Micromagnetics at High Temperature

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The method of micromagnetics is extended to the cases in which the magnitude of the magnetization cannot be considered constant throughout a ferromagnetic body. The nonlinear equations, corresponding to Brown's equations in the standard treatment of micromagnetics, have been deduced with the proper boundary conditions, as well as their linearized form suitable for evaluation of the nucleation field and mode. Some relationships have been found between the nucleation field deduced in this way and the nucleation field for constant magnitude, emphasizing that a detailed knowledge of the magnetic equation of state of the material is needed for the determination of the true nucleation field. The case of a boundless plate with an applied field normal to its boundary planes has been considered in detail, and the existence of one nucleation mode, termed as "waving," peculiar to this treatment, has been recognized; the conditions for this mode of reversal have been deduced, with the result that it can be expected near the Curie temperature.

1. INTRODUCTION

THE purpose of micromagnetics[1–3] is a description of ferromagnetic bodies by means of a vector field with constant magnitude and with direction varying continuously with position, this vector field representing the local magnetization. This description can be easily accepted at temperatures significantly lower than the Curie temperature, because in this range the susceptibility χ of the ferromagnetic material can be disregarded in a first approximation, and consequently the magnitude of the magnetization is determined only by the temperature. However, when the temperature is raised to a value near to the Curie point, the remanent magnetization decreases and the susceptibility increases, so that it becomes impossible to neglect the change in magnitude of the local magnetization due to the magnetic fields and to the exchange forces.

This paper extends the method of micromagnetics to the case in which the magnetization is described by a vector field $\mathbf{M}$, whose both direction and magnitude vary continuously with position. The other limitations of micromagnetics—namely, the disregard of thermal fluctuations, atomic structure, and magnetoelastic effects—will affect also the results of this work. Probably thermal fluctuations can play a more relevant role in the neighborhood of the Curie point than in the range of validity of standard micromagnetics; nevertheless, the theory presented here can help in evaluating their relevance, because it can be extended consistently to the case of local disuniformities in temperature distribution.

The new formulation of micromagnetics will be presented with particular emphasis on comparison with the standard treatment.[1–3] In particular, a new form of Brown's equation will be deduced without restriction on the magnitude of the magnetization; the theorems on overconstrained and underconstrained solutions will be proved also for this new formulation; finally, a complete treatment of the nucleation modes in a boundless plate with easy axis normal to its boundary planes will be given.

2. EXTENDED BROWN'S EQUATIONS

As in standard micromagnetics, the fundamental equations will be deduced by means of a variational procedure, which minimizes the total energy; this total energy will again be assumed to consist of four terms, namely, (a) the exchange energy, (b) the self-magnetostatic energy, (c) the magnetocrystalline energy, and (d) the energy of interaction with the external field.

If we designate as $d\tau$ the volume differential element of our ferromagnetic body and we understand that all integrals are extended to its whole volume unless otherwise stated, it follows from the elementary treatment of magnetostatics that the energy of interaction with the external field $\mathbf{H}$ is

$$E_H = -\int \mathbf{M}\cdot\mathbf{H}\,d\tau \tag{1}$$

and the self-magnetostatic energy E_m is given (in CGS system) by

$$E_m = -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{H}'\,d\tau = \frac{1}{8\pi}\int_{\text{whole space}} H'^2\,d\tau, \tag{2}$$

where $\mathbf{H}'$ is the magnetic field created by the free poles of the magnetization.

The exchange energy is defined of course for finite distance between nearest atoms or ions, but it has been already modified to a form suitable for the continuum approximation in the case of constant magnetization magnitude.[3] In Appendix A a new proof is given, without any restriction on the constancy of the magnetization amplitude, and it yields the result

$$E_x = \tfrac{1}{2}\alpha\int[(\text{grad}M_x)^2+(\text{grad}M_y)^2+(\text{grad}M_z)^2]\,d\tau. \tag{3}$$

[1] W. F. Brown, Jr., *Micromagnetics* (John Wiley & Sons, Inc., New York, 1963).
[2] S. Shtrikman and D. Treves, in *Magnetism*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1963), Vol. III, p. 395.
[3] W. Döring, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1966), Vol. XVIII/2, p. 341.

Constant[4] α is equal to C/M^2 [1] or to $2A/M^2$ [2,3] in the standard formulation of micromagnetics, and it will be, in a first approximation, dependent only upon the material; in Appendix A the more general formula, allowing for an explicit dependence of α upon the magnetization and the position, is also deduced. The general validity of the expression for E_x can however be stated in other ways.[5,6]

The magnetocrystalline energy can be defined in terms of an energy density depending only on **M**, as in the standard treatment, and we can write

$$E_k=\int\omega(\mathbf{M})d\tau. \tag{4}$$

Function ω, however, takes into account all contributions to the free energy depending by the local value of **M**. For instance, in an isotropic Weiss ferromagnet with spin $\frac{1}{2}$,[7]

$$\omega(\mathbf{M})=\omega(M)=\frac{kT}{2\mu_B}\left((M_s+M)\ln\frac{M_s+M}{M_s}+(M_s-M)\ln\frac{M_s-M}{M_s}\right)-\frac{kT_c}{2\mu_B}\frac{M^2}{M_s}, \tag{5}$$

where M_s is the saturation magnetization, T_c is the Curie temperature, and μ_B is the Bohr magneton.

Having in mind the expressions obtained for the different contributions to the total energy, we determine the equilibrium condition in the usual way, by equating to zero the first variation with respect to **M** of the total energy E:

$$E=E_x+E_m+E_k+E_H, \tag{6}$$

without any supplementary condition for **M**. It is easy to find[8] that **M** must satisfy the equation

$$\alpha\nabla^2\mathbf{M}-\frac{\partial\omega}{\partial\mathbf{M}}+\mathbf{H}+\mathbf{H}'=0 \tag{7}$$

with the boundary condition

$$\frac{\partial\mathbf{M}}{\partial n}=0. \tag{8}$$

In (7) $\partial\omega/\partial\mathbf{M}$ is a vector whose components are $\partial\omega/\partial M_x$, $\partial\omega/\partial M_y$, and $\partial\omega/\partial M_z$ respectively; in (8) $\partial/\partial n$ denotes differentiation along the outward normal of the surface of the body.

[4] L. Landau and E. Lifshitz, Physik Z. Sowjetunion **8**, 153 (1935).
[5] C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).
[6] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Ltd., London, 1960), p. 159.
[7] R. Brout, in Ref. 2, Vol. II A, p. 43.
[8] L. E. Elsgolc, *Calculus of Variations* (Pergamon Press, Ltd., Oxford, 1961).

Equations (7) and (8), which do not require any restriction on the magnitude of **M**, are the replacement for Brown's equations at high temperature.

3. NUCLEATION

In the present case the formulation of the nucleation problem is slightly different from the standard way, because the magnetization is a function of the applied field also before the actual nucleation. For sake of simplicity let us restrict our considerations to an ellipsoidal body in a field parallel to one axis, and let this axis be parallel to one of the magnetocrystalline symmetry axes at all values of M. In this case, Eqs. (7) and (8) are satisfied by all uniform vector fields parallel to this axis, say z, which satisfy the condition

$$\left(\frac{\partial\omega}{\partial M_z}\right)_{M_x=M_y=0}=H_z-N_cM_z, \tag{9}$$

where N_c is the demagnetization factor in z direction. Equation (9) will generally allow only one solution for $T\geqslant T_C$, one or three solutions (one is unstable) for $T<T_C$; if H_z is large enough, only one solution is possible. Then we will define our problem in this way: We choose our ellipsoid with so large a (positive) value of H_z, that only one solution to Eq. (9) exists, and this is the only stable solution of Eqs. (7) and (8); and thereafter we decrease H_z until this solution becomes unstable; the value of H_z at which this happens will be defined as the nucleation field H_n. It is superfluous to recall to mind that Eq. (9) is exactly the one met in the elementary theory of uniformly magnetized bodies.

A necessary condition for the onset of instability is the existence of another solution of the fundamental equation, presenting a small deviation **m** (function of the position) from the uniform magnetization $\mathbf{M}_0(H_z)$ parallel to the positive z axis and satisfying Eq. (9) This small deviation must solve the differential form of Eqs. (7) and (8), namely,

$$\alpha\nabla^2\mathbf{m}-\mathrm{grad}_{\mathbf{M}}(\mathbf{m}\cdot\mathrm{grad}_{\mathbf{M}}\omega)+\mathbf{h}'=0, \tag{10}$$

where $\mathrm{grad}_{\mathbf{M}}$ has to be evaluated at $\mathbf{M}_0$, $\mathbf{h}'$ is the field produced by the poles of **m**, and

$$\frac{\partial\mathbf{m}}{\partial n}=0 \tag{11}$$

at the boundary.

Equations (10) and (11) are written in a simple form, but it is easier to discuss them after reducing Eq. (10) to a form more directly comparable with standard Brown's equation. With this goal, let us decompose ω into two fractions, namely,

$$\omega(\mathbf{M})=\omega_0(M)+\omega_1(\mathbf{M}), \tag{12}$$

where ω_0 is isotropic; of course, this decomposition is not unique, and we can impose the supplementary

condition

$$\omega_1(\mathbf{M})=0 \quad \text{for} \quad \mathbf{M}\times\mathbf{M}_0=0. \tag{13}$$

Remember that $\mathbf{M}_0$ is always parallel to the z axis due to the previous assumptions. Then the differential form of Eq. (7) takes the form

$$\alpha\nabla^2\mathbf{m}-\frac{\mathbf{M}_0\times(\mathbf{m}\times\mathbf{M}_0)}{M_0^3}\frac{d\omega_0}{dM}-\frac{(\mathbf{m}\cdot\mathbf{M}_0)\mathbf{M}_0}{M_0^2}\frac{d^2\omega_0}{dM^2}$$
$$-\mathrm{grad}_{\mathbf{M}}(\mathbf{m}\cdot\mathrm{grad}_{\mathbf{M}}\omega_1)+\mathbf{h}'=0. \tag{14}$$

This equation can be split into two more significant equations, by considering separately its components normal and parallel to $\mathbf{M}_0$; in accomplishing this separation we will take advantage of two consequences of Eqs. (12) and (13), i.e.,

$$\frac{\mathbf{M}_0}{M_0}\frac{d\omega_0}{dM}=\mathbf{H}+\mathbf{H}' \tag{15}$$

and

$$\mathbf{M}_0\cdot\mathrm{grad}_{\mathbf{M}}(\mathbf{m}\cdot\mathrm{grad}_{\mathbf{M}}\omega_1)=0 \tag{16}$$

independently of $\mathbf{m}$. We obtain the set of equations

$$\mathbf{M}_0\times[\alpha\nabla^2\mathbf{m}-\mathrm{grad}_{\mathbf{M}}(\mathbf{m}\cdot\mathrm{grad}_{\mathbf{M}}\omega_1)+\mathbf{h}']$$
$$+\mathbf{m}\times(\mathbf{H}+\mathbf{H}')=0, \tag{17}$$

$$\mathbf{M}_0\cdot\left[\alpha\nabla^2\mathbf{m}-\mathbf{m}\frac{d^2\omega_0}{dM^2}+\mathbf{h}'\right]=0, \tag{18}$$

$$\frac{\partial\mathbf{m}}{\partial n}=0 \quad \text{at the boundary.} \tag{19}$$

In Eq. (17) one can immediately recognize the differential form of Brown's equation; Eq. (18) reduces to

$$\mathbf{M}_0\cdot\mathbf{m}=0, \quad \text{i.e.,} \quad M=\text{const} \tag{20}$$

at the limit $d^2\omega_0/dM^2\to\infty$, which is equivalent to $\chi\to 0$. The equivalence of the present formulation with the standard one in the case of vanishing susceptibility has been definitely established, as well as the need for a more general treatment when the susceptibility cannot be neglected.

At this point it is interesting to look for some relationship between the nucleation field deduced by the standard form of Brown's equations and the form presented here. For this purpose it is convenient to assess the validity of the theorems about overconstrained and underconstrained solutions; the proofs run exactly parallel to the standard treatment, and the extension to the present case is written down in Appendix B. As in the standard treatment, the theorem about the overconstrained solutions states that, if we try to determine H_n after subjecting $\mathbf{m}$ to some additional constraint, we will find a value not larger (algebraically) than the true H_n; the theorem about the underconstrained solutions states that, if we try to determine H_n after dropping a non-negative contribution from the expression for the total energy, we will find a value not smaller (algebraically) than the true H_n.

Then let us consider the constraint

$$\mathbf{m}\cdot\mathbf{M}_0=0. \tag{21}$$

Equation (19) reduces to

$$\mathbf{M}_0\cdot\mathbf{h}'=0. \tag{22}$$

This restriction is always satisfied by coherent rotation and curling. Therefore, we can infer that the highest instability field arising from coherent rotation and curling in standard Brown's equations is surely a lower boundary to the true nucleation field. An obvious corollary is: If the nucleation mode deduced through Brown's equations is either coherent rotation or curling, then the corresponding nucleation field is (algebraically) not larger than the true nucleation field.

Then let us drop from the expression of E the non-negative term

$$\frac{1}{8\pi}\int_{\text{space}} h_z'^2 d\tau;$$

then Eq. (17) remains unchanged and Eq. (18) becomes

$$\nabla^2(\mathbf{m}\cdot\mathbf{M}_0)-\frac{1}{\alpha}\frac{d^2\omega_0}{dM^2}(\mathbf{m}\cdot\mathbf{M}_0)=0. \tag{23}$$

Note that M_0 is a monotonic increasing function of H_z; $d^2\omega_0/dM^2$ is $1/\chi$ and will be an increasing function of M, because it generally becomes more difficult to change the amplitude M when this amplitude is larger. Therefore the highest value of $d^2\omega_0/dM^2$ satisfying Eq. (23) is the one corresponding to the highest M_0 and consequently to H_n. However Eq. (23) will be certainly satisfied if

$$\mathbf{m}\cdot\mathbf{M}_0=0; \tag{24}$$

for this reason we can conclude that the nucleation field deduced by Brown's equations is an upper boundary to the true nucleation field, if it corresponds to a value of $(1/\alpha)d^2\omega_0/dM^2$, higher than the highest eigenvalue of Eq. (23) for our ellipsoid. On the other hand, all these eigenvalues are negative, and we can establish the corollary: If the nucleation field deduced through Brown's equations corresponds to a non-negative value of $d^2\omega_0/dM^2$, then it is (algebraically) not lower than the true nucleation field.

From these two statements we reach immediately the conclusion: If the nucleation mode deduced through Brown's equations is either coherent rotation or curling and the associated nucleation field corresponds to a positive value of $d^2\omega_0/dM^2$, then this field is the true nucleation field, and the found mode is the true nucleation mode.

As a last particular case, let us impose the constraint

$$\mathbf{m}\times\mathbf{M}_0=0, \quad m_z \text{ const.} \tag{25}$$

This mode is a coherent inversion, and it occurs when

$$\frac{d^2\omega_0}{dM^2}=-N_c. \tag{26}$$

Therefore we can infer that the nucleation field is (algebraically) not lower than the field for which Eq. (26) is satisfied. Of course this holds only when $-N_c$ is in the field of variation of $d^2\omega_0/dM^2$.

At this point let us go more deeply into a specific case, namely the case of a shape for which the nucleation mode calculated at constant M is either coherent rotation or curling. This case, however, is not very restricted, because it is well known[9] that this is the situation occurring for all oblate spheroids. If the nucleation takes place at room temperature through a mode with

$$m_z=0, \tag{27}$$

we must expect that this will be the case also at higher T, unless $d^2\omega_0/dM^2$ becomes negative for the deduced H_n. Now we will assume a power dependence of the remanent magnetization upon T_C-T and a power dependence of the anisotropy constant K upon M; then we deduce for the calculated (negative) H_n an expression proportional to $(T_C-T)^a$; we can consistently assume for the (negative) field corresponding to

$$\frac{d^2\omega_0}{dM^2}=0 \tag{28}$$

an expression proportional to $(T_C-T)^b$. Then, if $a>b$ the nucleation will remain the same till the Curie point, but if $b<a$ we must expect a transition to another nucleation mode at some temperature. This will be actually the case for materials with $K\propto M^2$ and with equation of state according to Weiss's model: In fact, in this case exponent a turns out to be $\frac{1}{2}$ and exponent b turns out to be $\frac{3}{2}$; this quadratic dependence of K corresponds to an anisotropy due mainly to shape factors at microscopic level.

4. BOUNDLESS PLATE

The determination of all possible modes of instability for the uniform magnetization in a boundless plate subjected to an external field normal to its boundary planes has been performed for the standard form of Brown's equations,[10] with the resulting conclusion that the nucleation mode is coherent rotation. This conclusion could be reached by means of the simple remark that coherent rotation is an actual solution, and it is also *with the same applied field* the underconstrainted solution after dropping the exchange and self-magnetostatic terms. Nevertheless, determining the nucleation field in our more general case requires a complete discussion of the fundamental equations, Eqs. (10) and (11), and moreover a knowledge of the equation of state for the material under consideration.

We consider a boundless plate, limited by planes $z=l$ and $z=-l$, with magnetocrystalline symmetry axis parallel to z axis, namely,

$$\omega_1=R(M_x^2+M_y^2), \tag{29}$$

with R, defined as K/M^2 in the usual notation, possibly dependent on M. By indicating as U the magnetostatic potential, the set of equations to be solved is

$$\alpha M_0\nabla^2 m_x-[(2R-4\pi)M_0=H_z]m_x-M_0\frac{\partial U}{\partial x}=0,$$

$$\alpha M_0\nabla^2 m_y-[(2R-4\pi)M_0+H_z]m_y-M_0\frac{\partial U}{\partial y}=0,$$

$$\alpha\nabla^2 m_z-\frac{d^2\omega_0}{dM^2}m_z-\frac{\partial U}{\partial z}=0,$$

$$\nabla^2 U-4\pi\left(\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}+\frac{\partial m_z}{\partial z}\right)=0,$$

$$\text{for} \quad -l<z<l \tag{30}$$

$$\nabla^2 U=0, \quad \text{for} \quad z<-l \quad \text{and} \quad z>l \tag{31}$$

$$\frac{\partial m_x}{\partial z}=\frac{\partial m_y}{\partial z}=\frac{\partial m_z}{\partial z}=0,$$

U is continuous,

$$\left(\frac{\partial U}{\partial z}\right)_{\text{out}}=\left(\frac{\partial U}{\partial z}\right)_{\text{in}}-4\pi(M_0+m_z),$$

$$\text{at} \quad z=-l \quad \text{and} \quad z=l. \tag{32}$$

Let us define the dimensionless quantities

$$\xi=x(4\pi/\alpha)^{1/2}, \quad \eta=y(4\pi/\alpha)^{1/2},$$
$$\zeta=z(4\pi/\alpha)^{1/2}, \quad S=l(4\pi/\alpha)^{1/2}, \tag{33}$$

$$h=\frac{R-2\pi}{2\pi}+\frac{H_z}{4\pi M_0}, \tag{34}$$

$$k=\frac{1}{4\pi}\frac{d^2\omega_0}{dM^2}, \tag{35}$$

and to replace the magnetostatic potential by

$$u=(4\pi\alpha)^{-1/2}(U+4\pi M_0 l), \quad \text{for} \quad z\leqslant -l$$
$$u=(4\pi\alpha)^{-1/2}(U-4\pi M_0 z), \quad \text{for} \quad -l\leqslant z\leqslant l$$
$$u=(4\pi\alpha)^{-1/2}(U-4\pi M_0 l), \quad \text{for} \quad z\geqslant l. \tag{36}$$

[9] A. Aharoni, Phys. Status Solidi **16**, 3 (1966).
[10] F. Forlani, N. Minnaja and G. Sacchi, IEEE Trans. Magnetics **4**, 70 (1968).

Introducing the operator ∇'^2 defined as

$$\nabla'^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}, \tag{37}$$

Eqs. (30)–(32) are rewritten as follows:

$$\nabla'^2 m_x-hm_x-\frac{\partial u}{\partial\xi}=0,$$

$$\nabla'^2 m_y-hm_y-\frac{\partial u}{\partial\eta}=0,$$

$$\nabla'^2 m_z-km_z-\frac{\partial u}{\partial\zeta}=0,$$

$$-\frac{\partial m_x}{\partial\xi}-\frac{\partial m_y}{\partial\eta}-\frac{\partial m_z}{\partial\zeta}+\nabla'^2 u=0,$$

$$\text{for}\quad -S<\zeta<S \tag{38}$$

$$\nabla'^2 u=0,\quad \text{for}\quad |\zeta|>S, \tag{39}$$

$$\frac{\partial m_x}{\partial\zeta}=\frac{\partial m_y}{\partial\zeta}=\frac{\partial m_z}{\partial\zeta}=0,$$

u is continuous,

$$\left(\frac{\partial u}{\partial\zeta}\right)_{\text{out}}=\left(\frac{\partial u}{\partial\zeta}\right)_{\text{in}}-m_z. \qquad \text{at}\quad \zeta=\pm S \tag{40}$$

From the statements of Sec. 3 we can deduce that coherent rotation will certainly be the nucleation mode and H_n will be equal to $(4\pi-2R)M_0$ if it results in

$$k>0\quad\text{for}\quad h=0. \tag{41}$$

We can also infer that the nucleation mode will certainly be different from coherent rotation and H_n will be larger than $(4\pi-2R)M_0$ if it results in

$$k<-1\quad\text{for}\quad h=0. \tag{42}$$

The complete discussion of the set of Eqs. (38)–(40) will confirm these first conclusions.

The solution of Eqs. (38) and (39) is a linear combination of expressions such as

$$m_x=Xe^{in\xi+ip\eta+q\zeta},$$

$$m_y=Ye^{in\xi+ip\eta+q\zeta},$$

$$m_z=Ze^{in\xi+ip\eta+q\zeta},$$

$$u=We^{in\xi+ip\eta+q\zeta},\quad\text{for}\quad |\zeta|<S \tag{43}$$

$$u=W_+e^{in\xi+ip\eta-t\zeta},\quad\text{for}\quad \zeta<S \tag{44}$$

$$u=W_-e^{in\xi+ip\eta+t\zeta},\quad\text{for}\quad \zeta<-S. \tag{45}$$

In Eqs. (43)–(45) n and p are real, t is given by

$$t=(n^2+p^2)^{1/2}. \tag{46}$$

and n, t, q are mutually related by the secular equation

$$(q^2-t^2-h)[(q^2-t^2-h)(q^2-t^2-k)(q^2-t^2)+t^2(q^2-t^2-k)-q^2(q^2-t^2-h)]=0. \tag{47}$$

Therefore, for fixed values of n and p, we find eight linearly independent solutions of Eqs. (38), and we must determine whether one linear combination of them satisfies all boundary conditions (40). Fortunately, we reduce the difficulty of this problem, because it is easy to verify that Eqs. (40) can be satisfied separately by linear combinations of the two solutions with q satisfying

$$q^2-t^2-h=0 \tag{48}$$

and by linear combinations of the six solutions with q satisfying

$$(q^2-t^2-n)(q^2-t^2-k)(q^2-t^2)+t^2(q^2-t^2-k)-q^2(q^2-t^2-h)=0. \tag{49}$$

The proof of this statement can be deduced in a straightforward way from the case with $m_z=0$.[10]

Let us consider first the case of q satisfying Eq. (48); all the instability modes have

$$m_z=0,\quad u=0, \tag{50}$$

and the boundary conditions for m_x and m_y can be satisfied only if

$$t^2+h=-r^2\pi^2/4S^2\quad(r=0,1,2,\cdots). \tag{51}$$

It is obvious that the highest instability field of these modes corresponds to

$$h=0, \tag{52}$$

and the resulting mode turns out to be coherent rotation.[10]

Then we go to the other case, namely, to q given by Eq. (49). A common feature to all these solutions is that

$$m_x:n=m_y:p, \tag{53}$$

and therefore we can always perform a rotation around the z axis in order to reduce p and m_y to 0. The system derived from Eqs. (38)–(40) after performing this rotation is very similar to the one solved approxmately by Muller[11] and numerically by Brown[12] for the case of a boundless plate with an applied field parallel to its boundary planes. After performing carefully the algebraic changes due to different coordinates and normalization, it is easy to recognize that the only difference lies in the fact, that in those papers p is not necessarily 0; however this difference is not significant, because it is also proved in the same papers that in the nucleation mode p takes value 0. Therefore we can take advantage of the work already performed and reach the following

[11] M. W. Muller, Phys. Rev. **122**, 1485 (1961).
[12] W. F. Brown, Jr., Phys. Rev. **124**, 1348 (1961).

TABLE I. Values of h and $-k$ at nucleation for different S, and the corresponding values of n.

S^2	h	$-k$	n
1	0.16595	0.83405	0.3369
	1.16411	0.83589	0.3345
	9.16072	0.83928	0.3296
	19.15995	0.84005	0.3290
	99.15927	0.84073	0.3275
	∞	0.8409	0.3272
2	0	>0.5	
	0.26304	0.73696	0.3795
	4.24915	0.75085	0.3711
	9.24636	0.75364	0.3694
	49.24390	0.75610	0.3667
	∞	0.75675	0.3661
10	0	>0.2	
	0.07304	0.42696	0.3564
	0.53474	0.46526	0.3762
	1.50972	0.49028	0.3832
	9.48656	0.51344	0.3839
	∞	0.51964	0.3835
20	0.17698	0.32302	0.3246
	0.63733	0.36267	0.3502
	4.59321	0.40679	0.3644
	9.58677	0.41323	0.3654
	∞	0.41985	0.3659
40	0.25913	0.24067	0.2884
50	0.034434	0.165566	0.2275
60	0.29838	0.20162	0.2733
100	0.009661	0.090339	0.1627
	0.080827	0.119173	0.2047
	0.33978	0.16022	0.2499
	0.81299	0.18701	0.2726
	9.77381	0.22619	0.2963
	∞	0.23200	0.2986
150	0.101769	0.098231	0.1912
200	0.002812	0.047188	0.1144
	0.035598	0.064402	0.1487
	0.38369	0.11631	0.2192
	0.86295	0.13705	0.2393
	9.8309	0.1691	0.2632
	∞	0.17401	0.2659
250	0.123129	0.076871	0.1743
300	0.047152	0.052848	0.1395
400	0.016474	0.033526	0.10625
500	0.000606	0.019394	0.07173
	0.058836	0.041164	0.1274
	0.145052	0.054948	0.1521
600	0.022542	0.027458	0.1000
1000	0.000207	0.009793	0.05050
	0.006245	0.013755	0.06748
	0.028660	0.021340	0.09155
	0.070720	0.029280	0.1112
	0.935448	0.064552	0.1711
	∞	0.085826	0.1944
1500	0.008748	0.011252	0.06341
2000	0.000076	0.004924	0.03560
	0.003059	0.006941	0.04776
	0.034857	0.015143	0.08007
	0.95369	0.04631	0.1465
	∞	0.06193	0.1673
2500	0.011268	0.008734	0.05844
3000	0.004326	0.005674	0.04515
4000	0.001512	0.003488	0.03377
5000	0.0000226	0.0019774	0.02244
	0.005599	0.004401	0.04145
	0.013813	0.006187	0.05111
6000	0.002150	0.002850	0.03196
10 000	0.0000096	0.0009904	0.01585
	0.0006002	0.0013998	0.02136
	0.002790	0.002210	0.02936
	0.006884	0.003116	0.03627
15 000	0.0008564	0.0011436	0.02022
20 000	0.0002991	0.0007009	0.01510
	0.003436	0.001564	0.02569
25 000	0.011136	0.000864	0.01858
30 000	0.0004275	0.0005725	0.01430
50 000	0.0005564	0.0004436	0.01314
	0.0013730	0.0006270	0.0163
100 000	0.0006862	0.0003138	0.01150

conclusions:

(a) If $S \gg 1$,[11] nucleation different from coherent rotation happens, if h and k are related approximately by

$$k = -\frac{\pi}{S}\left(\frac{h}{h+1}\right)^{1/2} \tag{54}$$

with $h>0$; the corresponding value of n is

$$n = \left(\frac{\pi}{2S}\right)^{1/2}\left(\frac{h}{h+1}\right)^{1/4}. \tag{55}$$

(b) Without any restriction on S,[12] a numerical calculation yields pairs (h,k) corresponding to nucleation. All pairs deduced by the previous calculations[9] are listed in Table I with the associated value of n. It is immediately seen that $-k$ is an increasing function of h for given S and a decreasing function of S for given h, bounded between 0 (for $S \to \infty$) and -1 (for $S \to 0$), as was expected from the general results of Sec. 3.

(c) The nucleation mode is periodic along x axis, and the period is $(\pi\alpha)^{1/2}/n$. For a given x the maximum of m_z is reached for $z=0$, where m_x vanishes linearly. For giving a qualitative idea of the behavior, we have evaluated an approximate solution at the limit $h \to \infty$,

$S\to\infty$:

$$m_x=\left(\frac{\pi}{2S}\right)^{1/2}\frac{1}{h}\left[\frac{\pi}{S}e^{-S}\sinh\zeta+\sin\frac{\pi\zeta}{2S}\right]\times\cos\left[\left(\frac{\pi}{2S}\right)^{1/2}\xi\right],$$

$$m_z=\left[\frac{\pi}{S}e^{-S}\cosh\zeta+\cos\frac{\pi\zeta}{2S}\right]\sin\left[\left(\frac{\pi}{2S}\right)^{1/2}\xi\right], \tag{56}$$

$$u=\left[\frac{\pi}{S}e^{-S}\sinh\zeta+\sin\frac{\pi\zeta}{2S}\right]\sin\left[\left(\frac{\pi}{2S}\right)^{1/2}\xi\right].$$

This mode can be indicated as "waving."

The determination of the nucleation field requires also a knowledge of the dependence of ω on $\mathbf{M}$ and on T, or, according to our assumptions, of ω_0 and R on M and on T. If this dependence is known, we can easily deduce an equation of state relating h and k, because

$$h=\frac{R}{2\pi}+\frac{1}{4\pi M}\left(\frac{\partial\omega_0}{\partial M}\right)_T, \tag{57}$$

$$k=\frac{1}{4\pi}\left(\frac{\partial^2\omega_0}{\partial M^2}\right)_T. \tag{58}$$

Then it is possible to plot in the same (h,k) plane, the curves deduced from the numerical calculations for various S and the curves given for the chosen material by Eqs. (57) and (58) for various temperatures: The intersections give the nucleation fields for the different values of S and of T.

For sake of clearness, let us sketch one simple example choosing ω_0 according to Weiss's model with spin $\frac{1}{2}$.[7] If N_W is the Weiss constant, we can write

$$h=\frac{R}{2\pi}+\frac{N_W}{4\pi}\left[\frac{M_sT}{2MT_c}\ln\left(\frac{M_s+M}{M_s-M}\right)-1\right], \tag{59}$$

$$k=-\frac{N_W}{4\pi}\left(1-\frac{TM_s^2}{T_c(M_s^2-M^2)}\right). \tag{60}$$

It is immediately seen that the term in brackets in the right-hand side of Eq. (59) is negative if k given by Eq. (60) is negative; therefore, h can be positive only if the term in R predominates. On the other hand R is generally significantly smaller than N_W, and this means that h can be positive only if $M\ll M_s, T_C-T\ll T_C$. Then we can expand h and k as series of $(M/M_s)^2$; if we assume also that R is independent of M (anisotropy energy proportional to M^2) we can write

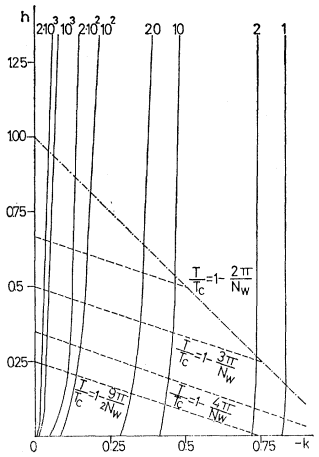

FIG. 1. Solid lines plot the relationship between h and $-k$ for several thicknesses, as given by the present treatment; they are labeled with the value of S^2. The dashed lines plot the relationship between h and $-k$ for several temperatures, as given by the equation of state in Weiss's model. The cross points of the two families correspond to nucleation according to the waving mode; the crossing points of the dashed lines with the dot-dashed line correspond to nucleation according to coherent inversion.

$$h\simeq\frac{R}{2\pi}-\frac{N_W}{6\pi}\left(1-\frac{T}{T_C}\right)+\tfrac{1}{3}k, \tag{61}$$

i.e., the relationship between h and k is linear, with the restriction

$$k\geqslant-\frac{N_W}{4\pi}\left(1-\frac{T}{T_C}\right). \tag{62}$$

In Fig. 1 some curves deduced by the data of Table I have been plotted together with some straight lines deduced by Eqs. (61) and (62) for the case $R=2\pi$, which corresponds to a shape anisotropy due to acicular particles at microscopic level. Inspection to Fig. 1 leads to the following conclusions:

(a) Nucleation is always due to coherent rotation if

$$T<T_C(1-3R/N_W). \tag{63}$$

(b) For temperatures in the range

$$T_C(1-3R/N_W)<T<T_C(1-2R/N_W) \tag{64}$$

nucleation takes place by waving for temperatures and thicknesses yielding intersections, otherwise by coherent rotation.

(c) For temperatures in the range

$$T_C(1-2R/N_W)<T<T_C \tag{65}$$

nucleation takes place by waving for temperatures and thicknesses yielding intersections, otherwise by coherent inversion.

5. CONCLUSION

The extension of the well-known Brown's equations to cases, in which a constant magnitude of the magnetization is questionable, has been deduced by the variational method. Some simple relationships between the nucleation field deduced through the standard form and the one deduced through the new form have been briefly discussed. By means of one simple example, the possibility of nucleation modes with nonconstant magnetization magnitude has been proved. The interest of this treatment for high temperatures (near the Curie point) has been emphasized.

APPENDIX A

Consistently with the continuum approximation, peculiar of micromagnetics, let us consider our ferromagnetic body as divided in many cubic cells with edge a, and to restrict the exchange interaction between the cell labeled with i and the cell labeled with k to the form

$$E_{x(ik)}=-2B_{(ik)}\mathbf{M}_{(i)}\cdot\mathbf{M}_{(k)}, \tag{A1}$$

where $\mathbf{M}_{(i)}$ and $M_{(k)}$ are the average magnetizations of cell i and of cell k, respectively.[3] If we assume also that $B_{(ik)}$ vanishes if the cells have no common face and is a constant $B(a)$ for nearest neighboring cells, the total exchange energy is

$$E_x=-\sum_i \sum_{k,\ \text{adj. to } i} B(a)\mathbf{M}_{(i)}\cdot\mathbf{M}_{(k)}. \tag{A2}$$

The Cartesian coordinates of the centers of cells i and k are $\mathbf{x}(i)$ and $\mathbf{x}(k)$; if a is small enough for the validity of Taylor expansion we have

$$E_x=-\sum_i \sum_{k,\ \text{adj. to } i} B(a)\mathbf{M}_{(i)}\cdot\left\{\mathbf{M}_{(i)}+\left(\frac{\partial\mathbf{M}}{\partial x_l}\right)_{(i)}\times[x_l(k)-x_l(i)]+\frac{1}{2}\left(\frac{\partial^2\mathbf{M}}{\partial x_l\partial x_m}\right)_{(i)}[x_l(k)-x_l(i)]\times[x_m(k)-x_m(i)]\right\}, \tag{A3}$$

where the sum symbol for l and m has been suppressed, l and m varying between 1 and 3. Performing the sum on index k, for instance under the assumption of a cubic body with edge multiple of a, we obtain, after suppression of the irrelevant term $M_{(i)}{}^2$,

$$E_x=\sum_{i,\ \text{surface}} B(a)\mathbf{M}_{(i)}\cdot\left(\frac{\partial\mathbf{M}}{\partial n}\right)_{(i)}a-\sum_{i,\ \text{volume}} B(a)\mathbf{M}_{(i)}\cdot(\nabla^2\mathbf{M})_{(i)}a^2, \tag{A4}$$

where the first sum is applied only to the cells at the surface of the body and the second to all other cells. Let us go to the limit $a\to 0$, keeping at a finite value $\frac{1}{2}\alpha$ the limit of the ratio $B(a)/a$; we obtain, if $d\sigma$ is the elementary surface element:

$$E_x=\int_S \tfrac{1}{2}\alpha\mathbf{M}\cdot\frac{\partial\mathbf{M}}{\partial n}d\sigma-\int_V \mathbf{M}\cdot\nabla^2\mathbf{M}d\tau$$

$$=\tfrac{1}{2}\alpha\int_V[(\mathrm{grad}M_x)^2+(\mathrm{grad}M_y)^2+(\mathrm{grad}M_z)^2]d\tau, \tag{A5}$$

under the assumption that α is independent of M and of the position; if this is not true, the last equality of Eq. (A5) does not hold, and must be replaced as follows:

$$E_x=\tfrac{1}{2}\int_V[\mathrm{grad}(\alpha M_x)\cdot\mathrm{grad}M_x+\mathrm{grad}(\alpha M_y)\cdot\mathrm{grad}M_y+\mathrm{grad}(\alpha M_z)\cdot\mathrm{grad}M_z]d\tau. \tag{A6}$$

APPENDIX B

The uniform magnetization $\mathbf{M}_0(\mathbf{H})$ parallel to z axis and therefore parallel or antiparallel to $\mathbf{H}$ is certainly one solution of Eqs. (7) and (8) under the conditions listed in Sec. 3 and provided that

$$\left(\frac{\partial\omega}{\partial M_z}\right)_{\substack{M_x=M_y=0\\ M_z=M_0}}=H_z+H_z'=H_z-N_cM_0. \tag{B1}$$

The second variation of the total energy in respect to an infinitesimal change $\mathbf{m}$ of the magnetization is given by

$$\delta^2E=\int\left\{\tfrac{1}{2}\alpha[(\mathrm{grad}m_x)^2+(\mathrm{grad}m_y)^2+(\mathrm{grad}m_z)^2]+\frac{1}{2}\left(\frac{\partial^2\omega}{\partial M_x{}^2}m_x{}^2+\frac{\partial^2\omega}{\partial M_y{}^2}m_y{}^2+\frac{\partial^2\omega}{\partial M_z{}^2}m_z{}^2+2\frac{\partial^2\omega}{\partial M_x\partial M_y}m_xm_y+2\frac{\partial^2\omega}{\partial M_y\partial M_z}m_ym_z+2\frac{\partial^2\omega}{\partial M_z\partial M_x}m_zm_x\right)-\tfrac{1}{2}\mathbf{m}\cdot\mathbf{h}'\right\}d\tau. \tag{B2}$$

If we split ω into the isotropic term $\omega_0(M)$ and the anisotropic one $\omega_1(\mathbf{M})$ as specified in Sec. 3, we can take advantage of the following relations, valid for the deriva-

tives at $\mathbf{M}=\mathbf{M}_0$:

$$\frac{\partial^2\omega_0}{\partial M_x^2}=\frac{\partial^2\omega_0}{\partial M_y^2}=\frac{1}{M_0}\frac{d\omega_0}{dM}, \tag{B3}$$

$$\frac{\partial^2\omega_0}{\partial M_z^2}=\frac{d^2\omega_0}{dM^2}, \tag{B4}$$

$$\frac{\partial^2\omega_0}{\partial M_x\partial M_y}=\frac{\partial^2\omega_0}{\partial M_y\partial M_z}=\frac{\partial^2\omega_0}{\partial M_z\partial M_x}=0, \tag{B5}$$

$$\frac{\partial^2\omega_1}{\partial M_x\partial M_z}=\frac{\partial^2\omega_1}{\partial M_y\partial M_z}=\frac{\partial^2\omega_1}{\partial M_z^2}=0, \tag{B6}$$

and we obtain the formula

$$\delta^2E=\int\Big\{\tfrac{1}{2}\alpha[(\mathrm{grad}m_x)^2+(\mathrm{grad}m_y)^2+(\mathrm{grad}m_z)^2]$$
$$+\frac{1}{2}\Big[\frac{H_z+H_z'}{M_0}(m_x^2+m_y^2)+\frac{d^2\omega_0}{dM^2}m_z^2+\frac{\partial^2\omega_1}{\partial M_x^2}m_x^2$$
$$+2\frac{\partial^2\omega_1}{\partial M_x\partial M_y}m_xm_y+\frac{\partial^2\omega_1}{\partial M_y^2}m_y^2\Big]-\tfrac{1}{2}\mathbf{m}\cdot\mathbf{h}'\Big\}d\tau. \tag{B7}$$

The uniform magnetization $\mathbf{M}_0$ satisfying Eq. (B1) is a stable solution if δ^2E is positive for each vector field $\mathbf{m}$ (not identically vanishing); remark that this is always true if H_z is positive and large enough, because of the presence of terms

$$\frac{H_z}{2M_0}\int(m_x^2+m_y^2)d\tau$$

and

$$\frac{1}{2}\int\frac{d^2\omega_0}{dM^2}m_z^2d\tau.$$

Then consider one definite vector field $\mathbf{m}$ and begin with H_z large enough to insure stability; if H_z is now decreased it will eventually reach a value H'' at which, for the given vector field $\mathbf{m}$, δ^2E vanishes. This value of H'' must satisfy the relation

$$-(H''+H_z')$$
$$=M_0\Big\{\alpha\int[(\mathrm{grad}m_x)^2+(\mathrm{grad}m_y)^2+(\mathrm{grad}m_z)^2]d\tau$$
$$+\int\Big(\frac{d^2\omega_0}{dM_z^2}m_z^2+\frac{\partial^2\omega_1}{\partial M_x^2}m_x^2+2\frac{\partial^2\omega_1}{\partial M_x\partial M_y}m_xm_y$$
$$+\frac{\partial^2\omega_1}{\partial M_y^2}m_y^2\Big)d\tau+\frac{1}{4\pi}\int_{\text{space}}h'^2d\tau\Big\}\Big/$$
$$\int(m_x^2+m_y^2)d\tau, \tag{B8}$$

if at least one of the components m_x and m_y does not vanish everywhere; otherwise it must be true that

$$\tfrac{1}{2}\alpha\int(\mathrm{grad}m_z)^2d\tau+\frac{1}{2}\frac{d^2\omega_0}{dM^2}\int m_z^2d\tau$$
$$+\frac{1}{4\pi}\int_{\text{space}}h'^2d\tau=0, \tag{B9}$$

where the dependence on H_z is implicit in function ω_0. The highest value reached by H'' among all values corresponding to the different vector fields $\mathbf{m}$ is the nucleation field.

At this point the proof of the theorems on overconstrained and underconstrained solutions is straightforward. If we impose to the vector-field $\mathbf{m}$ some additional constraint, the highest value reached by H'' for $\mathbf{m}$ belonging to the set defined by this constraint cannot exceed the true upper boundary of H'' for $\mathbf{m}$ completely free; a solution determined in this way is called overconstrained. On the other hand, let us drop from the expression of δ^2E some non-negative term: The resulting value of this reduced functional is not higher than the true value of δ^2E for the same field $\mathbf{m}$, and therefore it can happen that this functional takes on the value 0 for some nontrivial field at some value of H_z, at which the true δ^2E is positive for each nontrivial $\mathbf{m}$; a solution determined in this way is called underconstrained.