Electrical Conductivity in Pure Type-II Superconductors near H_{c2}†

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The electrical conductivity for a pure type-II superconductor in high magnetic fields near H_{c2} is calculated for frequencies and relaxation times that satisfy the limit $\omega\tau > 1$. The conductivity in the absence of fluctuations, i.e., with a static vortex lattice, is calculated in a manner similar to the procedure used by Brandt, Pesch, and Tewordt to obtain the density of states. This direct approach allows us to bypass an ansatz made by Maki for the purpose of calculating transport properties of pure type-II superconductors. The fluctuations of the order parameter are included in the theory by using the calculation by Caroli and Maki of the conductivity due to fluctuations. In the Pippard limit, the conductivity is calculated numerically. From the conductivity, results for the surface resistance are obtained. The theory predicts that, as the magnetic field is lowered below H_{c2}, a highly frequency-dependent decrease in the surface resistance occurs, especially for frequencies in the neighborhood of 0.5 GHz. This frequency dependence is not present in the surface resistance calculated by Caroli and Maki. For frequencies in the megahertz region, where experimental measurements have been made, the theory predicts a sharper decrease in R_s/R_n than is observed experimentally. It is pointed out, however, that for these frequencies the limit $\omega\tau > 1$ is not valid, and higher-frequency measurements are necessary for direct comparison to the theory. The effects on the surface resistance of a smearing-out of the vortex lattice near H_{c2} are also determined numerically by inserting a phenomenological delocalization factor.

I. INTRODUCTION

TRANSPORT properties of dirty type-II superconductors near the upper critical field H_{c2} are now well understood in terms of gapless superconductivity.[1] They have been extensively studied in the last few years from both an experimental and theoretical point of view. Pure type-II superconductors have been less commonly studied and it has turned out that theoretical predictions are much more difficult to obtain. This was first pointed out by Cyrot and Maki[2] by showing that the usual expansion in powers of the order parameter is not valid in the clean limit and leads to unphysical results for the ultrasonic attenuation.

Since then two different methods have been employed to gain a theoretical understanding of pure type-II superconductors. Maki, studying the different terms in the nonconverging expansion in powers of the order parameter, $\Delta(\mathbf{r})$, noticed a similarity with a Bardeen, Cooper, and Schrieffer (BCS) superconductor carrying a current, and made an ansatz[3] based on these similarities. He claimed to sum in this manner the most divergent terms of the divergent expansion in powers of $\Delta(\mathbf{r})$. The second attempt to handle this problem was made by Brandt, Pesch, and Tewordt[4] (BPT). They did not use an expansion but were able to calculate the Green's function of the system by using the periodicity of the known solution of the order parameter in the field region near H_{c2}. However, in their calculation they have to neglect the change in amplitude of the order parameter and only succeed in taking into account its phase variation.

Both methods have been used to test different kinds of experiments; both lead to a more complete agreement with experiment. Maki used his technique to calculate ultrasonic attenuation[3] and was able to predict a behavior near H_{c2} proportional to the square root of $(H_{c2}-H)$ in contrast to the dirty limit[5] which gives a linear dependence. This square-root dependence has been confirmed experimentally but the quantitative comparison is difficult and, in particular, experimental results show a dependence on the mean free path[6] which is not predicted in the Maki theory. Another success of the Maki theory is the prediction of the same square-root dependence in the thermal conductivity.[7] All of these square-root dependences in the Maki calculation occur because of the similarities of the Maki theory to the BCS theory, with its nonspatially varying order parameter, which yields a linear dependence upon Δ.

For the nuclear magnetic resonance, Maki[8] obtained the same result that was obtained previously in the large mean-free-path limit by the Orsay group.[9] These predictions have been confirmed experimentally in a qualitative way.[10] Pesch has also calculated the nuclear

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[1] See for example the chapter by K. Maki, in *Superconductivity*, edited by R. Parks (Marcell Dekker, Inc., New York, 1969).

[2] M. Cyrot and K. Maki, Phys. Rev. **156**, 433 (1967).

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[5] M. Cyrot, Thesis, Orsay (unpublished); K. Maki, Phys. Rev. **148**, 370 (1966).

[6] J. Vinen (private communication).

[7] K. Maki, Phys. Rev. **158**, 397 (1967).

[8] K. Maki (private communication) and Ref. 1.

[9] Orsay Group of Superconductivity, Phys. Kondensierten Materie **5**, 141 (1966).

[10] M. Cyrot, C. Froidevaux, and D. Rossier, Phys. Rev. Letters **19**, 647 (1967).

magnetic spin relaxation rate[11] using the BPT technique. This calculation has given a quantitative agreement with experiments but discrepancies[12] still exist for temperatures close to T_c, discrepancies which increase with an increasing mean free path.

Caroli and Maki have studied in great detail the electromagnetic properties.[13] They showed in particular that one has to take into account the dynamical fluctuations of the order parameter which are induced by the fluctuating vector potential. Their calculation is based on the description of the dynamical fluctuations in the framework of a simple time-dependent Ginzburg-Landau theory. In this framework one can show, as done by Caroli and Maki (CM), that the dynamical fluctuations are only present when the space-oscillating vector potential $\mathbf{A}_\omega$ is perpendicular to the static field $\mathbf{H}_0$. Moreover, the contribution of the fluctuations in the clean limit completely cancels the reactive part of the conductivity in this orientation. Due to this cancellation a simple expression for the ratio of the normal to superconducting surface resistance may be obtained in the orientation $\mathbf{A}_\omega \perp \mathbf{H}_0$, and for fields very near H_{c2} the surface resistance decreases as the square root of $H_{c2}-H_0$ ($\mathbf{A}_\omega$ is the vector potential of the perturbing field and $\mathbf{H}_0$ is the constant magnetic field). The Caroli-Maki theory also predicts an anisotropy in the surface impedance if, while $\mathbf{A}_\omega \perp \mathbf{H}_0$, one varies the propagation vector $\mathbf{q}$ from parallel to perpendicular to the magnetic field.

Using very pure niobium in the mixed superconducting state, Hibler and Maxfield[14] measured the surface resistance as a function of magnetic field in a geometry where $\mathbf{A}_\omega$ is perpendicular to $\mathbf{H}_0$ while varying the direction of $\mathbf{q}$. For $\mathbf{q}$ parallel to $\mathbf{H}_0$ they obtained a magnetic-field dependence of the surface resistance in qualitative agreement with the CM theory, but the anisotropy depending on the angle between $\mathbf{q}$ and $\mathbf{H}_0$ was opposite to the predicted anisotropy.

Due partially to this experimental discrepancy, but primarily to the necessity in the CM theory of making an assumption in order to calculate the response function, we were led to reexamine the Maki ansatz. While we adopt the general approach of CM, especially for the calculation of the dynamical fluctuations, we believe that the Maki ansatz overemphasizes the BCS behavior of the excitations parallel to the vortex. As one varies $\mathbf{q}$ from parallel to perpendicular to $\mathbf{H}_0$, one studies principally excitations perpendicular and parallel, respectively, to $\mathbf{H}_0$. This could be the reason for the failure of the Maki ansatz to explain the fine structure of the anisotropy in this geometry.

Based on this reasoning we recalculated the electromagnetic properties using the Brandt, Pesch, and Tewordt approximation which does not emphasize so much the BCS-like behavior of clean type-II superconductors. We calculate only the contribution of a static vortex lattice. For the contributions of the fluctuations we use the results of Caroli and Maki. This procedure bypasses the Maki ansatz because the ansatz is only used by CM to calculate the conductivity for a static vortex lattice. This calculation has been only partially successful. More precisely, agreement with the observed magnetic field dependence is not obtained unless the order parameter, $\Delta(\mathbf{r})$, is phenomenologically assumed to vary more slowly in space than is expected from the known Abrikosov GL solution for $\Delta(\mathbf{r})$.[15] If this is assumed, qualitative agreement with the anisotropy measurements is also obtained.

We note, however, that our theory may not be relevant to the experimental conditions of Ref. 14. This is because the experimental measurements of the surface impedance were made at frequencies in the megahertz region where the pure limit condition $\omega\tau > 1$ is probably not satisfied even for very pure niobium, although the limit $ql \gg 1$ should be valid. Current calculations[16] of the ultrasonic attenuation indicate that the numerical results for the type of calculation presented in this paper may be quite different for $\omega\tau < 1$. (τ and l are the electron collision time and mean free path; q and ω are the scales of space and time variation of the perturbing electromagnetic field and $q \sim 1/\lambda$ where λ is the penetration depth.)

On the other hand in the case of the CM theory, the relevance of the limit $\omega\tau > 1$ is not clear. This is because of the similarity of the Maki ansatz to the BCS calculation, as done by Mattis and Bardeen,[17] for example, and the fact, as noted by Miller,[18] that if $l \gtrsim 10^{-3}$ cm, then one may take $l \to \infty$ in the Mattis-Bardeen result.

In addition to the magnetic-field dependence, the other important result of our theory is the prediction of the frequency dependence of the surface impedance. Either with or without assuming a slowly varying order parameter our calculation predicts a fairly sharp decrease in the surface resistance near H_{c2} as the frequency increases above a certain cutoff frequency of order $\alpha\Delta^2$, where $\alpha = 1/v_F(2eH_0)^{1/2}$, and $\Delta^2 = \langle |\Delta(r)|^2 \rangle_{\text{av}}$. This sort of frequency dependence is not present in the CM theory. Moreover, this frequency dependence would be present in the gigahertz region where we would expect the limit $\omega\tau > 1$ to be valid.

This paper will be divided into two sections. In the first one we calculate the complex conductivity after the derivation of a general formula for the conductivity close to H_{c2}. We discuss the imaginary and real parts

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[13] C. Caroli and K. Maki, Phys. Rev. **159**, 316 (1967).

[14] W. D. Hibler and B. Maxfield, Phys. Rev. Letters **21**, 742 (1968).

[15] Even though T may not be near T_c, according to the work of Helfand and Werthamer $\Delta(r)$ has the form of Abrikosov's GL solution for H_0 near H_{c2}. E. Helfand and N. R. Werthamer, Phys. Rev. Letters **13**, 686 (1964).

[16] A. Houghton (private communication).

[17] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).

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of the conductivity. In the second section we give some numerical results for particular geometries and compare these results to experiment.

II. COMPLEX CONDUCTIVITY

In this section, we will calculate the electrical conductivity in the absence of fluctuations of the order parameter by considering a static lattice of vortices produced by a constant magnetic field near H_{c2}. The contribution of the fluctuations has already been calculated in the pure limit by Caroli and Maki and we will utilize their result for the conductivity due to fluctuations.

In Sec. II A we derive an expression for the response function of a pure type-II superconductor in magnetic fields near H_{c2}. In this derivation certain approximations are necessary which we show to be equivalent to the approximations used by Brandt, Pesch and Tewordt in obtaining the density of states. In Sec. II B, the imaginary part of the conductivity is calculated from the general response function and combined with the fluctuations of the order parameter as calculated by Caroli and Maki. In Sec. II C, the real part of the electrical conductivity is calculated in the Pippard, large q, limit. In this limit the integration over momentum may be carried out analytically leaving a frequency integral which may be evaluated numerically. We also find that in this calculation the Pippard limiting case is valid as long as $\lambda<(2eH_0)^{1/2}/\Delta^2$ where λ is the penetration depth, Δ^2 the spatial average of the square of the order parameter, H_0 is the magnetic field, and e the electronic charge. Near H_{c2} this limit is well satisfied (as long as T is not near T_c).

Throughout this calculation we take $\hbar=c=k_B=1$.

A. Response Function near H_{c2}

By using linear response theory,[19] the response of a small transverse electromagnetic field is described by introducing a response function $Q_{\mu\nu}(\mathbf{q},\omega)$ which relates the induced current to the perturbing EM field according to

$$J_\mu(\mathbf{q},\omega)=-\sum_\nu Q_{\mu\nu}(\mathbf{q},\omega)A_\nu(\mathbf{q},\omega)\,. \quad (1)$$

We take $Q_{\mu\nu}$ to be diagonal in momentum because the scale of $q\sim 1/\lambda$ and the system may be considered homogeneous since $\xi_0/\lambda<1$ (ξ_0 and λ are the coherence length and penetration depth of the pure superconductor). $Q_{\mu\nu}(\mathbf{x},\mathbf{x}',\omega)$ is given by the analytic continuation[19] of

$$K_{\mu\nu}(\mathbf{x},\mathbf{x}',\omega_\nu)=\frac{Ne^2}{m}-i\times\int_0^{-i/T}\langle TJ_\mu{}^p(\mathbf{x},t)J_\nu{}^p(\mathbf{x}',0)\rangle e^{i\omega_\nu t}dt\,, \quad (2)$$

where $\omega_\nu=2\pi\nu iT$ and ν is an integer. Since we are in the mixed superconducting state, $J_\mu{}^p(\mathbf{x},t)$ is the current in the presence of a constant external magnetic field characterized by the vector potential $\mathbf{A}_0(\mathbf{x},t)$:

$$J_\mu{}^p(\mathbf{x},t)=-[(\nabla_x-ie\mathbf{A}_0(\mathbf{x},t))-(\nabla_x{}'+ie\mathbf{A}_0(\mathbf{x}',t))]\times\frac{ei}{2m}\sum_s\psi_s{}^\dagger(\mathbf{x}',t)\psi_s(\mathbf{x},t)\,|_{\mathbf{x}'\to x}. \quad (3)$$

The thermal average must be evaluated for the mixed state of the superconductor in the presence of a constant magnetic field. So, in order to calculate the thermal product $P_{\mu\nu}(\mathbf{x},\mathbf{x}',t)=-i\langle TJ_\mu{}^p(\mathbf{x},t)J_\nu{}^p(\mathbf{x}',0)\rangle$ by means of the usual Hartree-Fock-Gorkov factorization, it is useful to introduce new Green's functions defined by the equations

$$G^B(\mathbf{x}_2,\mathbf{x}_1,\omega)=\exp\left(ie\int_{\mathbf{x}_2}^{\mathbf{x}_1}\mathbf{A}_0\cdot d\mathbf{l}\right)G(\mathbf{x}_2,\mathbf{x}_1,\omega)\,, \quad (4)$$

$$G^0(\mathbf{x}_2-\mathbf{x}_1,\,\omega)=\exp\left(ie\int_{\mathbf{x}_2}^{\mathbf{x}_1}\mathbf{A}_0\cdot d\mathbf{l}\right)\tilde{G}^0(\mathbf{x}_2,\mathbf{x}_1,\omega)\,. \quad (5)$$

$G(\mathbf{x}_2,\mathbf{x}_1,\omega)=-i\langle T\psi(\mathbf{x}_2,t)\psi^\dagger(\mathbf{x}_1,0)\rangle_\omega$ is the Green's function for the superconductor in the presence of a constant magnetic field and $\tilde{G}^0(\mathbf{x}_2,\mathbf{x}_1,\omega)$ is the normal-state Green's function in the presence of a constant magnetic field. $\mathbf{A}_0(\mathbf{x}',t)$ is the vector potential corresponding to the magnetic field applied along the z axis with magnitude close to H_{c2} and the line integral in the exponential is taken along a straight line connecting $\mathbf{x}_1$ and $\mathbf{x}_2$. As shown by Gorkov,[20] $G^0(\mathbf{x}_2-\mathbf{x}_1,\,\omega)$ is the translationally invariant normal state Green's function in the absence of a magnetic field. G^B is the BPT Green's function which was shown by BPT to be periodic in the sum of its spatial variables with respect to the vortex lattice. This periodicity was proven by BPT by noting that the usual Gorkov integral equation[20] for G could be written in the form

$$G^B(\mathbf{x},\mathbf{x}',\omega)=G^0(\mathbf{x}-\mathbf{x}',\,\omega)-\int d^3l d^3l' G^0(\mathbf{x}-\mathbf{l},\,\omega)\times G^0(\mathbf{l}'-\mathbf{l},\,-\omega)G^B(\mathbf{l}',\mathbf{x}',\omega)V(\mathbf{l},\mathbf{l}')\,, \quad (6)$$

where $V(\mathbf{l},\mathbf{l}')$ is given by

$$V(\mathbf{l},\mathbf{l}')=\exp\left(2ie\int_{\mathbf{l}}^{\mathbf{l}'}\mathbf{A}_0\cdot d\mathbf{s}\right)\Delta(\mathbf{l})\Delta^*(\mathbf{l}')\,. \quad (7)$$

By using in Eq. (7) the Abrikosov GL expression[15] for $\Delta(\mathbf{r})$

$$\Delta(\mathbf{r})=\sum_K C_K e^{iKy}\exp[-eH(x-K/2eH)^2]\,, \quad (8)$$

[19] See, for example, A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Inc., Englewood Cliffs, N. J., 1963).

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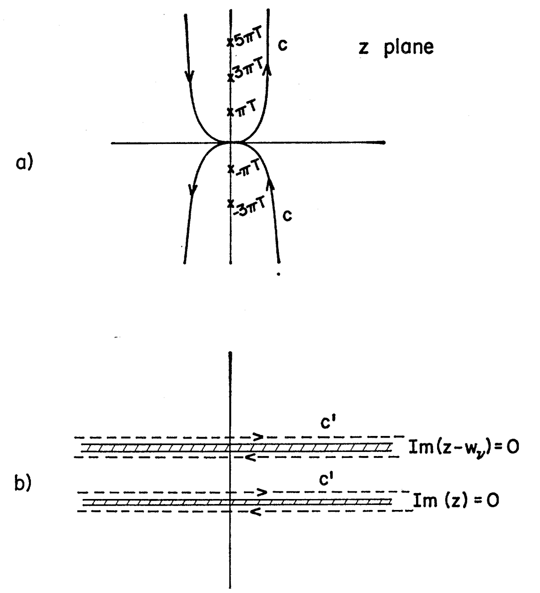

FIG. 1. (a) Contour for changing sum over discrete frequency variables to an integral. (b) Branch cuts of Green's functions.

where $K=(4\pi eH)^{1/2}n$ with n an integer, one sees that $V(1,1')$ has the periodicity of the vortex lattice with respect to its sum coordinates. Therefore from Eq. (6), G^B also has this periodicity.

Using these definitions and the Gorkov integral equations relating the F Green's function to the G Green's functions

$$F(\mathbf{x}_1,\mathbf{x}_2,\omega)=-i\langle T\psi_\uparrow(\mathbf{x}_1,t)\psi_\downarrow(\mathbf{x}_2,0)\rangle_\omega$$

$$=-\int\tilde{G}^0(\mathbf{x}_1,1,\omega)\Delta(1)G(\mathbf{x}_2,1,-\omega)d^3l, \qquad (9)$$

$$\bar{F}(\mathbf{x}_1,\mathbf{x}_2,\omega)=-i\langle T\psi_\downarrow{}^\dagger(\mathbf{x}_1,t)\psi_\uparrow{}^\dagger(\mathbf{x}_2,0)\rangle_\omega$$

$$=-\int\tilde{G}^0(1,\mathbf{x}_1,-\omega)\Delta^*(1)G(1,\mathbf{x}_2,\omega)d^3l, \qquad (10)$$

we get, by substituting Eq. (3) into Eq. (2) and by making the Hartree-Fock-Gorkov factorization, the following result:

$$K_{\mu\nu}(\mathbf{x},\mathbf{x}',\omega_\nu)=\frac{Ne^2}{m}\delta_{\mu\nu}-\frac{e^2T}{2m^2}\Big\{(\nabla_{x_1'}-\nabla_{x_1})(\nabla_{x_2'}-\nabla_{x_2})$$
$$\times\sum_{\zeta_l}\big[G^B(\mathbf{x}_2,\mathbf{x}_1',\zeta_l-\omega_\nu)G^B(\mathbf{x}_1,\mathbf{x}_2',\zeta_l)$$
$$-\int G_0(1',\mathbf{x}_1',-\zeta_l)G^B(1',\mathbf{x}_2',\zeta_l)$$
$$\times G^B(\mathbf{x}_2,1,\zeta_l-\omega_\nu)G_0(\mathbf{x}_1,1,\omega_\nu-\zeta_l)$$
$$\times V(1,1')d^3ld^3l'\big]\Big\}_{\mathbf{x}_1'\to\mathbf{x}_1,\mathbf{x}_2'\to\mathbf{x}_2}. \qquad (11)$$

In Eq. (11), $\zeta_l=(2l+1)\pi iT$, and the terms coming from the gradients operating on the magnetic-field phase factors have been cancelled by the diamagnetic terms in the current operators proportional to $\mathbf{A}_0$.

The spatial fourier transform is only tractable if we take G^B to be diagonal in momentum. More precisely, in the notation of BPT, we will neglect terms containing Green's functions $G^B(\mathbf{p},\mathbf{K},\zeta_l)$, with $\mathbf{K}\neq0$, where $\mathbf{p}$ is the normal fourier transform variable corresponding to the difference of the coordinates and $\mathbf{K}=(2\pi)^{1/2}\Lambda^{-1}\mathbf{n}$ is a reciprocal fluxoid lattice vector corresponding to the transform with respect to the sum of the spatial variables. [$\Lambda=(2eH_0)^{-1/2}$ and the components of the vector $\mathbf{n}$ are integers.] The Fourier coefficient $G^B(\mathbf{p},\mathbf{K},\zeta_l)$ is explicitly defined by the equation

$$G^B(\mathbf{x},\mathbf{x}',\zeta_l)=\sum_{\mathbf{K}}e^{i(1/2)\mathbf{K}\cdot(\mathbf{x}+\mathbf{x}')}$$
$$\times\int\frac{d^3p}{(2\pi)^3}G^B(\mathbf{p},\mathbf{K},\zeta_l)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}.$$

In particular we show in the Appendix that within the approximations BPT used to calculate $G^B(\mathbf{p},0,\zeta_l)$, the Green's function $G^B(\mathbf{p},\mathbf{K},\zeta_l)$ with $\mathbf{K}\neq0$ is small compared to $G^B(\mathbf{p},0,\zeta_l)$. Since this approximation involves only taking into account $V(\mathbf{p},\mathbf{K})$ for $\mathbf{K}=0$, the approximation is physically equivalent to taking into account only the phase variation of the order parameter and neglecting its variation in amplitude ($V(\mathbf{p},\mathbf{K})$ is the fourier transform of $V(1,1')$.)

Assuming G^B to be diagonal in momentum and Fourier transforming we obtain

$$K_{\mu\nu}(\mathbf{q},\omega_\nu)=\frac{Ne^2}{m}\delta_{\mu\nu}+\frac{2e^2}{m^2}\sum_{\mathbf{k}}k_\mu k_\nu T$$
$$\times\sum_{\zeta_l}\Big\{G^B(\mathbf{k}+\mathbf{q},\zeta_l+\omega_\nu)G^B(\mathbf{k},\zeta_l)$$
$$+\Big[\int du\,\rho_0(u,\Omega)G^B(\mathbf{k}+\mathbf{q},\zeta_l+\omega_\nu)$$
$$\times G^B(\mathbf{k},\zeta_l)G^0(\mathbf{k},-\zeta_l+2u)$$
$$\times G^0(\mathbf{k}+\mathbf{q},-(\zeta_l+\omega_\nu-2u))\Big]\Big\}. \qquad (12)$$

Following Cyrot and Maki we have defined and calculated a function $\rho_0(u,\Omega)$:

$$\Delta^2\rho_0(u,\Omega)=\int\frac{d^3k}{(2\pi)^3}V(\mathbf{k},0)\delta(2u-\mathbf{v}\cdot\mathbf{k})$$
$$=\frac{\Delta^2\exp[-(u/\epsilon\sin\theta)^2]}{\pi^{1/2}\epsilon\sin\theta}. \qquad (13)$$

(v is the Fermi velocity, $\epsilon=v(\frac{1}{2}eH)^{1/2}$ and θ, ϕ are polar coordinates describing $\mathbf{v}$ where the polar axis is taken along the z axis, i.e., in the field direction. Ω stands for the angular co-ordinates θ and ϕ.)

B. Imaginary Part of Conductivity

To proceed further with the calculation we need to perform the frequency sum and momentum integral in Eq. (12). For the real part of the response function (imaginary part of the conductivity) the expansion in powers of Δ as carried out by Cyrot and Maki[2] is assumed to be valid because there are no divergences in the real part of the response function for the BCS case. (The unphysical results in the Cyrot-Maki expansion in powers of Δ occurred only when there were divergences in the case of a non-spatially-varying order parameter.) In the limit of $vq<2\pi T$ and $\omega<2\pi T$ we obtain by expanding in powers of Δ the result[21]

$$\mathrm{Re}K_{\mu\nu}(\mathbf{q},\omega)=-\frac{2e^2}{m^2}N(0)\Delta^2\pi\int\frac{d\Omega}{4\pi}\delta_{\mu\nu}(k_F)_\mu{}^2\times\int du\rho_0(u,\Omega)T\sum_{\zeta_l}\frac{\zeta_l}{|\zeta_l|}\frac{1}{(\zeta_l-u)^3}.\quad(14)$$

This is the same result as obtained by the Maki ansatz and is shown by CM to be cancelled by the fluctuations of the order parameter for the orientation $\mathbf{A}_\omega\perp\mathbf{H}_0$. It is also shown by CM that the contribution of the fluctuations to the real part of the conductivity is of order Δ^2 and therefore near H_{c2} is small compared to the magnitude of the static response function we are calculating. Consequently the effect of the fluctuations may be included by taking the imaginary part of the conductivity to be zero in the pure limit, and calculating the real part from the static response function.

C. Real Part of Conductivity

As pointed out above, the contribution of the fluctuations of the order parameter to the real part of the conductivity is small compared to the contribution of the static response function, which we calculate in this subsection. Consequently the real part of the conductivity is given by the imaginary part of the response function derived in Sec. II A. To calculate the imaginary part of the static response function we note, by considering Eq. (12), that in general we need to consider an expression

$$T\int\frac{d^3k}{(2\pi)^3}k_\mu k_\nu\sum_{\zeta_l}G(\mathbf{k}+\mathbf{q},\zeta_l)G(\mathbf{k},\zeta_l-\omega_\nu)=\frac{-i}{4\pi}\int\frac{d^3k}{(2\pi)^3}k_\mu k_\nu\int_C\tanh\frac{z}{2T}G(\mathbf{k}+\mathbf{q},z)\times G(\mathbf{k},z-\omega_\nu)dz,\quad(15)$$

where G refers to both the first and second terms in the response function and the contour C is illustrated in Fig. 1. Deforming the contour about the branch cuts of the Green's function at $\mathrm{Im}(z-\omega_\nu)=0$, $\mathrm{Im}(z)=0$ and using the analytic continuation identity

$$\tanh\left(\frac{\omega\pm\omega_\nu}{2T}\right)=\tanh\left(\frac{\omega}{2T}\right),$$

we obtain, after analytically continuing ω_ν onto the real axis from above ($\omega_\nu\sim\omega_0+i\delta$), the expression

$$\mathrm{Im}K_{\mu\nu}(\mathbf{q},\omega_0)=\frac{e^2}{\pi m^2}\int\frac{d^3k}{(2\pi)^3}\left\{\int_{-\infty}^{\infty}d\omega\left[\tanh\left(\frac{\omega}{2T}\right)-\tanh\left(\frac{\omega+\omega_0}{2T}\right)\right]\mathrm{Im}G_+(\mathbf{k},\omega)\times\mathrm{Im}G_+(\mathbf{k}+\mathbf{q},\omega+\omega_0)\right\}k_\mu k_\nu,\quad(16)$$

where $G_+(\mathbf{k},\omega)[G_-(\mathbf{k},\omega)]$ refers to G analytic above (below) the branch cut and we have used the identity $G_-(\mathbf{k},\omega)=G_+{}^*(\mathbf{k},\omega)$.

To evaluate the expression in Eq. (16) it is necessary to exchange the order of integration and do the momentum integral first. For the momentum integral we employ the transformation

$$\int_0^\infty k^2dk\int_0^\pi\sin\theta d\theta=\frac{m^2}{q}\int_{-\mu}^\infty d\xi(k)\int_{\xi(k-q)}^{\xi(k+q)}d\xi(k'),\quad(17)$$

where $q^2=|\mathbf{k}-\mathbf{k}'|^2=|k^2+k'^2-2kk'\cos\theta|$ and $\xi(k)=k^2/2m-\mu$. Using the fact that the dominant contribution of the integral of $\mathrm{Im}G_+(\mathbf{k})$ comes at $k=k_F$ we may extend the lower limit of integration in the first integral to $-\infty$. Also,

$$\xi(k+q)\approx\xi(k)+qv,\quad \xi(k-q)\approx\xi(k)-qv,\quad(18)$$

so if qv is greater than the width of the nonzero part of $\mathrm{Im}G_+[\xi(k)]$ or ω_0, whichever is larger, then we may extend the limits of the second integral to $\pm\infty$. It is easy to see from the definition of $G^B(\mathbf{k},\zeta_l)$ which we will write down below, that the width of $\mathrm{Im}G_+{}^B(\xi)\sim\Delta^2\Lambda/v$ which is very small near H_{c2} and therefore the qv large

[21] In obtaining this result we have utilized the usual convergence trick of writing the response function in the form

$$K_{\mu\nu}(\mathbf{x},\mathbf{x}',\omega)=P_{\mu\nu}(\mathbf{x},\mathbf{x}',\omega)-P_{\mu\nu}(\mathbf{x},\mathbf{x}',0)|_{\Delta\to0},$$

which is valid if Landau diamagnetism is neglected. See Ref. 19, for example.

or Pippard limit appears quite good. Moreover, since both $\mathbf{k}$ and $(\mathbf{k}+\mathbf{q})$ must be near the Fermi surface the main contribution to the integral comes when $\mathbf{k}\cdot\mathbf{q}=0$, a fact we may take care of by inserting $\int\delta(\mathbf{q}\cdot\mathbf{v})qvd\theta$.

In light of the above comments we obtain for the imaginary part of the response function

$$\begin{aligned}\operatorname{Im}K_{\mu\nu}(\mathbf{q},\omega_0)&\\ =-\frac{e^2}{m}\frac{k_F^3}{\pi}\int_{-\infty}^{\infty}d\omega\left[\tanh\frac{\omega}{2T}-\tanh\left(\frac{\omega+\omega_0}{2T}\right)\right]&\\ \times\int d\Omega\delta(\mathbf{q}\cdot\mathbf{v})\frac{k_\mu k_\nu}{k_F^2}\frac{1}{(2\pi)^3}\bigg\{\int_{-\infty}^{\infty}d\xi(k)\operatorname{Im}G_+^B(\mathbf{k},\omega-\omega_0)&\\ \times\int_{-\infty}^{\infty}d\xi(k')\operatorname{Im}G_+^B(\mathbf{k}',\omega)+\int_{-\infty}^{\infty}du\,\rho_0(u,\Omega)&\\ \times\int_{-\infty}^{\infty}d\xi(k)\operatorname{Im}[G_+^B(k,\omega-\omega_0)G_+^0(k,-\omega+\omega_0+2u)]&\\ \times\int_{-\infty}^{\infty}d\xi(k')\operatorname{Im}[G_+^B(\mathbf{k},\omega)G_+^0(\mathbf{k}',-\omega+2u)]\bigg\}.&\end{aligned}\tag{19}$$

The integral $\int d\xi(k)\operatorname{Im}G_+(\mathbf{k},\omega)$ may be carried out by using the explicit form of the BPT Green's function (see Appendix)

$$G_+^B(\mathbf{k},\omega)=G_+^B(\mathbf{k},0,\omega)=[\omega-\xi+i\pi^{1/2}\alpha\Delta^2W(z)]^{-1},\tag{20}$$

where

$$z=[\omega+i\delta+\xi(k)]\alpha,$$

$$\alpha=\frac{\Lambda}{\sin\theta v},\quad\theta=\arccos(\hat{k}\cdot\hat{H}_0)$$

and

$$W(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{dte^{-t^2}}{z-t}.$$

[We note that contrary to the implication in Ref. 4, $W(z)=\exp(-z^2)\operatorname{erfc}(-iz)$ only for $\operatorname{Im}z>0$ and not for all z. The integral definition is the general definition for $W(z)$ as it must be for $G(z)$ to have a branch cut at $\operatorname{Im}z=0$.] As shown by BPT, $G_+^B(\xi,\omega)$ has only one pole in the upper-half energy plane, namely at $\xi=\xi_1$ where ξ_1 is a solution of the integral equation

$$\xi_1(\omega)=\omega+i\pi^{1/2}\Delta^2\alpha W[(\omega+\xi_1+i\delta)\alpha].\tag{21}$$

Using this result we obtain for the respective energy integrals[22]

$$-1/\pi\int_{-\infty}^{\infty}\operatorname{Im}G_+^B(\xi,\omega)d\xi=N(\omega,\theta)\tag{22}$$

$$-1/\pi\int_{-\infty}^{\infty}\operatorname{Im}[G_+^B(\xi,\omega)G_+^0(\xi,-\omega+2u)]d\xi=N_1(\omega,\theta,u),\tag{23}$$

where

$$N(\omega,\theta)=\operatorname{Re}\left\{\frac{2}{1-i\pi^{1/2}\Delta^2\alpha^2W'[(\xi_1+\omega)\alpha]}-1\right\}\tag{24}$$

$$\equiv\operatorname{Re}(2\bar{N}(\omega,\theta)-1)$$

and

$$N_1(\omega,\theta,u)=\operatorname{Re}\left(2\bar{N}(\omega,\theta)\frac{1}{2u-\omega-\xi_1}\right).\tag{25}$$

Thus, for the response function, we have

$$\begin{aligned}\operatorname{Im}K_{\mu\nu}(\mathbf{q},\omega_0)=-\frac{Ne^2}{m}\frac{3}{8}\int d\Omega\frac{k_\mu^2}{k_F^2}\delta(\mathbf{q}\cdot\mathbf{v})\delta_{\mu\nu}&\\ \times\int_{-\infty}^{\infty}\left[\tanh\frac{\omega}{2T}-\tanh\left(\frac{\omega-\omega_0}{2T}\right)\right]&\\ \times\bigg[N(\omega,\theta)N(\omega-\omega_0,\theta)+\Delta^2\int_{-\infty}^{\infty}du\,\rho_0(u,\Omega)&\\ \times N_1(\omega,\theta,u)N_1(\omega-\omega_0,\theta,u)\bigg],&\end{aligned}\tag{26}$$

where we have noted that by the symmetry of the integral $K_{\mu\nu}$ is diagonal, and we have used $N=k_F^3/3\pi^2$.

Dividing by the normal-state conductivity and using the facts that $N(\omega,\theta)=N(-\omega,\theta)$ and

$$\begin{aligned}\int\rho_0(u,\Omega)N_1(\omega,u,\theta)N_1(\omega+\omega_0,u,\theta)du&\\ =\int\rho_0(u,\Omega)N_1(-\omega,u,\theta)N_1(-\omega-\omega_0,u,\theta)du.&\end{aligned}\tag{27}$$

{These results follow from the definitions of $N(\omega,\theta)$ and $N_1(\omega,\theta,u)$ and the facts that $[W(z)]^*=W(-z^*)$ and $W'(z)=-2zW(z)+2i/\pi^{1/2}$} the $\operatorname{Im}K_{\mu\mu}(\mathbf{q},\omega_0)$ may be cast into a more useful form for doing calculations. Writing the result in terms of the conductivity we have [using the extreme anomalous limit result for the normalstate conductivity[23] $\sigma(q,\omega)=3\pi Ne^2/4mqv$, which

[22] In Eq. (22) we have used the easily proven identity

$$\int\operatorname{Im}G_+(\xi,\omega)d\xi=\operatorname{Im}\left\{\int\left[G_+(\xi,\omega)-\frac{1}{\omega-\xi+i\delta}\right]d\xi\right\}-\pi.$$

[23] M. Tinkham, in *Optical Properties and Electronic Structure of Metals and Alloys*, edited by F. Abeles (John Wiley & Sons, Inc., New York, 1966).

may be obtained by setting $\Delta=0$ in Eq. (26)]:

$$\frac{\mathrm{Re}\sigma_{\mu\mu}(\mathbf{q},\omega_0)}{\sigma_n(\mathbf{q},\omega_0)}=\frac{1}{2\pi T}\int d\Omega\, vq\delta(\mathbf{v}\cdot\mathbf{q})\frac{v_\mu^2}{v_F^2}\times\left\{\int_0^\infty d\omega\left(\cosh\frac{\omega}{2T}\right)^{-2}\times\left[N(\omega+\omega_0,\theta)N(\omega,\theta)+\Delta^2\int_{-\infty}^{\infty}du\,\rho_0(u,\Omega)\times N_1(\omega+\omega_0,u,\theta)N_1(\omega,u,\theta)\right]\right\},\quad(28)$$

where terms of higher order in $\omega_0/2T\ll1$ have been neglected in obtaining this result.

III. NUMERICAL RESULTS

To obtain numerical results it is necessary to evaluate the integral over u which may be done analytically. Using the above definition of $\bar{N}(\omega,\theta)$ and $W(z)$ this integral may be carried out straightforwardly and we obtain finally the expression for the real part of the conductivity:

$$\frac{\mathrm{Re}\sigma_{\mu\mu}(\mathbf{q},\omega_0)}{\sigma_n(\mathbf{q},\omega_0)}=\frac{1}{\pi}\int d\Omega\frac{v_\mu^2}{v^2}\delta(\mathbf{v}\cdot\mathbf{q})vq\times\int_0^\infty\frac{d\omega}{2T}\left(\cosh\frac{\omega}{2T}\right)^{-2}\Bigg\{[\mathrm{Re}(2\bar{N}(\omega,\theta)-1)]^2+2\,\mathrm{Re}[\bar{N}(\omega,\theta)(\bar{N}(\omega,\theta)-1)]+2\,\mathrm{Re}\left[\frac{\bar{N}(\omega,\theta)\bar{N}(\omega,\theta)[2i\,\mathrm{Im}\xi_1+2(\bar{N}(\omega,\theta)-1)\omega_0]}{2i\,\mathrm{Im}\xi_1+2\bar{N}(\omega,\theta)\omega_0}\right]\Bigg\}.\quad(29)$$

The imaginary part of the conductivity is given by $(1/\omega)\,\mathrm{Re}K_{\mu\nu}$ where $\mathrm{Re}K_{\mu\nu}$ is given by Eq. (14) for the orientation $\mathbf{A}_\omega\|\mathbf{H}_0$ and $\mathrm{Re}K_{\mu\nu}=0$ for $\mathbf{A}_\omega\perp\mathbf{H}_0$.

Of particular interest is the orientation $\mathbf{A}_\omega\perp\mathbf{H}_0$ and $\mathbf{q}\|\mathbf{H}_0$. In this orientation $\sin\theta=1$ for all angles integrated over. For Δ small (i.e., H near H_{c2}) and $T/2\epsilon<1$, we have, neglecting terms of order $(\Delta/\epsilon)^2$,

$$\sigma(\mathbf{q},\omega)=\sigma_n(\mathbf{q},\omega)\left\{1+2\,\mathrm{Re}\left\{\frac{i\pi^{1/2}\Delta/\epsilon}{\dfrac{\omega}{\Delta}+i\pi^{1/2}\Delta/\epsilon}\right\}\right\}.\quad(30)$$

As $\Delta\to0$ this result reduces to the normal-state conductivity [as does the more general expression in Eq. (19)] as it must. For $\omega<\Delta^2/\epsilon$, σ becomes approximately equal to $3\sigma_n$, and as ω approaches $\pi^{1/2}\Delta^2/\epsilon$, σ decreases and eventually approaches σ_n for $\omega\gg\Delta^2/\epsilon\pi^{1/2}$.

In general, the expression on the right-hand side of Eq. (29) must be evaluated numerically by evaluating the integral over ω and over angles by numerical quadrature.

From either the specular or diffuse scattering result for the surface impedance (using $\mathrm{Im}\sigma_s=0$) we obtain for the ratio of the mixed state to normal surface impedances[23]

$$\frac{Z_s}{R_n}=2\left[\frac{\sigma_n e^{i\pi}}{\sigma_s(\mathbf{q},\omega)}\right]^{1/3}.\quad(31)$$

So, by carrying out the numerical integration, we may obtain results for the surface resistance as a function of angle and of magnetic field. There is an angular dependence because, as $\mathbf{H}_0$ varies from parallel to perpendicular to $\mathbf{q}$, there is a change in the angles $\theta=\arccos(\hat{k}\cdot\hat{H}_0)$ which are integrated over.

We will also include numerical results where the order parameter is assumed to vary more slowly in space than predicted by the Abrikosov GL solution given by Eq. (8). Since the spatial variation of $\Delta(x)$ is of the form $\Delta(x)\sim\exp[-eHx^2]$ if the order parameter becomes less localized in space this can be taken into account by multiplying H by a delocalization factor δ so that

$$\Delta(x)\sim\exp[-eHx^2\delta^2].$$

We note that one motivation for arbitrarily assuming a large amount of delocalization (δ small) is that for very small δ our theory approaches the BCS result and is consequently similar to the result obtained by CM using the Maki ansatz. Numerical results for a small value of the delocalization factor will be given together with the normal numerical results.

To carry out numerical calculations for comparison with experiments on pure niobium we determined $K_2(T)$ from the results of McConville and Serin[24] and used $N(0)=5.6\times10^{34}$ (states/cm^3-erg), the result obtained from specific-heat measurements.[25] For v_F we used $v_F=3\times10^7$ cm/sec. From Eilenberger's work[26]

$$\Delta^2=-\frac{M(2\pi T)^2}{2eN(0)v_F^2}[g_1(H,T,l)]^{-1}.\quad(32)$$

Near H_{c2} the magnetization is given by

$$4\pi M=\frac{H-H_{c2}}{\beta[2K_2^2(T)-1]},$$

where $K_2(T)$ is the second Landau-Ginzburg parameter and $\beta\simeq1.2$. In the limit of $l\to\infty$

$$g_1(H,T,l)=-G(\rho)=\frac{1}{16}\int_0^1\tfrac{1}{2}dz(1-z^2)^{1/2}\times\int_{-\infty}^{\infty}\frac{dx}{\pi}e^{-x^2/(1-z^2)}\psi^{(2)}(\tfrac{1}{2}+i\rho x),\quad(33)$$

[24] T. McConville and B. Serin, Phys. Rev. **140**, 1169 (1965).
[25] H. A. Leupold and H. A. Boorse, Phys. Rev. **134**, A1322 (1964).
[26] G. Eilenberger, Phys. Rev. **153**, 584 (1967).

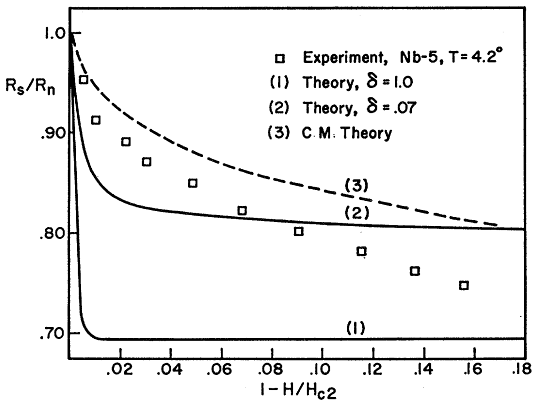

FIG. 2. Magnetic-field dependence of the normalized surface resistance for the orientation $\mathbf{H}_0 \perp \mathbf{A}_\omega$; $\mathbf{H}_0 \| \mathbf{q}$. The Fermi velocity is taken to be $v_F = 3.10^7$ cm/sec.

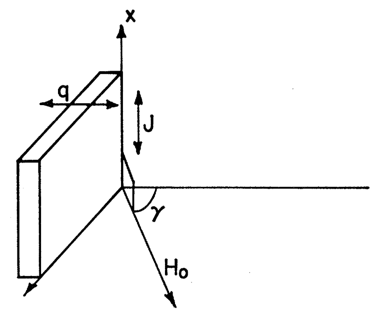

FIG. 4. Orientation for anisotropic predictions and measurements. $\mathbf{H}_0 \perp \mathbf{J}_{osc}$ for all values of γ.

where $\rho = \epsilon / 2\pi T$ and $\psi^{(2)}$ is the tetragamma function. We note that this expression for Δ^2, which may be deduced from Eilenberger's work and which is also used by Pesch,[11] differs somewhat from the Maki-Tsuzuki[27] expression for Δ^2. For T, we use 4.2°K.

Using these expressions we plot in Fig. 2 theoretical values for R_s/R_n as a function of magnetic field near H_{c2} for $\mathbf{H}_0$ parallel to $\mathbf{q}$, the direction of decay of the screening current. The experimental data points are taken from Ref. 14. The theoretical curve with no delocalization ($\delta = 1.0$) is not in good agreement, since when plotted as a function of magnetic field it shows a sharper drop off than is observed experimentally and then a leveling off. This leveling off may not be completely valid theoretically because Caroli and Maki proved the cancellation of the imaginary part of the conductivity only to order Δ^2. As we move away from H_{c2}, Imσ may become nonzero and this would decrease the surface resistance. The curve with the delocalization factor of $\delta = 0.07$ is in much better agreement. We also illustrate in this figure the CM prediction using the numerical parameters listed previously. With these parameters the Maki theory does not agree quantitatively with experiment although the functional form agrees well with the experimental results.

The anisotropic results for the surface resistance are given in Fig. 3 with the orientation described in Fig. 4. These anisotropic results were obtained[28] by expanding $N_1(\omega,\theta,u)$ and $N(\omega,\theta,u)$ to linear order in $\sin^2\theta$. This is of sufficient accuracy since we are primarily concerned with the qualitative anisotropic behavior and it simplifies the numerical calculations considerably.

As before the theoretical curve with a delocalization factor is in qualitative agreement with the experimental results, which in this case is in sharp contrast to the Caroli-Maki theory. The theoretical angular dependence with no delocalization effects is dependent upon the value of the frequency as can be seen. For high frequencies R_s/R_n decreases as the angle γ increases while for frequencies of the order of one megahertz R_s/R_n remains constant. The frequency at which R_s/R_n begins to decrease if no delocalization effects are included depends upon the value of the Fermi velocity, and for smaller values of v_F the higher this frequency

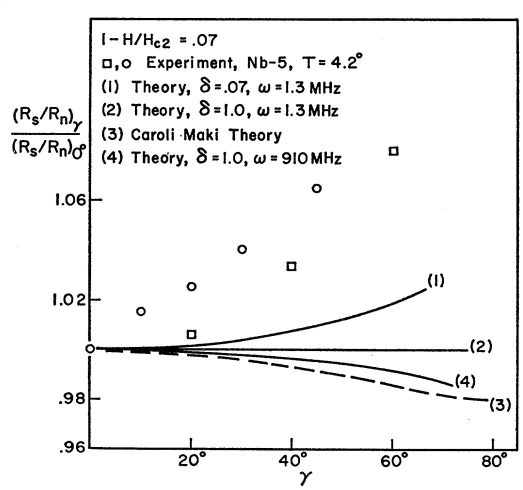

FIG. 3. Angular dependence of surface resistance for $\mathbf{H}_0 \perp \mathbf{A}_\omega$ and $1 - H/H_{c2} = 0.07$. The angle γ is the angle between $\mathbf{H}_0$ and $\mathbf{q}$ ($\mathbf{q}$ is the propagation vector of the perturbing field).

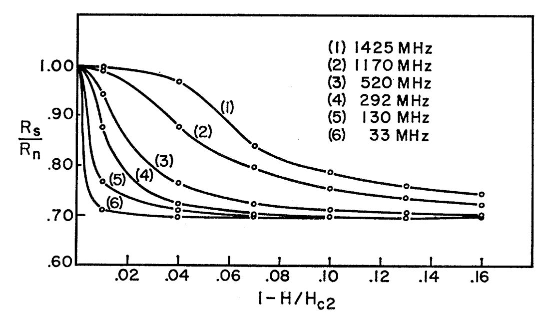

FIG. 5. Magnetic-field dependence of the normalized surface resistance for frequencies above 1 MHz. The orientation is $\mathbf{H}_0 \perp \mathbf{A}_\omega$, $\mathbf{H}_0 \| \mathbf{q}$; and the Fermi velocity is taken to be $v_F = 3.10^7$ cm/sec. The small circles were numerically computed from Eq. (29) and smooth curves were drawn through them.

[27] K. Maki and T. Tsuzuki, Phys. Rev. **139**, 868 (1965).

[28] W. D. Hibler, III, Thesis, Cornell University, 1969 (unpublished).

will be. At high enough frequencies, even with a delocalization factor added, we would expect R_s/R_n to decrease as a function of the angle γ.

The demagnetization factor has not been included in the theoretical curves but from the analysis of Cape and Zimmerman[29] it may easily be included by a change in the magnetization. For the numerical values of $K_2(T)$ used it will be a small effect. If included it would tend to cause the CM curve to disagree a little more strongly with experiment (i.e., decrease more as γ increases) and would cause the small delocalization factor curve to be slightly flatter than it appears in Fig. 3.

The final numerical result is the surface resistance as a function of frequency. The field sweep curves of R_s/R_n for different frequencies are given in Fig. 5. The change in the surface resistance is quite marked and above frequency $\omega_1=\alpha\Delta^2[\alpha=1/v\sin\theta(2eH)^{1/2}]$ the surface resistance begins to flatten off at H_{c2} and does not begin to decrease appreciably until $H<H_{c2}$. The cutoff frequency ω_1 is quite dependent upon the Fermi velocity so that it is not easy to predict it exactly although we expect it to be in the neighborhood of 500 MHz. This frequency dependence is not predicted by the CM theory as the dependence upon frequency is of the form $\ln(\Delta/\omega)$ and no sharp changes occur.

Concerning the temperature dependence, as can be seen from Eq. (32) for Δ^2 and Eq. (30), the theory predicts that at high frequencies R_s/R_n increases as T decreases. This follows because, for a fixed value of $1-H/H_{c2}$, Δ decreases with decreasing temperature.[30] At low frequencies around 1 MHz there is hardly any temperature dependence. The temperature dependence of the theory with a finite delocalization factor δ varies depending on the magnitude of δ. Since the amount of delocalization may be dependent upon temperature, it is difficult to make a definite statement about temperature dependence in this case.

IV. CONCLUSION

In conclusion, our theory predicts a strong frequency dependence of the conductivity and hence of the surface resistance for frequencies of the order of 500 MHz. These predictions at frequencies above 500 MHz are especially important because we expect the pure limit assumption, $\omega\tau>1$, to hold. It should be possible to observe this frequency dependence experimentally using many pure niobium samples.

At frequencies of the order of 1 MHz it is difficult to obtain agreement with experimental field dependence and angular measurements without making arbitrary changes in the parameters put in the theory. In particular if a slower variation of the order parameter is assumed, the low-frequency agreement is greatly improved. In this way one may obtain at least qualitative agreement with certain experimental measurements that at present are not explained by other theories. We point out, however, that the quantitative disagreement between our theory and experiments done in the megahertz region is not surprising because the limit $\omega\tau>1$, assumed in the theory, is probably not valid at such frequencies in the samples used.

ACKNOWLEDGMENTS

We would like to thank Professor V. Ambegaokar for many helpful discussions. Thanks go also to Professor B. Maxfield for discussions on experimental measurements and to Professor J. Wilkins for reading this manuscript.

APPENDIX

We wish to show here that as a lowest-order approximation we may neglect $G_\omega{}^B(\mathbf{p},\mathbf{K})$, for $\mathbf{K}\neq 0$, relative to $G_\omega{}^B(\mathbf{p},0)$. Without this approximation the calculation of the conductivity would be rather intractable. The procedure for proving the smallness of $G_\omega{}^B(\mathbf{p},\mathbf{K})$ is very similar to the approach used by BPT to obtain the density of states. The important difference is that BPT were only concerned with obtaining an expression for $G_\omega{}^B(\mathbf{p},0)$, and did not discuss $G_\omega{}^B(\mathbf{p},\mathbf{K})$.

To obtain an expression for $G_\omega{}^B(\mathbf{p},K)$, we Fourier transform Eq. (6) with respect to the sum and difference variables which yields

$$\begin{aligned}G_\omega{}^B(\mathbf{p},-\mathbf{K})=\delta_{\mathbf{K},0}G_\omega{}^0(\mathbf{p})-G_\omega{}^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K})\\ \times\sum_{\mathbf{K}'}G_\omega{}^B(\mathbf{p}-\mathbf{K}'/2,-\mathbf{K}-\mathbf{K}')\\ \times\int\frac{d^3p'}{(2\pi)^3}V(\mathbf{p}',\mathbf{K}')\\ \times G_{-\omega}{}^0(\mathbf{p}-\mathbf{p}'-\mathbf{K}/2-\tfrac{1}{2}\mathbf{K}'),\quad\text{(A1)}\end{aligned}$$

where $\mathbf{K}=\mathbf{n}(2\pi)^{1/2}\Lambda^{-1}$ is a reciprocal vortex lattice vector. Inserting Eq. (8) for the order parameter into Eq. (7), one finds the Fourier transform of the quantity $V(\mathbf{l},\mathbf{l}')$ to be given by

$$\begin{aligned}V(\mathbf{p}',\mathbf{K}')=\Delta^2\Lambda^2 2(2\pi)^2\delta(p_z')\exp[-\Lambda^2(p_x'^2+p_y'^2)]\\ \times\exp[i\Lambda^2(-p_x'K_y'+p_y'K_x'+\tfrac{1}{2}K_x'K_y')],\quad\text{(A2)}\end{aligned}$$

where we have assumed a square lattice of flux lines.

We note, as pointed out by BPT, that the order of magnitude of the coefficients multiplying the BPT Green's functions on the right-hand side of Eq. (A1) is determined by the magnitude of the p' integral, and from Eq. (A2) it is clear that the integrand is cut off at about $p'\geq\Lambda^{-1}$. On the other hand since $K\geq(2\pi)^{1/2}\Lambda^{-1}$, the order of magnitude of the p' integral is determined

[29] J. A. Cape and J. M. Zimmerman, Phys. Rev. **153**, 416 (1967).

[30] The temperature dependence of Δ^2 is not obvious from Eq. (32) because it turns out numerically that $g_1(H,T,l)$, at $H=H_{c2}$, varies on the order of T^2 with temperature. However, dM/dH (near H_{c2}) becomes smaller at lower temperatures which causes a net decrease in Δ^2 for a fixed value at $1-H/H_{c2}$.

by the expression

$$\int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}',\mathbf{K}') = \Delta^2 \exp[-\Lambda^2(\tfrac{1}{4}K'^2 - \tfrac{1}{2}iK_x'K_y')] , \quad \text{(A3)}$$

provided that $\mathbf{K}+\mathbf{K}' \neq 0$. Since for $\mathbf{K}' \neq 0$ and $\mathbf{K}+\mathbf{K}' \neq 0$, the coefficients are exponentially small, they may be neglected and therefore the dominant terms in the equation for $G_\omega^B(\mathbf{p}, -\mathbf{K})$ for $\mathbf{K} \neq 0$ are

$$\begin{aligned} G_\omega^B(\mathbf{p}, -\mathbf{K}) = & -G_\omega^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K}) G_\omega^B(\mathbf{p}, -\mathbf{K}) \\ & \times \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}',0) G_{-\omega}^0(\mathbf{p}-\mathbf{p}'-\tfrac{1}{2}\mathbf{K}) \\ & -G_\omega^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K}) G_\omega^B(\mathbf{p}+\tfrac{1}{2}\mathbf{K},0) \\ & \times \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}', -\mathbf{K}) G_{-\omega}^0(\mathbf{p}-\mathbf{p}') . \quad \text{(A4)} \end{aligned}$$

Solving for $G_\omega^B(\mathbf{p}, -\mathbf{K})$, we obtain

$$\begin{aligned} G_\omega^B(\mathbf{p}, -\mathbf{K}) = & -G_\omega^B(\mathbf{p}+\mathbf{K}/2, 0) G_\omega^B(\mathbf{p}-\mathbf{K}/2, 0) \\ & \times \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}', -\mathbf{K}) G_{-\omega}^0(\mathbf{p}-\mathbf{p}') , \quad \text{(A5)} \end{aligned}$$

where we have used the result

$$G_\omega^B(\mathbf{p}-\tfrac{1}{2}\mathbf{K}, 0) = \frac{G_\omega^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K})}{1+G_\omega^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K}) \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}',0) G_{-\omega}^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K}-\mathbf{p}')} . \quad \text{(A6)}$$

[This last equation defines the BPT Green's function and is obtained from Eq. (A1) by solving for $G_\omega^B(\mathbf{p}-\mathbf{K}/2, 0)$ and using the approximations mentioned above.]

From Eq. (A5) we may obtain an estimate of the magnitude of $G_\omega^B(\mathbf{p}, -\mathbf{K})$ for the discrete complex frequencies $\omega = \zeta_l$. To do this we note that choosing the p' coordinate system so that the p_z' axis is parallel to the magnetic field and the momentum $\mathbf{p}-\frac{1}{2}\mathbf{K}$ is in the $p_z'p_x'$ plane, we easily find that

$$\begin{aligned} & \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}',0) G^0(\mathbf{p}-\tfrac{1}{2}\mathbf{K}-\mathbf{p}', -\zeta_l) \\ & = \frac{i\pi^{1/2}\Delta^2\Lambda}{v \sin\theta} \left[\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{dt e^{-t^2}}{[\zeta_l + \xi(\mathbf{p}-\tfrac{1}{2}\mathbf{K})](\Lambda/v \sin\theta) - t} \right] , \quad \text{(A7)} \end{aligned}$$

where θ is the angle between the momentum $\mathbf{p}$ and the magnetic field, and $v = v_F$. For $\omega = \zeta_l$ we have used the notation $G_\omega^B(\mathbf{p}, -\mathbf{K}) = G^B(\mathbf{p}, -\mathbf{K}, \zeta_l)$. Substituting Eq. (A7) into Eq. (A8) reproduces the explicit form of the BPT Green's function as given in Eq. (20). The imaginary part of the quantity in Eq. (A7) always has the same sign as the sign of $\text{Im}\zeta_l$, so that an upper bound for the magnitude of $G^B(\mathbf{p},0,\zeta_l)$ is

$$\begin{aligned} |G^B(\mathbf{p},0,\zeta_l)| & \leq 1 \Big/ \left| \zeta_l + \int \frac{d^3p'}{(2\pi)^3} V(p',0) G^0(\mathbf{p}-\mathbf{p}', -\zeta_l) \right| \\ & \leq 1/|\zeta_l| \sim 1/\pi T . \quad \text{(A8)} \end{aligned}$$

Furthermore, using Eq. (A2), we see that

$$\begin{aligned} & \left| \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}', -\mathbf{K}) G^0(\mathbf{p}-\mathbf{p}', -\zeta_l) \right| \\ & \leq \int \frac{d^3p'}{(2\pi)^3} |V(\mathbf{p}', -\mathbf{K})| \, |G^0(\mathbf{p}-\mathbf{p}', -\zeta_l)| \\ & \leq \int \frac{d^3p'}{(2\pi)^3} \frac{V(\mathbf{p}',0)}{|\zeta_l|} = \Delta^2/|\zeta_l| . \quad \text{(A9)} \end{aligned}$$

Using the inequalities (A8) and (A9), and Eq. (A5) we have

$$|G^B(\mathbf{p}, -\mathbf{K}, \zeta_l)| \leq \frac{\Delta^2}{|\zeta_l|^2} |G^B(\mathbf{p}+\mathbf{K}/2, 0, \zeta_l)| . \quad \text{(A10)}$$

Consequently, since Δ/T is small near H_{c2}, the Green's function $G^B(\mathbf{p}, -\mathbf{K}, \zeta_l)$ is small compared to $G^B(\mathbf{p}+\mathbf{K}/2, 0, \zeta_l)$.

We note that in general $G^B(\mathbf{p}, -\mathbf{K}, \zeta_l)$ may be even smaller than indicated by the inequality (A10). This is because the maxima of the Green's function $G_\omega^B(\mathbf{p},0)$ occurs for $p \sim p_F$ so that $G_\omega^B(\mathbf{p}+\frac{1}{2}\mathbf{K}, 0)$ and $G_\omega^B(\mathbf{p}-\frac{1}{2}\mathbf{K}, 0)$ have their maxima displaced in momentum space by $K > (2\pi)^{1/2}/\Lambda$. Therefore one of the Green's functions will be of the order $\Lambda/v(2\pi)^{1/2}$ if $\Lambda/v(2\pi)^{1/2} < (2\pi T)^{-1}$ which for $T \sim 4.2°$ is the case for pure niobium.

As shown above, once Eq. (A5) is obtained, it may easily be proven that $G_\omega^B(\mathbf{p},\mathbf{K})$ is quite small compared to $G_\omega^B(\mathbf{p},0)$. The approximations made in obtaining Eq. (A5), on the other hand, are the same as employed by BPT in obtaining an expression for the density of states. Consequently to be consistent within the BPT approximation whenever one uses $G_\omega^B(\mathbf{p},0)$ as given in Eq. (20) for example, one must also neglect $G_\omega^B(\mathbf{p},\mathbf{K})$ for $\mathbf{K} \neq 0$.