Transport in a Magnetic Field. II. The Transport Equations for an Electron Gas*

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(Received 17 March 1969)

The gauge-independent formalism of the previous discussion is applied to the construction of transport equations on the density, spin density, and current density of a degenerate electron gas. Interactions are included in a self-consistent field approximation. Scattering events are treated in a relaxation-time approximation. Three kinds of rf driving fields are considered: transverse magnetic, longitudinal electric, and transverse electric.

I. INTRODUCTION

WRITING the energy of a many-body system in terms of the many coordinates and their momenta, we can, in principle, study the system dynamics by solving Hamilton's canonical equations. The information required for and obtained from this procedure is of such detail that the prescription cannot be followed. Also, this formulation of the many-body problem is not suited to the generation of approximations.

To express the plasma problem in a more practicable form, we focus our attention on more directly measurable dynamical variables such as the local particle density or current density. We attempt to construct a dynamics of densities in place of the dynamics of particles.

There are as many different independent densities as degrees of freedom of the problem, but because we need information only about one or a few densities to make contact with experiment, this point of view is far more useful. The generation of approximations is greatly facilitated.

A transport equation is a relationship between various time and space derivatives of some densities. Setting up a transport equation in classical physics starts with the consideration of the joint density of particles in coordinates and velocities, $f(\mathbf{r},\mathbf{v})$. In general, this function is coupled to more complicated functions, $f_1(\mathbf{r},\mathbf{r}',\mathbf{v})$, $f_2(\mathbf{r},\mathbf{r}',\mathbf{r}'',\mathbf{v})\ldots$, $f_3(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}')\ldots$, etc. Quite useful approximations can be obtained, however, by replacing all these effects by a coupling between $f(\mathbf{r},\mathbf{v})$ and $f(\mathbf{r}',\mathbf{v}')$ or even by a phenomenological constant. The construction of a transport equation then reduces to the kinematic observation that

$$-\frac{\partial f}{\partial t}=\mathbf{v}\cdot\nabla f+\frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}f \qquad (1)$$

and the identification of $m d\mathbf{v}/dt$ with the average force on a constituent particle. This is the method of the Boltzmann equation.

The quantum-mechanical problem is more difficult, so rendered by the uncertainty relation between $\mathbf{r}$ and $\mathbf{p}$. This problem is surmountable in the limit of long-wavelength variations in the densities. One considers the behavior of wave packets describing the motion of single particles. As long as the spread in the packet can be maintained well below the characteristic dimensions of the system, i.e., much less than the wavelengths of the important phenomena and the Fermi velocity, the failure of localization is unimportant. The classical dynamics of particles is replaced by the quantum mechanics of wave packets. Silin[1] has presented transport equations for the density and spin density based on this semiclassical approach, and these equations have recently been put to considerable use.[2–4]

We claim here that a fully quantum-mechanical treatment of many-body dynamics leads to equations which differ from Silin's in the effects of magnetic fields.

The reduction of the many-body problem to single-packet dynamics occurs at a disturbingly early point in the semiclassical treatment. Furthermore, the treatment of the forces on a wave packet suggests a different gauge choice for every particle of the plasma.[5] Thus even if no discrepancies arose, it would be of some interest to attempt the derivation of transport equations from a more completely quantum-mechanical viewpoint.

It is not sufficient, however, merely to discard the semiclassical approach and completely reformulate the plasma problem. For the same reasons that the Boltzmann equation approach is valuable in the classical problem, the semiclassical approximation, with its emphasis on the more directly measureable densities, is highly useful. We have, therefore, cast our reformulation in language which makes as close contact as possible with the semiclassical form. Our results appear as slightly different, Boltzmann-like transport equations.

It is also important that we maintain close correspondence between our approach and the semiclassical forms, because Silin's[1,6] equations are presented essentially without accompanying derivation.

The joint density function $f(\mathbf{r},\mathbf{v})$ occupies a central

* Supported by Advanced Research Projects Agency.

[1] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **33**, 495 (1957) [English transl: Soviet Phys—JETP 6, 945 (1958)].
[2] P. M. Platzman and P. A. Wolff, Phys. Rev. Letters **18**, 280 (1967).
[3] S. C. Ying and J. J. Quinn, Phys. Letters **26A**, 347 (1968).
[4] P. M. Platzman and W. M. Walsh, Jr., Phys. Rev. Letters **19**, 514 (1967).
[5] A. H. Wilson, *Theory of Metals* (Cambridge University Press, Cambridge, England, 1953), 2nd ed., p. 51ff.
[6] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **35**, 1253 (1958) [English transl.: Soviet Phys—JETP **8**, 870 (1959)].

position in the classical and semiclassical transport theories; physical, measurable numbers are expressed as integrals over the space of $\mathbf{v}$ with $f(\mathbf{r},\mathbf{v})$ as a weight function. The corresponding quantity in the quantum theory is the density matrix ρ. Physical numbers are expressed as traces of products of operators with the density matrix. The density matrix is therefore a matter of central concern.

The interactions make obtaining exactly correct transport equations humanly impossible. We begin, therefore, with an undisturbed, noninteracting plasma and add the interactions through an approximation scheme; for our discussion here we shall use a form of self-consistent field approximation.

In the absence of interactions, external driving, or scattering, the density matrix can be found exactly. The one-particle eigenstates are harmonic-oscillator wave functions on various arguments depending on the formulation of the problem. Frequently, they are oscillator functions of the hybrid variables, $\mathbf{p}\pm e\mathbf{A}/c$.[7] In this form, the particle states bear no resemblance to the particle packets of the semiclassical approximation. Furthermore, constructing matrix elements of interaction operators depending on $\mathbf{r}$ and $\mathbf{r}'$ between wave functions on the hybrid variables is awkward.

In the semiclassical approximation, we begin with a collection of states describing particles with various isotropically distributed velocities precessing around B at the cyclotron frequency under the influence of the Lorentz force. In application to degenerate Fermi gases, particles of all velocities up the Fermi velocity are present in the plasma; higher velocity particles are absent. Particles of all energies between zero and the Fermi energy are present.

In the quantum-mechanical states above, the isotropy in velocity is built in fundamentally; the states are eigenstates of v^2 but not of v_x or v_y, precession is meaningless, and only certain discrete particle energies are possible.

It is possible to reformulate the quantum-mechanical Liouville equation, however, in order to achieve a much closer correspondence with the semiclassical form. We have previously developed this reformulation for the interacting many-body system[8] by application of a transformation of Thomas[9] useful for the free-particle problem. As we shall discuss, a certain "overlocalization" approximation leads to the semiclassical form. This reformulation is entirely gauge independent, thus eliminating one troublesome aspect of the semiclassical approach.

In Sec. II, we present solutions of the reformulated Liouville equation for the case of ideal plasmas. We discuss the nature of the approximation leading to the semiclassical form.

In Sec. III we include the effects of interactions and external driving fields. The resulting treatment of the interactions is of the form of a time-dependent self-consistent field approximation.

In Sec. IV, we proceed to the actual generation of transport equations on the particle current and spin density. To obtain a transport equation, it is necessary to consider a hierarchy of expectations of various operators. For instance, to obtain a complete description of the particle density, we shall consider the density itself, the current density, the current-density current, and so forth. The reason for this is that our theory makes its predictions in terms of traces of various operators multiplied by ρ, not in terms of $f(\mathbf{r},\mathbf{v})$ directly. Effectively, we build up $f(\mathbf{r},\mathbf{v})$ by generating its moments.

II. SOLUTION OF MODIFIED LIOUVILLE EQUATIONS IN "LOCALIZED VELOCITY" APPROXIMATION

Our starting point is the series of basic equations developed in Ref. 8. Let the Hamiltonian of a many-body system be given by

$$\mathcal{H}=\sum_{s=1}^{N}\left\{\frac{1}{2m}\left(\mathbf{p}_s-\frac{e}{c}\mathbf{A}_s\right)^2+U(\mathbf{r}_s,\boldsymbol{\sigma}_s)+\tfrac{1}{2}\sum_{s'\neq s}^{N}V(\mathbf{r}_s-\mathbf{r}_{s'})\right\}. \quad (2)$$

The density matrix satisfies Liouville's equation,

$$i\hbar\frac{\partial\rho}{\partial t}=[\mathcal{H},\rho]. \quad (3)$$

We define a new operator $\tilde{\rho}$:

$$\tilde{\rho}=\exp\left\{\sum_s\frac{e}{c}\mathbf{A}_s\cdot\left(\frac{\partial}{\partial\mathbf{p}_s}\right)_{op}\right\}\rho, \quad (4)$$

in which $(\partial/\partial p_{sx})_{op}{}^n\rho$ means

$$(i\hbar)^n[x_s,[x_s\cdots n \text{ times}\ldots,\rho]]\cdots n \text{ times}\cdots]. \quad (5)$$

The operator $\tilde{\rho}$ satisfies a modified equation of motion obtained from the Liouville equation

$$i\hbar\frac{\partial\tilde{\rho}}{\partial t}=[\tilde{\mathcal{H}},\tilde{\rho}]-\sum_{s=1}^{N}\frac{e}{2mc}[\mathbf{p}_s\times\mathbf{B}\cdot[\mathbf{r}_s,\tilde{\rho}]+[\mathbf{r}_s,\tilde{\rho}]\cdot\mathbf{p}_s\times\mathbf{B}], \quad (6)$$

in which $\tilde{\mathcal{H}}$ is $\mathcal{H}$ but with $\mathbf{p}_s-(e/c)\mathbf{A}_s$ replaced by $\mathbf{p}_s$.

Values of physical quantities are found by taking traces with ρ. Let $g[\mathbf{r}_s,m\mathbf{v}_s]$ be some interesting operator. In terms of $\tilde{\rho}$,

$$\langle g\rangle=\mathrm{Tr}g\rho=\mathrm{Tr}g\left[\mathbf{r}_s,\mathbf{p}_s-i\hbar\frac{e\mathbf{B}}{2c}\times\left(\frac{\partial}{\partial\mathbf{p}_s}\right)_{op}\right]\tilde{\rho}. \quad (7)$$

[7] For example, R. E. Peierls, *Quantum Theory of Solids* (Clarendon Press, Oxford, England, 1955), p. 144ff.
[8] L. L. Van Zandt, preceding paper, Phys. Rev. B 1, 3217 (1970).
[9] R. B. Thomas, Jr., Phys. Rev. **171**, 827 (1968).

Let us consider $\tilde{\rho}$ in the representation described by basis functions which are determinants of plane-wave states,

$$\tilde{\rho}=\sum_{\{\mathbf{k}\},\{\mathbf{k}'\}}|D\{\mathbf{k}'\}\rangle\rho(\{\mathbf{k}'\},\{\mathbf{k}\})\langle D\{\mathbf{k}\}|\,, \tag{8}$$

in which $\{\mathbf{k}\}$ means a set of vectors $\mathbf{k}_1\cdots\mathbf{k}_N$ describing occupied plane-wave states.

Consider the operator $\sum_s \mathbf{p}_s\times\mathbf{B}\cdot\mathbf{r}_s\tilde{\rho}$. To find matrix elements of this, we construct

$$\sum_{\{\mathbf{k}''\}}\langle D\{\mathbf{k}'\}|\sum_s \mathbf{p}_s\times\mathbf{B}\cdot\mathbf{r}_s|D(\{\mathbf{k}''\})\rangle\rho(\{\mathbf{k}''\},\{\mathbf{k}\}), \tag{9}$$

since $\langle D(\{\mathbf{k}'\})|D(\{\mathbf{k}\})\rangle=\delta(\{\mathbf{k}',\}\{\mathbf{k}\})$. To give meaning to matrix elements of $\mathbf{r}_s$ between plane-wave states, we write

$$x_s=\lim_{\gamma\to 0}\frac{e^{i\gamma x_s}-1}{i\gamma}, \tag{10}$$

with a similar expression for y_s. Taking $\mathbf{B}$ in the z direction ensures only x_s and y_s need be considered. Multiplying the plane-wave state $e^{i\mathbf{k}\cdot\mathbf{r}_s}$ by $e^{i\gamma x_s}$ changes it to $e^{i(\mathbf{k}+\gamma\bar{x})\cdot\mathbf{r}_s}$. This plane wave is an eigenstate of $\mathbf{p}_s$. Carrying out the indicates sum over $\{\mathbf{k}''\}$ yields

$$\sum_{\mathbf{k}'\epsilon\{\mathbf{k}'\}} i\hbar\mathbf{k}'\times\mathbf{B}\cdot\nabla_{\mathbf{k}'}\rho(\{\mathbf{k}'\},\{\mathbf{k}\}). \tag{11}$$

If we now take the operator $\sum_s \mathbf{p}_s\times\mathbf{B}\cdot(\tilde{\rho}\mathbf{r}_s)$ and contruct its matrix elements, we obtain

$$-\sum_{\mathbf{k}\epsilon\{\mathbf{k}\}} i\hbar\mathbf{k}'\times\mathbf{B}\cdot\nabla_{\mathbf{k}}\rho(\{\mathbf{k}'\},\{\mathbf{k}\}). \tag{12}$$

Combining (11) and (12) yields

$$\begin{aligned}&(\textstyle\sum\mathbf{p}_s\times\mathbf{B}\cdot[\mathbf{r}_s,\rho])_{\{\mathbf{k}'\},\{\mathbf{k}\}}\\&=\sum_{\mathbf{k}\epsilon\{\mathbf{k}\},\mathbf{k}'\epsilon\{\mathbf{k}'\}} i(\hbar\mathbf{k}'\times\mathbf{B}\cdot\nabla_{\mathbf{k}}+\hbar\mathbf{k}'\times\mathbf{B}\cdot\nabla_{\mathbf{k}'})\\&\qquad\times\rho(\{\mathbf{k}'\},\{\mathbf{k}\}).\end{aligned} \tag{13}$$

By a similar set of manipulations, we obtain a similar expression for

$$\begin{aligned}&(\textstyle\sum_s[\mathbf{r}_s,\tilde{\rho}]\cdot\mathbf{p}_s\times\mathbf{B})_{\{\mathbf{k}'\},\{\mathbf{k}\}}\\&=\sum_{\mathbf{k}\epsilon\{\mathbf{k}\},\mathbf{k}'\epsilon\{\mathbf{k}'\}} i(\hbar\mathbf{k}\times\mathbf{B}\cdot\nabla_{\mathbf{k}'}+\hbar\mathbf{k}\times\mathbf{B}\cdot\nabla_{\mathbf{k}})\\&\qquad\times\rho(\{\mathbf{k}'\},\{\mathbf{k}\}).\end{aligned}$$

We can write $(\partial\tilde{\rho}/\partial t)_{\{\mathbf{k}'\},\{\mathbf{k}\}}$ as two terms:

$$\begin{aligned}&i\hbar\left(\frac{\partial\tilde{\rho}}{\partial t}\right)_{\{\mathbf{k}'\},\{\mathbf{k}\}}\\&=i\hbar\left(\frac{\partial\tilde{\rho}}{\partial t}\right)_{\{\mathbf{k}'\},\{\mathbf{k}\}\ \text{fixed k's}}+\sum_{\mathbf{k}\epsilon\{\mathbf{k}\},\mathbf{k}'\epsilon\{\mathbf{k}'\}}\frac{e}{2mc}(\mathbf{k}+\mathbf{k}')\\&\qquad\times\mathbf{B}\cdot(\nabla_{\mathbf{k}}+\nabla_{\mathbf{k}'})\rho(\{\mathbf{k}'\},\{\mathbf{k}\}).\end{aligned} \tag{14}$$

Thus we may rewrite Eq. (6) as

$$i\hbar\left(\frac{\partial\tilde{\rho}}{\partial t}\right)_{\text{fixed k's}}=[\tilde{\mathfrak{IC}},\tilde{\rho}], \tag{15}$$

by writing ρ as $\rho(\{\mathbf{k}+\mathbf{k}'\},\{\mathbf{k}-\mathbf{k}'\})$ and making the $\mathbf{k}+\mathbf{k}'$ part time dependent with

$$\frac{d}{dt}(\mathbf{k}+\mathbf{k}')=-\frac{e}{mc}(\mathbf{k}+\mathbf{k}')\times\mathbf{B}. \tag{16}$$

The problem of the plasma in the magnetic field has now been reduced to the solution of Eq. (15) from which not only the gauge, but even reference to the field has been eliminated. That the problem can be cast into the zero-field form, however, presents something of a paradox, since there are characteristic differences in the field and no-field problems which should survive any series of nonsingular transformations; as examples of such differences, we have the differing lower bounds and characteristic structure of the energy spectra.

The resolution of this paradox lies in considering the conditions which ρ and $\tilde{\rho}$ must satisfy besides the Liouville equation. To represent physically meaningful states, ρ must have all real non-negative eigenvalues. This condition implies that the eigenvalues of $\tilde{\rho}$ are real, but it does *not* make them positive. The operation defined in (4) is not a similarity transformation and thus can change the character of the spectrum. If we treat $\tilde{\rho}$ like an ordinary density matrix, we are led through the transformation (4) back to a ρ with nonphysical eigenstates. Conversely, insisting that ρ have only nonnegative eigenvalues leads to a $\tilde{\rho}$ with negative eigenvalues. The distinctions between field and no-field cases have vanished from the differential equation for the density matrix, but survive in the form of initial conditions. If, in addition, we need to consider pure states $\rho^2=\rho$, we are led to ensembles in the space of transformed states $\tilde{\rho}^2<\tilde{\rho}$.

The importance of the negative eigenvalues of $\tilde{\rho}$ and the extent to which they depart from unity or zero depends on the degree of localization of the states in $\mathbf{k}$ space. We may consider forming packets in the space of functions obtained by solving (15), applying the transformation, (4), to the resulting $\tilde{\rho}$ and so obtaining ρ. As long as the initial packet is broad compared to the separation of states in neighboring Landau levels, ρ approximately describes a pure state and has eigenvalues 1,0,0,0,.... Only when we attempt to make the packet too small in $\tilde{\rho}$ do the eigenvalues of ρ become pathological. These results are developed for a specific example in Appendix A.

Thus as long as our results are not sensibly dependent on a tight "localization" of states in $\mathbf{k}$ space we may ignore the actual necessity of using broadened states in solving (15). This is equivalent in the cases we shall con-

sider to ignoring the condensation of energy eigenvalues into Landau levels. It is precisely the approximation we need to obtain the semiclassical formalism. We shall now proceed with the solving of (15) [or equivalently (16)], in an "extreme localization" approximation. Of course we obtain no de Haas–van Alphen effect nor related phenomena, exactly as in the semiclassical approach.

III. SELF-CONSISTENT FIELD APPROXIMATION

We are interested in a homogeneous electron gas driven by some external rf field. We shall consider separately two types of driving field: a magnetic field driving the electron spins and an electric field driving the orbital motion. We begin with the spin-driving magnetic field because it is the simpler. In our Hamiltonian (1), we take

$$U(\mathbf{r}_s,\sigma_s)=c_0\{e^{i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}\sigma_s^{-}+e^{-i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}\sigma_s^{+}\}. \quad (17)$$

The wave vector and frequency of the driving field are given by $\mathbf{q}$ and ω, the amplitude by c_0.

Our unperturbed starting state is described by an ensemble of Slater determinants of plane-wave states with time-dependent wave vectors. The driving field in (17) takes a state $\mathbf{k}\uparrow$ and converts it to the state $\mathbf{k}+\mathbf{q}\downarrow$ and vice versa. (It is this reciprocity which makes the spin-wave case easier.) We therefore take linear combinations of states, $b_1\varphi_{\mathbf{k}\uparrow}+b_2\varphi_{\mathbf{k}+\mathbf{q}\downarrow}$ as our approximate one particle states in the presence of the driving field and form Slater determinants from them to obtain approximate wave functions.

From these wave functions, we construct $\tilde{\rho}$. To determine the $b_1(\mathbf{k})$, $b_2(\mathbf{k})$, we could use the variational principle on $\partial\tilde{\rho}/\partial t$ developed in Ref. 8. The time variation of the various $\mathbf{k}$ could likewise be obtained. However, our result in (15) and (16) of the preceeding sections makes this more general but more cumbersome procedure unnecessary. Instead, we use (15) to write

$$\mathfrak{K}\Phi=i\hbar\left(\frac{\partial\Phi}{\partial t}\right)_{\text{fixed k's}} \quad (18)$$

as an effective Schrödinger equation for the eigenstates of $\tilde{\rho}$, and (18) may then be solved approximately by the use of the "time-dependent Hartree-Fock" approximation. We have elsewhere[10] discussed this approximation at length including its derivation from Frenkel's[11,12] variational principle.

We first assume that c_0 is small so that we may expand in powers of c_0 about the undriven starting state. We then apply the variation equations[10]

$$0=\int\left\{\frac{\delta}{\delta b(\mathbf{k})}\left(\frac{\partial\Phi^*}{\partial t}\right)_{\text{fixed k's}}\right\}\times\left\{\mathfrak{K}\Phi-i\hbar\left(\frac{\partial\Phi}{\partial t}\right)_{\text{fixed k's}}\right\}d\mathbf{r}^{(n)}. \quad (19)$$

Using our assumptions about the form of Φ in (19), we obtain a set of self-consistent field equations for the $b(\mathbf{k})$. This derivation is presented in detail in an appendix, and a similar one has been presented elsewhere,[10]

$$(\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\downarrow})b(\mathbf{k}\uparrow)+i\hbar\left(\frac{\partial}{\partial t}\right)_{\text{fixed k}}b(\mathbf{k}\uparrow)=\sum_{\mathbf{k}'}\{V(\mathbf{k}-\mathbf{k}')(n_{\mathbf{k}'+\mathbf{q}\downarrow}-n_{\mathbf{k}'\uparrow})\times[(b(\mathbf{k}'\uparrow)-b(\mathbf{k}\uparrow))]\}+c_0e^{-i\omega t}. \quad (20)$$

This expression is composed of readily identifiable parts. The kinetic-energy difference and the time-derivative terms are frequently encountered in self-consistent field calculations as the energy denominator $\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}-\epsilon_{\mathbf{k}\uparrow}+\hbar\omega$. The interaction term can be broken into two parts. One part is proportional to $b(\mathbf{k}\uparrow)$ and can be added to $\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}$ to correct the single-particle kinetic energies by the addition of the Hartree-Fock exchange energy. The second part involves a sum over "other" states of terms proportional to the response of the other states to the disturbing rf field. This part assumes the role of a correction to the driving field by the addition of an internal polarization.

We obtain $\tilde{\rho}$ by solving (20), using the b's to construct single-particle wave functions, combining these wave functions in Slater determinant many-particle functions, and constructing an appropriate ensemble from many-body functions of differing $n_{\mathbf{k},\sigma}$.

The construction of single-particle wave functions from combinations of plane waves, $\mathbf{k}\uparrow$ and $\mathbf{k}+\mathbf{q}\downarrow$, as well as the derivation of (20) assumes that $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ are separated in $\mathbf{k}$ space, that the required packet size or spreading of the wave functions may be ignored. This assumption also enters the semiclassical approximation when the constituent packets on which the Lorentz force is calculated are eventually replaced by well-defined plane waves. We are here in a position to place quantitative limits on the validity of that approximation. Quoting from the Appendix, the wave vector $\mathbf{q}$ must be large compared to $\sqrt{\beta^{-1}}$ which measures the extent to which $\tilde{\rho}$ is not diagonal in the "extreme localized" functions. Now $\sqrt{\beta^{-1}}$ may be made small only if simultaneously $\sqrt{\alpha^{-1}}$, the parameter measuring the ensemble width (as opposed to packet width), is made large. This ensemble width should be kept much less than the Fermi radius to preserve a well-defined Fermi surface.

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[10] L. L. Van Zandt, Phys. Rev. 172, 372 (1968).
[11] J. Frenkel, *Wave Mechanics, Advanced General Theory* (Clarendon Press, Oxford, England, 1934), p. 253.
[12] See also A. D. McLachlan and M. A. Ball, Rev. Mod. Phys. 36, 844 (1964).

From Appendix A we have

$$\frac{m\omega_c}{\hbar}(\alpha\beta)^{1/2}\ll 1, \tag{21}$$

and since $q\gg 1/\sqrt{\beta}$ and $k_F\gg 1/\sqrt{\alpha}$, we have

$$\hbar\omega_c\left[\frac{\hbar^2 q k_F}{2m}\right]^{-1}\ll 1 \tag{22}$$

as the condition for the validity of the semiclassical approximation. Outside this range, explicit account must be taken of the dispersion of the individual single-particle states about their mean $\mathbf{k}$ vector. In other words, the uncertainty principle operating between v_x and v_y may not be ignored if (22) is not satisfied. We shall point out in a subsequent paper where this condition has been violated in recent publications.

We consider now longitudinal density waves. Equations analogous to (20) may be obtained by taking

$$U(\mathbf{r}_s,\boldsymbol{\sigma}_s)=d_0[e^{i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}+e^{-i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}] \tag{23}$$

and assuming wave functions of the form

$$\Psi=\varphi_{\mathbf{k},\sigma}+d_1\varphi_{\mathbf{k}+\mathbf{q},\sigma}+d_2\varphi_{\mathbf{k}-\mathbf{q},\sigma}, \tag{24}$$

where the $\varphi_{\mathbf{k}}$ are plane-wave functions. Self-consistent field equations on the d's have been derived by many authors. Reference 10 includes a derivation in the notation and language used here,

$$\begin{aligned}(\epsilon_{\mathbf{k},\sigma}-\epsilon_{\mathbf{k}+\mathbf{q},\sigma})d_1(\mathbf{k},\boldsymbol{\sigma})+i\hbar\left(\frac{\partial}{\partial t}\right)_{\text{fixed k}}d_1(\mathbf{k},\boldsymbol{\sigma})\\ =V(\mathbf{q})\sum_{\mathbf{k}',\sigma'}(n_{\mathbf{k}',\sigma'}-n_{\mathbf{k}'+\mathbf{q},\sigma'})d_1(\mathbf{k}',\boldsymbol{\sigma}')\\ +\sum_{\mathbf{k}'}V(\mathbf{k}-\mathbf{k}')(n_{\mathbf{k}'+\mathbf{q},\sigma}-n_{\mathbf{k}',\sigma})[d_1(\mathbf{k}',\boldsymbol{\sigma})-d_1(\mathbf{k},\boldsymbol{\sigma})]\\ +d_0e^{-i\omega t}\end{aligned} \tag{25}$$

and

$$d_2(\mathbf{k}+\mathbf{q},\boldsymbol{\sigma})+d_1^*(\mathbf{k},\boldsymbol{\sigma})=0. \tag{26}$$

The only substantial difference between (25) and (20) is the appearance of the "direct" term, proportional to $V(\mathbf{q})$, as well as the exchange terms.

Finally, we consider the case of transverse electromagnetic driving fields. To treat these, we set $U(\mathbf{r}_s,\boldsymbol{\sigma}_s)=0$ and take

$$\sum_s\frac{1}{2m}\left[\mathbf{p}_s-\frac{e\mathbf{a}(\mathbf{r}_s,t)}{c}\right]^2 \tag{27}$$

for the kinetic-energy part of $\mathfrak{H}$. We relate $\mathbf{a}(\mathbf{r}_s)$ to the rf fields by

$$\begin{aligned}\mathbf{B}_{\text{rf}}(\mathbf{r})&=\nabla\times\mathbf{a}(\mathbf{r},t),\\ \mathbf{E}_{\text{rf}}&=-\frac{1}{c}\frac{\partial\mathbf{a}(\mathbf{r},t)}{\partial t},\end{aligned} \tag{28}$$

and assuming $\mathbf{a}$ small obtain

$$\sum_s\frac{e\mathbf{p}_s}{mc}\cdot\mathbf{a}(\mathbf{r}_s,t)=\sum_s\frac{e\mathbf{p}_s\cdot\mathbf{a}_0}{mc}[e^{i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}+e^{-i(\mathbf{q}\cdot\mathbf{r}_s-\omega t)}] \tag{29}$$

for the driving field. This has a form similar to (23), so we construct similar wave functions to obtain

$$\begin{aligned}(\epsilon_{\mathbf{k},\sigma}-\epsilon_{\mathbf{k}+\mathbf{q},\sigma})d_1(\mathbf{k},\boldsymbol{\sigma})+i\hbar\left(\frac{\partial}{\partial t}\right)_{\text{fixed k}}d_1(\mathbf{k},\boldsymbol{\sigma})\\ =V(\mathbf{q})\sum_{\mathbf{k}',\sigma'}(n_{\mathbf{k}',\sigma'}-n_{\mathbf{k}'+\mathbf{q},\sigma'})d_1(\mathbf{k}',\boldsymbol{\sigma}')\\ +\sum_{\mathbf{k}'}V(\mathbf{k}-\mathbf{k}')(n_{\mathbf{k}'+\mathbf{q},\sigma}-n_{\mathbf{k}',\sigma})[d_1(\mathbf{k}',\boldsymbol{\sigma})-d_1(\mathbf{k},\boldsymbol{\sigma})]\\ +\frac{e\hbar\mathbf{a}_0}{mc}\cdot(\mathbf{k}+\mathbf{q})e^{-i\omega t}.\end{aligned} \tag{30}$$

The only difference between (30) and (25) is in the form of the deriving terms reflecting the differences in the fields (23) and (29).

This completes the assembly of the self-consistent field relations necessary to construct our transport equations.

IV. TRANSPORT EQUATIONS

A transport equation is a relation between various time and space derivatives of some physical quantity. The physical quantities for which we shall construct transport equations are densities and current densities; the simplest of these is the particle density itself,

$$n(\mathbf{r})\equiv\mathrm{Tr}\sum_{s=1}^{N}\delta(\mathbf{r}-\mathbf{r}_s)\rho. \tag{31}$$

Physically meaningful numbers are obtained as traces of the appropriate operators multiplied by the density matrix.[13] We shall consider, besides the density itself, the spin density:

$$\boldsymbol{\sigma}(\mathbf{r})\equiv\mathrm{Tr}\sum_{s=1}^{N}\boldsymbol{\sigma}_s\delta(\mathbf{r}-\mathbf{r}_s)\rho, \tag{32}$$

the particle-current density:

$$\mathbf{j}(\mathbf{r})\equiv\mathrm{Tr}\sum_{s=1}^{N}\tfrac{1}{2}\{\mathbf{v}_s\delta(\mathbf{r}-\mathbf{r}_s)+\delta(\mathbf{r}-\mathbf{r}_s)\mathbf{v}_s\}\rho, \tag{33}$$

the spin-current density:

$$\mathbf{S}(\mathbf{r})=\tfrac{1}{2}\,\mathrm{Tr}\sum_{s=1}^{N}\boldsymbol{\sigma}_s\{\mathbf{v}_s\delta(\mathbf{r}-\mathbf{r}_s)+\delta(\mathbf{r}-\mathbf{r}_s)\mathbf{v}_s\}\rho, \tag{34}$$

[13] The transport equation of Ref. 9 is obtained by considering a certain set of matrix elements of $\tilde{\rho}$. We shall see, Eq. (61), that in the self-consistent field approximation these elements actually are the appropriate quantities to study. This does not seem to be an obvious result, however, and we feel that the following discussion may clarify the relation of these matrix elements to actual physical quantities.

the current-current density:

$$J_{ij}(\mathbf{r})=\tfrac{1}{2}\,\mathrm{Tr}\sum_{s=1}^{N}\{v_{si}v_{sj}\delta(\mathbf{r}-\mathbf{r}_s)+\delta(\mathbf{r}-\mathbf{r}_s)v_{si}v_{sj}\}\rho\,, \tag{35}$$

and a hierarchy of tensor current densities obtained from powers of the velocity operators.

We obtain equations of motion for the operators by differentiating by time

$$\frac{\partial\langle O\rangle}{\partial t}=\frac{\partial}{\partial t}\mathrm{Tr}O\rho$$

$$=\frac{\partial}{\partial t}\mathrm{Tr}\tilde{O}\tilde{\rho} \tag{36}$$

from Eq. (7). We shall not consider operators which are explicit functions of time. Thus

$$i\hbar\frac{\partial\langle O\rangle}{\partial t}=i\hbar\,\mathrm{Tr}\tilde{O}\frac{\partial\tilde{\rho}}{\partial t}\,. \tag{37}$$

Using Eq. (5),

$$i\hbar\frac{\partial\langle O\rangle}{\partial t}=\mathrm{Tr}\tilde{O}\Big\{[\tilde{\mathcal{H}},\tilde{\rho}]-\frac{e}{2mc}\sum_{s=1}^{N}\mathbf{p}_s\times\mathbf{B}\cdot[\mathbf{r}_s,\tilde{\rho}]$$
$$+[\mathbf{r}_s,\tilde{\rho}]\cdot\mathbf{p}_s\times\mathbf{B}\Big\}\,. \tag{38}$$

We construct $\tilde{\rho}$ to be diagonal in a representation built up from single-particle states as obtained in the last section. Let $D\{\mathbf{k}\}$ be a determinant of such states. In terms of plane-wave states $D\{\mathbf{k}\}$, $\tilde{\rho}$ has large diagonal elements, $\rho(\{\mathbf{k}\},\{\mathbf{k}\})$, plus off-diagonal elements $\rho(\{\mathbf{k}'\},\{\mathbf{k}\})$, which are of higher order in the b's or d's. In particular, $\rho_{\mathbf{k}_0\pm\mathbf{q};\mathbf{k}_0}$ is of first order, where this symbol means $\{\mathbf{k}'\}$ is formed from $\{\mathbf{k}\}$ by replacing $\mathbf{k}_0$ with $\mathbf{k}_0\pm\mathbf{q}$. Similarly, $\rho_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0'\pm\mathbf{q};\mathbf{k}_0,\mathbf{k}_0'}$ is of second order, and so forth; we discard these parts. The operator $\tilde{\mathcal{H}}$ has diagonal elements in the $D\{\mathbf{k}\}$ coming from the kinetic-energy parts and part of the interactions; it has off-diagonal elements of first order in the driving fields of the form $\tilde{\mathcal{H}}_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}$, and elements of the form $\tilde{\mathcal{H}}_{\mathbf{k}_0+\mathbf{k},\mathbf{k}_0'-\mathbf{k};\mathbf{k}_0',\mathbf{k}_0}$ from the interaction terms. The operator O may have diagonal elements but as long as we consider only single-particle operators, as in (31)–(35), the off-diagonal elements of O can only be of the form $\tilde{O}_{\mathbf{k}_0,\mathbf{k}_0'}$. Elements of the form $\tilde{O}_{\mathbf{k}_0,\mathbf{k}_1;\mathbf{k}_1',\mathbf{k}_0'}$ require O to contain forms $\mathbf{r}_s\mathbf{r}_{s'}$, $\mathbf{v}_s\mathbf{v}_{s'}$, $\mathbf{v}_s\mathbf{r}_{s'}$, or the like. Thus, when we evaluate the trace in (38) only those parts of $\partial\tilde{\rho}/\partial t$ which are of the form $(\partial\tilde{\rho}/\partial t)_{\mathbf{k}_0',\mathbf{k}_0}$ or are diagonal need to be considered.

The diagonal elements of $[\tilde{\mathcal{H}},\tilde{\rho}]$ vanish. We can form "one-particle off-diagonal" elements of this from the diagonal parts of $\tilde{\mathcal{H}}$ and $\rho_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}$, from the driving terms in $\tilde{\mathcal{H}}$ and diagonal parts of $\tilde{\rho}$, or from the "two-particle off-diagonal" elements of $\tilde{\mathcal{H}}$ and $\rho_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}$. In these last, $\rho_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}$ must restore to the diagonal one of the two states changed by $\tilde{\mathcal{H}}$. We then have $\{[\tilde{\mathcal{H}},\tilde{\rho}]\}_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}$ for the only significant parts of $[\tilde{\mathcal{H}},\tilde{\rho}]$ in the trace. We may evaluate the various necessary matrix elements by procedures similar to those used in Appendix B to obtain

$$\mathrm{Tr}\tilde{O}[\tilde{\mathcal{H}},\tilde{\rho}]$$
$$=\sum_{\mathbf{k}_0,\sigma_0}\tilde{O}_{\mathbf{k}_0,\sigma_0;\mathbf{k}_0+\mathbf{q},\sigma_0}d_1(\mathbf{k}_0,\boldsymbol{\sigma}_0)(n_{\mathbf{k}_0+\mathbf{q},\sigma_0}-n_{\mathbf{k}_0,\sigma_0})$$
$$\times\Big\{\epsilon_{\mathbf{k}_0,\sigma_0}-\epsilon_{\mathbf{k}_0+\mathbf{q},\sigma_0}-V(\mathbf{q})\sum_{\mathbf{k}'\sigma'}(n_{\mathbf{k}'}-n_{\mathbf{k}'+\mathbf{q}})\frac{d_1(\mathbf{k}',\boldsymbol{\sigma}')}{d_1(\mathbf{k}_0,\boldsymbol{\sigma}_0)}$$
$$-\sum_{\mathbf{k}'}V(\mathbf{k}_0-\mathbf{k}')(n_{\mathbf{k}'+\mathbf{q},\sigma_0}-n_{\mathbf{k}',\sigma_0})$$
$$\times\frac{[d_1(\mathbf{k}',\boldsymbol{\sigma}_0)-d_1(\mathbf{k}_0,\boldsymbol{\sigma}_0)]}{d_1(\mathbf{k}_0,\boldsymbol{\sigma}_0)}-\frac{d_0e^{-i\omega t}}{d_1(\mathbf{k}_0,\boldsymbol{\sigma}_0)}\Big\}+\text{c.c.} \tag{39}$$

for the case of longitudinal driving fields, (3), and analogous expressions for the other cases.

Let us consider now the terms of (38) in $\mathbf{p}_s\times\mathbf{B}$. Using the procedure of Sec. II, we have

$$\mathrm{Tr}\tilde{O}\Big(\frac{\partial\rho}{\partial t}\Big)_{\text{fixed }d\text{'s}}$$
$$=\sum_{\{\mathbf{k}\}\{\mathbf{k}'\}}\langle D\{\mathbf{k}\}|\tilde{O}|D\{\mathbf{k}'\}\rangle$$
$$\times\frac{e}{2mc}\Big\{\sum_{\substack{\mathbf{k}'\epsilon\{\mathbf{k}'\}\\ \mathbf{k}\epsilon\{\mathbf{k}\}}}\mathbf{B}\times(\mathbf{k}+\mathbf{k}')\cdot\Big(\frac{\partial}{\partial\mathbf{k}}+\frac{\partial}{\partial\mathbf{k}'}\Big)$$
$$\times\rho(\{\mathbf{k}'\},\{\mathbf{k}\})\Big\}\,. \tag{40}$$

The diagonal elements of the second factor are independent of the amplitude of the driving field, which means equal to their equilibrium values. For these elements, we have

$$\frac{e}{mc}\sum_{\mathbf{k}\epsilon\{\mathbf{k}\}}\mathbf{B}\times\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{k}}\rho(\{\mathbf{k}\},\{\mathbf{k}\})=0\,. \tag{41}$$

Also,

$$\mathbf{B}\times\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{k}}\rho_{\mathbf{k}_0\pm\mathbf{q},\mathbf{k}_0}=0,\quad\text{if}\quad\mathbf{k}\neq\mathbf{k}_0,\mathbf{k}_0\pm\mathbf{q}\,.$$

The factor in braces in (40) thus yields

$$\frac{e}{2mc}\sum_{\substack{\mathbf{k}_0'\epsilon\{\mathbf{k}'\}\\ \mathbf{k}_0\epsilon\{\mathbf{k}\}}}\mathbf{B}\times(\mathbf{k}_0+\mathbf{k}_0')\cdot\Big(\frac{\partial}{\partial\mathbf{k}_0}+\frac{\partial}{\partial\mathbf{k}_0'}\Big)\rho_{\mathbf{k}_0',\mathbf{k}_0}\,. \tag{42}$$

Now for $\rho_{\mathbf{k}_0',\mathbf{k}_0}$, we have

$$\rho_{\mathbf{k}_0',\mathbf{k}_0}=-d_1(\mathbf{k}_0)(n_{\mathbf{k}_0'}-n_{\mathbf{k}_0})\delta(\mathbf{k}_0'-\mathbf{k}_0-\mathbf{q})$$
$$-d_2(\mathbf{k}_0)(n_{\mathbf{k}_0'}-n_{\mathbf{k}_0})\delta(\mathbf{k}_0'-\mathbf{k}_0+\mathbf{q})\,.$$

Then for (40), we obtain

$$\frac{e}{mc}\int d\mathbf{k}_0\, d\mathbf{k}_0'\, \tilde{O}_{\mathbf{k}_0,\mathbf{k}_0'}\mathbf{B}\times\tfrac{1}{2}(\mathbf{k}_0'+\mathbf{k}_0)\cdot\left(\frac{\partial}{\partial\mathbf{k}_0}+\frac{\partial}{\partial\mathbf{k}_0'}\right)\rho_{\mathbf{k}_0',\mathbf{k}_0}. \quad (43)$$

We perform the indicated differentiations to obtain

$$=\int \tilde{O}_{\mathbf{k}_0,\mathbf{k}_0'}\left\{\tfrac{1}{2}(\dot{\mathbf{k}}_0'+\dot{\mathbf{k}}_0)\cdot\left(\frac{\partial}{\partial\mathbf{k}_0'}n_{\mathbf{k}_0'}-\frac{\partial}{\partial\mathbf{k}_0}n_{\mathbf{k}_0}\right)\right.$$
$$\times d_1(\mathbf{k}_0)\delta(\mathbf{k}_0'-\mathbf{k}_0-\mathbf{q})$$
$$+\tfrac{1}{2}(\dot{\mathbf{k}}_0'+\dot{\mathbf{k}}_0)(n_{\mathbf{k}_0'}-n_{\mathbf{k}_0})\left(\frac{\partial}{\partial\mathbf{k}_0}d_1(\mathbf{k}_0)\right)\delta(\mathbf{k}_0'-\mathbf{k}_0-\mathbf{q})$$
$$+\tfrac{1}{2}(\dot{\mathbf{k}}_0'+\dot{\mathbf{k}}_0)(n_{\mathbf{k}_0'}-n_{\mathbf{k}_0})d_1(\mathbf{k}_0)$$
$$\left.\times\left(\frac{\partial}{\partial\mathbf{k}_0}+\frac{\partial}{\partial\mathbf{k}_0'}\right)\delta(\mathbf{k}_0'-\mathbf{k}_0-\mathbf{q})\right\}, \quad (44)$$

plus a similar expression in $\delta(\mathbf{k}_0'-\mathbf{k}_0+\mathbf{q})$. Now $\dot{\mathbf{k}}_0'$ differs from $\dot{\mathbf{k}}_0$ by $(e/mc)\mathbf{B}\times\mathbf{q}$. Hence the first two terms give

$$\frac{e}{mc}\int \tilde{O}_{\mathbf{k}_0,\mathbf{k}_0+\mathbf{q}}\left[\mathbf{B}\times(\mathbf{k}_0+\tfrac{1}{2}\mathbf{q})\cdot\frac{\partial}{\partial\mathbf{k}_0}d_1(\mathbf{k}_0)(n_{\mathbf{k}_0+\mathbf{q}}-n_{\mathbf{k}_0})\right]d\mathbf{k}_0. \quad (45)$$

Also,

$$\left(\frac{\partial}{\partial\mathbf{k}_0}+\frac{\partial}{\partial\mathbf{k}_0'}\right)\delta(\mathbf{k}_0-\mathbf{k}_0'\pm\mathbf{q})=0, \quad (46)$$

so that (45) is the complete expression. We have also

$$\frac{e}{mc}\int \tilde{O}_{\mathbf{k}_0,\mathbf{k}_0-\mathbf{q}}\left[\mathbf{B}\times(\mathbf{k}_0-\mathbf{q}/2)\cdot\frac{\partial}{\partial\mathbf{k}_0}d_2(\mathbf{k}_0)(n_{\mathbf{k}_0-\mathbf{q}}-n_{\mathbf{k}_0})\right]d\mathbf{k}_0. \quad (47)$$

The terms in $\mathbf{B}\times\frac{1}{2}\mathbf{q}$ in (45) and (47) are some of the small correction terms not obtained in the semiclassical approximation. Observe that $n_{\mathbf{k}_0\pm\mathbf{q}}-n_{\mathbf{k}_0}$ is already of order q/k_F. These correction terms are of order $(q/k_F)^2$.

Now let us consider the structure of $\tilde{O}$ in more detail. All of the O's we consider will be densities,

$$O_0=\sum_s \delta(\mathbf{r}-\mathbf{r}_s). \quad (48)$$

Furthermore, the δ functions in $\mathbf{r}_s$ are the only $\mathbf{r}_s$ dependences in O,

$$\mathbf{O}_1=m\sum_s \mathbf{v}_s\delta(\mathbf{r}-\mathbf{r}_s). \quad (49)$$

We convert $\mathbf{O}_1$ to $\tilde{O}_1$ by replacing $m\mathbf{v}_s$ by $\mathbf{p}_s-(ie\mathbf{B}\hbar/2c)\times(\partial/\partial\mathbf{p}_s)_{op}$. But

$$\mathrm{Tr}\left(\frac{\partial}{\partial\mathbf{p}_s}\right)_{op}\delta(\mathbf{r}-\mathbf{r}_s)\frac{\partial\tilde{\rho}}{\partial t}=0, \quad (50)$$

so that only $\mathbf{p}_s$ survives in $\tilde{O}_1$. If we form O_{2ij},

$$O_{2ij}=\tfrac{1}{2}m^2(v_{si}v_{sj}+v_{sj}v_{si})\delta(\mathbf{r}-\mathbf{r}_s), \quad (51)$$

again we can effectively replace mv_i by p_i in writing O_{2ij}. In fact, if $F(\mathbf{v}_s,\boldsymbol{\sigma}_s)$ is a completely symmetric function of v_{xs}, v_{ys}, and if

$$O_{sy}=\sum_s F(m\mathbf{v}_s,\boldsymbol{\sigma}_s)\delta(\mathbf{r}-\mathbf{r}_s), \quad (52)$$

then we always have

$$\tilde{O}_{sy}=\sum_s F(\mathbf{p}_s,\boldsymbol{\sigma}_s)\delta(\mathbf{r}-\mathbf{r}_s)+\text{terms of vanishing trace}. \quad (53)$$

To obtain complete symmetry, we might also consider

$$O_{sy}{}^{\dagger}=\sum_s \delta(\mathbf{r}-\mathbf{r}_s)F(m\mathbf{v}_s,\boldsymbol{\sigma}_s), \quad (54)$$

but this will not be necessary.

We expand O_{sy} as a Fourier transform,

$$O_{sy}(\mathbf{r})=\frac{1}{(2\pi)^3}\int d\mathbf{q}'O_s(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{r}}, \quad (55)$$

$$O_{sy}(\mathbf{q}')=\sum_s F(m\mathbf{v}_s,\boldsymbol{\sigma}_s)e^{-i\mathbf{q}'\cdot\mathbf{r}_s}. \quad (56)$$

Thus,

$$O_{sy}(\mathbf{r})=\frac{1}{(2\pi)^3}\int d\mathbf{q}'\sum_s F(m\mathbf{v}_s,\boldsymbol{\sigma}_s)e^{i\mathbf{q}'\cdot(\mathbf{r}-\mathbf{r}_s)}. \quad (57)$$

To construct a transport equation for $O_{sy}(\mathbf{q}')$, we need $(\tilde{O}_{sy}(\mathbf{q}'))_{\mathbf{k}_0,\mathbf{k}_0'}$, and from (56) we have immediately

$$\tilde{O}_{sy}(\mathbf{k}_0-\mathbf{k}_0')_{\mathbf{k}_0,\mathbf{k}_0'}=F(\hbar\mathbf{k}_0,\boldsymbol{\sigma}_{\mathbf{k}_0}). \quad (58)$$

We now have enough information to write down a transport equation. We use Eq. (38), (39), (45), and (58) to obtain

$$i\hbar\frac{\partial}{\partial t}\langle O_{sy}(\mathbf{q})\rangle$$
$$=-i\hbar\frac{\partial}{\partial t}\int d\mathbf{k}_0\, F(\hbar\mathbf{k}_0,\boldsymbol{\sigma}_{\mathbf{k}_0})d_1(\mathbf{k}_0)(n_{\mathbf{k}_0+\mathbf{q}}-n_{\mathbf{k}_0})$$
$$=\int d\mathbf{k}_0\, F(\hbar\mathbf{k}_0,\boldsymbol{\sigma}_{\mathbf{k}_0})d_1(\mathbf{k}_0)[n_{\mathbf{k}_0+\mathbf{q}}-n_{\mathbf{k}_0}]$$
$$\times\left\{\epsilon_{\mathbf{k}_0}-\epsilon_{\mathbf{k}_0+\mathbf{q}}-V(\mathbf{q})\int d\mathbf{k}'(n_{\mathbf{k}'}-n_{\mathbf{k}'+\mathbf{q}})\frac{d_1(\mathbf{k}')}{d_1(\mathbf{k}_0)}\right.$$
$$-\int V(\mathbf{k}_0-\mathbf{k}')(n_{\mathbf{k}'+\mathbf{q}}-n_{\mathbf{k}'})\left[\frac{d_1(\mathbf{k}')}{d_1(\mathbf{k}_0)}-1\right]d\mathbf{k}'$$
$$\left.-\frac{d_0e^{-i\omega t}}{d_1(\mathbf{k}_0)}\right\}+i\hbar\int d\mathbf{k}_0F(\hbar\mathbf{k}_0,\boldsymbol{\sigma}_{\mathbf{k}_0})\frac{e}{mc}(\mathbf{k}_0+\mathbf{q}/2)$$
$$\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{k}_0}\{d_1(\mathbf{k}_0)[n_{\mathbf{k}_0+\mathbf{q}}-n_{\mathbf{k}_0}]\}. \quad (59)$$

It is clear from its construction that $F(\hbar\mathbf{k}_0,\sigma_{\mathbf{k}_0})$ may be any function of $\hbar\mathbf{k}_0$. In particular, we may summarize the result of taking many different $F(\hbar\mathbf{k}_0,\sigma_{\mathbf{k}_0})$, obtaining transport relations for various currents by setting

$$F(\hbar\mathbf{k}_0,\sigma_{\mathbf{k}_0})=\delta(\mathbf{k}_0-\mathbf{k}). \tag{60}$$

We perform the indicated integrations in (59). For clarity, we define

$$\mathfrak{D}_{\mathbf{k}}=d_1(\mathbf{k})[n_{\mathbf{k}+\mathbf{q}}-n_{\mathbf{k}}], \tag{61}$$

$$\mathfrak{N}_{\mathbf{k}}=n_{\mathbf{k}+\mathbf{q}}-n_{\mathbf{k}}, \tag{62}$$

and write

$$i\hbar\frac{\partial}{\partial t}\mathfrak{D}_{\mathbf{k}}+\left(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}+\mathbf{q}}-i\hbar\frac{e}{mc}\mathbf{B}\times(\mathbf{k}+\mathbf{q}/2)\cdot\nabla_{\mathbf{k}}\right)\mathfrak{D}_{\mathbf{k}}$$
$$=\mathfrak{N}_{\mathbf{k}}d_0e^{-i\omega t}+\sum_{\mathbf{k}'}V(\mathbf{k}\mathbf{k}')(\mathfrak{D}_{\mathbf{k}'}\mathfrak{N}_{k}-\mathfrak{D}_{\mathbf{k}}\mathfrak{N}_{k'})$$
$$-V(q)\mathfrak{N}_{\mathbf{k}}\sum_{\mathbf{k}',\sigma'}\mathfrak{D}_{\mathbf{k}'}. \tag{63}$$

If we write $\mathfrak{D}_k$ in terms of its time Fourier components, in (63) we can set

$$i\hbar\frac{\partial\mathfrak{D}_{\mathbf{k}}}{\partial t}=\hbar\omega\mathfrak{D}_{\mathbf{k}}(\mathbf{q},\omega). \tag{64}$$

We have not considered here the effects of collisions with impurities or of the two-particle scattering events dropped from the problem by the single-determinant assumption, made in Sec. III, the "time-dependent Hartree-Fock" approximation. We may crudely allow for these effects by the inclusion of a relaxation-time term in (63)

$$\left(\frac{\partial\mathfrak{D}_{\mathbf{k}}}{\partial t}\right)_{\text{collisions}}=-\frac{\mathfrak{D}_{\mathbf{k}}}{\tau}. \tag{65}$$

We have finally, setting $\tilde{\omega}=\omega-i(1/\tau)$,

$$\left\{\epsilon_{\mathbf{k},\sigma}-\epsilon_{\mathbf{k}+\mathbf{q},\sigma}+\hbar\tilde{\omega}-i\hbar\frac{e}{mc}\mathbf{B}\times(\mathbf{k}+\mathbf{q}/2)\cdot\nabla_{\mathbf{k}}\right\}\mathfrak{D}_{\mathbf{k}}$$
$$=\mathfrak{N}_{\mathbf{k},\sigma}d_0-V(\mathbf{q})\mathfrak{N}_{\mathbf{k},\sigma}\sum_{\mathbf{k}',\sigma}\mathfrak{D}_{\mathbf{k}',\sigma'}$$
$$+\sum_{\mathbf{k}'}V(\mathbf{k}-\mathbf{k}')(\mathfrak{D}_{\mathbf{k}',\sigma}\mathfrak{N}_{\mathbf{k},\sigma}-\mathfrak{D}_{\mathbf{k},\sigma}\mathfrak{N}_{\mathbf{k}',\sigma}). \tag{66}$$

Equation (66) applies to the case of longitudinal density waves. As we have seen, the treatment of transverse electric driving fields requires only the modification of the driving term,

$$d_0\rightarrow\mathbf{d}_0\cdot(\mathbf{k}+\mathbf{q}). \tag{67}$$

The manipulations necessary to obtain the analogue of (66) for the case of longitudinal spin-density oscillations are like those we have exhibited. One obtains

$$\mathfrak{N}_{\mathbf{k}}\equiv b(\mathbf{k})(n_{\mathbf{k}+\mathbf{q}\downarrow}-n_{\mathbf{k}\uparrow}), \tag{68}$$

$$\mathfrak{N}_{\mathbf{k}}\equiv n_{\mathbf{k}+\mathbf{q}\downarrow}-n_{\mathbf{k}\uparrow} \tag{69}$$

$$\left\{\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}+\hbar\tilde{\omega}-i\hbar\frac{e}{mc}\mathbf{B}\times(\mathbf{k}+\mathbf{q}/2)\cdot\nabla_{\mathbf{k}}\right\}\mathfrak{N}_{\mathbf{k}}$$
$$=\mathfrak{N}_{\mathbf{k}}c_0+\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')(\mathfrak{N}_{\mathbf{k}'}\mathfrak{N}_{\mathbf{k}}-\mathfrak{N}_{\mathbf{k}}\mathfrak{N}_{\mathbf{k}'}). \tag{70}$$

Equations (66), (67), and (70), together with the condition (22), constitute our results. These equations are not yet in a form which makes easy the comparison with the transport equations of Ref. 1. We defer the explicit comparisons until the following paper in this sequence in which we consider a specific physical situation and the effects of various possible functional forms of $V(\mathbf{k},\mathbf{k}')$. In this way we can show not only the similarities and the differences, but also the results of the differences.

V. CONCLUSIONS

We have derived transport equations appropriate to a charged plasma driven by three possible types of fields. We have as yet made no assumptions about the form of the interactions.

We have made a self-consistent field approximation, similar to the Hartree-Fock approximation; we have made a semiclassical approximation which eliminates the effects of the Landau-level structure on the states; and we have made a relaxation-time approximation appropriate to the case $\omega\tau\gg1$. We have not had to make assumptions about single-particle wave packets, Lorentz forces on localized particles, or special gauges. Our results are in conflict with previous results. We have deferred explorations of the conflicts to the following paper where in the specific case of paramagnetic spin waves is treated.

ACKNOWLEDGMENT

It is a pleasure to acknowledge interesting conversations with S. Gartenhaus on the Wigner density operator.

APPENDIX A: PROPERTIES OF BASIC TRANSFORMATION

For simplicity, we consider only a single particle.

We have been led to consider a modified density matrix $\tilde{\rho}$ obtained from the actual density matrix ρ by the transformation

$$\rho=\tilde{U}\tilde{\rho}, \tag{A1}$$

in which

$$\tilde{U}=e^{-ax\hbar(\partial/\partial p_y)_{0p}e^{ay\hbar(\partial/\partial p_x)_{0p}}} \tag{A2}$$

and

$$i\hbar\left(\frac{\partial}{\partial p_x}\right)_{0p}O\equiv[x,O]. \tag{A3}$$

This transformation is interesting because $\tilde{\rho}$ is much closer to the classical joint probability density than ρ.

In a plane-wave representation, we write

$$\tilde{\rho}=\sum_{\mathbf{k},\mathbf{k}'}|\mathbf{k}'\rangle\mathfrak{N}_{\mathbf{k}'\mathbf{k}}\langle\mathbf{k}|\,. \tag{A4}$$

Then

$$i[x,\tilde{\rho}]=\sum_{\mathbf{k}\mathbf{k}'}\left(\frac{\partial}{\partial k_x'}|\mathbf{k}'\rangle\right)\mathfrak{N}_{\mathbf{k}'\mathbf{k}}\langle\mathbf{k}|+|\mathbf{k}'\rangle\mathfrak{N}_{\mathbf{k}'\mathbf{k}}\frac{\partial}{\partial k_x}\langle\mathbf{k}|\,. \tag{A5}$$

Integrating by parts yields

$$i[x,\tilde{\rho}]=-\sum_{\mathbf{k}\mathbf{k}'}|\mathbf{k}'\rangle\left[\left(\frac{\partial}{\partial k_x'}+\frac{\partial}{\partial k_x}\right)\mathfrak{N}_{\mathbf{k}'\mathbf{k}}\right]\langle\mathbf{k}|\,. \tag{A6}$$

Hence

$$e^{a\hbar y(\partial/\partial p_x)_{0p}}\tilde{\rho}=\sum_{\mathbf{k},\mathbf{k}',l}\frac{a^l y^l}{l!}|\mathbf{k}'\rangle\times\left[\left(\frac{\partial}{\partial k_x'}+\frac{\partial}{\partial k_x}\right)^l\mathfrak{N}_{\mathbf{k}'\mathbf{k}}\right]\langle\mathbf{k}|\,. \tag{A7}$$

We expand $\mathfrak{N}_{\mathbf{k}'\mathbf{k}}$ in a double Fourier integral:

$$\mathfrak{N}_{\mathbf{k}'\mathbf{k}}=\int\mathfrak{N}(\mathbf{q}'\mathbf{q})e^{i\mathbf{q}'\cdot\mathbf{k}'}e^{i\mathbf{k}\cdot\mathbf{q}}d\mathbf{q}'d\mathbf{q}\,, \tag{A8}$$

thus obtaining for ρ

$$\rho=\sum_{\mathbf{k},\mathbf{k}'}\int e^{iay(q_x+q_x')}e^{-iax(q_y+q_y')}|\mathbf{k}'\rangle\mathfrak{N}(\mathbf{q}'\mathbf{q})\times e^{i\mathbf{q}'\cdot\mathbf{k}'}e^{i\mathbf{q}\cdot\mathbf{k}}\langle\mathbf{k}|\,d\mathbf{q}d\mathbf{q}'\,. \tag{A9}$$

Now

$$e^{iay(q_x+q_x')-iax(q_y+q_y')}|\mathbf{k}'\rangle=|\mathbf{k}'+\bar{\mathbf{q}}+\bar{\mathbf{q}}'\rangle\,, \tag{A10}$$

where

$$\bar{q}_x=-aq_y,\quad \bar{q}_y=aq_x\,. \tag{A11}$$

Therefore

$$\rho=\int\sum_{\mathbf{k},\mathbf{k}'}|\mathbf{k}'\rangle\mathfrak{N}(\mathbf{q}'\mathbf{q})e^{i\mathbf{q}'\cdot\mathbf{k}'}e^{i\mathbf{q}\cdot\mathbf{k}}e^{-i\mathbf{q}'\cdot\bar{\mathbf{q}}}\langle\mathbf{k}|\,d\mathbf{q}d\mathbf{q}'\,. \tag{A12}$$

Let us consider a family of possible $\tilde{\rho}$ more or less diagonal in the plane-wave representation,

$$\tilde{\rho}_{\mathbf{k}'\mathbf{k}}=\rho_0 e^{-\beta(\mathbf{k}-\mathbf{k}')^2}e^{-\alpha(\mathbf{k}+\mathbf{k}'-2\mathbf{k}_0)^2}\,. \tag{A13}$$

As β is allowed to become large, the eigenstates of $\tilde{\rho}$ approach plane waves with eigenvalues given by the second factor in (A13). As α is allowed to become large, the ensemble of (A13) approaches a "pure" state described by wave vector $\mathbf{k}_0$. We shall let $\mathbf{k}_0$ be zero for simplicity and without loss of generality.

The Fourier analysis is straightforward and yields

$$\mathfrak{N}(\mathbf{q}'\mathbf{q})=\rho_0'e^{-q^2(\alpha+\beta)/16\alpha\beta}e^{-q'^2(\alpha+\beta)/16\alpha\beta}e^{\mathbf{q}\cdot\mathbf{q}'(\alpha-\beta)/16\alpha\beta}\,. \tag{A14}$$

We substitute into (A12) and compute the inverse double transforms to obtain $\rho_{\mathbf{k}'\mathbf{k}}$,

$$\rho_0''\exp\left\{\left(-\frac{\alpha+\beta}{16\alpha\beta}(k'^2+k^2)-\frac{\alpha-\beta}{8\alpha\beta}\mathbf{k}\cdot\mathbf{k}'+ia^2\mathbf{k}\cdot\tilde{\mathbf{k}}'\right)\times16\alpha\beta(1+16a^2\alpha\beta)^{-1}\right\}\,, \tag{A15}$$

in which

$$\tilde{k}_x'=\frac{k'_y}{a},\quad \tilde{k}_y'=\frac{-k'_x}{a}\,. \tag{A16}$$

We need to know the eigenvalues of this transformed ρ. These are not the same as the eigenvalues of $\tilde{\rho}$ because (A1) is not a similarity transformation.

We proceed by first exhibiting an eigenstate of (A15) and then constructing raising and lowering operators to generate other eigenstates and their associated eigenvalues. It is first convenient, however, to simplify (A15) by redefining the variables.

$$\rho_{\mathbf{p},\mathbf{p}'}=\rho_0''e^{-p^2/2}e^{-p'^2/2}e^{i\theta\mathbf{p}\cdot\tilde{\mathbf{p}}'}\,. \tag{A17}$$

The variables $\mathbf{p}$ and $\mathbf{p}'$ are obtained from $\mathbf{k}$ and $\mathbf{k}'$.

$$\mathbf{p}=(\mathbf{k}-\mathbf{k}')\left(\frac{2\beta}{1+16a^2\alpha\beta}\right)^{1/2},\quad \mathbf{p}'=(\mathbf{k}+\mathbf{k}')\left(\frac{2\alpha}{1+16a^2\alpha\beta}\right)^{1/2},\quad \rho_0''=\frac{4\alpha\beta}{\pi^2(1+16a^2\alpha\beta)},\quad \theta=a^2 4(\alpha\beta)^{1/2}\,. \tag{A18}$$

Actually, we can anticipate most of the final results from (A17) and (A18). The parameters ρ_0'' and θ are all that determine the eigenvalues of ρ. θ is proportional to a^2, hence to H^2, and thus we would anticipate that the semiclassical approximation is valid so long as θ does not become too large. Indeed, as $\theta\to0$, the eigenvalues of (A17) are 1 and 0 by inspection. However, we want β to become large in order that $\tilde{\rho}$ shall be diagonal in the plane-wave representation. To avoid large θ in this case, we will have to have α small. But small α means a large spread of the ensemble state described by $\tilde{\rho}$ from (A13). Thus to solve our equations of motion of $\tilde{\rho}$ exactly produces an infinite spread of states through the plane-wave space of functions.

An eigenfunction of (A17) is

$$\Psi=\int e^{-\lambda p^2/2}e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{p}\,,\quad \lambda=(1+16a^2\alpha\beta)^{1/2}\,, \tag{A19}$$

with eigenvalue $\rho_0''\pi/(1+\lambda)$.

From one eigenstate, we can construct the rest by finding raising and lowering operators. Let us search for "ladder" operators of the form

$$O=\alpha_1\mathbf{p}+\alpha_2\frac{\partial}{\partial\mathbf{p}}+\alpha_3\tilde{\mathbf{p}}+\alpha_4\frac{\partial}{\partial\tilde{\mathbf{p}}}. \tag{A20}$$

Of course, $\mathbf{p}$ and $\tilde{\mathbf{p}}$ are not independent, but they are mutually perpendicular:

$$\rho\Psi=\mu_0\Psi \tag{A21}$$

and therefore,

$$O\rho\Psi=\mu_0 O\Psi. \tag{A22}$$

We seek to learn if there exist α's such that

$$\rho O\Psi=\mu_1 O\Psi. \tag{A23}$$

Of course if we generalize O to include $\mathbf{p}^2$, $\mathbf{p}\partial/\partial\mathbf{p}$, $\partial^2/\partial\mathbf{p}^2$, $\mathbf{p}^3\cdots$ and there must exist O satisfying (A23). As it develops, however, (A20) is altogether sufficient.

From (A23) and (A22), we have

$$(O\rho-\rho O)\Psi=(\mu_0-\mu_1)\Psi=\frac{\mu_0-\mu_1}{\mu_0}O\rho\Psi \tag{A24}$$

or

$$\left(\frac{\mu_1}{\mu_0}O\rho-\rho O\right)\Psi=0. \tag{A25}$$

Now $\rho\Psi$ means $\int d\mathbf{p}'\rho(\mathbf{p},\mathbf{p}')\Psi(\mathbf{p}')$ and therefore by partial integration

$$\rho\frac{\partial}{\partial\mathbf{p}'}\Psi=-\frac{\partial\rho}{\partial\mathbf{p}'}\Psi, \tag{A26}$$

so that

$$\rho O\Psi=\left[\alpha_1\mathbf{p}'-\alpha_2\frac{\partial}{\partial\mathbf{p}'}+\alpha_3\tilde{\mathbf{p}}'-\alpha_4\frac{\partial}{\partial\tilde{\mathbf{p}}'}\right]\rho\Psi, \tag{A27}$$

and since

$$O\rho\Psi=\left[\alpha_1\mathbf{p}+\alpha_2\frac{\partial}{\partial\mathbf{p}}+\alpha_3\tilde{\mathbf{p}}+\alpha_4\frac{\partial}{\partial\tilde{\mathbf{p}}}\right]\rho\Psi, \tag{A28}$$

we can obtain μ_1/μ_0 without reference to Ψ.

$$\frac{\partial\rho}{\partial\mathbf{p}}=(-\mathbf{p}+i\theta\tilde{\mathbf{p}}')\rho, \tag{A29}$$

$$\frac{\partial\rho}{\partial\tilde{\mathbf{p}}}=(-a^2\mathbf{p}-i\theta\mathbf{p}')\rho, \tag{A30}$$

$$\frac{\partial\rho}{\partial\mathbf{p}'}=(-\mathbf{p}'-i\theta\tilde{\mathbf{p}})\rho, \tag{A31}$$

$$\frac{\partial\rho}{\partial\tilde{\mathbf{p}}'}=(-a^2\tilde{\mathbf{p}}'+i\theta\mathbf{p})\rho. \tag{A32}$$

Both $\mathbf{p}$ and $\mathbf{p}'$ have two components giving four equations in the α's. Setting the determinant of these equations equal to zero yields

$$\frac{\mu_1}{\mu_0}-\frac{\mu_0}{\mu_1}=\pm\frac{4a}{\theta}=\pm\frac{1}{a(\alpha\beta)^{1/2}}\equiv\frac{\eta}{a(\alpha\beta)^{1/2}}, \tag{A33}$$

from which we obtain

$$\frac{\mu_1}{\mu_0}=\frac{\eta}{a(\alpha\beta)^{1/2}}\pm\left(\frac{1}{a^2\alpha\beta}+1\right)^{1/2}, \tag{A34}$$

or zero at the bottom of the ladders. Also, clearly,

$$\mu_n/\mu_{n-1}=\mu_1/\mu_0. \tag{A35}$$

We have two independent choices of sign to make. The eigenvalues of our ρ are thus arrayed on a two-dimensional net. There are two independent sets of raising and lowering operators for a total of four operators corresponding to the four possible sign choices in (A34). Some of the eigenvalues, however, are negative and these states are "unphysical." If the magnitudes of these unphysical eigenvalues can be made to vanish, or nearly so, our initial $\tilde{\rho}$ can then be considered physically meaningful.

We expand the square root in (A34) about small $a^2\alpha\beta$.

$$\frac{\mu_1}{\mu_0}\cong\frac{\eta}{a(\alpha\beta)^{1/2}}+\frac{1}{a(\alpha\beta)^{1/2}}(1+\tfrac{1}{2}a^2\alpha\beta), \tag{A36}$$

and making opposite sign choices in the two places, we obtain

$$\mu_1/\mu_0\cong\pm\tfrac{1}{2}a(\alpha\beta)^{1/2}. \tag{A37}$$

Thus at the bottom of the ladder, we have one large eigenvalue, and all other eigenvalues are down at least by a factor $\frac{1}{2}a(\alpha\beta)^{1/2}$. This parameter must remain small if our $\tilde{\rho}$ is to make physical sense. We may, therefore, make β large, thereby diagonalizing $\tilde{\rho}$ in the plane-wave representation, only at the cost of making α small and spreading out the "ensemble" over many plane waves. The condition we must maintain is

$$1\gg a(\alpha\beta)^{1/2}=\frac{eB}{2c\hbar}\frac{2}{\Delta\mathbf{k}_1\Delta\mathbf{k}_2}=\tfrac{1}{2}\hbar\omega_c\frac{2m}{\hbar^2\Delta\mathbf{k}_1\Delta\mathbf{k}_2}, \tag{A38}$$

in which $\Delta\mathbf{k}_1$ and $\Delta\mathbf{k}_2$ are the rms "spreads" of the ensemble about the center, $\mathbf{k}_0$. Equation (A38) reads, "the spread in energy of the states described by $\tilde{\rho}$ must exceed the zero-point energy of the lowest Landau level."

We could have obtained this result immediately. Since v_x and v_y do not commute, there exists an uncertainty relation between them. Their commutator is proportional to the magnetic field strength. The product

of the mean dispersions, $\Delta v_x \Delta v_y$ must exceed some constant proportional to the field. Our formalism allows us to use $\hbar\mathbf{k}/m$ for the velocity, but maintains the uncertainty relation between the components. We have shown here that the uncertainty need not be maintained in individual wave packets describing "pure" states, but many find expression in an "ensemble" spreading.

APPENDIX B: DERIVATION OF SELF-CONSISTENT SPIN-WAVE FIELD

We seek a solution of the variational equations (18) or, rather, their reexpression in terms of the $b(\mathbf{k},\boldsymbol{\sigma})$. Since the explicit use of determinantal wave functions is unnecessarily awkward, we shall cast our expressions into "second quantized" form,

$$\mathcal{H}=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma}n_{\mathbf{k},\sigma}+\tfrac{1}{2}\sum_{\mathbf{k},\mathbf{k}';\mathbf{K};\sigma,\sigma'}V(\mathbf{K})C_{\mathbf{k}+\mathbf{K},\sigma}{}^{\dagger}C_{\mathbf{k}'-\mathbf{K},\sigma'}{}^{\dagger}C_{\mathbf{k}',\sigma'}C_{\mathbf{k},\sigma}+c_0\sum_{\mathbf{k}}e^{-i\omega t}C_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}C_{\mathbf{k}\uparrow}+e^{i\omega t}C_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}C_{\mathbf{k}\downarrow}, \quad \text{(B1)}$$

$$\Phi=\prod_{\mathbf{k}\epsilon\{\mathbf{k}\}}\tilde{C}_{\mathbf{k},\sigma}{}^{\dagger}|0\rangle e^{-i\Gamma t}, \quad \text{(B2)}$$

where

$$\epsilon_{\mathbf{k},\sigma}=\frac{\hbar^2k^2}{2m}+\mu B\sigma_z,$$

$$n_{\mathbf{k},\sigma}=C_{\mathbf{k},\sigma}{}^{\dagger}C_{\mathbf{k},\sigma}, \quad \text{(B3)}$$

$$\tilde{C}_{\mathbf{k}\uparrow}{}^{\dagger}=b_1(\mathbf{k}\uparrow)C_{\mathbf{k}\uparrow}{}^{\dagger}+b_2(\mathbf{k}\uparrow)C_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}\cong C_{\mathbf{k}\uparrow}{}^{\dagger}+b_2(\mathbf{k}\uparrow)C_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}, \quad \text{(B4)}$$

$$\tilde{C}_{\mathbf{k}\downarrow}{}^{\dagger}=b_2(\mathbf{k}\downarrow)C_{\mathbf{k}\downarrow}{}^{\dagger}+b_1(\mathbf{k}\downarrow)C_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}\cong C_{\mathbf{k}\downarrow}{}^{\dagger}+b_1(\mathbf{k}\downarrow)C_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}. \quad \text{(B5)}$$

The mixing of the states by the rf field must still perserve their orthogonality:

$$\{\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow},\tilde{C}_{\mathbf{k}\uparrow}{}^{\dagger}\}=0, \quad \text{(B6)}$$

from which we obtain

$$b_1{}^{*}(\mathbf{k}+\mathbf{q}\downarrow)+b_2(\mathbf{k}\uparrow)=0. \quad \text{(B7)}$$

The creation and anihilation operators appearing in the wave functions are different from those in the Hamiltonian. With about the same amount of difficulty, we can either convert those in the wave functions or the energy; we choose to change the Hamiltonian.

We consider first the kinetic-energy terms. Working only to first order in the b's, we invert (B4) and (B5),

$$C_{\mathbf{k}\uparrow}{}^{\dagger}=\tilde{C}_{\mathbf{k}\uparrow}{}^{\dagger}-b_2(\mathbf{k}\uparrow)\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}, \quad \text{(B8)}$$

$$C_{\mathbf{k}\downarrow}{}^{\dagger}=\tilde{C}_{\mathbf{k}\downarrow}{}^{\dagger}-b_1(\mathbf{k}\downarrow)\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}. \quad \text{(B9)}$$

Thus the kinetic energy becomes

$$\begin{aligned}T=&\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma}\tilde{n}_{\mathbf{k},\sigma}-b_2(\mathbf{k}\uparrow)\epsilon_{\mathbf{k}\uparrow}\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\uparrow}\\&-b_2{}^{*}(\mathbf{k}\uparrow)\epsilon_{\mathbf{k}\uparrow}\tilde{C}_{\mathbf{k}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}-b_1(\mathbf{k}\downarrow)\epsilon_{\mathbf{k}\downarrow}\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\downarrow}\\&-b_1{}^{*}(\mathbf{k}\downarrow)\epsilon_{\mathbf{k}\downarrow}\tilde{C}_{\mathbf{k}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}\\=&\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma}\tilde{n}_{\mathbf{k},\sigma}\\&-\sum_{\mathbf{k}}[b_2(\mathbf{k}\uparrow)\epsilon_{\mathbf{k}\uparrow}+b_1{}^{*}(\mathbf{k}+\mathbf{q}\downarrow)\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}]\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\uparrow}\\&-\sum_{\mathbf{k}}[b_2{}^{*}(\mathbf{k}\uparrow)\epsilon_{\mathbf{k}\uparrow}+b_1(\mathbf{k}+\mathbf{q}\downarrow)\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}]\tilde{C}_{\mathbf{k}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow},\end{aligned} \quad \text{(B10)}$$

and by using (B7)

$$T=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k},\sigma}\tilde{n}_{\mathbf{k},\sigma}-\sum_{\mathbf{k}}b_2(\mathbf{k}\uparrow)(\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\downarrow})\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\uparrow}-\sum_{\mathbf{k}}b_1(\mathbf{k}\downarrow)(\epsilon_{\mathbf{k}\downarrow}-\epsilon_{\mathbf{k}-\mathbf{q}\uparrow})\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\downarrow}. \quad \text{(B11)}$$

The interaction potential energy is found by a similar, but more arduous procedure,

$$\begin{aligned}V=&\tfrac{1}{2}\sum_{\mathbf{k},\mathbf{k};\mathbf{K}}V(\mathbf{K})\tilde{C}_{\mathbf{k}+\mathbf{K},\sigma}{}^{\dagger}\tilde{C}_{\mathbf{k}'-\mathbf{K},\sigma'}{}^{\dagger}\tilde{C}_{\mathbf{k}',\sigma'}\tilde{C}_{\mathbf{k},\sigma}\\&-\sum_{\mathbf{k},\mathbf{k}';\mathbf{K};\sigma'}V(\mathbf{K})\tilde{C}_{\mathbf{k}+\mathbf{q}+\mathbf{K}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}'-\mathbf{K},\sigma'}{}^{\dagger}\tilde{C}_{\mathbf{k}',\sigma'}\tilde{C}_{\mathbf{k}\uparrow}b_2(\mathbf{k}+\mathbf{K}\uparrow)\\&-\sum_{\mathbf{k},\mathbf{k}';\mathbf{K};\sigma'}V(\mathbf{K})\tilde{C}_{\mathbf{k}-\mathbf{q}+\mathbf{K}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}'-\mathbf{K},\sigma'}{}^{\dagger}\tilde{C}_{\mathbf{k}',\sigma'}\tilde{C}_{\mathbf{k}\downarrow}b_1(\mathbf{k}+\mathbf{K}\downarrow)\\&-\sum_{\mathbf{k},\mathbf{k}';\mathbf{K};\sigma}V(\mathbf{K})\tilde{C}_{\mathbf{k}+\mathbf{K},\sigma}{}^{\dagger}\tilde{C}_{\mathbf{k}'+\mathbf{K}\uparrow}\tilde{C}_{\mathbf{k}'+\mathbf{q}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k},\sigma}b_2{}^{*}(\mathbf{k}'\uparrow)\\&-\sum_{\mathbf{k},\mathbf{k}';\mathbf{K};\sigma}V(\mathbf{K})\tilde{C}_{\mathbf{k}+\mathbf{K},\sigma}{}^{\dagger}\tilde{C}_{\mathbf{k}'-\mathbf{K}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}'-\mathbf{q}\uparrow}\tilde{C}_{\mathbf{k},\sigma}b_1{}^{*}(\mathbf{k}'\downarrow).\end{aligned} \quad \text{(B12)}$$

Finally, the driving terms in (B1) are already of first order in c_0 (hence b).

$$D=c_0\sum_{\mathbf{k}}e^{-i\omega t}\tilde{C}_{\mathbf{k}+\mathbf{q}\downarrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\uparrow}+e^{i\omega t}\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}{}^{\dagger}\tilde{C}_{\mathbf{k}\downarrow}. \quad \text{(B13)}$$

The variational equations require us to construct

$$\left(\frac{\partial\Phi}{\partial t}\right)_{\text{fixed k's}},$$

$$\frac{\delta}{\delta b(\mathbf{k})}\left(\frac{\partial\Phi^{\dagger}}{\partial t}\right)_{\text{fixed k's}},$$

and

$$\mathcal{H}\Phi.$$

In calculating $\mathcal{H}\Phi$, however, since we are going to project onto $\delta\Phi^{\dagger}$, only a limited number of the many different matrix elements of (B12) need to be con-

sidered. To see this, we calculate $\dot{\Phi}$:

$$\left(\frac{\partial\Phi}{\partial t}\right)_{\text{fixed k's}} = -i\Gamma\Phi + \sum_{\mathbf{k}'} \frac{\partial b_2(\mathbf{k}'\uparrow)}{\partial t}\tilde{C}_{\mathbf{k}'+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}'\uparrow}\Phi + \frac{\partial b_1(\mathbf{k}'\downarrow)}{\partial t}\tilde{C}_{\mathbf{k}'-\mathbf{q}\uparrow}^{\dagger}\tilde{C}_{\mathbf{k}'\downarrow}\Phi. \quad \text{(B14)}$$

Taking the functional derivative with respect to $b_2(\mathbf{k}_0\uparrow)$ yields two kinds of terms: those of zero order in the b's,

$$\left\{-i\Gamma\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}\uparrow} + \left[\frac{\delta}{\delta b_2(\mathbf{k}_0\uparrow)}\frac{\partial b_2(\mathbf{k}_0\uparrow)}{\partial t}\right]\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}\right\}\Phi, \quad \text{(B15)}$$

and terms of first order in the b's which we discard. Thus $\delta\Phi/\delta b_0$ contains only functions of the form

$$\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}\Phi, \quad \tilde{C}_{\mathbf{k}_0-\mathbf{q}\uparrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\downarrow}\Phi, \quad \text{and} \quad \Phi. \quad \text{(B16)}$$

Thus from all the terms in (B12), we need keep only those which fit the patterns of (B16).

Consider the first term in (B12); it leaves the total number of spins up and down unchanged. Therefore, only that part of it need be kept for which either

$$\text{(i)} \quad \mathbf{k}+\mathbf{K}=\mathbf{k}, \quad \mathbf{k}'-\mathbf{K}=\mathbf{k}'$$

or

$$\text{(ii)} \quad \mathbf{k}+\mathbf{K}=\mathbf{k}' \quad \text{and} \quad \sigma=\sigma', \quad \mathbf{k}'-\mathbf{K}=\mathbf{k}.$$

The first possibility gives $V(\mathbf{0})\tilde{n}_{\mathbf{k},\sigma}\tilde{n}_{\mathbf{k}'\sigma'}$ which we assume to vanish, cancelling $V(\mathbf{0})$ against a uniform positive background. The second possibility gives

$$-\sum_{\mathbf{k},\mathbf{k}',\sigma} V(\mathbf{k}-\mathbf{k}')\tilde{n}_{\mathbf{k},\sigma}\tilde{n}_{\mathbf{k}',\sigma}. \quad \text{(B17)}$$

The second term of (B12) turns one spin down. We therefore need only keep terms of the form $\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}$. Now we have three possibilities:

$$\text{(i)} \quad \mathbf{k}+\mathbf{q}+\mathbf{K}=\mathbf{k}_0+\mathbf{q}, \quad \mathbf{k}'-\mathbf{K}=\mathbf{k}',$$

which we discard,

$$\text{(ii)} \quad \mathbf{k}+\mathbf{q}+\mathbf{K}=\mathbf{k}_0+\mathbf{q}, \quad \mathbf{k}'-\mathbf{K}=\mathbf{k}; \ \sigma'=\uparrow,$$

which yields

$$\sum_{\mathbf{k}} V(\mathbf{k}_0-\mathbf{k})\tilde{n}_{\mathbf{k}\uparrow}b_2(\mathbf{k}_0\uparrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow} \quad \text{(B18)}$$

and

$$\text{(iii)} \quad \mathbf{k}+\mathbf{q}+\mathbf{K}=\mathbf{k}'; \ \mathbf{k}'-\mathbf{K}=\mathbf{k}_0+\mathbf{q}; \ \sigma'=\downarrow,$$

which yields

$$\sum_{\mathbf{k}'} V(\mathbf{k}'-\mathbf{k}_0-\mathbf{q})\tilde{n}_{\mathbf{k}'\downarrow}b_2(\mathbf{k}'-\mathbf{q}\uparrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow} = \sum_{\mathbf{k}'} V(\mathbf{k}'-\mathbf{k}_0)\tilde{n}_{\mathbf{k}'+\mathbf{q}\downarrow}b_2(\mathbf{k}'\uparrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}. \quad \text{(B19)}$$

The fifth term of (B12) also turns one spin down, and we have three possible terms: (i) $\mathbf{k}+\mathbf{K}=\mathbf{k}$; $\mathbf{k}'-\mathbf{K}=\mathbf{k}_0+\mathbf{q}\rightarrow\mathbf{K}=0$, which we discard, (ii) $\mathbf{k}+\mathbf{K}=\mathbf{k}'-\mathbf{q}$; $\sigma=\uparrow$, $\mathbf{k}'-\mathbf{K}=\mathbf{k}_0+\mathbf{q}$, yielding

$$\sum_{\mathbf{k}'} V(\mathbf{k}'-\mathbf{k}_0-\mathbf{q})\tilde{n}_{\mathbf{k}'-\mathbf{q}\uparrow}b_1^*(\mathbf{k}'\downarrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow} = \sum_{\mathbf{k}'} V(\mathbf{k}'-\mathbf{k}_0)\tilde{n}_{\mathbf{k}'\uparrow}b_1^*(\mathbf{k}'+\mathbf{q}\downarrow)\tilde{C}_{\mathbf{k}_0-\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}, \quad \text{(B20)}$$

and (iii) $\mathbf{k}+\mathbf{K}=\mathbf{k}_0+\mathbf{q}$; $\sigma=\downarrow$; $\mathbf{k}'-\mathbf{K}=\mathbf{k}$, yielding

$$\sum_{\mathbf{k}} V(\mathbf{k}_0+\mathbf{q}-\mathbf{k})\tilde{n}_{\mathbf{k}\downarrow}b_1^*(\mathbf{k}_0+\mathbf{q}\downarrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow} = \sum_{\mathbf{k}} V(\mathbf{k}_0-\mathbf{k})\tilde{n}_{\mathbf{k}+\mathbf{q}\downarrow}b_1^*(\mathbf{k}_0+\mathbf{q}\downarrow)\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}C_{\mathbf{k}_0\uparrow}. \quad \text{(B21)}$$

We collect (B18)–(B21) and obtain

$$\sum_{\mathbf{k}'} V(\mathbf{k}_0-\mathbf{k}')[\tilde{n}_{\mathbf{k}'\uparrow}b_2(\mathbf{k}_0\uparrow)+\tilde{n}_{\mathbf{k}'+\mathbf{q}\downarrow}b_2(\mathbf{k}'\uparrow) + n_{\mathbf{k}'\uparrow}b_1^*(\mathbf{k}'+\mathbf{q}\downarrow)+\tilde{n}_{\mathbf{k}'+\mathbf{q}\downarrow}b_1^*(\mathbf{k}_0+\mathbf{q}\downarrow)]\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow} = \sum_{\mathbf{k}'} V(\mathbf{k}_0-\mathbf{k}')[b_2(\mathbf{k}_0\uparrow)-b_2(\mathbf{k}'\uparrow)] \times[\tilde{n}_{\mathbf{k}'\uparrow}-\tilde{n}_{\mathbf{k}'+\mathbf{q}\downarrow}]\tilde{C}_{\mathbf{k}_0+\mathbf{q}\downarrow}^{\dagger}\tilde{C}_{\mathbf{k}_0\uparrow}. \quad \text{(B22)}$$

The third and fourth terms of (B12) yield a similar expression in $b_1(\mathbf{k}\downarrow)$ and $\tilde{C}_{\mathbf{k}_0-\mathbf{q}}^{\dagger}\tilde{C}_{\mathbf{k}_0\downarrow}$.

We gather up the terms from $-i\hbar\partial\Phi/\partial t$, the kinetic, potential, and driving terms to obtain

$$\left\{b_2(\mathbf{k}\uparrow)(\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}-\epsilon_{\mathbf{k}\uparrow}) + \sum_{\mathbf{k}'} V(\mathbf{k}-\mathbf{k}')[(b_2(\mathbf{k}\uparrow)-b_2(\mathbf{k}'\uparrow)](\tilde{n}_{\mathbf{k}'\uparrow}-\tilde{n}_{\mathbf{k}'+\mathbf{q}\downarrow}) + c_0e^{-i\omega t}-i\hbar\frac{\partial b_2(\mathbf{k}\uparrow)}{\partial t}\right\}=0, \quad \text{(B23)}$$

by applying the variational equations (18). A similar set of manipulations, together with the orthogonality relation (B8) gives another expression in $\tilde{C}_{\mathbf{k}-\mathbf{q}\uparrow}^{\dagger}\tilde{C}_{\mathbf{k}\downarrow}$ which, however, is identical to (B23). These are the self-consistent field equations we seek.