Condensation of the Rotating Two-Dimensional Ideal Bose Gas*

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The two-dimensional ideal Bose gas in a circular container rotating at fixed rim velocity has some unexpected and instructive properties. In the thermodynamic limit, the system undergoes a phase transition, as argued by Widom. We find that the transition is a type of Bose-Einstein condensation, in which, however, infinitely many one-particle states are macroscopically occupied. The system evades general theorems forbidding Bose-Einstein condensation in two dimensions, not merely because it is inhomogeneous, as Widom suggests, but because the equilibrium density is unbounded near the circumference in the thermodynamic limit. We point out that, independently of the particular model, Bose-Einstein condensation into any single-particle state (or states), spatially uniform or nonuniform, is forbidden in one and two dimensions provided only that (a) the Hamiltonian is real, and (b) the equilibrium density is everywhere bounded in the thermodynamic limit.

THE properties of a two-dimensional ideal Bose gas, contained in a circular vessel of radius R, rotating with angular velocity $\omega=v/R$ have been investigated by Widom.[1] In the thermodynamic limit $R\to\infty$, $\omega\to 0$, v fixed, he has argued that this model will exhibit Bose-Einstein condensation, a rather unexpected result in view of the variety of recent theorems excluding such condensation in two dimensions.[2] We wish to make the following additional points about the model:

(1) Although there is a gap in Widom's analysis, it can be circumvented, and his conclusion that a phase transition occurs is correct.

(2) The condensate has a most remarkable structure: Below T_c *infinitely many* single-particle levels are macroscopically occupied.

(3) The inapplicability of the general theorems prohibiting condensation in two dimensions can be traced to the fact that below T_c the density of particles near the rim of the circle becomes unbounded in the thermodynamic limit. Since this concentration of particles at the rim is clearly an artifact of the ideal gas, there is no reason to expect condensation in the presence of interactions, even in homogeneous systems. However, the model has conceptual value, revealing that a restriction of bounded density is an essential part of the most general arguments forbidding condensation in two dimensions.

(1) Widom's analysis is based on his evaluation in the thermodynamic limit of $\gamma(t)=\mathrm{Tr}e^{-Ht}$:

$$\lim\left[\gamma(t)/\pi R^2\right]=(M/2\pi\hbar^2 t)(e^{Mv^2t/2}-1)/\tfrac{1}{2}Mv^2t. \quad (1)$$

(We shall always understand "lim" to mean the limit $R\to\infty$, with $\omega=v/R$, v fixed.) Equation (1) is derived by constructing the Green's function $G_0(\mathbf{r},\mathbf{r}',t)$ from an application of the rotation operator $e^{\omega Lt}$ to the nonrotating Green's function $g_0(\mathbf{r},\mathbf{r}',t)$, and evaluating $\gamma(t)$ from the identity

$$\gamma(t)=2\pi\int_0^R G_0(\mathbf{r},\mathbf{r},t)rdr. \quad (2)$$

However, in evaluating the thermodynamic limit of (2), Widom *first* replaces the nonrotating Green's function g_0 by the much simpler form it assumes in the limit $R\to\infty$, then evaluates the integral for finite R, and finally takes the thermodynamic limit of the remaining R and ω dependence. This procedure seems quite reasonable as long as nothing exceptional is happening near the rim; however, it is easy to show that *all* the condensate particles move out to infinity with the rim of the circle in the thermodynamic limit. It is, therefore, far from obvious that Eq. (1) is correct in the thermodynamic limit.

To establish Eq. (1) more rigorously, and to explore some of the rather remarkable properties of the condensate, we must consider the explicit wave functions and energy levels of the rotating two-dimensional ideal Bose gas. The single-particle Hamiltonian is

$$H=p^2/2M-\omega L, \quad (3)$$

which, together with the boundary condition that the wave functions vanish at $r=R$, leads to the single-particle stationary states

$$\Phi_{mn}(r,\varphi)=J_m(x_{mn}r/R)e^{im\varphi} \quad (4)$$

and energy levels

$$E_{mn}=\hbar^2 x_{mn}^2/2MR^2-\hbar mv/R, \quad (5)$$

where x_{mn} is the nth zero of the mth-order Bessel function J_m. For large m, the first few zeros are given by[3]

$$x_{mn}=m+\alpha_n m^{1/3}+O(m^{-1/3}), \quad (6)$$

where $\alpha_1=1.8558$, $\alpha_2=3.2447$, $\alpha_3=4.3817$, ...

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[1] A. Widom, Phys. Rev. **168**, 150 (1968); **176**, 254 (1968).

[2] See, for example, P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); G. V. Chester, in *Lectures in Theoretical Physics*, edited by K. T. Mahanthappa (Gordon and Breach Science Publishers, Inc., New York, 1968), Vol. 11, p. 253; G. V. Chester, M. E. Fisher, and N. D. Mermin, Phys. Rev. **185**, 760 (1969).

[3] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945), p. 143; *Royal Society Mathematical Tables*, edited by F. W. J. Olver (Cambridge University Press, London, 1960), Vol. 7, Part III.

We write $\gamma(t)$ as follows:

$$\lim\frac{\gamma(t)}{\pi R^2}=\lim\sum_{mn}\frac{e^{-E_{mn}t}}{\pi R^2}$$
$$=\lim\sum_{m=-\infty}^{\infty}e^{\hbar vmt/R}\int_{C'}e^{-\hbar^2z^2t/2MR^2}\frac{J_m'(z)dz}{J_m(z)2\pi^2iR^2} \quad (7)$$
$$=\lim\int_{-\infty}^{\infty}d\eta e^{\hbar v\eta t}\int_C e^{-\hbar^2\xi^2t/2M}\frac{J_{R\eta}'(R\xi)}{J_{R\eta}(R\xi)}\frac{d\xi}{2\pi^2 i},$$

where C is a contour surrounding the positive zeros of $J_{R\eta}(R\xi)$ in a counterclockwise sense. In the Appendix, it is shown that this contour can be so placed that except in regions where the contribution to the integral is negligible, the asymptotic evaluation

$$J_{R\eta}'(R\xi)/J_{R\eta}(R\xi)\underset{R\to\infty}{\longrightarrow}-i(1-\eta^2/\xi^2)^{1/2}\operatorname{sgn}(\operatorname{Im}\xi) \quad (8)$$

may be used, leading in the thermodynamic limit to

$$\lim\frac{\gamma(t)}{\pi R^2}=\int_{-\infty}^{\infty}d\eta e^{\hbar v\eta t}\int_{|\eta|}^{\infty}e^{-\hbar^2\xi^2t/2M}\left(1-\frac{\eta^2}{\xi^2}\right)^{1/2}\frac{d\xi}{\pi^2}. \quad (9)$$

Equation (1) follows from a direct evaluation of the integral in Eq. (9).

(2) From Eq. (1), Widom deduces that at any temperature there will be Bose-Einstein condensation if the number density exceeds the critical density ρ_c:

$$\rho_c=\frac{1}{\pi\hbar^2\beta^2v^2}\sum_{s=1}^{\infty}\frac{1-e^{-s\beta Mv^2/2}}{s^2}. \quad (10)$$

To this we wish to add the observation that condensation occurs not only into the lowest one-particle state, but into *infinitely many* one-particle states. To see this, we expand the energy E_{mn} about its minimum ($m=m_n$, not necessarily an integer) using (6):

$$E_{mn}=E_0+(\alpha_n-\alpha_1)(\hbar^2Mv^4)^{1/3}/R^{2/3}+(\hbar^2/2MR^2)$$
$$\times\{(m-m_n)^2-(m_1-[m_1])^2\}+O(R^{-8/3}), \quad (11)$$

where

$$E_0=-\tfrac{1}{2}Mv^2+\alpha_1(\hbar^2Mv^4)^{1/3}/R^{2/3}$$
$$+(\hbar^2/2MR^2)(m_1-[m_1])^2+O(R^{-8/3}) \quad (12)$$

and $[m_n]$ is the nearest integer to m_n.

The contribution to the number density from the one-particle state with energy E_{mn} is

$$\rho_{mn}=(1/\pi R^2)1/(e^{\beta(E_{mn}-\mu)}-1). \quad (13)$$

Therefore, when $E_0-\mu=O(1/R^2)$, the contribution of the lowest one-particle state ($n=1$, $m=[m_1]$) does not vanish for large R. Furthermore, it follows from (11) that *every* one-particle state with radial quantum number $n=1$ and $m-m_1$ finite makes a nonvanishing contribution to the density. Since, however, $\alpha_2-\alpha_1=1.389$ and since α_n increases with n, states with radial quantum numbers greater than unity will contribute terms of at most $O(R^{-4/3})$ to the density.

We conclude that at densities ρ above the critical density ρ_c all states with radial quantum number $n=1$ and finite $m-m_1$ are macroscopically occupied. The contribution ρ_{m1} of each such state to the density ρ is given for large R by

$$\rho_{m1}\to(2Mk_BT/\pi\hbar^2)1/[(m-m_1)^2+a], \quad (14)$$

where

$$a=2MR^2(E_0-\mu)/\hbar^2-(m_1-[m_1])^2. \quad (15)$$

The quantity a can be determined at fixed R by the condition[4]

$$\rho-\rho_c=\sum_m\rho_{m1}=\frac{2Mk_BT}{\hbar^2a^{1/2}}\frac{\sinh(2\pi a^{1/2})}{\cosh(2\pi a^{1/2})-\cos(2\pi m_1)}. \quad (16)$$

Although it is interesting to see so simple a model displaying condensation into not only more than one, but infinitely many levels, there is little doubt (see point 3 below) that none of these levels will remain macroscopically occupied as soon as any reasonable interactions are added. Furthermore, it is hard to imagine, even within the model, any thermodynamic properties that are dramatically affected by the multitude of condensate levels. One might think, for example, that the condensate entropy could play a novel and important role in the thermodynamics; however, it can be shown that the contribution of the condensate to the entropy per particle remains vanishingly small in the thermodynamic limit.

(3) The multiplicity of the condensate levels is an entertaining curiosity, but the more significant feature of Widom's model is its bearing on the rigorous arguments excluding Bose-Einstein condensation in one and two dimensions.[5] It might appear (and Widom suggests) that there is no inconsistency since the best known of such arguments, due to Hohenberg,[2] only excludes condensation into the zero-momentum one-particle level, and the condensate levels in the model have nonuniform spatial wave functions. However, Chester[2] has argued that once condensation into the $\mathbf{k}=0$ state is excluded, there cannot be condensation into any one-particle state at all. His argument relies on a criterion for Bose-Einstein condensation established

[4] Note that both a and $\rho_{[m_1]+\Delta m,1}$ for fixed integral Δm oscillate without approaching unique limits as $R\to\infty$. Nevertheless, $\rho_{[m_1]+\Delta m,1}$ is bounded below, so these states are correctly described as macroscopically occupied.

[5] In this regard, we should mention a second closely related model of Widom's (see Ref. 1): a one-dimensional ideal Bose gas in a box of length L subject to a uniform gravitational field with a fixed potential difference from one end of the box to the other. Although his analysis there is subject to the same criticism as in the two-dimensional model, his conclusion that condensation occurs is again correct. This time, however, there is condensation into only a single one-particle state.

by Penrose and Onsager,[6] the derivation of which assumes that the spatial density of the system is everywhere bounded in the thermodynamic limit. In Widom's model, the maximum spatial density satisfies[7]

$$\max[\sigma(r,r)]=\max\left[\sum_{mn}\rho_{mn}\frac{J_m^2(x_{mn}r/R)}{J_m'^2(x_{mn})}\right]$$

$$>\rho_{m1}\left[\frac{J_m^2(x_{m1}')}{J_m'^2(x_{m1})}\right]_{m=[m_1]}$$

$$=O(m_1^{2/3}), \tag{17}$$

[where x_{m1}' is the first zero of $J_m'(z)$] and, therefore, diverges as $R^{2/3}$ in the thermodynamic limit. This (and the fact that the Hamiltonian is not real[8]) makes Chester's argument inapplicable.

Since any system of real bosons will have bounded density due to the short range repulsion, it seems most unlikely that the model has any implications for real systems. It is, however, of conceptual interest, in that it sets limits to the generality with which Bose-Einstein condensation can be excluded; i.e., a correct statement of Chester's theorem is that any system of interacting bosons in one or two dimensions with a real Hamiltonian cannot have a condensate *unless the density of particles is somewhere unbounded.*

APPENDIX

It is convenient to calculate Eq. (7) using the contour in Fig. 1.

We can easily estimate the contribution to (7) from contour I by converting the integral back to a sum and noting that $J_{R\eta}(R\xi)$ has fewer than RX zeros within the contour,[9] all of which exceed η. It then follows that the net contribution from contour I is $O(R^{-2/3+\delta})$.

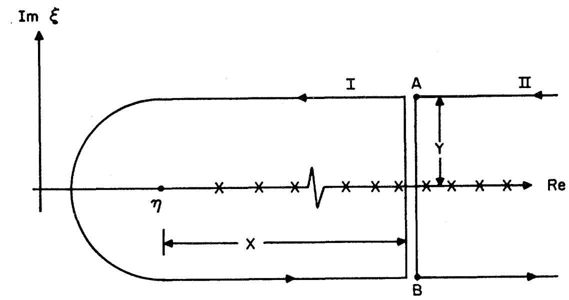

FIG. 1. Contour for Eq. (7). The distances X and Y are such that $|X+iY|=O(R^{-2/3+\delta})$ for small positive δ, and $\arg(X+iY)<60°$.

Along contour II, we can use the Debye asymptotic expansions of $J_m(z)$. These give an asymptotic expansion of $J_m'(z)/J_m(z)$ from which we obtain an asymptotic expansion for the integral along contour II.[10] We note that the first term in the expansion of $J_{R\eta}'(R\xi)/J_{R\eta}(R\xi)$ is

$$J_{R\eta}'(R\xi)/J_{R\eta}(R\xi) \underset{R\to\infty}{\longrightarrow} -i(1-\eta^2/\xi^2)^{1/2}\tanh\psi, \tag{18}$$

where $\psi=iR[(\xi^2-\eta^2)^{1/2}-\eta\cos^{-1}\eta/\xi]-i\pi/4$. Along the horizontal portions of contour II,

$$\tanh\psi \underset{R\to\infty}{\longrightarrow} \mathrm{sgn}(\mathrm{Im}\xi).$$

Furthermore, we can choose the contour between points A and B in Fig. 1 so that $|\tanh\psi|=1$. [We simply maintain $\mathrm{Im}\psi=\frac{1}{4}\pi$ (modulo 2π).] Then the net contribution to $\gamma(t)/\pi R^2$ from the path A → B can be shown to be $O(R^{-1+3\delta/2})$. Similar calculations indicate that the remaining terms in the asymptotic expansion of $J_{R\eta}'(R\xi)/J_{R\eta}(R\xi)$ give no contribution to $\lim\gamma(t)/\pi R^2$.

We conclude that, in the exact calculation of Eq. (7), it is sufficient to replace $J_{R\eta}'(R\xi)/J_{R\eta}(R\xi)$ by its asymptotic form

$$J_{R\eta}'(R\xi)/J_{R\eta}(R\xi) \to -i(1-\eta^2/\xi^2)^{1/2}\,\mathrm{sgn}(\mathrm{Im}\,\xi). \tag{19}$$

[6] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1954).

[7] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publication, Inc., New York, 1965), p. 371.

[8] We suspect that the lack of time-reversal invariance is not essential to Chester's argument, but have been unable to generalize it to complex Hamiltonians. In this respect, the one-dimensional model of Widom's (Ref. 5) is more clear-cut—there, the Hamiltonian is real and it is *only* the failure of the density to be bounded that makes the condensation possible.

[9] This follows from the interlacing theorem and the fact that all positive zeros of $J_m(z)$ exceed m. See G. N. Watson, *Bessel Functions* (Cambridge University Press, London, 1948), p. 479f.

[10] We use the identity $2J_m'(z)=J_{m-1}(z)-J_{m+1}(z)$ and the asymptotic expansions $2J_m(z)=S_m^{(1)}(z)+S_m^{(2)}(z)$ on *p.* 263 of Watson (Ref. 9). Term-by-term integration of asymptotic expansions is shown to be valid by E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, London, 1965), Chap. VIII. Similarly, division of these asymptotic expansions is easily justified.