] R. E. Glover III, Phys. Letters **25A**, 542 (1967).
[14] M. Strongin, O. F. Kammerer, J. Crow, R. S. Thompson, and H. L. Fine, Phys. Rev. Letters **20**, 922 (1968).
[15] W. E. Masker and R. D. Parks, Phys. Rev. (to be published).
[16] C. Caroli, M. Cyrot, and P. G. de Gennes, Solid State Commun. **4**, 17 (1966).
[17] J. A. Cape and J. M. Zimmerman, Phys. Rev. **153**, 416 (1967).

we expand in powers of $\omega/\pi T_{c0}$:

$$\mathrm{Re}Q'_{\parallel a}=-4eM\frac{1}{2}\frac{\omega^2}{\omega^2+\epsilon_0^2},$$
$$\mathrm{Im}Q'_{\parallel a}=4eM\left(\frac{\omega}{8\pi T_{c0}}\frac{\psi''(\frac{1}{2}+\rho)}{t\psi'(\frac{1}{2}+\rho)}+\frac{1}{2}\frac{\omega\epsilon_0}{\omega^2+\epsilon_0^2}\right). \tag{5}$$

If we require the additional condition (footnote 23, Ref. 2) that the temperature not be too close to T_{c0} so that $\omega\ll\pi(T_{c0}-T)$, Eqs. (5) may be expanded in powers of ω/ϵ_0. Then the sum $Q_{\parallel}$ of Q_n, Eq. (2), and Eq. (5) is the same as the expansion derived in Ref. 1, Sec. 3 and Ref. 2, Eqs. (34) and (34'). However, they could have noted in connection with the function $C_{\parallel}$, defined as $\mathrm{Im}Q'_{\parallel}=4eM(\omega/8\pi T_{c0})C_{\parallel}(t)$ and shown on Fig. 2 of Ref. 2, that as $T\rightarrow T_{c0}$, the function $C_{\parallel}$ only increases like $(T_{c0}-T)^{-1}$ until $\omega=\pi(T_{c0}-T)$. Then $C_{\parallel}$ reaches a maximum and decreases to $3\psi''(\frac{1}{2})/\psi'(\frac{1}{2})$ at T_{c0}. Also $\mathrm{Re}Q'_{\parallel}$ goes from $-4eM$ to $\frac{3}{2}(-4eM)$ at T_{c0}.

Because no explicit use has been made of the spatial dependence of Δ, we may compare our results with the response calculated by GE in the limit $T\rightarrow 0$. We must only change the pair-breaking interaction from a magnetic field to spin-flip scattering on magnetic impurities by letting $\epsilon_0=2/\tau_s$, where τ_s is the time between spin-reversal scatterings. In the limit $T\rightarrow 0$, $\psi'(\frac{1}{2}+\rho)\rightarrow 1/\rho$ and $\psi''(\frac{1}{2}+\rho)\rightarrow -1/\rho^2$. The anomalous contributions to Q' cancel out as required to get the simple form of the time-dependent Ginzberg-Landau equations GE obtained. However, it is clear that, as they stated, this cancellation occurs, and the simple time-dependent Ginzburg-Landau theory is valid, only when $T\ll$ the pair-breaking energy $\epsilon_0\approx T_{c0}-T$. The real part of $Q_{\parallel r}$ becomes $2\sigma\tau_s|\Delta|^2$, which is the usual static value. The imaginary part of $Q'_{\parallel r}$ becomes $\sigma\tau_s^2|\Delta|^2\omega$, which was ignored by GE, since it is of order $\tau_s\omega$ or ω/T_{c0} with respect to the real part. However, it gives the leading correction to the normal-state absorption and will sometimes be important. The presence of this leading correction $\mathrm{Re}\sigma_s/\sigma=1-\tau_s^2|\Delta|^2$ can easily be seen on the $\omega=0$ axis of Fig. 11 of Skalski *et al.*[18] It is interesting to note from Figs. 2 of Refs. 1 and 2 that the leading absorptive correction changes sign at $t=0.6$ and is positive for higher temperatures.

More recently Éliashberg[19] has extended the GE theory to the case of small magnetic impurity concentrations and found singular behavior in the time-dependent equation for the order parameter in the terms of order Δ^3, similar to those we find here for the current. Although such terms are very important in the case of magnetic impurities, they are not important in the case of external magnetic fields which we are interested in here. A typical denominator in the time-dependent perturbation theory for Δ involves the inverse of this time-dependent equation, $[-i\omega+\delta E+O(\Delta^2)]^{-1}$. Unlike Éliashberg's case, here the excited states have energies which are separated by a finite interval $\delta E=4eDH$ from the ground-state energy. We can ignore the higher-order terms in Δ, subject to the restriction on the size of Δ necessary for the validity of the theory, which we give at the end of this section, $\Delta\ll 2eDH$.

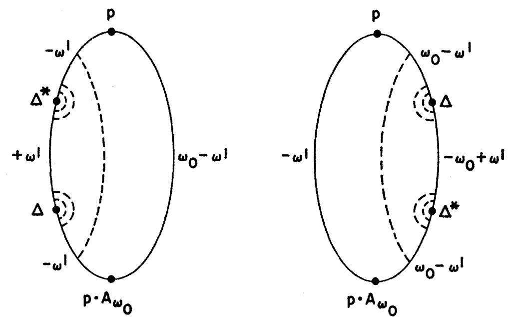

FIG. 1. Two additional diagrams contributing to the response $Q'_{\parallel a}$ in the anomalous regime ($\omega_0>\omega'>0$). The solid lines are propagating electrons, while the dotted lines represent impurity scattering. Where two dashed lines are shown crossing a vertex, a summation over all possible numbers of such dashed lines is implied. Where only a single dashed line is shown, such a summation is not possible because the two poles lie in the same half-plane.

If the rf electric field is oriented perpendicular to the static magnetic field, the response $Q_{\perp}$ is composed of the response for parallel orientation $Q_{\parallel}$, which we have already given, plus additional contributions of two types. The first type $Q'_{\perp 1}$ arises from the changes in the order parameter induced by the rf field and was calculated in Ref. 2, Eq. (31). Again as in the case of $Q'_{\parallel a}$, when expanding $Q'_{\perp 1}$ in powers of the frequency ω one must specify whether the condition $\omega\ll\pi(T_{c0}-T)$ is satisfied. If so, the results of the expansion are given by their Eq. (35);

$$\mathrm{Re}Q'_{\perp 1}=4eM\frac{\psi(\frac{1}{2}+3\rho)-\psi(\frac{1}{2}+\rho)}{2\rho\psi'(\frac{1}{2}+\rho)},$$
$$\mathrm{Im}Q'_{\perp 1}=4eM\frac{\omega}{8\pi T_{c0}}\frac{1}{\rho t}\left(2-\frac{\psi'(\frac{1}{2}+3\rho)}{\psi'(\frac{1}{2}+\rho)}\right). \tag{6}$$

If only the condition $\omega\ll\pi T_{c0}$ is satisfied, Eqs. (6) will not be valid as $T\rightarrow T_{c0}$ and a different expansion should be used in this limit;

$$\mathrm{Re}Q'_{\perp 1}=4eM\frac{4\epsilon_0^2+[\psi''(\frac{1}{2})/\psi'(\frac{1}{2})]\rho(4\epsilon_0^2+3\omega^2)}{4\epsilon_0^2+\omega^2},$$
$$\mathrm{Im}Q'_{\perp 1}=4eM\frac{2\epsilon_0\omega(1-2\rho[\psi''(\frac{1}{2})/\psi'(\frac{1}{2})])}{4\epsilon_0^2+\omega^2}. \tag{7}$$

Thus the function $C_{\perp}$, defined analogously to $C_{\parallel}$, also will not diverge indefinitely as $(T_{c0}-T)^{-1}$ in the limit $T\rightarrow T_{c0}$ for any finite frequency, but will reach a peak

[18] S. Skalski, O. Betbeder-Matibet, and P. R. Weiss, Phys. Rev. **136**, A1500 (1964).
[19] G. M. Éliashberg, Zh. Eksperim. i Teor. Fiz. **55**, 2443 (1968) [English transl.: Soviet Phys.—JETP **28**, 1298 (1969)].

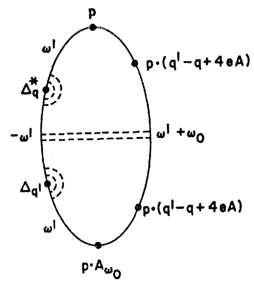

FIG. 2. A typical diagram giving additional contributions $Q'_{\perp 2}$ to the response when $E \perp H$. Other diagrams of this class are obtained by interchanging the positions of the order parameter Δ_q, the rf vector potential A_{ω_0}, and the gauge-invariant derivative $(q+2eA)$ of Δ_q. A vertex correction as shown across the center of the diagram would vanish when the angular average over the momenta p at the top and bottom is taken separately, if we had not expanded the electron propagators in the momentum change q occurring at Δ_q.

near $\omega=\pi(T_{c0}-T)$ and go to the same limit as $C_{\parallel}$ at T_{c0}.

Even when we ignore the finite-frequency difficulty associated with the limit $T\rightarrow T_{c0}$, there are more serious difficulties with the results of Ref. 2. Adding expressions (6) for $Q'_{\perp 1}$ to the results for $Q'_{\parallel}$, they obtained

$$\begin{aligned} &\mathrm{Re}Q'_{\perp}=-4eMA(t)\,,\\ &\quad A(t)=1-[\psi(\tfrac{1}{2}+3\rho)-\psi(\tfrac{1}{2}+\rho)]/2\rho\psi'(\tfrac{1}{2}+\rho)\,,\\ &\mathrm{Im}Q'_{\perp}=4eM(\omega/8\pi T_{c0})C_{\perp}(t)\,,\\ &\quad C_{\perp}(t)=C_{\parallel}(t)-\frac{1}{\rho t}\left(\frac{\psi'(\frac{1}{2}+3\rho)}{\psi'(\frac{1}{2}+\rho)}-2\right),\\ &\quad C_{\parallel}(t)=\frac{1}{t}\left(\frac{1}{\rho}+\frac{3\psi''(\frac{1}{2}+\rho)}{\psi'(\frac{1}{2}+\rho)}\right). \end{aligned} \tag{8}$$

$A(t)$ vanishes only at the point $T=T_{c0}$, as illustrated in their Fig. 1. This result contradicts the well-known results of flux-flow experiments, which show that the conductivity

$$\sigma_s=\lim_{\omega\rightarrow 0}\frac{Q_{\perp}}{-i\omega}$$

is finite for $E\perp H$. Furthermore, the difficulty with their results is not concerned so much with incorrect handling of the time dependence but with the $\omega=0$ limit.

They left out of their calculation certain important contributions, which we call $Q'_{\perp 2}$ and illustrate in Fig. 2. This type of diagram contributes only when $E\perp H$, because if $E\|H$ then $A_{\omega_0}\perp(q+2eA)$ and the average over angles gives zero. As usual, the real ω is replaced by an imaginary discrete frequency $i\omega_0=i2n\pi T$ during the calculation. Before analytic continuation, the sum of the additional contributions is

$$Q'_{\perp 2}(\omega_0)=-\sigma|\Delta|^2 8\pi T\Big(\sum_{\omega'<-\omega_0}+\sum_{\omega'>0}\Big)\times\left(\frac{1}{2|\omega'|+\epsilon_0}+\frac{1}{2|\omega'+\omega_0|+\epsilon_0}\right)^2\frac{\epsilon_0}{|2\omega'+\omega_0|+3\epsilon_0}\,. \tag{9}$$

The sum over frequencies is obtained in terms of the ψ functions by separating the various types of denominators into partial fractions:

$$\begin{aligned} Q'_{\perp 2}(\omega_0)&=-\sigma|\Delta|^2\frac{\epsilon_0}{\pi T}\Bigg\{\frac{1}{2\epsilon_0+\omega_0}\psi'(\tfrac{1}{2}+\rho)\\ &+\frac{1}{2\epsilon_0-\omega_0}\psi'\left(\frac{1}{2}+\frac{\omega_0}{2\pi T}+\rho\right)-\frac{4\pi T}{(2\epsilon_0+\omega_0)^2}\\ &\times\left[\psi\left(\frac{1}{2}+\frac{\omega_0}{4\pi T}+3\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]-\frac{4\pi T}{(2\epsilon_0-\omega_0)^2}\\ &\times\left[\psi\left(\frac{1}{2}+\frac{\omega_0}{4\pi T}+3\rho\right)-\psi\left(\frac{1}{2}+\frac{\omega_0}{2\pi T}+\rho\right)\right]\Bigg\}\,. \end{aligned} \tag{10}$$

The analytic continuation to real frequencies may now be made simply by replacing ω_0 by $-i\omega$. Then Eq. (10) may be expanded in powers of ω/T_{c0} (with no modifications in the limit $T\rightarrow T_c$ for $\omega\neq 0$),

$$\mathrm{Im}Q'_{\perp 2}=\frac{\omega}{8\pi T}\frac{\sigma|\Delta|^2}{\pi T}\Bigg\{2\psi''(\tfrac{1}{2}+\rho)-\frac{1}{\rho}[\psi'(\tfrac{1}{2}+3\rho)-\psi'(\tfrac{1}{2}+\rho)]\Bigg\}\,. \tag{11}$$

When these new contributions are added to those given in Eqs. (8), we obtain the necessary cancellation of the real part so that $\mathrm{Re}Q'_{\perp}=0$ and a corrected value for $C_{\perp}$;

$$\begin{aligned} &\mathrm{Re}Q'_{\perp 2}=-\frac{\sigma|\Delta|^2}{\pi T}\psi'(\tfrac{1}{2}+\rho)A(t)\,,\\ &\quad C_{\perp}(t)=C_{\parallel}(t)+\frac{1}{\rho t}-\frac{2\psi''(\frac{1}{2}+\rho)}{\psi'(\frac{1}{2}+\rho)}\,,\\ &\quad C_{\perp}(t)=\frac{2}{\rho t}+\frac{\psi''(\frac{1}{2}+\rho)}{t\psi'(\frac{1}{2}+\rho)}\,. \end{aligned} \tag{12}$$

From this value of $C_{\perp}$, the film conductivity σ_s is obtained

$$\begin{aligned} &\sigma_s=\sigma-(4eM/8\pi T_{c0})C_{\perp}(t)\,,\\ &\sigma_s=\sigma\left[1+\frac{4\kappa_1^2(0)L_D(t)}{1.16[2\kappa_2^2(t)-1]+n}\left(1-\frac{H}{H_{c2}(t)}\right)\right], \end{aligned} \tag{13}$$

where $\kappa_1(0)=1.20\kappa$ and $L_D(t)=\rho t C_{\perp}$ is plotted in Fig. 1 of Ref. 3. For finite $\omega\ll\pi T_{c0}$, Eqs. (2), (5), (7), and (11) may be summed to obtain the correct frequency-dependent expressions in the limit $T\rightarrow T_{c0}$.

Unfortunately, the value of the flux-flow resistivity we have obtained by correcting Ref. 2 differs from that obtained in Ref. 3 by the factor $L_D(t)$. The reason is that the anomalous contributions given by the small ω limit of Eqs. (5) were left out of the calculation in Ref. 3. The calculation in Ref. 3 was performed in the gauge $\phi=Ex$. The anomalous contributions in this gauge arise from diagrams of the type shown in Fig. 3. One might think such diagrams, which contribute only in the range $0<\omega'<\omega_0$, could be ignored as being of order $\omega_0\phi$ if ϕ was assumed to be time-independent ($\omega_0\rightarrow 0$). However, GE showed that due to the singular vertex corrections a factor of $1/\omega_0$ arises from the sum of all corrections on the lower end of the diagram, resulting in a finite $\omega_0\rightarrow 0$ limit. The Coulomb force has been included by requiring there to be no fluctuations in the density of electrons. Terms of order Δ^2 of the type shown in Fig. 3 were not explicitly included in GE because they vanish in the $T=0$ limit. The addition of these anomalous terms to the results of Ref. 3 gives exactly Eq. (13).[19a]

The revised prediction for the initial slope of the flux-flow resistance is shown in Fig. 4. It is the same as given in Ref. 3 at $T=0$, and twice as steep near $T=T_{c0}$. However, it is necessary to note that this expansion has a very narrow range of validity as $T\rightarrow T_{c0}$. The static calculation of the penetration depth (ReQ) is valid in the range $\Delta\ll\pi T_{c0}$. However, the dynamic calculation of the conductivity (ImQ') is only valid in the range $\Delta\ll\epsilon_0\approx\pi(T_{c0}-T)$ (or $\Delta\ll\omega$ for finite frequencies when $\omega>\epsilon_0$). This restriction on expansions in powers of Δ can be inferred from the observation that if all frequencies $\omega\rightarrow 0$ in the expressions for the Green's functions, the only remaining parameter for expansions for small Δ is Δ/ϵ_0, as is verified by direct calculation of the anomalous terms of order Δ^4. The restriction on magnetic field thus becomes increased from $H_{c2}(t)-H \ll H_{c2}(0)$ to $H_{c2}(t)-H\ll H_{c2}{}^2(t)/H_{c2}(0)$. We illustrate the different regions of validity in Fig. 5. When account is taken of the large fluctuations near the transition,[20] one would expect experimental comparison with this theory to be quantitative only for intermediate and low reduced temperatures t. The data of Kim *et al.*[6] indicate approximate agreement with the theory, showing slopes of the resistance of the order of 2–4. However, we cannot determine very precisely the temperature dependence of the slope near the transition from their published curves.

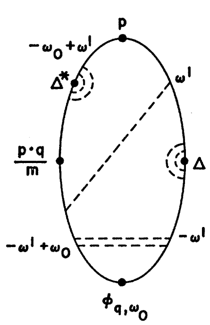

FIG. 3. A typical diagram giving anomalous contributions to the current in the gauge $\phi=Ex$. The only nonvanishing regular contributions in this gauge come from the time dependence of Δ induced by E.

[19a] H. Ebisawa and H. Takayama (unpublished) have found the time-dependent Ginzburg-Landau equation used in Ref. 3 incorrect at low temperatures owing to the noncommutivity of the momentum operator with the scalar potential. However, they found that due to a cancellation of errors the current derived in Ref. 3 is the correct regular part, as we stated.

[20] K. Maki, Phys. Letters **27A**, 481 (1968).

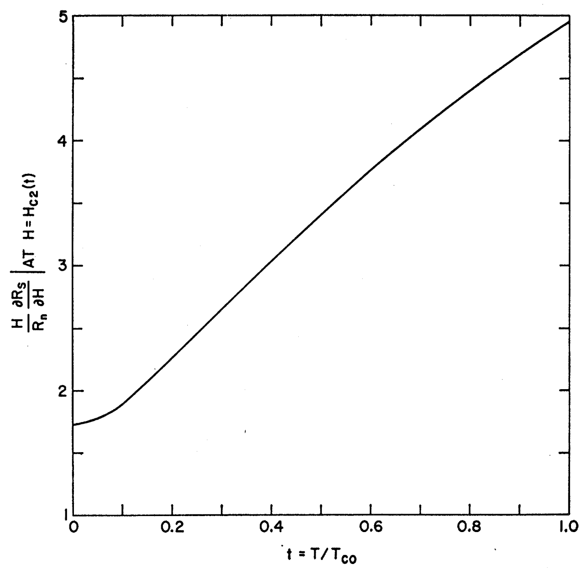

FIG. 4. Slope of the flux-flow resistance at $H_{c2}(t)$ as a function of the reduced temperature $t=T/T_{c0}$ for dirty materials with $\kappa\gg1$.

As Maki has pointed out[21] these calculations are easily extended to a semi-infinite superconductor with a magnetic field near H_{c3} parallel to its surface. One simply uses the appropriate static value for Δ^2 given by Maki and replaces the ground-state energy $\epsilon_0=2eDH_{c2}$ by $0.59(2eDH_{c3})$ and the first-excited-state energy $3\epsilon_0$ by $5.62\epsilon_0$. In addition, in the $E\perp H$ orientation the matrix element squared of A between the ground and excited states

$$\langle 0|Hx|1\rangle^2\approx\langle 0|H^2x^2|0\rangle-\langle 0|Hx|0\rangle^2,$$

which gives a contribution $\frac{1}{2}H_{c2}$ to the numerator of $Q'_\perp$, must be replaced by approximately $\frac{1}{2}(0.59H_{c3})$. Higher-order excited states, which make no contributions in the bulk type-two case, do contribute here, but computer solutions show that the error involved in neglecting them is small $\leq 7\%$ (see also Fink[22]). Re$Q'_\perp$ no longer vanishes in the small ω limit for $E\perp H$. The ratio of ReQ' for $E\perp H$ to its value for $E\|H$, which is a measure of the anisotropy, becomes 0.586 (independent of temperature). We have made detailed calculations of the response for all film thicknesses, including the regime of flux entry[23] for films of thickness approximately equal to the coherence length, and will present our results in a separate paper.

III. FLUCTUATION RESISTANCE

Recently, a great deal of attention has been paid to the rounding of the resistive transition due to fluctuations. The most popular theory has been that of AL,[9]

[21] K. Maki, Progr. Theoret. Phys. (Kyoto) **39**, 1165 (1968); G. Fischer and K. Maki, Phys. Rev. **176**, 581 (1968).

[22] H. J. Fink, Phys. Rev. **177**, 732 (1969).

[23] E. Guyon, F. Meunier, and R. S. Thompson, Phys. Rev. **156**, 452 (1967).

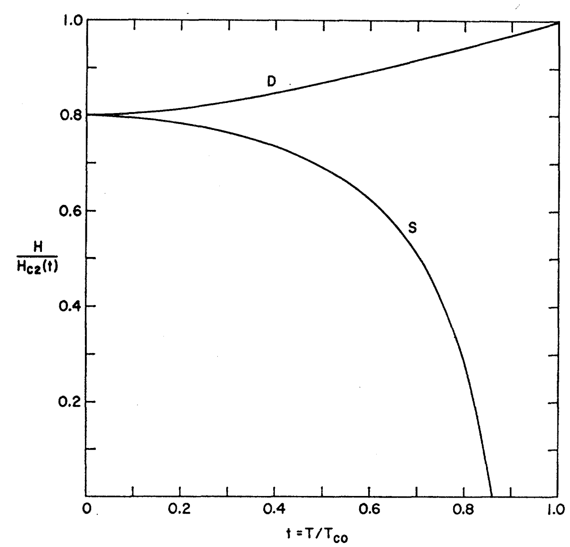

FIG. 5. Static generalized Ginzburg-Landau theory for the penetration depth (ReQ) is expected to be valid to better than about 20% accuracy in the entire region above and to the right of curve S. The dynamic theory for the resistance (ImQ') in the small frequency limit applies only in that part of the first region which lies above the curve D.

which, as has been emphasized by Abrahams and Woo,[24] is based on the addition of a time derivative of the order parameter to the Ginzburg-Landau equation with the proper sign to restore deviations from equilibrium. When an electric field is applied an extra current arises from the time derivative of the fluctuations of the order parameter, giving rise to a "Curie-Weiss" form of the conductivity $\sigma_s=\sigma(1+\tau_0/\tau)$ in a two-dimensional film. AL obtained $\tau_0=\frac{1}{16}e^2R_\square=(1.52\times10^{-5}\ \Omega^{-1})R_\square$, where $R_\square$ is the film resistance per square area. The response calculated by AL is similar to the $Q'_{\text{I}1}$ part of the calculation in Sec. II.

Maki[10] has proposed a different theory of the fluctuation resistance based on the anomalous terms $Q'_{\text{II}a}$ of Sec. II. Maki applied his theory only to the three-dimensional case, perhaps because it is divergent in lower dimensions. Even so, we now understand that these anomalous terms appear in all dynamic calculations near T_{c0}, so we must deal with these divergences for films and wires. We agree with Maki's note added in proof to Ref. 10 that the two types of fluctuation conductivity must be summed.

Introducing the temperature-independent coherence length $\xi=0.85(l\xi_0)^{1/2}=[\psi'(\frac{1}{2})D/4\pi T_{c0}]^{1/2}$, the AL result for a fluctuation of momentum q is given in a simple form. We may call this contribution the regular one σ_r':

$$\sigma'_r(q)=\pi e^2\xi^4q^2\cos^2\theta\,\{1/[\tau+\xi^2q^2]^3\}. \tag{14}$$

θ is the angle between q and E. To sum over all fluctuations, this expression must be intergrated over an appropriate momentum element. For a two-dimensional film of thickness $d<\xi(t)=\xi/|\tau|^{1/2}$, the momentum element is

$$\frac{1}{d}\int\frac{d^2q}{(2\pi)^2}=\frac{1}{d}\int_0^\infty\frac{dq^2}{4\pi}\int_0^{2\pi}\frac{d\theta}{4\pi}.$$

The AL result follows integration:

$$\sigma'_r(2D)=e^2/16d\tau. \tag{15}$$

For bulk samples the three-dimensional momentum element is

$$\int\frac{d^3q}{(2\pi)^3}=\int_0^\infty\frac{q^2\,dq}{2\pi^2}\frac{1}{2}\int_{-1}^{1}d\cos\theta.$$

In agreement with Schmidt[25] we obtain

$$\sigma'_r(3D)=e^2/32\xi\tau^{1/2}. \tag{16}$$

In practical units $e^2=2.43\times10^{-4}\ \Omega^{-1}$. The coherence length may be obtained by measuring $H_{c2}=|\tau|/2e\xi^2$ or $H_{c3}=1.69H_{c2}$ for precise comparison with theory.[26] The general result which interpolates between Eqs. (15) and (16) for any film thickness may easily be obtained by summing discrete prependicular momenta $n\pi/d$, corresponding to the allowed eigenvalues of the cosine function, which satisfies the boundary condition of vanishing derivative;

$$\sigma'_r(2-3D)=(e^2/32\xi\tau^{1/2})[\coth(d\tau^{1/2}/\xi)+\xi/d\tau^{1/2}]. \tag{17}$$

In one dimension, the momentum element is $(1/S)\times\int_{-\infty}^{\infty}(dq/2\pi)$ where S is the cross-sectional area of the wire;

$$\sigma'_r(1D)=\pi e^2\xi/16S\tau^{3/2}. \tag{18}$$

The anomalous response may be calculated similarly from the most singular part of Eqs. (4) and (5);

$$\sigma'_a=\sigma\psi'(\tfrac{1}{2})|\Delta|^2/2\pi T\epsilon_0 \tag{19}$$

by replacing ϵ_0 by Dq^2 and $|\Delta|^2$ by the fluctuation probability given by Maki[10]

$$|\Delta|^2=[T/N(0)]1/[\tau+(\xi q)^2]; \tag{20}$$

$$\sigma'_a=\tfrac{1}{2}\pi e^2(1/q^2)1/(\tau+\xi^2q^2). \tag{21}$$

Only the zero-frequency part of $|\Delta|^2$ has been included here. A careful consideration of the analytic continuation, including the frequency dependence of $|\Delta|^2$, shows that the leading singular contribution to σ'_a is correctly obtained in this simple way in this particular case.[27,28]

[24] E. Abrahams and J. W. F. Woo, Phys. Letters **27A**, 117 (1968).

[25] H. Schmidt, Z. Physik **216**, 336 (1968).

[26] J. I. Gittleman, R. W. Cohen, and J. J. Hanak, Phys. Letters **29A**, 56 (1969). These authors did not determine the coherence length precisely from the critical fields and could only compare their results roughly with theory.

[27] We wish to thank A. I. Larkin for discussions of the analytic continuations involved here. AL now agree that these additional Maki contributions must in general be added to their previous theory.

[28] Recently, E. Abrahams, M. Redi, and J. W. F. Woo (unpublished) have shown that taking only the zero-frequency part of $|\Delta|^2$ gives the wrong answer for the fluctuation contribution to the density of states $N(\omega)$ [Phys. Rev. B 1, 208 (1970)].

The same momentum elements apply as before. In agreement with Maki, we obtain

$$\sigma'_a(3D)=e^2/8\xi\tau^{1/2}, \tag{22}$$

although for the total $\sigma'(3D)$ we get $5/4\,\sigma'_a(3D)$ and not the factor 7/4 mentioned by Maki.

In two dimensions σ'_a is logarithmically singular. In terms of a low-momentum cutoff $q_c=\xi^{-1}\tau_c^{1/2}$, we obtain

$$\sigma'_a(2D)=(e^2/8d\tau)\ln[(\tau+\tau_c)/\tau_c]. \tag{23}$$

The interpolation formula between Eqs. (22) and (23) is obtained by again summing over discrete momenta $n\pi/d$;

$$\sigma'_a(2-3D)=\frac{e^2}{8\tau d}\left[\ln\left(\frac{\xi}{d\tau_c^{1/2}}\sinh\frac{d(\tau+\tau_c)^{1/2}}{\xi}\right)+\tfrac{1}{2}\ln\left(\frac{\tau+\tau_c}{\tau_c}\right)\right]. \tag{24}$$

The divergence problem is unaffected by the presence of the third dimension when $d<\xi/(\tau+\tau_c)^{1/2}$.

τ_c may arise from the change in T_c due to a pair-breaking interaction. For example, if a magnetic field is applied parallel to the film $\tau_c=\delta=[T_c(0)-T_c(H)]/T_c=\frac{1}{3}(eHd\xi)^2$. Then the total conductivity due to fluctuations is

$$\sigma'(2D)=\tau_0/(\tau+\delta)+(2\tau_0/\tau)\ln[(\tau+\delta)/\delta]. \tag{25}$$

Near the transition this result tends to AL, but far away it has values typically an order of magnitude larger. A perpendicular field also has a pair-breaking influence. Because of the discrete nature of the excitation spectrum, $q^2=(2n+1)2eH$, the results are different. Far from the transition ($\tau\gg\tau_c$), the limiting value is

$$\sigma'(2D)=(\tau_0/\tau)(1+2\ln 2\gamma\tau/\delta), \tag{26}$$

where $\delta=2eH\xi^2$ and Euler's constant $\ln\gamma=0.5772$.

The cutoff of the divergence may also be obtained from the frequency ω if $\omega/T_c>\tau_c$. In this case by referring to Eq. (5), we see that the denominator q^{-2} in Eq. (21) is replaced by $\xi^2/[(\pi\omega/8T_c)^2+(\xi q)^2]$. Following Schmidt,[25] we introduce the notation $\tilde{\omega}=\pi\omega/16T_c\tau$. Then the frequency-dependent anomalous response which should be added to Schmidt's result for the AL contribution is

$$\sigma'_a(2D)=(2\tau_0/\tau)[\tilde{\omega}\pi-\ln(2\tilde{\omega})]. \tag{27}$$

Both contributions σ'_r and σ'_a are always positive, so it is somewhat puzzling that significant changes in the conductivity were not observed in one reported microwave experiment,[29] especially since corrections to the static conductivity were expected only very near the transition. A more recent experiment by another group[30] shows good agreement with Schmidt's theory, as expected for lead films, in which the static conductivity is dominated by the AL contribution.

For wires the divergence is much stronger, going as q_c^{-1};

$$\sigma'_a(1D)=(e^2\xi/2S)(1/\tau\tau_c^{1/2}-\pi/2\tau^{3/2}). \tag{28}$$

If the cutoff is due to a pair-breaking interaction like that resulting from the application of an external magnetic field, the total conductivity due to fluctuations may be expressed in terms of $\delta=[T_c(0)-T_c(H)]/T_c$;

$$\sigma'(1D)=\frac{\pi e^2\xi}{16S}\left[\frac{1}{(\tau+\delta)^{3/2}}+\frac{4}{\tau}\left(\frac{1}{\delta^{1/2}}-\frac{1}{(\tau+\delta)^{1/2}}\right)\right]. \tag{29}$$

The answers we have obtained for the first corrections to the conductivity are valid only as long as they are small compared with higher-order corrections. For the regular terms the requirement is met simply if $\sigma'\ll\sigma$. However, the anomalous terms are more badly behaved. As we noted at the end of Sec. II the expansion of the anomalous terms goes as $(\Delta/Dq^2)^2$. Substituting Eq. (20) for $|\Delta|^2$ and integrating over the appropriate momentum element, we obtain the requirement $\sigma'\ll\sigma\tau_c$ in one and two dimensions. Even in the three-dimensional case, the higher-order dynamic response is singular, requiring $\sigma'\ll\sigma(\tau\tau_c)^{1/2}$. These results may be verified by examining a typical anomalous term of order Δ^4.

In general, we do not know how large τ_c is, nor do we have a rigorous calculation of the response to all orders in Δ, which would be important if τ_c is small. However, starting from any small τ_c we can increase it by applying an additional pair-breaking interaction. Therefore, it would be particularly interesting to study the effects of magnetic fields on those films which show fluctuation conductivities larger than AL.

The results given in this section are valid for not too large shifts of T_c, $\delta\lesssim\frac{1}{10}$, and for perpendicular fields only if $\tau\gg\delta$. We have also calculated more general results not subject to these limitations and will present them in a separate publication.[31]

IV. CONCLUSION

We have recalculated the dynamic response of superconductors to electric fields, removing some inconsistencies and obtaining significantly different results from previous calculations. Further experimental work might test the validity of the details of the theory in the microwave, flux-flow, and fluctuation regimes.

[29] R. V. D'Aiello and S. J. Freedman, Phys. Rev. Letters **22**, 515 (1969).

[30] S. L. (A.) Lechoczky and C. V. Briscoe, Phys. Rev. Letters **23**, 695 (1969).

[31] International Conference on the Science of Superconductivity, Stanford, 1969, Physica (to be published).