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Lindhard Dielectric Function in the Relaxation-Time Approximation*

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It is well known that the effects of collisions in a degenerate electron gas cannot be taken into account merely by replacing ω by $\omega+i/\tau$, in the collisionless longitudinal dielectric constant of Lindhard[1]:

$$\epsilon^0(\mathbf{q},\omega)=1+(4\pi e^2/q^2)B(\mathbf{q},\omega)\,, \tag{1}$$

$$B(\mathbf{q},\omega)=\int\frac{d\mathbf{p}}{4\pi^3}\frac{f_{\mathbf{p}+\mathbf{q}/2}-f_{\mathbf{p}-\mathbf{q}/2}}{\omega-\epsilon_{\mathbf{p}+\mathbf{q}/2}+\epsilon_{\mathbf{p}-\mathbf{q}/2}}\,. \tag{2}$$

This is because such an extension is tantamount to treating the collisions in a relaxation-time approximation that fails to conserve local electron number. The simplest way to remove this defect is to use a relaxation-time approximation in which the collisions relax the electronic density matrix ρ not to its uniform equilibrium value, but to a local equilibrium density matrix $\bar{\rho}$:

$$\begin{gathered}\bar{\rho}=1/[e^{\beta(\epsilon-\mu-\delta\mu)}+1]\,,\\ \langle\mathbf{k}|\bar{\rho}|\mathbf{k}'\rangle=f_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}-\frac{f_{\mathbf{k}}-f_{\mathbf{k}'}}{\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}}\delta\mu(\mathbf{k}-\mathbf{k}')+o(\delta\mu)^2.\end{gathered} \tag{3}$$

The dielectric constant resulting from this simple extension of the random-phase approximation is apparently not widely known[2]; in particular, a recent paper attempts to construct it by guessing the generalization of the apparently better known classical dielectric constant (as calculated with a number conserving relaxation-time approximation to the Boltzmann equation).[3] Without going into the more fundamental question of whether any relaxation-time approximation (or, for that matter, any model which deals with collisions between well defined quasiparticles) can be trusted to describe the effects of collisions on the short-wavelength Friedel oscillations, it nevertheless seems important to point out that the problem posed in Ref. 3 does have a unique solution, which is not the one suggested there.

The correct procedure is to calculate the first-order density matrix $\rho^{(1)}$ by adding to the collisionless equation of motion a term that relaxes it to the local equilibrium distribution (3):

$$\begin{aligned}\omega\langle\mathbf{k}|\rho^{(1)}|\mathbf{k}'\rangle={}&(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'})\langle\mathbf{k}|\rho^{(1)}|\mathbf{k}'\rangle+(f_{\mathbf{k}'}-f_{\mathbf{k}})U(\mathbf{k}-\mathbf{k}')\\ &-\frac{i}{\tau}\left(\langle\mathbf{k}|\rho^{(1)}|\mathbf{k}'\rangle+\frac{f_{\mathbf{k}}-f_{\mathbf{k}'}}{\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}}\delta\mu(\mathbf{k}-\mathbf{k}')\right).\end{aligned} \tag{4}$$

Here U is the total electrostatic potential energy. Equation (4) asserts that electrons evolve in time as free particles in the presence of U, except that in an interval dt, a fraction dt/τ of them collide, immediately after which they are distributed according to the local equilibrium density matrix (3). The local chemical potential is determined by requiring that number be locally conserved:

$$\begin{gathered}\omega\delta n(\mathbf{q},\omega)=\mathbf{q}\cdot\delta\mathbf{j}(\mathbf{q},\omega)\,,\\ \begin{Bmatrix}\delta n(\mathbf{q},\omega)\\ \delta\mathbf{j}(\mathbf{q},\omega)\end{Bmatrix}=\int\frac{d\mathbf{p}}{4\pi^3}\langle\mathbf{p}+\tfrac{1}{2}\mathbf{q}|\rho^{(1)}|\mathbf{p}-\tfrac{1}{2}\mathbf{q}\rangle\begin{Bmatrix}1\\ \mathbf{p}/m\end{Bmatrix}.\end{gathered} \tag{5}$$

It follows from (4) that if (5) is to be satisfied then

$$\delta\mu(\mathbf{q})=\delta n(\mathbf{q},\omega)/B(\mathbf{q},0)\,. \tag{6}$$

With the aid of (6), it is not difficult to verify that the solution to (4) implies

$$\delta n(\mathbf{q},\omega)=\frac{B(\mathbf{q},\omega+i/\tau)U(\mathbf{q})}{1-[1/(1-i\omega\tau)][1-B(\mathbf{q},\omega+i/\tau)/B(\mathbf{q},0)]}\,. \tag{7}$$

Since $\epsilon(\mathbf{q},\omega)=1+(4\pi e^2/q^2)\delta n(\mathbf{q},\omega)/U(\mathbf{q})$,

$$\epsilon(\mathbf{q},\omega)=1+\frac{(1+i/\omega\tau)(\epsilon^0(\mathbf{q},\,\omega+i/\tau)-1)}{1+(i/\omega\tau)(\epsilon^0(\mathbf{q},\,\omega+i/\tau)-1)/(\epsilon^0(\mathbf{q},0)-1)}\,. \tag{8}$$

The dielectric constant arrived at in Ref. 3 [Eq. (39)] is just (8) with the denominator replaced by its classical limit. It is not easy to isolate a single error in a frankly conjectural analysis, but it appears to have been overlooked that the explicit q dependence in the denominator of the classical result [Ref. 3, Eq. (25)] requires generalization along with the other terms, in reconstructing the quantum result, since it can be traced [from (8)] back to

$$\lim_{\hbar\to 0}\epsilon^0(q,0)=1+3\omega_p{}^2/q^2v_F{}^2. \tag{9}$$

Equation (8) has the limiting properties (i)–(v) listed in Ref. 3, but also yields the static limit

$$\epsilon(\mathbf{q},0)=\epsilon^0(\mathbf{q},0)\,. \tag{10}$$

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[1] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **28**, No. 8 (1954).

[2] See N. W. Ashcroft and N. D. Mermin (unpublished) for an elementary discussion.

[3] K. L. Kliewer and Ronald Fuchs, Phys. Rev. **181**, 552 (1969).

In particular, the singularity in the static dielectric constant at $q=2k_F$ is unaltered by collisions in the relaxation-time approximation. This is not surprising, since the static limit of any calculation in the relaxation-time approximation only yields whatever has been put into it—in this case, Eq. (3), which already implies this singularity. The extent to which collisions might soften the singularity in the equilibrium distribution requires an analysis going well beyond the simple phenomenology of either this paper or Ref. 3.[4]

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[4] Such an analysis has been given by P. G. de Gennes, J. Phys. Radium **23**, 630 (1962).

Erratum

Self-Consistent Many-Electron Theory of Electron Work Functions and Surface Potential Characteristics for Selected Metals, John R. Smith [Phys. Rev. **181**, 522 (1969)]. Table I contains a typographical error. The designation "Neglecting correlation energies" belongs only with the row of results 0.978, ~1.