Time Scale of Intrinsic Resistive Fluctuations in Thin Superconducting Wires

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A thermal-activation theory of intrinsic fluctuations in thin superconducting wires has been proposed by Langer and Ambegaokar (LA). Their fluctuation rate equals an exponential activation factor $e^{-\Delta F/k_BT}$ times a prefactor Ω which fixes the fluctuation time scale. Using a model based upon a time-dependent Ginzburg-Landau equation, we obtain a new estimate of Ω which is different in functional form from the LA estimate, and smaller than that estimate by more than 10 orders of magnitude for the conditions in recent experiments. To within corrections which are roughly of order unity, our expression is $\Omega=(L/\xi)(\Delta F/k_BT)^{\frac{1}{2}}/\tau$, where (L/ξ) is the length of the sample in units of the Ginzburg-Landau coherence length ξ, $(\Delta F/k_BT)^{\frac{1}{2}}$ is a correction for overlap of fluctuations at different places along the wire, and $\tau\approx10^{-8}$ sec is the relaxation time in the Ginzburg-Landau equation. Although our specific expressions have been derived from a time-dependent Ginzburg-Landau theory, we expect from general physical arguments that they are relatively insensitive to the starting model.

I. INTRODUCTION

EXPERIMENTS by Webb and Warburton[1] on the intrinsic resistive transition of thin superconducting whisker-crystal wires of Sn for T near but below T_c(bulk) have been successfully interpreted in terms of a theory developed by Langer and Ambegaokar (LA).[2] This theory derives from a suggestion by Little[3] that dissipation occurs in superconducting wires when thermal fluctuations cause the magnitude of the order parameter to vanish briefly at some point along the wire. This model was refined by LA, who expressed the dissipation rate as the product of a time-scale prefactor which they related to electronic properties in the normal state and a thermal-activation exponential which they computed from a novel utilization of the Ginzburg-Landau theory. Using a time-dependent Ginzburg-Landau equation,[4,5] we have calculated a revised prefactor which in the Webb-Warburton geometry is more than 10 orders of magnitude smaller than the LA estimate. The reported shifts ΔT_c in the effective transition temperature[1] are about twice those predicted with the new prefactor.

It is useful to compare the two prefactors using a statistical description appropriate to the LA model. In that model resistance is a manifestation of thermal fluctuations which cause the total phase change along the wire of the superconductor order parameter to jump by 2π. The important fluctuations extend over a length of the wire comparable to the coherence length $\xi(T)$,[2] because a greater or lesser length is energetically more expensive. If $\Delta F(T)\gg k_BT_c$ is the free-energy barrier for a particular transition in the LA model, that transition occurs with an average rate

$$\Gamma(T)=\Omega(T)\exp[-\Delta F(T)/k_BT], \tag{1.1}$$

where the prefactor $\Omega(T)$ fixes the fluctuation time scale and is the focus of our present attention. LA argue that $\Delta F(T)$ can be correctly calculated using the ordinary (time-independent) Ginzburg-Landau equations. We accept their arguments and use their results for $\Delta F(T)$ in this paper.

The prefactor $\Omega(T)$ is related to the frequency of random excursions in the order-parameter function space of the LA model and is only weakly sensitive to $\Delta T\equiv T_c(\text{bulk})-T$ and to the current in the wire. The exponential factor in (1.1) varies by several orders of magnitude for small changes in ΔT or in current, so $\Omega(T)$ need only be known to within that latitude for accurate computation of fluctuation-induced temperature shifts.

LA proposed that[2]

$$\Omega(T)=N_e/\tau_e, \tag{1.2}$$

where N_e is the number of conduction electrons in the wire and $\tau_e\approx10^{-12}$ sec is a typical electron scattering time in the normal state. Our own result can be written in the form

$$\Omega(T)=N(T)/\tau(T), \tag{1.3}$$

where $N(T)$ is in effect the number of statistically independent subsystems along the length of the wire[6,7] and

$$\tau(T)=\pi\hbar/8k_B\Delta T \tag{1.4}$$

is a microscopic diffusion time inversely proportional to ΔT. Under the conditions of the Webb-Warburton experiment, the effective number $N(T)$ of coarse-grained phase-space elements[7] is approximately equal to the length of the wire measured in units of the

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[5] E. Abrahams and T. Tsuneto, Phys. Rev. **152**, 416 (1966); E. Abrahams and J. W. F. Woo, Phys. Letters **27A**, 117 (1968).
[6] D. L. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958), pp. 1–8.
[7] D. ter Haar, *Elements of Thermostatistics* (Holt, Rinehart and Winston, Inc., New York, 1966), 2nd ed., pp. 149–157, 201–204, and 229–231.

coherence length $\xi(T)$,[8]

$$N(T)\approx l\equiv L/\xi(T). \tag{1.5}$$

For $\Delta T=1$ m°K, $\tau(T)=3.00\times10^{-9}$ sec, which to within the accuracy we require is comparable to τ_e. Taking $\tau(T)=\tau_e$ and using (1.5), we find that the prefactor (1.2) is greater than (1.3) by a factor

$$N_\xi\equiv N_e/N(T), \tag{1.6}$$

approximately equal to the number of conduction electrons in a length $\xi(T)$ of the wire. In the Webb-Warburton geometry $N_\xi\gtrsim10^{10}$.

Although the specific expression (1.4) for $\tau(T)$ and those to be derived below for $N(T)$ follow from the time-dependent Ginzburg-Landau theory, rather general arguments intrinsic to the LA statistical model suggest that, even if the time-dependent Ginzburg-Landau theory is not exactly correct (as recent results suggest[9–13]), the prefactor (1.2) is still much too large. Independent of the detailed form of the dynamics, we expect that, when the statistical theory obtains, the prefactor $\Omega(T)$ should have the form (1.3), with $N(T)$ approximately as in (1.5) and $\tau(T)$ some microscopic time possibly different from (1.4). If we rewrite (1.2) in the form (1.3), we are led to a time

$$\tau=\tau_e/N_\xi, \tag{1.7}$$

which in the Webb-Warburton geometry ($\tau_e/N_\xi\lesssim10^{-22}$ sec) is much less than any reasonable microscopic relaxation time in solids. A similar statement holds for the wirelike thin films studied by Lukens and Goodkind.[14]

The available experimental evidence is not decisive, although the alternatives (1.2) and (1.3) differ by more than 10 orders of magnitude for both the Webb-Warburton and the Lukens-Goodkind samples.[1,14] The Webb-Warburton measurements of critical current versus ΔT in whisker crystals of Sn favor the LA estimate (1.2), but experimental uncertainties are such that the smaller prefactor (1.3) cannot be definitely excluded.[15]

Webb and Warburton measured the onset of dissipation by detecting a small finite voltage (~2 nV$=10^6$ transitions/sec). Lukens and Goodkind measured the transition rate $\Gamma(T)$ directly (for 0.1–10 transitions/sec) by monitoring the flux through evaporated-film Sn rings containing a narrow wirelike weak-link section. Fitting the Lukens-Goodkind data to theory,[16] one obtains a prefactor which is approximately one order of magnitude more than (1.3) but 12 orders less than (1.2). Although the reported geometry only marginally approximates a long thin wire and evaporated-film measurements of critical fluctuations may be less trustworthy than those on carefully selected single-crystal specimens (evaluated by the width of the transition region[1]), these measurements support the revised prefactor (1.3). The theory can be fully tested only with more extensive experimental data.

As we have indicated above, one cannot interpret subsequent support for the revised prefactor (1.3) as quantitative experimental verification of the time-dependent Ginzburg-Landau theory. If the statistical fluctuation idea is correct, we expect $\Omega(T)$ to have the form (1.3) with $N(T)$ approximately equal to its value in the time-dependent theory and $\tau(T)$ some effective microscopic relaxation time is not too different from τ_e or (1.4). If experiments confirm the original prefactor (1.2), we will be left with a puzzling dilemma. It might then be necessary to reexamine the use of the static Ginzburg-Landau theory to calculate the free-energy barrier $\Delta F(T)$, since a factor-of-2 error in $\Delta F(T)$ could counteract a factor 10^{10} in the prefactor.[16a]

In Sec. II we outline the time-dependent Ginzburg-Landau theory and its relation to the fluctuation problem, describe the free-energy extrema appropriate to the statistical LA model, and derive a formal general expression for the prefactor $\Omega(\kappa,T)$. In Sec. III we evaluate that expression for the special case of a long wire carrying no current. The more complicated case of a long wire carrying a finite current is treated in Sec. IV. A brief concluding discussion is contained in Sec. V. The validity of the Ginzburg-Landau free-energy functional and the relationship of the present work to some other calculations using the time-dependent Ginzburg-Landau model, are discussed in the Appendix.

II. TIME-DEPENDENT GINZBURG-LANDAU MODEL

A. Free-Energy Considerations

According to the usual (time-independent) Ginzburg-Landau theory, the Helmholtz free energy[17] of a thin

[8] The dimensionless length $l=L/\xi(T)$ should not be confused with carrier mean free path, which we never explicitly consider.

[9] W. E. Masker and R. D. Parks, Phys. Rev. (to be published).

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[11] L. P. Gor'kov and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **54**, 612 (1968) [Soviet Phys.—JETP **27**, 328 (1968)].

[12] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **55**, 2443 (1968) [Soviet Phys.—JETP **28**, 1298 (1969)].

[13] R. V. D'Aiello and S. J. Freedman, Phys. Rev. Letters **22**, 515 (1969).

[14] J. E. Lukens and J. M. Goodkind, Phys. Rev. Letters **20**, 1363 (1968).

[15] W. W. Webb, in *Fluctuations in Superconductors*, edited by W. S. Goree and F. Chilton (Stanford Research Institute, Menlo Park, Calif., 1968), p. 159; and (private communication).

[16] D. E. McCumber, Phys. Rev. **181**, 716 (1969).

[16a] *Note added in proof.* More recent experiments on whiskers suggest that the resistive transition may in fact be fit to the formulas of the present paper, using the correct prefactor, if one properly chooses "T_c (bulk)." [R. J. Warburton, B. R. Patton, W. W. Webb, and J. W. Wilkins, in Proceedings of the Conference on the Science of Superconductivity, Stanford, 1969 (unpublished).] Furthermore, there is reason for questioning the validity of the previous methods of finding "T_c (bulk)" by extrapolation. With further experiments it may even be possible to check the magnitude of the prefactor with some degree of accuracy. [See also the experimental results of D. R. Overcash and M. Skove (to be published).]

[17] D. E. McCumber, Phys. Rev. **172**, 427 (1968).

superconducting wire at a temperature T near, but below, T_c can be expressed as a functional $F[\psi(x)]$ of the normalized superconducting order parameter $\psi(x)$ according to the equation

$$F[\psi(x)]=[\sigma H_c^2(T)\xi(T)/4\pi]\times\int_{-l/2}^{l/2}dx\Big[-|\psi|^2+\tfrac{1}{2}|\psi|^4+\Big|\Big(\frac{\partial}{\partial x}-i\xi(T)\frac{2e}{\hbar c}A_x\Big)\psi\Big|^2\Big]. \quad (2.1)$$

In Eq. (2.1), σ is the cross-sectional area of the wire, x is the position along the wire measured in units of the coherence length $\xi(T)$, l is the length (1.5) of the wire measured in the same units, and $H_c(T)$ is the bulk critical field at temperature T. We have assumed that the wire diameter is small compared to $\xi(T)$ and to the London penetration depth, so that ψ can be treated as uniform across the cross section of the wire. Under these circumstances we may set the vector potential $\mathbf{A}$ equal to zero, provided the current carried by the wire is not too great and provided there are no external magnetic fields.

When interpreting (2.1), one should not consider that the value of $\psi(x)$ has been independently measured at every point of the wire. Rather, one should imagine that the value of the order parameter has been measured at a discrete set of points, closer together than $\xi(T)$ but further apart than $\xi(0)$. [The Ginzburg-Landau equation is not valid for fluctuations on a scale short compared to $\xi(0)$.] We could be more precise by defining, for example, $\psi(x)$ to include just its first $(2N+1)$ Fourier components,

$$\psi(x)=\textstyle\sum_k \psi_k e^{ikx}, \quad k=2\pi n/l \quad (2.2)$$

with $|n|\leq N$ integral and $1\ll N/l\ll\xi(T)/\xi(0)$. [Strictly speaking, ψ_k in (2.2) should be the quantum-mechanical operator $\Psi_{\rm op}(k)$ for the kth Fourier component of the order parameter, but we assert that a classical description is adequate because the quantum-mechanical uncertainty in (2.2) is very much less than $\psi(x)$ in the temperature range of interest.] Specification of $\psi(x)$ is to be interpreted in this picture as specification of the set of $(4N+2)$ real parameters $\{\mathrm{Re}\psi_k, \mathrm{Im}\psi_k\}$, that is, as the specification of $(4N+2)$ constraints on the system. The free-energy functional $F[\psi(x)]$ would then be

$$F[\psi(x)]=F_0(T)-k_BT\ln\{\mathrm{tr}[\exp(-\mathcal{H}/k_BT)]|_\psi\}, \quad (2.3)$$

where $F_0(T)$ is some reference energy (possibly temperature-dependent), $\mathcal{H}$ is the system Hamiltonian, and the trace is over all states consistent with the $(4N+2)$ constraints implicit in $\psi(x)$.

It is well known that the Ginzburg-Landau formula (2.1) is a very good approximation to the free energy (2.3) for $|T-T_c|/T_c\ll 1$, except in a very small region of temperature near T_c. For the case of zero current, the Ginzburg-Landau theory is believed to be valid for the one-dimensional wire near T_c as long as

$$\sigma H_c^2(T)\xi(T)/2\pi\gg k_BT_c. \quad (2.4)$$

(This condition is discussed further in the Appendix.) LA have argued[2] that the temperature where resistive fluctuations become important is sufficiently below T_c that the Ginzburg-Landau theory remains valid. Under the experimental conditions where experiments have been made,[1,14] the inequality (2.4) is sometimes satisfied.[16a] We shall therefore follow LA and shall base our calculations upon the Ginzburg-Landau free-energy formula (2.1).

Extrema of the Ginzburg-Landau functional (2.1) are solutions of the time-independent Ginzburg-Landau equation

$$0=(1-|\psi|^2)\psi+\Big(\frac{\partial}{\partial x}-i\xi(T)\frac{2e}{\hbar c}A_x\Big)^2\psi. \quad (2.5)$$

With $A_x=0$, it is convenient to take $\psi(x)=f(x)e^{i\phi(x)}$ and rewrite (2.5) as a pair of equations

$$d^2f(x)/dx^2=-f(x)+f^3(x)+J^2/f^3(x), \quad (2.6a)$$

$$J=f^2(x)d\phi(x)/dx, \quad (2.6b)$$

where J is dimensionless, independent of x, and related to the supercurrent I in the sample by

$$I=Jc\sigma H_c^2(T)\xi(T)/\Phi_0, \quad (2.7)$$

where Φ_0 is the flux quantum $hc/2e$. It is convenient to express J in terms of a new parameter κ such that

$$J=\kappa(1-\kappa^2), \quad \kappa^2<\tfrac{1}{3}. \quad (2.8)$$

We assume that the one-dimensional superconducting wire is sufficiently long ($l\gg 1$) that the actual boundary conditions at the ends of the wire are not important.[17] It is most convenient to impose periodic boundary conditions on $\psi(x)$. We specify the total phase difference along the wire,

$$\phi(\tfrac{1}{2}l)-\phi(-\tfrac{1}{2}l)=\Delta\phi, \quad (2.9a)$$

and require that $f(x)$ be periodic,

$$f(\tfrac{1}{2}l)=f(-\tfrac{1}{2}l). \quad (2.9b)$$

There are two kinds of solutions to (2.5) of interest to us. The first kind corresponds to a free-energy minimum and is given by

$$\psi(x)=(1-\kappa^2)^{1/2}e^{i(\phi_0+\kappa x)}, \quad (2.10a)$$

with ϕ_0 an arbitrary phase reference. For this solution

$$\Delta\phi=\kappa l. \quad (2.10b)$$

The second kind of solution corresponds to a free-energy

saddle point and is given by

$$\psi(x)=\{(1-3\kappa^2)^{1/2}\tanh[x(1-3\kappa^2)^{1/2}/\sqrt{2}] -i\sqrt{2}\kappa\}e^{i(\phi_0+\kappa x)}, \quad (2.11a)$$

or by any uniform translation of (2.11a). For this solution

$$\Delta\phi=\kappa l+2\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa]. \quad (2.11b)$$

The order parameters (2.10a) and (2.11a) are identical (to within a constant phase shift) except in a region Δx of order unity—that is, except over a physical length of the order of the coherence length $\xi(T)$.

It is useful conceptually to consider the decay of persistent current in a closed superconducting loop. The order parameter $\psi(x)$ must be periodic in the length l of the loop perimeter, so that the phase change $\Delta\phi$ around the loop must be an integral multiple of 2π.[18] The parameter κ in (2.10) is quantized with values

$$\kappa^{(n)}=2\pi n/l, \quad n \text{ integral}. \quad (2.12)$$

The parameter κ in (2.11) is also quantized, and each of its values $\bar{\kappa}^{(n)}$ is such that $\kappa^{(n+1)}>\bar{\kappa}^{(n)}>\kappa^{(n)}$, $(\kappa^{(n+1)})^2<\frac{1}{3}$; that is, the quantum values of κ at the free-energy minima and saddle points interlace.

For the parameters of interest in this paper the thermal energy k_BT is small compared to $\sigma H_c^2\xi/2\pi$, and the order parameter $\psi(x)$ will almost always be near one of the minimum-free-energy solutions (2.10). If large noise-produced fluctuations in $\psi(x)$ never occurred, the winding number n could never change with time and supercurrents in the wire would persist forever. Large fluctuations in $\psi(x)$ introduce the possibility of a transition from the vicinity of one minimum to another, and it is the rate of these infrequent transitions which we must calculate. They are ultimately responsible for the decay of persistent currents in a superconducting loop.

If the order parameter $\psi(x)$ is considered to be a continuous function of time, as is implied by the treatment of $\psi(x)$ as a classical variable, the free energy $F=F[\psi(x,t)]$ must also be a continuous function of time. As the order parameter $\psi(x,t)$ varies from one minimum-F configuration to the next, the value of F must first increase, pass through a maximum, and then decrease again. One can rigorously show that the value of F at this maximum must be equal to or larger than the value of the free energy at the saddle point (2.11) lying between the two minima of F. Furthermore, one can choose a continuous path for $\psi(x,t)$ such that the maximum value of F occurs when $\psi(x,t)$ has the saddle-point configuration (2.11). The most probable actual paths connecting the two minima will pass close to this saddle point.

[18] We restrict ourselves to loops for which $L_sI_c\ll1$, where L_s is the self-inductance of the loop and I_c the critical current of the wire. For this case the vector potential $\mathbf{A}(x)$ remains negligible, as assumed in (2.6). More general cases are discussed in Ref. 16.

The free-energy barriers for $\psi(x,t)$ to leave the neighborhood of a minimum-F configuration (2.10) and to pass over one of its two adjacent saddle points (2.11) are[2,17]

$$\Delta F_-(\kappa,T)=[\sigma H_c^2(T)\xi(T)/8\pi]\times\{(8/3)\sqrt{2}(1-3\kappa^2)^{1/2}-8\kappa(1-\kappa^2)\times\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa]\}, \quad (2.13a)$$

$$\Delta F_+(\kappa,T)=[\sigma H_c^2(T)\xi(T)/8\pi]\times(8/3)\sqrt{2}(1-3\kappa^2)^{1/2}+8\kappa(1-\kappa^2)\times\{\pi-\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa]\}), \quad (2.13b)$$

for fluctuations which for $\kappa\geq0$ tend, respectively, to decrease and increase the current magnitude. Because $\Delta F_-(\kappa,T)\leq\Delta F_+(\kappa,T)$, fluctuations tend on the average to reduce the current magnitude—that is, they are dissipative.

Our analysis is restricted to the limit $\Delta F_-\gg k_BT$. For zero current, this is equivalent to (2.4), but for currents much larger than $ek_BT_c/\hbar$, this condition can be more restrictive than (2.4).

B. Time Dependence of Order Parameter

In order to calculate the required transition rates, one must make some assumption regarding the time dependence of the order parameter $\psi(x)$. We use the simplest possible assumption, a purely dissipative time-dependent Ginzburg-Landau equation.

Although the time-independent Ginzburg-Landau equation (2.5) was originally derived from a postulated free energy of the form (2.1),[19] subsequently Gor'kov demonstrated that Eq. (2.5) follows from the microscopic BCS theory, if Δ_{BCS}/k_BT_c is treated as a small expansion parameter with slow spatial variations.[20] Using an expansion technique similar to that used by Gor'kov in Ref. 20, Schmid[4] and others[5] have extended Eq. (2.5) to include the simplest description of temporal relaxation. In this time-dependent Ginzburg-Landau theory, small fluctuations in the order parameter $\psi(x,t)$ about a minimum-F configuration decay according to

$$\tau(T)\left(\frac{\partial}{\partial t}+i\frac{2eV}{\hbar}\right)\psi=(1-|\psi|^2)\psi+\left(\frac{\partial}{\partial x}-i\xi(T)\frac{2e}{\hbar c}A_x\right)^2\psi, \quad (2.14)$$

where $V(x,t)$ is the electrochemical potential, $\mathbf{A}(x,t)$ is the vector potential, and $\tau(T)$ is the relaxation time defined in Eq. (1.4). In what follows we assume that the electromagnetic fields are negligible and take $V(x,t)=A_x(x,t)=0$. The assumption $A_x(x,t)=0$ was

[19] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[20] L. P. Gor'kov, Zh Eksperim. i Teor. Fiz. **36**, 1918 (1959); **37**, 1407 (1959) [Soviet Phys.—JETP **9**, 1364 (1959); **10**, 998 (1960)].

already justified, but the choice $V(x,t)=0$ requires discussion.

The total current I carried by the wire consists of a supercurrent (or pair current) plus a quasiparticle current. The latter is analogous to the current carried by the wire in its normal state and is characterized for T near T_c by a conductivity comparable to that in the normal state. The current at point x at time t can thus be written approximately in the form

$$I(x,t)=(\sigma H_c^2\xi^2 c/\Phi_0)|\psi(x,t)|^2\nabla\phi(x,t) + LG_n\nabla V(x,t), \quad (2.15)$$

where $\phi(x,t)$ is the phase of the order parameter, $\Phi_0=ch/2e$ is the flux quantum, G_n is the normal-state conductance of the wire, L is its length, and the remaining parameters are defined in Eq. (2.1). The fluctuation time scale is sufficiently slow that it is correct to use the dc normal conductance for G_n in (2.15) and to require that charge neutrality be maintained along the wire. The latter condition requires that any spatial variation in the supercurrent be canceled by an opposite variation in the normal current—that is, $\nabla I(x,t)=0$. Equation (2.15) thus gives

$$\nabla^2 V=-(\sigma H_c^2\xi^2 c/LG_n\Phi_0)\nabla(|\psi|^2\nabla\phi), \quad (2.16)$$

so that the electrochemical potential difference across a resistance-producing fluctuation is, with $\partial\nabla\phi/\partial t=(2e/h)\nabla V$ and $|\psi|^2\approx 1$,

$$\nabla V\lesssim\Phi_0/\tau_V c, \quad (2.17)$$

where τ_V is the relaxation time

$$\tau_V(T)=\frac{LG_n\Phi_0^2}{2\pi\sigma H_c^2(T)\xi^2(T)c^2}\propto(\Delta T)^{-1}. \quad (2.18)$$

For T near T_c, the ratio $\tau(T)/\tau_V(T)$ computed from Eqs. (1.4) and (2.18) is independent of ΔT. For a superconductor with mean free path comparable to the zero-temperature coherence length, this ratio is of order unity, so that on the $\tau(T)$ time scale the corrections in (2.3) produced by local variations in the electrochemical potential are not generally negligible. On the other hand, for $\psi(x)$ at the saddle point of the Ginzburg-Landau free energy, the right-hand side of (2.16) vanishes identically, so the term proportional to V in (2.14) will not make a qualitative difference in the rate at which $\psi(x)$ crosses the free-energy barrier. Insofar as we only seek an order-of-magnitude estimaet of the prefactor Ω, and insofar as the time-dependent Ginzburg-Landau equation is probably not strictly correct in any case, we shall replace $V(x,t)$ by its time average $\langle V(x)\rangle$. In a superconducting ring geometry, $\langle V(x)\rangle$ is a constant, which by suitable choice of reference potential can be set equal to zero. (For a thin superconducting wire in an external circuit with an imposed dc current, $\langle V(x)\rangle$ would be a linear function of position along the wire.) The situation could be different for T well below T_c, because the quasiparticle conductance is there substantially less than G_n. This case does not concern us here, although it might be relevant, for example, to the description of multiple-quantum transitions in point-contact weak links.[21]

C. Noise-Driven Fluctuatious

The time-dependent Ginzburg-Landau equation (2.14) is alleged to describe the mean motion of $\psi(x)$ in the neighborhood of a minimum-F configuration. Because we do not expect our final results to depend sensitively upon the detailed dynamics, we simply assume that this same equation holds everywhere in the order-parameter Hilbert space. With $A_x(x,t)=V(x,t)=0$, Eq. (2.14) has the explicitly dissipative form

$$\frac{\partial\psi(x)}{\partial t}=-\frac{4\pi}{\sigma H_c^2(T)\xi(T)\tau(T)}\frac{\delta F(\psi,\psi^*)}{\delta\psi^*(x)}, \quad (2.19)$$

where we consider F as an analytic functional of $\psi(x)$ and $\psi^*(x)$. To describe spontaneous fluctuations in the order parameter, we must add a Langevin noise term $\mathfrak{L}(x,t)$ to (2.19). The simplest assumption is that $\mathfrak{L}(x,t)$ is a complex Gaussian stochastic variable with autocorrelation function

$$\langle\mathfrak{L}(x,t)\mathfrak{L}(x',t')\rangle=0, \quad \langle\mathfrak{L}(x,t)^*\mathfrak{L}(x',t')\rangle=C\delta(x-x')\delta(t-t'), \quad (2.20)$$

where the constant C is determined by the requirement that the equilibrium distribution of $\psi(x)$ be proportional to $\exp\{-F[\psi(x)]/k_BT\}$.

Rather than use this Langevin formulation of the problem, we shall actually work with an equivalent Fokker-Planck formulation, focusing our attention upon the probability-density functional $\rho[\psi(x),t]$ in the Hilbert space. Using a formalism similar to that outlined by Landauer and Swanson[22] and by Langer,[23] we need to specify the Fokker-Planck equation explicitly only in the neighborhood of the different free-energy extrema. In the neighborhood of the extremum (2.10) or (2.11), we can conveniently express the order parameter in the form

$$\psi(x)=[r_1(x)+ir_2(x)+u(x)+iv(x)]e^{i(\phi_0+\kappa x)}, \quad (2.21)$$

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[21] J. E. Zimmerman and A. H. Silver, Phys. Rev. Letters **19**, 14 (1967); A. H. Silver and J. E. Zimmerman, Phys. Rev. **157**, 317 (1967).

[22] R. Landauer and J. A. Swanson, Phys. Rev. **121**, 1668 (1961).

[23] J. S. Langer, Phys. Rev. Letters **21**, 973 (1968). [The ξ_1' integrals in Eqs. (11) and (13) should extend from $(-\infty)$, which reduces the final rate R by a factor of 2.]

where $u(x)=v(x)=0$ at the extremum and where

$$r_1(x)=(1-\kappa^2)^{1/2}, \quad r_2(x)=0 \tag{2.22a}$$

for the F-minimum solution (2.10) and

$$r_1(x)=(1-3\kappa^2)^{1/2}\tanh[x(1-3\kappa^2)^{1/2}/\sqrt{2}], \quad r_2(x)=-\kappa\sqrt{2} \tag{2.22b}$$

for the F-saddle solution (2.11). Substituting (2.21) into (2.1) and retaining terms through second order in $u(x)$ and $v(x)$, we obtain

$$F=F_e(\kappa,T)+\left(\frac{\sigma H_c^2(T)\xi(T)}{4\pi}\right)\times\int_{-l/2}^{l/2}dx\,\mathbf{\Psi}(x)^T\cdot\mathbf{M}_e(x)\cdot\mathbf{\Psi}(x), \tag{2.23}$$

where $F_e(\kappa,T)$ is the value of $F[\psi(x)]$ at the $u=v=0$ extremum [extremum (e)=minimum (m) with (2.22a) and saddle (s) with (2.22b)], $\mathbf{M}_e(x)$ is the 2×2 self-adjoint matrix differential operator

$$\mathbf{M}_e(x)=\begin{bmatrix}\left(-\frac{d^2}{dx^2}-(1-\kappa^2)+3r_1^2+r_2^2\right) & \left(2r_1r_2+2\kappa\frac{d}{dx}\right)\\ \left(2r_1r_2-2\kappa\frac{d}{dx}\right) & \left(-\frac{d^2}{dx^2}-(1-\kappa^2)+r_1^2+3r_2^2\right)\end{bmatrix}, \tag{2.24}$$

and $\mathbf{\Psi}(x)$ is the real two-component vector function

$$\mathbf{\Psi}(x)=\begin{pmatrix}u(x)\\ v(x)\end{pmatrix}. \tag{2.25}$$

The boundary conditions associated with (2.24) must be chosen so that $\mathbf{M}_e(x)$ represents the second variational derivative of F with respect to ψ, with the boundary conditions (2.9) imposed on ψ. Terms linear in $u(x)$ and $v(x)$ are absent from (2.23) because $u=v=0$ is a free-energy extremum.

The eigenfunctions $\mathbf{\Psi}_{en}(x)$ of the operator $\mathbf{M}_e(x)$ satisfy the eigenvalue equation

$$\mathbf{M}_e(x)\cdot\mathbf{\Psi}_{en}(x)=\epsilon_{en}\mathbf{\Psi}_{en}(x) \tag{2.26}$$

with real eigenvalues ϵ_{en} and form a real orthonormal complete set. [The boundary conditions on $\mathbf{\Psi}_{en}(x)$ implied by Eqs. (2.9) will be explicitly stated in Sec. III.] If δ_{nm} is the Kronecker δ symbol, the orthonormality condition is

$$\delta_{nm}=\int_{-l/2}^{l/2}dx\,\mathbf{\Psi}_{en}{}^T(x)\cdot\mathbf{\Psi}_{em}(x). \tag{2.27}$$

The completeness property implies that, for any real vector function $\mathbf{\Psi}(x)$, Eq. (2.23) can be written in the form

$$F=F_e(\kappa,T)+[\sigma H_c^2(T)\xi(T)/4\pi]\sum_{n=1}^{\infty}\epsilon_{en}\eta_n^2, \tag{2.28}$$

where the ϵ_{en} are the eigenvalues in (2.26) and the η_n are the real expansion coefficients

$$\eta_n=\int_{-l/2}^{l/2}dx\,\mathbf{\Psi}_{en}{}^T(x)\cdot\mathbf{\Psi}(x). \tag{2.29}$$

In the neighborhood of the extremum, where Eqs. (2.23) and (2.28) accurately approximate the free energy, we can usefully describe the different order-parameter functions (2.21) by the set $\{\eta_n\}$ of expansion coefficients (2.29). Because $\psi(x)$ as defined in Eq. (2.2) has a finite number of degrees of freedom, only the first $(4N+2)$ coefficients η_n are truly independent variables, but, because N/l is large and the results are convergent for $N/l\to\infty$, only a negligible error will result if we assume that all of the η_n are independent.

Using the representation (2.21) in (2.14), we can rewrite the latter equation for (u,v) small in the form

$$\tau(T)(\partial/\partial t)\mathbf{\Psi}(x,t)=-\mathbf{M}_e(x)\cdot\mathbf{\Psi}(x,t), \tag{2.30}$$

where $\tau(T)$ is the relaxation time (1.4), $\mathbf{\Psi}(x,t)$ is the vector function (2.25), and $\mathbf{M}_e(x)$ is the matrix operator (2.24). If $\{\eta_n(t)\}$ is the set of expansion coefficients (2.29) appropriate to $\mathbf{\Psi}(x,t)$, Eq. (2.30) can equivalently be written

$$\tau(T)[\partial\eta_n(t)/\partial t]=-\epsilon_{en}\eta_n(t), \tag{2.31}$$

where ϵ_{en} is the eigenvalue from (2.26).

Converting to a Fokker-Planck description at this stage, we introduce the probability density $\rho_e(\{\eta_n\},t)$ appropriate to the particular extremum $(e=m$ or $s)$ and construct the Fokker-Planck equation[24]

$$\tau(T)\frac{\partial\rho_e}{\partial t}=\sum_n\left(\epsilon_{en}\frac{\partial}{\partial\eta_n}\eta_n+\frac{2\pi k_BT}{\sigma H_c^2(T)\xi(T)}\frac{\partial^2}{\partial\eta_n^2}\right)\rho_e. \tag{2.32}$$

This equation gives the same mean motion as (2.31)

[24] In that we have neglected velocity coordinates conjugate to the η_n, Eq. (2.25) might more appropriately be called a Smoluchowski equation. For discussions of the more general case, see Ref. 22 and S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943), especially pp. 63–68.

and yields a steady-state Boltzmann distribution $\rho_e(\{\eta_n\}) \propto \exp[-F(\{\eta_n\})/k_BT]$, with $F(\{\eta_n\})$ as in (2.28). It is, moreover, the simplest possible Fokker-Planck equation consistent with these conditions.

D. Thermal-Activation Rate

Thermally activated processes of the type with which we are here concerned have been studied by Landauer and Swanson[22] and by Langer[23] in systems with extremum Fokker-Planck equations of the form (2.32). Two new features require special consideration, the extension from a finite to a denumerably infinite number of degrees of freedom $\{\eta_n\}$ and the treatment of those variables η_n associated with zero eigenvalues ($\epsilon_{en}=0$). Both features have previously been considered in another context by Langer.[25]

Using mathematics outlined in Secs. III and IV below, one can easily verify for the infinitely long wire ($l\to\infty$) that all of the eigenvalues ϵ_{en} of (2.26) are positive except for one zero eigenvalue (which we label $\epsilon_{m1}=0$) at the free-energy minimum (2.22a) and for one negative eigenvalue ($\epsilon_{s1}<0$) and two zero eigenvalues ($\epsilon_{s2}=\epsilon_{s3}=0$) at the free-energy saddle (2.22b). The negative eigenvalue ϵ_{s1} is associated with that degree of freedom corresponding to motion over the saddle point. The zero eigenvalues ($\epsilon_{m1},\epsilon_{s2},\epsilon_{s3}$) are associated with symmetry properties of the system; the free energy (2.28) is invariant with respect to changes in the corresponding coordinates η_n. In particular, the eigenvalues $\epsilon_{m1}=\epsilon_{s2}=0$ are associated with variations (u,v) in (2.21) equivalent to a trivial displacement of the phase reference ϕ_0; their associated coordinates (η_{m1},η_{s2}) produce exactly compensating effects in the final transition rate and will hereinafter be ignored. The eigenvalue $\epsilon_{s3}=0$ is associated with translations of the saddle solution (2.22b) along the wire; the contribution of its coordinate to the transition rate cannot be ignored.

If the free-energy barriers (2.13) are much larger than k_BT, we may reasonably assume that the effective domain of all coordinates η_n associated with *positive* eigenvalues ϵ_{en} is limited by free-energy considerations to values of η_n for which the quadratic approximation (2.28) is accurate and for which no other physical constraints enter. For different but related reasons, a similar assumption pertains to the coordinate η_{s1} associated with the negative eigenvalue ϵ_{s1}.[22,23] The assumption is clearly not valid for the zero-eigenvalue coordinates. In particular, the domain of the translation coordinate η_{s3} is not limited by free-energy considerations but rather by the physical length $l\gg1$ of the wire. [Consistent with (2.28), the exact free energy is independent of η_{s3} for all values in this domain.[26]] The translational symmetry can be exhibited explicitly in (2.22b) by replacing x with $(x-\bar{x})$, $-\frac{1}{2}l<\bar{x}<\frac{1}{2}l$. If $\bar{x}$ becomes the infinitesimal $d\bar{x}$, we can alternatively represent this translation, as in (2.14), with [cf. (2.18)]

$$d\Psi(x)=-d\bar{x}[(1-3\kappa^2)/\sqrt{2}]\times\operatorname{sech}^2[x^2(1-3\kappa^2)/2]^{1/2}\begin{pmatrix}1\\0\end{pmatrix} \quad (2.33a)$$

$$=-d\bar{x}[8(1-3\kappa^2)^3/9]^{1/4}\Psi_{s3}(x), \quad (2.33b)$$

where $\Psi_{s3}(x)$ is the eigenfunction appropriate to ϵ_{s3} in (2.26) [obtained by applying the normalization condition (2.27) to (2.33a)]. Rewriting (2.33b) as $d\eta_{s3}\Psi_{s3}(x)$ and recalling that $\bar{x}$ is physically constrained to an interval of length l, we learn immediately that η_{s3} ranges over an interval of length

$$\Lambda_{s3}(\kappa)=\int d\eta_{s3}=l[8(1-3\kappa^2)^3/9]^{1/4}. \quad (2.34)$$

If all delicate questions of convergence and of one-to-one correspondence between eigenvalues ϵ_{mn} and ϵ_{sn} are suitably resolved (cf. Secs. III and IV below), we can use these results in the formalism of Landauer and Swanson[22] and of Langer[23] to obtain a transition rate

$$\Gamma_\pm(\kappa,T)=\Omega_\pm(\kappa,T)\exp[-\Delta F_\pm(\kappa,T)/k_BT], \quad (2.35)$$

where

$$\Omega_\pm(\kappa,T)=\frac{\Lambda_{s3}(\kappa_s)}{4\pi^2\tau(T)}\left(\frac{\sigma H_c^2(T)\xi(T)}{k_BT}\right)^{1/2}|\epsilon_{s1}(\kappa_s)|\times\left(\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}\right)^{1/2} \quad (2.36)$$

and $\Delta F_\pm(\kappa,T)$ are the barriers defined in (2.13). Implicit in these results is the assumption $\Delta F_\pm(\kappa,T)\gg k_BT$.

The arguments κ_m and κ_s in (2.36) are the values of the parameter κ appropriate to the fixed-$\Delta\phi$ boundary condition at the free-energy minimum and saddle point, respectively. The difference $\kappa_s-\kappa_m$ is of order l^{-1} and depends upon whether the saddle point is taken in the direction to increase or to decrease the current. Although a correction $\Delta\kappa\sim l^{-1}$ is negligible for $l\gg1$ in any one factor of (2.36), the number of relevant factors in the infinite product is of order l, so that corrections due to $\kappa_s-\kappa_m$ can be of order unity.

In the following two sections we compute the eigenvalue $\epsilon_{s1}(\kappa_s)$ and the eigenvalue product ratio $[\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)/|\epsilon_{s1}(\kappa_s)|]\prod_n[\epsilon_{mn}(\kappa_m)/\epsilon_{sn}(\kappa_s)]$, first in Sec. III for the special case $\kappa=0$ and then in Sec. IV for general $\kappa^2<\frac{1}{3}$. The results of the complicated but rigorous mathematics are summarized in Sec. V, which also contains a brief physical description of the components of the prefactor (2.36).

[25] J. S. Langer, Ann. Phys. (N. Y.) **41**, 108 (1967).

[26] We of course neglect corrections which might be necessary in the immediate neighborhood of the wire ends.

III. EVALUATION OF PREFACTOR

A. General Considerations

Before we can calculate the prefactor (2.36), we must fix the extremum parameters (κ_m,κ_s) and discuss the boundary conditions on (u,v) at $x=\pm\frac{1}{2}l$.

Holding the phase difference (2.9) constant (modulo 2π), we find from (2.10b) and (2.11b) that $(\kappa\geq 0)$

$$\kappa_s-\kappa_m=-(2/l)\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa] \quad (3.1a)$$

for current-decreasing transitions and

$$\kappa_s-\kappa_m=(2\pi/l)-(2/l)\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa] \quad (3.1b)$$

for current-increasing transitions. A uniform shift of both κ_s and κ_m by $\Delta\kappa\sim\pi/l$ will change the eigenvalue product ratio in (2.36) by an amount of relative order l^{-1}, which is negligible because $l\gg 1$. Using this flexibility, we choose $\kappa_s=\kappa$ and fix κ_m from Eqs. (3.1). We rewrite the eigenvalue product ratio of Eq. (2.36) in the form

$$\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}$$

$$=g_1(\kappa)\frac{\epsilon_{m2}(\kappa)\epsilon_{m2}(\kappa)}{|\epsilon_{s1}(\kappa)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa)}{\epsilon_{sn}(\kappa)}, \quad (3.2)$$

where

$$g_1(\kappa)=\prod_{n=2}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{mn}(\kappa_s)}. \quad (3.3)$$

In the ratio on the left-hand side of (3.2), the parameters κ_m and κ_s are different, but the order-parameter phase difference $\Delta\phi$ is the same (modulo 2π) at the free-energy minimum and saddle. In the ratio on the right-hand side of (3.2), the value of κ is the same at the minimum and saddle, but the phase differences $\Delta\phi$ differ (modulo 2π) by $2\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa]$.

To proceed further, we require (u,v) boundary conditions for the eigenvalue equation (2.26). Order parameters $\psi(x)$ described as in Eq. (2.21) satisfy the conditions (2.9) if the magnitude $(u^2+v^2)^{1/2}$ of the two-component vector function (u,v) is the same at the two ends of the wire $(x=-\frac{1}{2}l$ and $\frac{1}{2}l)$ and if that vector rotates through the same angle (modulo 2π) as does the vector (r_1,r_2) when x increases from $-\frac{1}{2}l$ to $\frac{1}{2}l$, where r_1 and r_2 are the functions (2.22). At the free-energy minimum, where (2.22a) obtains, it follows that we must associate periodic boundary conditions with the operator $\mathbf{M}_m(x)$ in order that it represent the second variational derivative of $F[\psi(x)]$ with respect to admissible functions $\psi(x)$. The eigenfunctions $\mathbf{\Psi}_{mn}(x)=(u_{mn},v_{mn})$ at the free-energy minimum satisfy the periodic boundary conditions

$$\begin{aligned}\mathbf{\Psi}_{mn}(\tfrac{1}{2}l)&=\mathbf{\Psi}_{mn}(-\tfrac{1}{2}l),\\ \mathbf{\Psi}_{mn}'(\tfrac{1}{2}l)&=\mathbf{\Psi}_{mn}'(-\tfrac{1}{2}l),\end{aligned} \quad (3.4)$$

where the prime denotes differentiation with respect to x. At the free-energy saddle point, where (2.22b) obtains, we must associate modified periodic boundary conditions with the operator $\mathbf{M}_s(x)$. The eigenfunctions $\mathbf{\Psi}_{sn}(x)$ at the saddle point satisfy the boundary conditions

$$\begin{aligned}\mathbf{S}^{-1}\cdot\mathbf{\Psi}_{sn}(\tfrac{1}{2}l)&=-\mathbf{S}\cdot\mathbf{\Psi}_{sn}(-\tfrac{1}{2}l),\\ \mathbf{S}^{-1}\cdot\mathbf{\Psi}_{sn}'(\tfrac{1}{2}l)&=-\mathbf{S}\cdot\mathbf{\Psi}_{sn}'(-\tfrac{1}{2}l),\end{aligned} \quad (3.5)$$

where

$$\mathbf{S}=\begin{bmatrix}\sin\delta_0 & \cos\delta_0\\ -\cos\delta_0 & \sin\delta_0\end{bmatrix}$$

$$=\frac{1}{(1-\kappa^2)^{1/2}}\begin{bmatrix}(1-3\kappa^2)^{1/2} & \sqrt{2}\kappa\\ -\sqrt{2}\kappa & (1-3\kappa^2)^{1/2}\end{bmatrix}, \quad (3.6)$$

with

$$\delta_0=\tan^{-1}[(1-3\kappa^2)^{1/2}/\sqrt{2}\kappa]. \quad (3.7)$$

{We have assumed here and throughout this paper that l is so large that $\tanh[l(1-3\kappa^2)^{1/2}/2\sqrt{2}]=1$.} With these boundary conditions the operators $\mathbf{M}_e(x)$ are self-adjoint; they have real eigenvalues and complete sets of real eigenfunctions.

To evaluate the function $g_1(\kappa)$, it is convenient to list explicitly the eigenvalues $\epsilon_{mn}(\kappa)$ of $\mathbf{M}_m(x)$. Because the functions $r_1(x)$ and $r_2(x)$ in (2.22a) are constant, we can use complex eigenfunctions

$$\mathbf{\Psi}_{mn}(x)=e^{ik_\mu x}\begin{pmatrix}u_{\mu\pm}\\ v_{\mu\pm}\end{pmatrix}, \quad (3.8)$$

where $(u_{\mu\pm},v_{\mu\pm})$ are κ-dependent constants and $k_\mu=2\pi\mu/l$, μ integral. These functions satisfy the boundary conditions (3.4). Substituting these functions into the appropriate form of (2.26), we find for each k_μ a pair of eigenvalues

$$\begin{aligned}\epsilon_+(k_\mu^2)&=1-\kappa^2+k_\mu^2+[(1-\kappa^2)^2+4\kappa^2k_\mu^2]^{1/2},\\ \epsilon_-(k_\mu^2)&=1-\kappa^2+k_\mu^2-[(1-\kappa^2)^2+4\kappa^2k_\mu^2]^{1/2}.\end{aligned} \quad (3.9)$$

The eigenvalues for μ and $-\mu$ are degenerate. Real eigenfunctions $\mathbf{\Psi}_{mn}(x)$ can be constructed from the real and imaginary parts of the complex functions (3.8).

We introduce a function

$$S(\kappa)\equiv s_0-\tfrac{1}{2}k_B\ln\prod_{n=2}^{\infty}\frac{\epsilon_{mn}(\kappa)}{\epsilon_{mn}(0)} \quad (3.10a)$$

$$=s_0-\tfrac{1}{2}k_B\Big[\ln(1-\kappa^2)+\sum_{\mu\neq 0}\ln\Big(\frac{k_\mu^2[k_\mu^2+2(1-3\kappa^2)]}{k_\mu^2(k_\mu^2+2)}\Big)\Big], \quad (3.10b)$$

$$S(\kappa)=s_0-\frac{lk_B}{4\pi}\int_{-\infty}^{\infty}dk\ln\Big(\frac{k^2+2(1-3\kappa^2)}{k^2+2}\Big) \quad (3.10c)$$

$$=s_0+(lk_B/\sqrt{2})[1-(1-3\kappa^2)^{1/2}], \quad (3.10d)$$

where s_0 is a constant. In (3.10b) we used the eigenvalues (3.9), which are such that

$$\epsilon_+(k^2)\epsilon_-(k^2)=k^2[k^2+2(1-3\kappa^2)];$$

in (3.10c) we used the fact $l\gg 1$ to replace the sum by an integral. It follows directly from Eqs. (3.3) and (3.10) that

$$g_1(\kappa)=\exp\{2[S(\kappa_m)-S(\kappa_s)]/k_B\} \quad (3.11a)$$

$$=\exp\{\sqrt{2}l[(1-3\kappa_s^2)^{1/2}-(1-3\kappa_m^2)^{1/2}]\} \quad (3.11b)$$

$$=\exp[3\sqrt{2}\kappa(1-3\kappa^2)^{-1/2}(\kappa_s-\kappa_m)l]. \quad (3.11c)$$

For $\kappa=0$, it follows from (3.1) that $(\kappa_s-\kappa_m)l=\pm\pi$ and from (3.11) that $g_1(\kappa)=1$, to within negligible corrections of relative order l^{-1}.

B. Zero-Current Case, $\kappa=0$

With $g_1(\kappa)$ determined, we turn next to the remaining factors in (3.2). The mathematical problem is considerably simpler for $\kappa=0$ than for general $\kappa^2<\frac{1}{3}$. For both choices (2.22), the differential operator $\mathbf{M}_e(x)$ is diagonal when $\kappa=0$, and the matrix eigenvalue problem (2.26) reduces to a pair of uncoupled scalar eigenvalue equations. At the minimum these are

$$(-d^2/dx^2+2)u_{mn}(x)=\epsilon_{mn}u_{mn}(x), \quad (3.12a)$$

$$-(d^2/dx^2)v_{mn'}(x)=\epsilon_{mn'}v_{mn'}(x), \quad (3.12b)$$

and at the saddle they are

$$[-d^2/dx^2+2-3\,\mathrm{sech}^2(x/\sqrt{2})]u_{sn}(x)=\epsilon_{sn}u_{sn}(x), \quad (3.13a)$$

$$[-d^2/dx^2-\mathrm{sech}^2(x/\sqrt{2})]v_{sn'}(x)=\epsilon_{sn'}v_{sn'}(x). \quad (3.13b)$$

One component of the vector eigenfunction $\mathbf{\Psi}_{en}(x)$ is identically zero; the other satisfies one of these differential equations and boundary conditions derived from (3.4) and (3.5), where, at $\kappa=0$, $\delta_0=\frac{1}{2}\pi$ and $\mathbf{S}^2=1$.

Eigenfunctions and eigenvalues appropriate to the minimum have been described in Eqs. (3.8) and (3.9). With $\kappa=0$, the eigenvalues $\epsilon_+(k_\mu^2)\geq 2$ pertain to (3.12a) and the eigenvalues $\epsilon_-(k_\mu^2)\geq 0$ to (3.12b). The single zero eigenvalue belongs to the phase-shift solution

$$\epsilon_{m1}=0, \quad v_{m1}(x)=l^{-1/2}. \quad (3.14)$$

The saddle-point eigenvalues and eigenfunctions are more difficult to determine; however, we can easily verify four special cases[27]:

$$\epsilon_{s1}=-\tfrac{1}{2}, \quad v_{s1}(x)=2^{-3/4}\,\mathrm{sech}(x/\sqrt{2}); \quad (3.15a)$$

$$\epsilon_{s2}=0, \quad v_{s2}(x)=l^{-1/2}\tanh(x/\sqrt{2}); \quad (3.15b)$$

$$\epsilon_{s3}=0, \quad u_{s3}(x)=(9/32)^{1/4}\,\mathrm{sech}^2(x/\sqrt{2}); \quad (3.15c)$$

$$\epsilon_{s4}=\tfrac{3}{2}, \quad u_{s4}(x)=(9/8)^{1/4}\tanh(x/\sqrt{2})\times\mathrm{sech}(x/\sqrt{2}). \quad (3.15d)$$

[27] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958), p. 69, problem 4.

The first is the expected negative-eigenvalue solution, the second is the phase-shift solution analogous to (3.14), and the third is the translation solution described in conjunction with Eqs. (2.33). The fourth solution (3.15d) has no particular physical importance but, together with (3.15a) and (3.15b), completes the set of all localized discrete-eigenvalue solutions of Eqs. (3.13). All other solutions, including the continuum-edge solution (3.15b), are nonlocalized continuum-eigenvalue solutions, as are all solutions of Eqs. (3.12).

To complete our evaluation of (3.2), we do not require a list of the remaining eigenvalues ϵ_{sn}, but rather it is sufficient to determine directly the ratio of the positive-eigenvalue products. We can do this using techniques suggested by Langer,[25] although the differing boundary conditions (3.4) and (3.5) cause complications.

We introduce a differential operator

$$\mathcal{O}(\beta)=-d^2/dx^2-\beta\,\mathrm{sech}^2(x/\sqrt{2}), \quad (3.16)$$

β real, associated with the boundary conditions (3.5). We introduce real eigenfunctions $f_\mu(x,\beta)$ and eigenvalues $\xi_\mu(\beta)$ such that

$$\mathcal{O}(\beta)f_\mu(x,\beta)=\xi_\mu(\beta)f_\mu(x,\beta) \quad (3.17)$$

and

$$f_\mu(\tfrac{1}{2}l,\beta)=-f_\mu(-\tfrac{1}{2}l,\beta), \qquad f_\mu'(\tfrac{1}{2}l,\beta)=-f_\mu'(-\tfrac{1}{2}l,\beta), \quad (3.18)$$

which is the form of the boundary conditions (3.5) for $\kappa=0$. Comparing Eqs. (3.13) and (3.17), one can easily verify that the eigenvalues of (3.13a) and (3.13b) are, respectively,

$$\epsilon_{sn(\mu)}=2+\xi_\mu(3), \quad (3.19a)$$

$$\epsilon_{sn'(\mu)}=\xi_\mu(1). \quad (3.19b)$$

Thus, for each integer μ we have two eigenvalues of (3.13) which we have indexed by $n(\mu)$ and $n'(\mu)$. One might conclude upon comparing Eqs. (3.12) and (3.17) that the eigenvalues of (3.12) are $\epsilon_{mn}=2+\xi_\mu(0)$ and $\epsilon_{mn'}=\xi_\mu(0)$, but this is not correct because the boundary conditions (3.4) are different from (3.18).

It is useful to introduce a new function $g_2(\epsilon)$ which for $\kappa=0$ and $\epsilon>0$ is defined as the limit of a convergent product,

$$g_2(\epsilon)=\lim_{l\to\infty}\left(\prod_\mu\frac{\epsilon+\epsilon_{mn}}{\epsilon+2+\xi_\mu(0)}\prod_\mu\frac{\epsilon+\epsilon_{mn'}}{\epsilon+\xi_\mu(0)}\right). \quad (3.20)$$

The eigenvalues $(\epsilon_{mn},\epsilon_{mn'})$ follow from (3.9) with $\kappa=0$; the eigenvalues $\xi_\mu(0)$ are

$$\xi_\mu(0)=(k_\mu+\pi/l)^2, \quad (3.21)$$

with k_μ as in (3.8). For $\epsilon>0$, we can proceed much as in

Eqs. (3.10) and (3.11) to show that

$$g_2(\epsilon)=\lim_{l\to\infty}\exp\left[\sum_{\mu=-\infty}^{\infty}\ln\left(\frac{(\epsilon+2+k_\mu^2)(\epsilon+k_\mu^2)}{[\epsilon+2+(k_\mu+\pi/l)^2][\epsilon+(k_\mu+\pi/l)^2]}\right)\right] \tag{3.22a}$$

$$=\lim_{l\to\infty}\exp\left[\frac{l}{2\pi}\int_{-\infty}^{\infty}dk\,\ln\left(\frac{(\epsilon+2+k^2)(\epsilon+k^2)}{[\epsilon+2+(k+\pi/l)^2][\epsilon+(k+\pi/l)^2]}\right)\right] \tag{3.22b}$$

$$=1. \tag{3.22c}$$

Using the result (3.22), the eigenvalues (3.15) and (3.19), and our previous evaluation of $g_1(0)$, we can rewrite the product ratio (3.2) for $\kappa=0$ and $l\to\infty$ in the form

$$\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}=\lim_{l\to\infty}\lim_{\epsilon\to0^+}\left(-\epsilon\prod_{n=1}^{\infty}\frac{[\epsilon+\epsilon_{mn}(\kappa)]}{[\epsilon+\epsilon_{sn}(\kappa)]}\right) \tag{3.23a}$$

$$=\lim_{\epsilon\to0^+}\lim_{l\to\infty}\left[-\epsilon\prod_{\mu}\left(\frac{[\epsilon+2+\xi_\mu(0)][\epsilon+\xi_\mu(0)]}{[\epsilon+2+\xi_\mu(3)][\epsilon+\xi_\mu(1)]}\right)\right] \tag{3.23b}$$

$$=\lim_{\epsilon\to0^+}[-\epsilon r(\epsilon+2,3)r(\epsilon,1)], \tag{3.23c}$$

where, for $\xi>0$,

$$r(\xi,\beta)=\lim_{l\to\infty}r_l(\xi,\beta), \tag{3.24a}$$

$$r_l(\xi,\beta)=\prod_{\mu}\frac{\xi+\xi_\mu(0)}{\xi+\xi_\mu(\beta)}. \tag{3.24b}$$

In going from (3.23a) to (3.23b), we have interchanged the order of taking the $l\to\infty$ and $\epsilon\to0^+$ limits in (3.23a). This is permissible in (3.23a) because of an exact cancellation of the low-lying continuum eigenvalues ϵ_{mn} and ϵ_{sn}. The order of limits cannot be interchanged in (3.23b); in fact, that product diverges for $\epsilon\to0^+$ with l large but constant.

If ξ is sufficiently positive, each factor in (3.24b) is positive and

$$\ln r_l(\xi,\beta)=-\sum_{\mu}\{\ln[\xi+\xi_\mu(\beta)]-\ln[\xi+\xi_\mu(0)]\}$$

$$=-\mathrm{tr}\{\ln[\xi+\mathcal{O}(\beta)]-\ln[\xi+\mathcal{O}(0)]\}. \tag{3.25a}$$

Because both operators $\mathcal{O}(\beta)$ and $\mathcal{O}(0)$ are associated with the same boundary conditions (3.5) [or, equivalently, (3.18)], we can use an operator identity to rewrite (3.25a) in the form

$$\ln r_l(\xi,\beta)=-\mathrm{tr}(\ln\{1+[\xi+\mathcal{O}(0)]^{-1}\beta V\}), \tag{3.25b}$$

where

$$V=[\mathcal{O}(\beta)-\mathcal{O}(0)]/\beta=-\mathrm{sech}^2(x/\sqrt{2}). \tag{3.26}$$

If we introduce a set of eigenvalues $\Lambda_n(\xi,\beta)$ such that

$$\{1+[\xi+\mathcal{O}(0)]^{-1}\beta V\}g_n(x,\xi)=\Lambda_n(\xi,\beta)g_n(x,\xi), \tag{3.27}$$

where the eigenfunctions $g_n(x,\xi)$ satisfy the boundary conditions (3.18), it follows from (3.25) that

$$r_l(\xi,\beta)=\prod_{n}\Lambda_n^{-1}(\xi,\beta). \tag{3.28}$$

Multiplying both sides of (3.27) by $[\xi+\mathcal{O}(0)]$ and rearranging terms, we can rewrite (3.27) in the form

$$\mathcal{O}[\bar\beta_n(\xi)]g_n(x,\xi)=-\xi g_n(x,\xi), \tag{3.29a}$$

with

$$\bar\beta_n(\xi)=\beta/[1-\Lambda_n(\xi,\beta)]. \tag{3.29b}$$

This is similar to (3.17) but has a different interpretation. Whereas in (3.17) we fix the strength parameter β and find the eigenvalues $\xi_\mu(\beta)$, in (3.29) we fix the eigenvalue $(-\xi)$ and find a consistent set of strength parameters $\bar\beta_n(\xi)$. For $\xi>0$, the eigenfunctions $g_n(x,\xi)$ are localized and the parameters $\bar\beta_n(\xi)$ are positive.

Solving (3.29b) for $\Lambda_n(\xi,\beta)$ and using the result in (3.28), we obtain

$$r_l(\xi,\beta)=\prod_{n}\frac{\beta_n(\xi)}{[\bar\beta_n(\xi)-\beta]}. \tag{3.30}$$

If in computing the parameters $\bar\beta_n(\xi)$ we neglect exponentially small boundary corrections to localized eigenfunctions [as we did, for example, in Eqs. (3.15a), (3.15c), and (3.15d)], we have in effect taken the $l\to\infty$ limit of (3.24a), and (3.30) becomes an expression for $r(\xi,\beta)$.

The localized solutions of (3.29a) for $\xi>0$ and l infinite are well understood.[25,27] If we define $y=x/\sqrt{2}$ and $\bar g_n(y,\xi)=g_n(x,\xi)$, we may rewrite (3.29a) in the form

$$[d^2/dy^2-2\xi+2\bar\beta_n(\xi)\,\mathrm{sech}^2y]\bar g_n(y,\xi)=0. \tag{3.31}$$

For $\xi>0$, the localized solutions are (integer $n\geq1$)

$$\bar g_n(y,\xi)=\mathrm{sech}^{s(n)}y w_n(\sinh y), \tag{3.32a}$$

$$\bar\beta_n(\xi)=\tfrac{1}{2}s(n)[s(n)+1], \tag{3.32b}$$

with $w_n(t)$ a polynomial of degree $(n-1)$ and

$$s(n)=(2\xi)^{1/2}+n-1. \tag{3.32c}$$

Using these expressions in (3.30), we find that

$$r(\xi,\beta)=\prod_{n=1}^{\infty}\frac{[n-1+(2\xi)^{1/2}][n+(2\xi)^{1/2}]}{[n-1+(2\xi)^{1/2}][n+(2\xi)^{1/2}]-2\beta} \tag{3.33}$$

and, in particular, that

$$r(\xi,1)=[(2\xi)^{1/2}+1]/[(2\xi)^{1/2}-1], \tag{3.34a}$$

$$r(\xi,3)=[(2\xi)^{1/2}+2][(2\xi)^{1/2}+1]/[(2\xi)^{1/2}-2][(2\xi)^{1/2}-1]. \tag{3.34b}$$

Combining these results in (3.23b), we find finally that

$$\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}=24 \tag{3.35}$$

for $\kappa=0$.

If $\Omega(T)$ is the value at $\kappa=0$ of the prefactor $\Omega_{\pm}(\kappa,T)$ in (2.35), it follows from Eqs. (2.34), (2.36), (3.15a), and (3.35) that

$$\Omega(T)=[l/2\pi^2\tau(T)][\sqrt{2}\sigma H_c^2(T)\xi(T)/k_BT]^{1/2}, \tag{3.36}$$

where l is the length of the wire in units of the coherence length $\xi(T)$ and $\tau(T)$ is the relaxation time (1.4). The value at $\kappa=0$ of the free-energy barriers (2.13) is

$$\Delta F(T)=\sqrt{2}\sigma H_c^2(T)\xi(T)/3\pi, \tag{3.37}$$

so that (3.36) can alternatively be written

$$\Omega(T)=[l\sqrt{3}/2\pi^{3/2}\tau(T)][\Delta F(T)/k_BT]^{1/2}. \tag{3.38}$$

IV. FINITE CURRENTS, $\kappa^2<\frac{1}{3}$

We now turn to the more difficult case of finite currents, assuming for definiteness that $0\leq\kappa<1/\sqrt{3}$. Some of the results we require in order to evaluate $\Omega_{\pm}(\kappa,T)$ from Eq. (2.36) are already available in Sec. III.

Again choosing $\kappa_s=\kappa$, we fix κ_m from Eqs. (3.1) and rewrite the eigenvalue product ratio of (2.36) in the form (3.2). The function $g_1(\kappa)$ follows from Eqs. (3.1) and (3.11c). There are two cases corresponding to current-decreasing and current-increasing transitions,

$$g_1^{-}(\kappa)=\exp[-6\sqrt{2}\kappa\delta_0/(1-3\kappa^2)^{1/2}], \tag{4.1a}$$

$$g_1^{+}(\kappa)=\exp[6\sqrt{2}\kappa(\pi-\delta_0)/(1-3\kappa^2)^{1/2}], \tag{4.1b}$$

where δ_0 is the rotation angle defined in (3.7).

Solutions of the eigenvalue problem (2.26) appropriate to the free-energy minimum have been described in Eqs. (3.8) and (3.9). They derive from the pair of coupled second-order differential equations

$$[-d^2/dx^2+2(1-\kappa^2)]u+2\kappa(dv/dx)=\epsilon u, \tag{4.2a}$$

$$-2\kappa(du/dx)-d^2v/dx^2=\epsilon v, \tag{4.2b}$$

plus the boundary conditions (3.4). It is useful to distinguish two classes of eigenfunctions according to whether they belong to the set of eigenvalues $\epsilon_+(k_n^2)\geq 2(1-\kappa^2)$ or to the set $\epsilon_-(k_n^2)\geq 0$. Using the behavior when $\kappa\rightarrow 0$ as a label (Sec. III), we call the former u-type solutions and the latter v-type. The single zero-eigenvalue solution is v-type and corresponds, as in (3.14), to an order-parameter phase shift:

$$\epsilon_{m1}=0, \quad \Psi_{m1}(x)=\begin{pmatrix}0\\ l^{-1/2}\end{pmatrix}. \tag{4.3}$$

At the free-energy saddle point (2.22b), the eigenvalue problem (2.26) involves the pair of coupled second-order differential equations

$$[-d^2/dx^2+(1-3\kappa^2)(2-3\,\mathrm{sech}^2y)]u -2\kappa[\sqrt{2}(1-3\kappa^2)^{1/2}\tanh y-d/dx]v=\epsilon u, \tag{4.4a}$$

$$-2\kappa[\sqrt{2}(1-3\kappa^2)^{1/2}\tanh y+d/dx]u +[-d^2/dx^2+4\kappa^2-(1-3\kappa^2)\,\mathrm{sech}^2y]v=\epsilon v, \tag{4.4b}$$

where here and throughout this section we use [cf. Eq. (3.31)]

$$y=y(x)=(1-3\kappa^2)^{1/2}x/\sqrt{2}, \tag{4.5}$$

and assume that at the ends of the wire $|y(\pm l/2)|\gg 1$. The eigenfunctions $\Psi_{sn}(x)$ satisfy the boundary conditions (3.5). We can verify four special solutions:

$\epsilon_{s1}=-\frac{1}{2}\{[(1+\kappa^2)^2+3(1-3\kappa^2)^2]^{1/2}-(1+\kappa^2)\}<0$,

$$\Psi_{s1}(x)=\frac{[9(1-3\kappa^2)/32(1-4\kappa^2+7\kappa^4)]^{1/4}}{[(1-4\kappa^2+7\kappa^4)^{1/2}-(1-5\kappa^2)]^{1/2}}\begin{pmatrix}[(1-4\kappa^2+7\kappa^4)^{1/2}-(1-5\kappa^2)]\,\mathrm{sech}y\tanh y\\ \sqrt{2}(1-3\kappa^2)^{1/2}\kappa\,\mathrm{sech}y\end{pmatrix}; \tag{4.6a}$$

$\epsilon_{s2}=0$,

$$\Psi_{s2}(x)=[(1-\kappa^2)l]^{-1/2}\begin{pmatrix}\sqrt{2}\kappa\\ (1-3\kappa^2)^{1/2}\tanh y\end{pmatrix}; \tag{4.6b}$$

$\epsilon_{s3}=0$,

$$\Psi_{s3}(x)=[9(1-3\kappa^2)/32]^{1/4}\begin{pmatrix}\mathrm{sech}^2y\\ 0\end{pmatrix}; \tag{4.6c}$$

$\epsilon_{s4}=\frac{1}{2}\{1+\kappa^2+[(1+\kappa^2)^2+3(1-3\kappa^2)^2]^{1/2}\}$,

$$\Psi_{s4}(x)=\frac{[9(1-3\kappa^2)/32(1-4\kappa^2+7\kappa^4)]^{1/4}}{[(1-4\kappa^2+7\kappa^4)^{1/2}+(1-5\kappa^2)]^{1/2}}\begin{pmatrix}[(1-4\kappa^2+7\kappa^4)^{1/2}+(1-5\kappa^2)]\,\mathrm{sech}y\tanh y\\ -\sqrt{2}(1-3\kappa^2)^{1/2}\kappa\,\mathrm{sech}y\end{pmatrix}. \tag{4.6d}$$

$\epsilon_{s5}=2(1-\kappa^2)$,

When $\kappa \to 0$, these reduce, respectively, to the $\kappa=0$ solutions described in (3.15). The solutions (4.6a), (4.6c), and (4.6d) are localized discrete-eigenvalue solutions; it can be shown that all other solutions are nonlocalized continuum-eigenvalue solutions. To obtain the required properties of those continuum eigenvalues, we here proceed somewhat differently from Sec. III, concerning ourselves initially with the asymptotic phase shift produced by the order-parameter fluctuation near $x=0$.

Let us introduce a differential operator [cf. (3.6)]

$$\mathcal{O}(\beta) = -d^2/dx^2 - \beta(1-3\kappa^2)\,\mathrm{sech}^2 y. \tag{4.7}$$

For the present we will not associate any particular boundary conditions with $\mathcal{O}(\beta)$. The function v can be eliminated from (4.2) or (4.4) to obtain the fourth-order differential equation

$$\{\epsilon^2 - 2\epsilon[(1-\kappa^2)+\mathcal{O}(\beta)] + \mathcal{O}(\beta)[\mathcal{O}(\beta)+2(1-3\kappa^2)]\}u = 0, \tag{4.8a}$$

with $\beta=0$ for (4.2) and $\beta=3$ for (4.4). Similarly, we may eliminate u from (4.2) or (4.4) to obtain

$$\{\epsilon^2 - 2\epsilon[(1-\kappa^2)+\mathcal{O}(\beta)] + \mathcal{O}(\beta)[\mathcal{O}(\beta)+2(1-3\kappa^2)]\}v = 0, \tag{4.8b}$$

with $\beta=0$ for (4.2) and $\beta=1$ for (4.4).

Making use of Eqs. (4.8), one can readily show that the eigenfunctions $\mathbf{\Psi}_{sn}(x)=(u_{sn},v_{sn})$ of (4.4) can be chosen to satisfy the equations

$$\mathcal{O}(3)u_{sn} = \xi_{sn}u_{sn}, \tag{4.9a}$$

$$\mathcal{O}(1)v_{sn} = \xi_{sn}v_{sn}, \tag{4.9b}$$

with ξ_{sn} related to ϵ_{sn} by one or the other of the equations

$$\epsilon = \epsilon_+(\xi), \tag{4.10a}$$

$$\epsilon = \epsilon_-(\xi), \tag{4.10b}$$

where $\epsilon_\pm(\xi)$ are defined in (3.9). Again using the behavior when $\kappa \to 0$ as a label, we call solutions u-type when (4.10a) holds and v-type when (4.10b) holds. We further classify the eigenfunctions according to their behavior under reflection about $x=0$. Thus, we separate the eigenfunctions into four classes:

$$\begin{aligned} &\alpha=1, \quad u\text{-type with } u \text{ even}, v \text{ odd};\\ &\alpha=2, \quad u\text{-type with } u \text{ odd}, v \text{ even};\\ &\alpha=3, \quad v\text{-type with } v \text{ even}, u \text{ odd};\\ &\alpha=4, \quad v\text{-type with } v \text{ odd}, u \text{ even}. \end{aligned} \tag{4.11}$$

It is convenient to index the ξ_{sn} as $\xi_\mu^\pm$, where (+) refers to u-type and (−) to v-type functions and where the index μ runs from $-\infty$ to ∞. Further, we will use negative μ for the "odd" solution types $\alpha=2$ and 4, while the non-negative μ will be used for the "even" solution types $\alpha=1$ and 3.

The four solutions (4.6) may be readily identified as belonging to α classes 3, 4, 1 and 2, respectively. Specifically, we have

$$\epsilon_{s1} = \epsilon_-(\xi_0^-), \qquad \xi_0^- = -\tfrac{1}{2}(1-3\kappa^2), \tag{4.12a}$$

$$\epsilon_{s2} = \epsilon_-(\xi_{-1}^-), \quad \xi_{-1}^- = 0, \tag{4.12b}$$

$$\epsilon_{s3} = \epsilon_+(\xi_0^+), \qquad \xi_0^+ = -2(1-3\kappa^2), \tag{4.12c}$$

$$\epsilon_{s4} = \epsilon_+(\xi_{-1}^+), \quad \xi_{-1}^+ = -\tfrac{1}{2}(1-3\kappa^2). \tag{4.12d}$$

The remaining solutions all correspond to continuum eigenvalues and the appropriate $\xi_\mu^\pm$'s are conveniently expressed in terms of eigenfunction phase shifts. Let us define phase shifts $\delta_\alpha(k)$ according to the asymptotic behavior for $x \to +\infty$,

$$\mathbf{\Psi}(x) \propto \mathbf{S}\begin{pmatrix} \cos(kx+\delta_\alpha) \\ r_\pm \sin(kx+\delta_\alpha) \end{pmatrix} \quad \text{for } \alpha=1,4, \tag{4.13a}$$

$$\mathbf{\Psi}(x) \propto \mathbf{S}\begin{pmatrix} \sin(kx+\delta_\alpha) \\ -r_\pm \cos(kx+\delta_\alpha) \end{pmatrix} \quad \text{for } \alpha=2,3, \tag{4.13b}$$

where $k=\xi^{1/2}>0$, $\mathbf{S}$ is given by (3.6), and

$$r_+ = -\frac{1}{r_-} = \frac{2k\kappa}{(1-\kappa^2)+[(1-\kappa)^2+4\kappa^2k^2]^{1/2}}. \tag{4.14}$$

The choices r_+ and r_- are to be used with u-type and v-type solutions, respectively. The behavior for $x \to -\infty$ is determined by (4.13) and (4.11). The functions (4.13) satisfy the differential equations (4.4) and (4.9) for large $|x|$ for all values of δ, but the asymptotic forms will match up with the solution of (4.4) in the neighborhood of $x=0$ only for certain specific choices of δ. The definition of the phase shift implicit in (4.13) is unique only up to an additive integral multiple of π. This ambiguity is removed by requiring that $\delta_\alpha(k)$ be a continuous function of k and that $|\delta_\alpha(k)| < \frac{1}{2}\pi$ in the limit $k \to \infty$. With this convention, the phase shifts at $k=0$ can be shown to be

$$\delta_1(0) = \delta_2(0) = \pi, \quad \delta_3(0) = \delta_4(0) = \tfrac{1}{2}\pi. \tag{4.15}$$

The modified periodic boundary conditions (3.5) will be satisfied for a given value of $k>0$ and of α, if and only if $kl+2\delta_\alpha(k)$ is a half-integral multiple of 2π. If we index these values of k by

$$\begin{aligned} &k_\mu^+ l + 2\delta_1 = 2\pi(\mu+\tfrac{1}{2}), \quad \mu=1,2,\ldots\\ &k_\mu^+ l + 2\delta_2 = 2\pi|\mu+\tfrac{1}{2}|, \quad \mu=-2,-3,\ldots\\ &k_\mu^- l + 2\delta_3 = 2\pi(\mu+\tfrac{1}{2}), \quad \mu=1,2,\ldots\\ &k_\mu^- l + 2\delta_4 = 2\pi|\mu+\tfrac{1}{2}|, \quad \mu=-2,-3,\ldots \end{aligned} \tag{4.16}$$

then the corresponding values of $\xi_\mu^\pm$, together with the four eigenstates (4.12), give us a complete list of the eigenvalues.

The product (3.2) may now be written

$$\lim_{l\to\infty}\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}=g_1^{\pm}(\kappa)\lim_{\epsilon\to 0^+}\lim_{l\to\infty}\left[-\epsilon\prod_{\mu}\left(\frac{[\epsilon+\epsilon_+(k_\mu^2)][\epsilon+\epsilon_-(k_\mu^2)]}{[\epsilon+\epsilon_+(\xi_\mu^+)][\epsilon+\epsilon_-(\xi_\mu^-)]}\right)\right]\equiv g_1^{\pm}(\kappa)\lim_{\epsilon\to 0^+}\mathfrak{R}(\epsilon,\kappa), \quad (4.17)$$

where $k_\mu=2\pi\mu/l$. Just as in (3.23a), it is permissible to interchange the order of limits $\epsilon\to 0^+$ and $l\to\infty$ in (4.17) because of the exact cancellation between continuum eigenvalues at the minimum and saddle point for the eigenvalues near zero.

In order to evaluate $\mathfrak{R}(\epsilon,\kappa)$ it is most convenient to relate the phase shifts $\delta_\alpha(k)$ to new phase shifts $\bar{\delta}_\alpha(k)$, which are the even-parity and odd-parity phase shifts of the differential operator $\mathcal{O}(3)$ for u-type solutions and phase shifts of $\mathcal{O}(1)$ for v-type solutions. The new phase shifts $\bar{\delta}_\alpha$ can be defined in terms of the asymptotic form of the wave functions for $x\to+\infty$ according to

$$\begin{aligned} u(x)&\propto\cos(kx+\bar{\delta}_\alpha), \quad \alpha=1\\ u(x)&\propto\sin(kx+\bar{\delta}_\alpha), \quad \alpha=2\\ v(x)&\propto\cos(kx+\bar{\delta}_\alpha), \quad \alpha=3\\ v(x)&\propto\sin(kx+\bar{\delta}_\alpha), \quad \alpha=4 \end{aligned} \quad (4.18)$$

where the constants of proportionality are all nonzero. The arbitrary additive constant in $\bar{\delta}_\alpha$ of an integral multiple of π is chosen in the usual way, namely, that $\bar{\delta}_\alpha$ be a continuous function of k and go to zero for $k\to\infty$. Using (4.13) and (4.14), one can show that the $\bar{\delta}_\alpha$ are related to the δ_α by

$$\begin{aligned} \bar{\delta}_\alpha(k)&=\delta_\alpha(k)-\chi(k) \quad \text{for } \alpha=1,2,\\ \bar{\delta}_\alpha(k)&=\delta_\alpha(k)+\chi(k) \quad \text{for } \alpha=3,4, \end{aligned} \quad (4.19)$$

where

$$\chi(k)\equiv\tan^{-1}\left(\frac{2\kappa^2k\sqrt{2}}{(1-3\kappa^2)^{1/2}\{(1-\kappa^2)+[(1-\kappa^2)^2+4\kappa^2k^2]^{1/2}\}}\right). \quad (4.20)$$

With the conventions we are using for δ_α and $\bar{\delta}_\alpha$, the arc-tangent function in (4.20) has its principal value lying between zero and $\frac{1}{2}\pi$. If we impose antiperiodic boundary conditions on the eigenfunctions of the operators $\mathcal{O}(\beta)$, then the eigenvalues $\xi_\mu(\beta)$, for $\beta=1$ and 3, relate to the eigenvalue $\xi_\mu^{\pm}$ for the actual boundary conditions by

$$\xi_\mu(3)=[(\xi_\mu^+)^{1/2}+2\chi(k)/l]^2\equiv\bar{\xi}_\mu^+, \quad (4.21a)$$

$$\xi_\mu(1)=[(\xi_\mu^-)^{1/2}-2\chi(k)/l]^2\equiv\bar{\xi}_\mu^-, \quad (4.21b)$$

where χ is given by (4.20) with $k=(\xi_\mu^{\pm})^{1/2}$ for the continuum eigenvalues, and χ is taken to be zero for the bound states (4.12). For $\beta=0$, we have

$$\xi_\mu(0)=[k_\mu+\pi/l]^2\equiv\bar{\xi}_\mu^0. \quad (4.21c)$$

It follows that

$$\epsilon_\pm(\xi_\mu^{\pm})=\epsilon_\pm(\bar{\xi}_\mu^{\pm})\mp\frac{4k}{l}\chi(k)\frac{d\epsilon_\pm(k^2)}{dk^2}, \quad (4.22a)$$

$$[\epsilon_\pm(k_\mu^2)]^2=\epsilon_\pm(\bar{\xi}_\mu^0)\epsilon_\pm(\bar{\xi}_{-\mu}^0), \quad (4.22b)$$

to within corrections of order l^{-2}. Substituting (4.22) into the definition (4.17) of $\mathfrak{R}(\epsilon,\kappa)$, we see that

$$\lim_{\epsilon\to 0}\mathfrak{R}(\epsilon,\kappa)=g_3(\kappa)g_4(\kappa), \quad (4.23)$$

where

$$g_3(\kappa)=\lim_{\epsilon\to 0^+}\lim_{l\to\infty}\left[-\epsilon\prod_{\mu}\left(\frac{\{\epsilon+\epsilon_+[\xi_\mu(0)]\}\{\epsilon+\epsilon_-[\xi_\mu(0)]\}}{\{\epsilon+\epsilon_+[\xi_\mu(3)]\}\{\epsilon+\epsilon_-[\xi_\mu(1)]\}}\right)\right], \quad (4.24a)$$

$$g_4(\kappa)=\lim_{l\to\infty}\exp\left[-\frac{l}{\pi}\int_0^{\infty}dk\,\ln\left(1-\frac{4k\chi(k)}{l}\frac{d}{dk^2}\ln[\epsilon_+(k^2)/\epsilon_-(k^2)]\right)\right]$$

$$=\exp\left(\frac{4}{\pi}\int_0^{\infty}dk\,k\chi(k)\frac{d}{dk^2}\ln[\epsilon_+(k^2)/\epsilon_-(k^2)]\right), \quad (4.24b)$$

with $\epsilon_\pm(k^2)$ defined in (3.9) and $\chi(k)$ in (4.20). The function $g_4(\kappa)$ decreases monotonically from unity at $\kappa^2=0$ to zero at the critical-current value $\kappa^2=\frac{1}{3}$. For $\kappa^2\ll\frac{1}{3}$, Eq. (4.24b) gives

$$g_4(\kappa)\underset{\kappa^2\ll 1/3}{\sim}1-4\kappa^2+\cdots; \quad (4.25a)$$

for $\kappa^2\to\frac{1}{3}$, it gives

$$g_4(\kappa)\underset{\kappa^2\to 1/3}{\sim}9(1-3\kappa^2)^2. \quad (4.25b)$$

The function $g_3(\kappa)$ can be evaluated using techniques similar to those used in Sec. III. We introduce functions $r(\xi,\beta)$ and $r_l(\xi,\beta)$ as in (3.24), where the eigenvalues $\xi_\mu(\beta)$ of (3.17) and (3.18) now pertain to the operator (4.7). Proceeding much as in Sec. III, we find that

$$r(\xi,\beta)=\prod_{n=1}^{\infty}\frac{\{n-1+[2\xi/(1-3\kappa^2)]^{1/2}\}\{n+[2\xi/(1-3\kappa^2)]^{1/2}\}}{\{n-1+[2\xi/(1-3\kappa^2)]^{1/2}\}\{n+[2\xi/(1-3\kappa^2)]^{1/2}\}-2\beta} \tag{4.26}$$

and, in particular,

$$r(\xi,1)=\frac{[2\xi/(1-3\kappa^2)]^{1/2}+1}{[2\xi/(1-3\kappa^2)]^{1/2}-1}, \tag{4.27a}$$

$$r(\xi,3)=r(\xi,1)\frac{[2\xi/(1-3\kappa^2)]^{1/2}+2}{[2\xi/(1-3\kappa^2)]^{1/2}-2}. \tag{4.27b}$$

We introduce the function $\Delta\rho(\xi,\beta)$ which is the difference in the density of eigenstates for $\mathcal{O}(0)$ and $\mathcal{O}(\beta)$. For all ξ greater than $|\xi_0(\beta)|$, we have

$$\ln r(\xi,\beta)=\int_{-\infty}^{\infty}d\bar{\xi}\,\Delta\rho(\bar{\xi},\beta)\ln(\xi+\bar{\xi}). \tag{4.28}$$

It follows from Eqs. (4.27) that

$$\Delta\rho(\xi,1)=\frac{\theta_+(\xi)}{\pi[2\xi/(1-3\kappa^2)]^{1/2}[\xi+\frac{1}{2}(1-3\kappa^2)]}-\delta[\xi+\tfrac{1}{2}(1-3\kappa^2)], \tag{4.29a}$$

$$\Delta\rho(\xi,3)=\Delta\rho(\xi,1)+\frac{2\theta_+(\xi)}{\pi[2\xi/(1-3\kappa^2)]^{1/2}[\xi+2(1-3\kappa^2)]}-\delta[\xi+2(1-3\kappa^2)], \tag{4.29b}$$

where the step function $\theta_+(\xi)=0$ for $\xi<0$ and 1 for $\xi>0$. Both of the functions (4.29) satisfy the expected sum rule

$$\int_{-\infty}^{\infty}d\xi\,\Delta\rho(\xi,\beta)=0. \tag{4.30}$$

Returning at this point to the definition (4.24a), we use these and previous results to obtain

$$g_3(\kappa)=\lim_{\epsilon\to0^+}(-\epsilon)\exp\left(\int_{-\infty}^{\infty}d\xi\{\Delta\rho(\xi,3)\times\ln[\epsilon+\epsilon_+(\xi)]+\Delta\rho(\xi,1)\ln[\epsilon+\epsilon_-(\xi)]\}\right) \tag{4.31a}$$

$$=\exp\left(\int_{-\infty}^{\infty}d\xi\,\Delta\rho(\xi,1)\ln|\epsilon_+(\xi)\epsilon_-(\xi)|+\frac{2}{\pi}\int_0^{\infty}d\xi\frac{\ln[\epsilon_+(\xi)]}{[2\xi/(1-3\kappa^2)]^{1/2}[\xi+2(1-3\kappa^2)]}\right) \tag{4.31b}$$

$$=3\exp\left(\frac{2}{\pi}\int_0^{\infty}dz\frac{1}{z^{1/2}(z+4)}\times\ln\{1-\kappa^2+\tfrac{1}{2}(1-3\kappa^2)z+[(1-\kappa^2)^2+2\kappa^2(1-3\kappa^2)z]^{1/2}\}\right), \tag{4.31c}$$

where in going from (4.31b) to (4.31c) we introduce the new variable $z=2\xi/(1-3\kappa^2)$ in the second integral and for the first integral use

$$\int_{-\infty}^{\infty}d\xi\,\Delta\rho(\xi,1)\ln|\epsilon_+(\xi)\epsilon_-(\xi)|=\int_{-\infty}^{\infty}d\xi\,\Delta\rho(\xi,1)[\ln|\xi|+\ln|\xi+2(1-3\kappa^2)|]=\ln|r(0,1)r[2(1-3\kappa^2),1]|=\ln3, \tag{4.32}$$

as follows from (4.28) and (4.27a). The function $g_3(\kappa)$ decreases monotonically from 24 at $\kappa^2=0$ to 4 at $\kappa^2=\frac{1}{3}$. For $\kappa^2\ll\frac{1}{3}$, Eq. (4.31d) gives

$$g_3(\kappa)\underset{\kappa^2\ll1/3}{\sim}24(1-\kappa^2); \tag{4.33a}$$

for $\kappa^2\to\frac{1}{3}$, it gives

$$g_3(\kappa)\underset{\kappa^2\to1/3}{\sim}4+8\sqrt{3}(1-3\kappa^2)^{1/2}. \tag{4.33b}$$

It follows from Eqs. (2.34), (2.36), (4.6a), (4.17), and (4.23) that the prefactor $\Omega_\pm(\kappa,T)$ in the rate expression (2.35) is

$$\Omega_\pm(\kappa,T)=\{[(1+\kappa^2)^2+3(1-3\kappa^2)^2]^{1/2}-(1+\kappa^2)\}\times[(1-3\kappa^2)^{3/2}g_1^{\pm}(\kappa)g_3(\kappa)g_4(\kappa)/24]^{1/2}\Omega(T), \tag{4.34}$$

where $\Omega(T)$ is the $\kappa=0$ prefactor (3.36) [or (3.38)]. The functions $g_1^{\pm}(\kappa)$, $g_3(\kappa)$, and $g_4(\kappa)$ are given by Eqs. (4.1), (4.31c), and (4.24b). The expression (4.34) correctly reduces to $\Omega(T)$ in the limit $\kappa^2\to0$. For $\kappa\to1/\sqrt{3}$, it gives

$$\Omega_-(\kappa,T)\underset{\kappa\to1/\sqrt{3}}{\sim}\frac{1}{8\sqrt{6}}\left(\frac{3}{e}\right)^3(1-3\kappa^2)^{15/4}\Omega(T) \tag{4.35a}$$

and

$$\Omega_+(\kappa,T)\underset{\kappa\to1/\sqrt{3}}{\sim}\Omega_-(\kappa,T)\exp\frac{\pi\sqrt{6}}{(1-3\kappa^2)^{1/2}}. \tag{4.35b}$$

Over the entire range $0\leq\kappa<1/\sqrt{3}$, $\Omega_-(\kappa,T)$ may be approximated by

$$\Omega_-(\kappa,T)\approx(1-\kappa\sqrt{3})^{15/4}(1+\tfrac{1}{4}\kappa^2)\Omega(T). \tag{4.36}$$

V. DISCUSSION

Using the time-dependent Ginzburg-Landau theory described in Sec. II, we have derived explicit expressions (3.36) and (4.34) for the time-scale prefactor $\Omega_\pm(\kappa,T)$ of the LA thermal-activation model. Although our formulas are specific to the time-dependent Ginzburg-Landau theory, we expect from the general

physical arguments of Sec. I that very similar results would follow from any starting theory consistent with the thermal-activation hypothesis. Our principal conclusions remain even if the simple time-dependent Ginzburg-Landau starting point (2.14) should prove to be inadequate.[9–13]

We undertook the elaborate mathematical analysis of Secs. II–IV to ensure that no unforeseen mathematical singularities destroyed the qualitative physical arguments of Sec. I. We have reproduced that analysis here to substantiate our assertions and to enunciate the underlying assumptions. The mathematical methods may be applicable to other systems.

Apart from numerical factors particular to our model, the general expression (2.36) contains four basic factors which we can expect in all models. The first factor, $\Lambda_{s3}(\kappa)/\tau(T)$, is in essence the approximation to $\Omega(T)$ implicit in Eqs. (1.3) and (1.5). As defined in (2.34), $\Lambda_{s3}(\kappa)$ is the length of the superconducting wire as measured in units of a typical dissipative fluctuation and equals in first approximation the effective number $N(T)$ of coarse-grained phase-space elements along the wire.

The second factor $[\sigma H_c^2(T)\xi(T)/k_BT]^{1/2}$ is a concomitant of the translational invariance (reflected in the zero eigenvalue ϵ_{s3}) and is a subtle scaling correction to the length of the typical dissipative fluctuation implicit in $\Lambda_{s3}(\kappa)$. In brief, the relevant energy interval for enumerating the distinct phase-space cells for a thermal-activation process is the thermal energy k_BT rather than the barrier height $\Delta F(T)$. In the Webb-Warburton and Lukens-Goodkind experiments,[1,14] this second factor is approximately 10, so that its presence in $\Omega_\pm(\kappa,T)$ does not qualitatively alter the general arguments of Sec. I.

The third factor of (2.36), the eigenvalue magnitude $|\epsilon_{s1}|$, is inversely proportional to the square of the width in phase space of the free-energy saddle. It is proportional to the rate at which systems diffuse in phase space over the saddle-point barrier. Except near $\kappa^2=\frac{1}{3}$, where $|\epsilon_{s1}(\kappa)|$ vanishes as $(1-3\kappa^2)^2$, the factor $|\epsilon_{s1}(\kappa)|$ is of order unity.

The final factor in (2.36), the eigenvalue-product ratio, can be viewed physically as an entropy correction. As is clear from the discussion in Sec. II A, the Ginzburg-Landau functional $F[\psi(x)]$ is an *energy* with respect to the $(4N+2)$ macroscopic degrees of freedom implicit in the order-parameter field $\psi(x)$, but it is a *free energy* with respect to all other (microscopic) degrees of freedom.[22] Let the macroscopic state $[n]$ of a wire loop denote the ensemble of states described by a temperature T and a discrete value $2\pi n$ of the total phase change. Let $\psi_m{}^{(n)}(x)$ be the order parameter of the corresponding free-energy minimum and let $F_m{}^{(n)}\equiv F[\psi_m{}^{(n)}(x)]$. In thermal equilibrium at temperature T, the probability $P^{(n)}$ that the system is in state $[n]$, at a given instant of time, is then proportional to the Boltzmann factor $\exp[-(F_m{}^{(n)}-TS^{(n)})/k_BT]$, where $S^{(n)}$ is the entropy associated with (macroscopic) order-parameter fluctuations in the neighborhood of $\psi_m{}^{(n)}(x)$. In this picture $(F_m{}^{(n)}-TS^{(n)})$, rather than $F_m{}^{(n)}$ alone, is the true free energy of the system in state $[n]$.

The entropy $S^{(n)}$ is a measure of the $\psi_m{}^{(n)}(x)$ neighborhood in function space for which $F[\psi(x)]$ is within k_BT of the minimum $F_m{}^{(n)}$. The size of this neighborhood depends upon the curvature of $F[\psi(x)]$ at $\psi_m{}^{(n)}(x)$ in the function space and varies with the parameter $\kappa^{(n)}=2\pi n/l$. If we assume that (2.28) is an adequate approximation to $F[\psi(x)]$ in that neighborhood, it is easy to demonstrate for a long wire ($l\gg1$) that $S^{(n)}=S(\kappa^{(n)})$, where $S(\kappa)$ is the function defined in (3.10).[28]

One can define a corresponding entropy $\bar{S}(\kappa)$ for the free-energy saddle point, but this quantity has less direct physical relevance than $S(\kappa)$ and has an inherent ambiguity intrinsic to the negative eigenvalue ϵ_{s1} and the zero eigenvalue ϵ_{s3}. One possibly useful definition is

$$\bar{S}(\kappa)-S(\kappa)=-\tfrac{1}{2}k_B\ln\left(\frac{\epsilon_{m2}(\kappa)\epsilon_{m3}(\kappa)}{|\epsilon_{s1}(\kappa)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa)}{\epsilon_{sn}(\kappa)}\right) \quad (5.1a)$$

$$=-\tfrac{1}{2}k_B\ln[g_3(\kappa)g_4(\kappa)], \quad (5.1b)$$

where $g_3(\kappa)$ and $g_4(\kappa)$ are defined in Eqs. (4.24). With this definition the eigenvalue-product ratio in (2.36) can conveniently be represented as an entropy correction,

$$\left(\frac{\epsilon_{m2}(\kappa_m)\epsilon_{m3}(\kappa_m)}{|\epsilon_{s1}(\kappa_s)|}\prod_{n=4}^{\infty}\frac{\epsilon_{mn}(\kappa_m)}{\epsilon_{sn}(\kappa_s)}\right)^{1/2}=\exp\left(\frac{S(\kappa)_m-\bar{S}(\kappa_s)}{k_B}\right). \quad (5.2)$$

This factor could be absorbed into the activation exponential in (2.35) by shifting from the "energy" barriers $\Delta F_\pm(\kappa,T)$ to new "free-energy" barriers $[\Delta F_\pm(\kappa,T)-T\Delta S_\pm(\kappa,T)]$, but there is really little to be gained from this procedure because the definition (5.1) is somewhat artificial.

One generally overlooked but important consequence of the order-parameter entropy-$S(\kappa)$ correction to the Ginzburg-Landau free energy is a correction to the expression (2.7) for the average supercurrent carried by the wire. If the phase difference $\Delta\phi$ is the inde-

[28] For the one-dimensional wire an artificial cutoff is not required for the momentum integral (3.10c); the integrand derived from the Ginzburg-Landau theory decreases sufficiently rapidly to assure proper convergence. This is not the case in two or three dimensions, where artificial cutoffs and special T_c-renormalization procedures are required, as discussed in the literature cited in the Appendix. Even in one dimension we do not actually neglect the entropy contribution from fluctuations with wavelengths short compared to $\xi(0)$, where the Ginzburg-Landau functional is no longer valid; these fluctuations are responsible for the temperature dependence of the coefficients in the Ginzburg-Landau free-energy functional.

pendent thermodynamic variable (as we have assumed above[17]) and if $\kappa=\Delta\phi/l$ as in (2.10b), the Ginzburg-Landau minimum free energy consistent with this constraint is

$$F_{\rm GL}(\kappa)=-(1-\kappa^2)^2\sigma l H_c^2(T)\xi(T)/8\pi, \quad (5.3)$$

to within an unimportant additive constant. If we add to this the entropy contribution from order-parameter fluctuations about the solution (2.10), we obtain the true total free energy

$$F_{\rm tot}(\kappa)=F_{\rm GL}(\kappa)-TS(\kappa) \quad (5.4a)$$

$$=-(1-\kappa^2)^2\sigma l H_c^2(T)\xi(T)/8\pi + lk_B(1-3\kappa^2)^{1/2}/\sqrt{2}. \quad (5.4b)$$

The average current follows from the thermodynamic relation[6,17]

$$\bar{I}=(2\pi c/l\Phi_0)(\partial/\partial\kappa)F_{\rm tot}(\kappa) \quad (5.5a)$$

$$=\kappa(1-\kappa^2)\sigma H_c^2(T)\xi(T)c/\Phi_0 - 3\pi k_B T\sqrt{2}\kappa c/\Phi_0(1-3\kappa^2)^{1/2}, \quad (5.5b)$$

where $\Phi_0=ch/2e$ is the flux quantum. The first term is just the current (2.7) appropriate to the particular minimum-energy configuration (2.10). The second term is the first correction to this current due to macroscopic thermal fluctuations of the order parameter about the solution (2.10) but with $\Delta\varphi$ constrained to have its initial value. For $\kappa^2<\frac{1}{3}$, the second term of (5.5b) will be much less than the first term as long as $\Delta F_-\gg k_BT$.

It is interesting to note from Eqs. (2.13), (2.35), and (4.35) that the transition-rate ratio

$$\Gamma_+(\kappa,T)/\Gamma_-(\kappa,T) =[\Omega_+(\kappa,T)/\Omega_-(\kappa,T)] \times\exp\{-[\Delta F_+(\kappa,T)-\Delta F_-(\kappa,T)]/k_BT\} \quad (5.6a)$$

$$=\exp(-\bar{I}\Phi_0/ck_BT), \quad (5.6b)$$

where $\bar{I}$ is the average current (5.5). The relative proportion of current-increasing and current-decreasing transitions is determined by the *actual* system current $\bar{I}$, including fluctuation corrections.

The rate of decay of a current in a closed superconducting loop can be readily converted to give the voltage drop across a wire in an external circuit with a given dc current $\bar{I}$. This voltage drop is simply

$$V=\Phi_0\Gamma_-(\kappa,T)[1-\exp(-\bar{I}\Phi_0/ck_BT)], \quad (5.7)$$

with κ related to $\bar{I}$ by 5.5.

In conclusion, the qualitative arguments of Sec. I summarize the most important components of the prefactor Ω which, to within a few orders of magnitude, is $\Omega=l/\tau(T)$, where l is the length of the wire in units of the coherence length $\xi(T)$ and $\tau(T)$ is some microscopic time. For the time-dependent Ginzburg-Landau model described in Sec. II, $\tau(T)$ is defined in (1.4), but slightly different definitions might be relevant in other models.

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APPENDIX: GINZBURG-LANDAU FREE ENERGY

The criterion (2.4) for the validity of the Ginzburg-Landau free-energy functional is essentially the modification for a one-dimensional system of the criterion given by Ginzburg.[29] Ginzburg's argument was based on self-consistency and was not really rigorous; nevertheless, a number of calculations have supported the Ginzburg hypothesis. Thouless has calculated the first correction to the specific heat of a superconductor above T_c and found a significant enhancement of the specific heat if, and only if, $|T-T_c|$ fell within the range determined by the Ginzburg criterion.[30] More recent calculations of the free energy of an Ising model with long-range forces as a function of temperature and uniform external magnetic field have also supported the Ginzburg criterion.[31]

The Ginzburg-Landau theory is valid for temperatures very close to T_c, for the superconductor, and for the Ising model with long-range forces, because for those systems the zero-temperature coherence length $\xi(0)$ is very large compared to the interatomic spacing. The Ginzburg-Landau parameters for the long superconducting wire can be calculated in principle from the band structure and effective electron-electron interaction in the usual BCS or strong-coupling manner, if $\xi(0)$ is large compared to the wire diameter. If the diameter is larger than $\xi(0)$, the temperature for breakdown of the Ginzburg-Landau theory given by (2.4) moves closer to T_c, and it is probably necessary to consider the renormalization of T_c due to fluctuations not included in the BCS theory.

There has also been considerable theoretical and experimental investigation of the increase in electrical conductivity of superconductors for T just above T_c.[32] At least in the case of a dirty superconductor, the temperature at which the enhancement of the electrical conductivity becomes comparable to the normal conductivity is essentially the same as the temperature given by the Ginzburg criterion.[32]

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[29] V. L. Ginzburg, Fiz. Tverd. Tela **2**, 2031 (1960) [Soviet Phys.—Solid State **2**, 1824 (1960)].
[30] D. J. Thouless, Ann. Phys. (N. Y.) **10**, 553 (1960).
[31] D. J. Thouless, Phys. Rev. **181**, 954 (1969).
[32] Compare the review by P. C. Hohenberg, in *Proceedings of the Eleventh International Conference on Low Temperature Physics, 1968*, edited by J. F. Allen, D. M. Finlayson, and D. M. McCall (University of St. Andrews Press, St. Andrews, Scotland, 1968), Vol. 1, p. 33.

The contribution of fluctuations in $\psi(x)$ to the electrical conductivity of a wire above T_c has been calculated from the time-dependent Ginzburg-Landau equation (2.14) by Schmidt[33] and by Abrahams and Woo.[5] Their results agree with the Green's function calculations of Aslamazov and Larkin.[34] In principle, at least, the Fokker-Planck equation resulting from (2.14) and the associated Langevin noise source (2.20) could be solved to obtain the electrical conductivity of the time-dependent Ginzburg-Landau model at intermediate values of the temperature as well, i.e., for $|T-T_c|$ so small that the Ginzburg criterion is violated. The resulting conductivity would clearly vary smoothly between the Aslamazov-Larkin form well above T_c and the results of the present paper well below T_c. In the absence of such a calculation, one may still interpolate graphically between the two asymptotic forms, and a reasonable approximation to the answer may be obtained. By contrast, it is difficult to find a simple interpolation between the results of the LA theory and the conductivity predicted above T_c (see Fig. 1 of Ref. 32).

[33] H. Schmidt, Z. Physik **216**, 336 (1968). See also A. Schmid, Phys. Rev. **180**, 527 (1969).

[34] L. G. Aslamazov and A. I. Larkin, Fiz. Tverd. Tela **10**, 1104 (1968) [Soviet Phys.—Solid State **10**, 875 (1968)]; Phys. Letters **26A**, 238 (1968).

Marčelja[35] has given an approximate treatment of the time-dependent Ginzburg-Landau model, which enables him to calculate the electrical resistance in a one-dimensional geometry for all temperatures in the vicinity of T_c. Marčelja's approximation, which may be described as an effective linearization of the Ginzburg-Landau equation, agrees with the Aslamazov-Larkin results above T_c, but predicts a conductivity proportional to $(T_c-T)^3$, for a thin wire below T_c, in disagreement with the present results. (Marčelja's formulas, of course, vary smoothly between the asymptotic forms for $T>T_c$ and $T<T_c$.) We believe that Marčelja's results are incorrect in the asymptotic region below T_c because his effective linearization eliminates the free-energy barrier that is a necessary consequence of the nonlinear Ginzburg-Landau free-energy functional below T_c.

It is also worth pointing our that the calculations of the electrical conductivity above T_c require not only that one can use the time-dependent Ginzburg-Landau equation to calculate the contribution to the conductivity of fluctuations in the order parameter, but that one can neglect any temperature dependence of the background "normal" or "quasiparticle" conductivity. In the asymptotic region below T_c, the conductivity is much larger then the quasiparticle conductivity, and we need not consider any temperature variations of the latter quantity.

[35] S. Marčelja, Phys. Letters **28A**, 180 (1968); W. E. Masker, S. Marčelja, and R. D. Parks, Phys. Rev. (to be published).