

Direct Reconstruction of Two-Dimensional Currents in Thin Films from Magnetic-Field Measurements

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An accurate determination of microscopic transport and magnetization currents is of central importance for the study of the electric properties of low-dimensional materials and interfaces, of superconducting thin films, and of electronic devices. Current distribution is usually derived from the measurement of the perpendicular component of the magnetic field above the surface of the sample, followed by numerical inversion of the Biot-Savart law. The inversion is commonly obtained by deriving the current stream function g , which is then differentiated in order to obtain the current distribution. However, this two-step procedure requires filtering at each step and, as a result, oversmooths the solution. To avoid this oversmoothing, we develop a direct procedure for inversion of the magnetic field that avoids use of the stream function. This approach provides enhanced accuracy of current reconstruction over a wide range of noise levels. We further introduce a reflection procedure that allows for the reconstruction of currents that cross the boundaries of the measurement window. The effectiveness of our approach is demonstrated by several numerical examples.

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I. INTRODUCTION

Determination of two-dimensional current distribution from measurement of the normal component of a magnetic field is an important and commonly used tool for the investigation of a wide range of physical systems, including high-temperature superconductors [1–6], topological states of matter [7,8], oxide heterostructures [9,10], and carbon nanotubes and nanostructures [11–13], as well as for the nondestructive evaluation of semiconductor circuits [14]. Mapping of the local magnetic fields is commonly attained by scanning Hall probes [1,15–20], Hall-probe arrays [21,22], magneto-optical imaging [2–4,23–31], and scanning superconducting quantum-interference devices (SQUIDs) [7,16,32–39]. These techniques generate micrometer-to-millimeter-scale two-dimensional images of the normal component of the magnetic field $B_z(x, y)$ above a sample. Recently, however, nanoscale magnetic imaging has become a rapidly developing area of metrology, based on technological advances in scanning nitrogen-vacancy centers in nanodiamonds [11,40–43], nano-SQUIDs [44–52], and cold atom chips [53–55]. These techniques have the potential to provide higher spatial resolution, nanoscale proximity to the sample surface, improved field sensitivity, and lower measurement noise.

To take advantage of these recent developments in nanoscale magnetic imaging, accurate analytical methods for the reconstruction of electric currents are required. Reconstruction of current distribution from the measured

out-of-plane magnetic field requires inversion of the Biot-Savart law, which poses a number of challenges [1–3, 56–59]. First, the inversion equation, formulated as a Fredholm integral equation of the first kind, is ill posed, resulting in amplification of the high-spatial-frequency components during the inversion process. In fact, high frequencies are never negligible in practice and therefore dominate the solution unless they are damped during the inversion [60]. Thus, a naive inversion of the Biot-Savart law is unstable and must be regularized. The second complication arises from the long-range nature of current-induced magnetic fields. The magnetic field in the imaged area can be affected by currents flowing outside the field of view, making the inversion equation underdetermined. Therefore, in order to obtain an accurate and unique solution, one must make assumptions about the behavior of the current outside the measurement window. This problem is usually resolved by assuming that the entire current distribution is encompassed in the measurement window, similarly to the case where magnetization currents flow in closed loops. However, in the case of externally applied transport currents that significantly contribute to the measured field and necessarily cross the boundaries of the imaged area, this assumption is invalid and does not constitute a good approximation.

Various approaches that address the instability of the inversion of the Biot-Savart law have been utilized to date. These approaches introduce additional control parameters such as a cutoff frequency [3,56] or a limitation on the number of numerical iterations [57,58]. However, none of these methods provide a systematic means for determining the optimal control parameters, with the exception of the

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work of Feldmann [59], who recognized the inversion problem as mathematically ill posed. To overcome this difficulty, Feldmann used the Tikhonov regularization scheme [61] for current reconstruction, in which a free regularization parameter is used for controlling the smoothness of the solution. He then applied the generalized-cross-validation (GCV) method [62] for methodical determination of the regularization parameter. However, the method of Feldmann presented in Ref. [59] is not accurate at low heights above the sample. This lack of accuracy is particularly disadvantageous for next-generation techniques, which aim to provide magnetic imaging for current reconstruction at nanometer heights above the sample surface in order to improve the sensitivity and the spatial resolution [11,45,55].

The instability problem impacting commonly used inversion methods for the Biot-Savart law is further exacerbated by the use of an auxiliary stream function for the inversion [3,57–59]. Specifically, these methods determine the current distribution $\mathbf{J}(x, y)$ in the sample by means of a two-step procedure. First, the stream function $g(x, y)$ is derived by inversion from the measured magnetic field $B_z(x, y)$, and the current is then determined from the g function using the relation

$$\mathbf{J} = \nabla \times [g(x, y)\hat{z}]. \quad (1.1)$$

In these two-step inversion methods (which we term GI methods), each of the steps is able to amplify the noise. In the first step, the noise is controlled by a regularization procedure that filters high spatial frequencies from the reconstruction. The resulting reconstructed g , however, is usually not sufficiently smooth to be differentiated with a regular numerical differentiation, which significantly amplifies any remaining noise. Consequently, it is necessary to apply an additional smoothing filter to g , such as the Savitzky-Golay filter proposed by Feldmann [59], prior to differentiation. Application of a second filter in addition to the Tikhonov filter results in the smoothing of fine details in the solution that would have been otherwise preserved. Preservation of fine details is important in many cases. One such example is the reconstruction of sharp one-dimensional paths of higher current density at oxide interfaces [9], where oversmoothing can lead to inconclusive results. An exception to the above methodology in which a one-step procedure is used was proposed in Ref. [56]. However, the solution of the direct problem given in this paper differs from ours and does not use a rigorous regularization.

As mentioned above, the presence of external contributions to the magnetic field makes the inversion procedure more difficult. As far as we know, this problem has not yet been addressed systematically, and all of the methods cited above assume that the entire current distribution is contained in the measurement window. This is a severe restriction for most experimental setups, even in the

absence of an external transport current that requires the use of an enlarged measurement window to ensure enclosure of all the currents. In this work, the instability of the inverse problem and the presence of currents crossing the image boundary, which challenge the magnetic-field inversion schemes, are addressed by the introduction of a number of alternative procedures, as detailed below.

- (i) We introduce an inversion method utilizing the Tikhonov regularization in which the current distribution is obtained through a single-step inversion of the measured $B_z(x, y)$, without the need for the intermediate derivation and differentiation of the stream function g . We show that this direct-inversion (DI) scheme provides substantial improvement in the accuracy of current reconstruction over a wide range of noise levels. We also find that the quality of the reconstructions is not very sensitive to the exact value of the imaging height above the sample. This property is important, as the exact height is usually not known in practice.
- (ii) We develop two systematic procedures to determine the free Tikhonov regularization parameter in conjunction with the DI method based on GCV (DI GCV) and on Stein’s unbiased risk estimate (SURE) [63–65], which we refer to as DI SURE. For $B_z(x, y)$ imaged at low heights h above the sample, the two procedures give comparable results; however, at larger h values, DI SURE is preferable.
- (iii) We introduce a reflection procedure addressing the transport-current challenge. By symmetrically extending the $B_z(x, y)$ image, we show that a reliable inversion can be attained in the case of currents crossing the boundary of the magnetic image. This reflection procedure performs best in conjunction with the DI-SURE regularization.
- (iv) We improve the existing GI-GCV method and develop an alternative GI-SURE method, both of which can handle transport currents.
- (v) The four schemes DI GCV, DI SURE, GI GCV, and GI SURE are applied to solve specific numerical examples, and their solutions are analyzed and compared showing the advantages and limitations of each method.
- (vi) A user-friendly code is provided for all four inversion methods [66].

This paper is organized as follows. In Sec. II, we briefly describe the GI method. In Sec. III, we present our DI method for current reconstruction. In order to recover currents crossing the image boundary, we introduce the reflection scheme in Sec. III C and the DI-SURE method in Sec. III D. In Sec. IV, we present and discuss numerical results of two-dimensional current reconstruction using the GI and DI methods. An algorithm for the noise-variance estimation required for the SURE parameter-choice method is presented in the Appendix.

II. STREAM-FUNCTION GI METHOD

A. Forward problem

We begin by defining the problem and summarizing the stream-function GI method [59]. The current \mathbf{J} flows in a three-dimensional thin film of thickness d , bounded in space by $-d/2 \leq z \leq d/2$, $0 \leq x \leq w$, and $0 \leq y \leq l$. The measurement plane is parallel to the surface of the sample and to the x - y plane. Inside the film, \mathbf{J} is static, depends on x and y , and is uniform along the z axis inside the sample. The magnetic field is measured at height $h = z - d/2$ above the sample, where z is the z coordinate of the measurement plane. We assume the field detector to be sensitive only to the z component of the magnetic field, and small enough that its nonzero sensing area does not distort the reconstructed current.

The experimentally measured field distribution $B_z(x, y)$ at height z is related to the true currents in sample \mathbf{J}_{true} and their corresponding stream function g_{true} through

$$B_z(x, y, z) = K(x, y, z) * g_{\text{true}}(x, y) + N(x, y), \quad (2.1)$$

where $N(x, y)$ is an additive noise of zero mean and constant variance σ^2 , the kernel K can take different forms depending on the assumptions of the problem, and the convolution of $F(x, y)$ and $f(x, y)$ is given by

$$F(x, y) * f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x - x', y - y') f(x', y') dx' dy'. \quad (2.2)$$

For reconstruction of volume currents in a thin film with a non-negligible thickness d , the kernel is given by

$$K(x, y, z, d) = \frac{\mu_0}{4\pi} \left(\frac{z - d/2}{[x^2 + y^2 + (z - d/2)^2]^{3/2}} - \frac{z + d/2}{[x^2 + y^2 + (z + d/2)^2]^{3/2}} \right), \quad (2.3)$$

where μ_0 is the permeability of free space. We define the two-dimensional Fourier transform and its inverse as

$$\hat{f}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i(ux+vy)} dx dy, \quad (2.4)$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(u, v) e^{2\pi i(ux+vy)} du dv, \quad (2.5)$$

respectively, abbreviated as $\hat{f} = \mathcal{F}[f]$ and $f = \mathcal{F}^{-1}[\hat{f}]$. The Fourier transform of Eq. (2.3) can be evaluated analytically as

$$\hat{K}(u, v, z, d) = \mu_0 e^{-2\pi\sqrt{u^2+v^2}z} \sinh\left(\pi d\sqrt{u^2+v^2}\right). \quad (2.6)$$

If $d \ll h$, we can use the concept of sheet current, which assumes that currents are confined to an infinitesimally thin film, with the corresponding kernel given by

$$K(x, y, z) = \frac{\mu_0}{4\pi} \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}, \quad (2.7)$$

and its Fourier transform given by

$$\hat{K}(u, v, z) = \mu_0 \pi \sqrt{u^2 + v^2} e^{-2\pi\sqrt{u^2+v^2}z}. \quad (2.8)$$

Both kernels (2.3) and (2.7) can be thought of as low-pass filters with a cutoff frequency governed by the imaging height z . As such, they make the problem of approximating $g_{\text{true}}(x, y)$ in Eq. (2.1) ill posed and require regularization for a proper reconstruction [59].

Note that kernels (2.3) and (2.7) and their matching stream functions have different dimensions. In the case of a film of thickness d , the current density \mathbf{J} is given in units of A/m², g in units of A/m, and K in units of T/A m. In the case of sheet currents, \mathbf{J} , g , and K are given in units of A/m, A, and T/A m², respectively.

B. Inverse problem

The approximation of $g_{\text{true}}(x, y)$ in Eq. (2.1) by Tikhonov regularization for a measured magnetic field B_z consists of finding the g_λ that solves the problem

$$\min_{g_\lambda} (\|K * g_\lambda - B_z\|_2^2 + \lambda \|Lg_\lambda\|_2^2) \quad (2.9)$$

for a given regularization parameter λ and regularization operator L , where the 2-norm is defined as

$$\|f(x, y)\|_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 dx dy. \quad (2.10)$$

The regularization parameter λ in Eq. (2.9) sets the balance between a solution dominated by noise for small λ values and an oversmoothed solution for large λ values. In order to penalize nonsmooth solutions, we define $L = \nabla^2$ by following Ref. [59]. It can be shown that the minimizer of Eq. (2.9) is given by

$$g_\lambda(x, y) = \mathcal{F}^{-1} \left[\frac{\bar{\hat{K}}(u, v) \hat{B}_z(u, v)}{|\hat{K}(u, v)|^2 + \lambda (2\pi)^4 (u^2 + v^2)^2} \right], \quad (2.11)$$

where a bar denotes complex conjugation. The current distribution can be found similarly to Eq. (1.1) using

$$\mathbf{J}_\lambda = \nabla \times [g_\lambda(x, y) \hat{z}]. \quad (2.12)$$

We note that the stream function g_λ is defined up to a gradient term, whereas the current \mathbf{J}_λ is unique [58].

The regularization parameter λ can be estimated using the GCV method [62], which seeks to approximately minimize the predictive mean-square error (PMSE), $\|K * g_{\text{true}} - K * g_\lambda\|_2^2$, where g_{true} is the unknown true stream function. Since g_{true} is not known, the GCV method minimizes a slightly different function than the PMSE and is given by

$$G_1(\lambda) = \frac{\|B_z - K * g_\lambda\|_2^2}{[T_1(\lambda)]^2}, \quad (2.13)$$

where $T_1(\lambda)$ indicates the residual effective degrees of freedom used in regression analysis (see p. 63 of Ref. [62]) and, in our case, it formally equals

$$T_1(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1 - \frac{|\hat{K}(u, v)|^2}{|\hat{K}(u, v)|^2 + \lambda(2\pi)^4(u^2 + v^2)^2} dudv. \quad (2.14)$$

A more intuitive presentation of the GCV method and its connection to the PMSE can be found in Ref. [67].

C. Numerical implementation

In practice, the magnetic field is sampled on a rectangular grid with N points in the x direction, distanced Δx units apart, and M points in the y direction, distanced Δy units apart. Thus, the physical space grid consists of the points $(n\Delta x, m\Delta y)$ for $n = 0, 1, \dots, N-1$ and $m = 0, 1, \dots, M-1$ and the frequency space grid of the points $[(k/N\Delta x), (l/M\Delta y)]$ for $k = 0, 1, \dots, N-1$ and $l = 0, 1, \dots, M-1$. We can approximate Eq. (2.11) on the discrete grid by using the discrete Fourier transform (DFT) and its inverse, defined as

$$\hat{f}_{kl} = \Delta x \Delta y \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{nm} e^{-i2\pi kn/N - i2\pi lm/M}, \quad (2.15)$$

$$f_{nm} = \frac{1}{\Delta x \Delta y} \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}_{kl} e^{i2\pi kn/N + i2\pi lm/M}, \quad (2.16)$$

respectively, abbreviated as

$$\hat{f}_{kl} = \text{DFT}[f_{nm}]_{kl}, \quad (2.17)$$

$$f_{nm} = \text{DFT}^{-1}[\hat{f}_{kl}]_{nm}. \quad (2.18)$$

Then we can approximate Eq. (2.11) as

$$g_{nm} = \text{DFT}^{-1} \left[\frac{\hat{K}_{kl} \hat{B}_{kl}}{|\hat{K}_{kl}|^2 + \lambda |\hat{\nabla}_{kl}^2|^2} \right]_{nm}, \quad (2.19)$$

where $\hat{K}_{kl} = \hat{K}[(k/N\Delta x), (l/M\Delta y), z]$ is defined in either Eq. (2.6) or Eq. (2.8), the Laplacian is approximated by the second-order central finite difference and $\hat{B}_{kl} = \text{DFT}[B_{nm}]_{kl}$. Note also that because Eq. (2.19) employs DFT for the inversion, it implicitly assumes periodic boundary conditions at the boundaries of the measurement window. In the presence of currents crossing the boundaries, this assumption leads to highly inaccurate reconstructions, as discussed in Sec. III C, making this inversion method inapplicable in such cases.

The stream function g_{nm} reconstructed from a noisy measurement of B_{nm} is not smooth. Therefore, estimation of electric current using Eq. (2.12) by a simple numerical differentiation is not accurate and will amplify any noise left in g_{nm} . A more appropriate method for computation of the derivatives in this case is the Savitzky-Golay differentiation filter [68], which fits a polynomial of degree p to each set of $2q + 1$ successive data points by least squares. In this paper, the current \mathbf{J}_λ is estimated by the differentiation of the fitted polynomial, using $p = 2$ and $q = 2$, as suggested in Ref. [59].

For the GI-GCV method, the regularization parameter λ in Eq. (2.19) is found using the GCV scheme (2.13). The discrete version of the function $G_1(\lambda)$ is given by [69]

$$G_1(\lambda) = \frac{\sum_{k=0}^{N-1} \sum_{l=0}^{M-1} (1 - \hat{Z}_{kl})^2 |\hat{B}_{kl}|^2}{(NM - \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{Z}_{kl})^2}, \quad (2.20)$$

where

$$\hat{Z}_{kl} = \frac{|\hat{K}_{kl}|^2}{|\hat{K}_{kl}|^2 + \lambda |\hat{\nabla}_{kl}^2|^2}. \quad (2.21)$$

The regularization parameter λ is then estimated as the minimizer of the function $G_1(\lambda)$.

It is important to note that evaluation of the kernel using the discrete transform

$$\hat{K}_{kl} = \text{DFT}[K(n\Delta x, m\Delta y, z)]_{kl}, \quad (2.22)$$

as suggested in Ref. [59], should be avoided due to the large inaccuracy of this approximation compared to the exact expressions in Eqs. (2.6) and (2.8), especially for small heights. This difference in accuracy is demonstrated in Fig. 1, where we measure the accuracy of current reconstruction in the absence of noise by the mean-square deviation (MSD) defined as

$$\begin{aligned} \text{MSD} &= \frac{\|\mathbf{J}_{\text{true}}(x, y) - \mathbf{J}_\lambda(x, y)\|_2^2}{\|\mathbf{J}_{\text{true}}(x, y)\|_2^2} \\ &= \frac{\|j_{x,\text{true}} - j_{x,\lambda}\|_2^2 + \|j_{y,\text{true}} - j_{y,\lambda}\|_2^2}{\|j_{x,\text{true}}\|_2^2 + \|j_{y,\text{true}}\|_2^2}. \end{aligned} \quad (2.23)$$

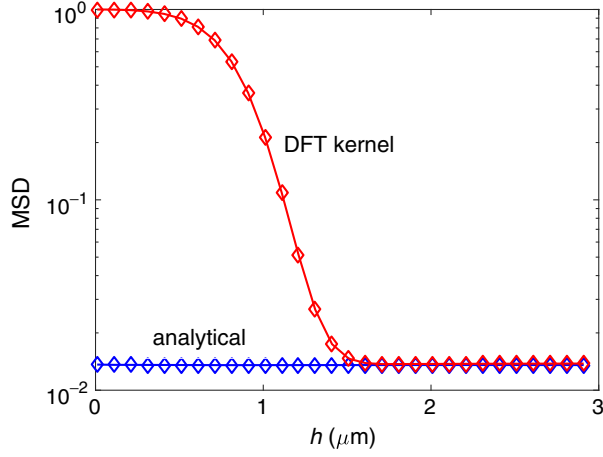


FIG. 1. The MSD of current reconstruction vs the field measurement height h , obtained using the analytical kernel (2.6) and the DFT kernel (2.22). The calculation is performed on sample A (as described in Sec. IV) with no additive noise, using the GI-GCV scheme with grid size $\Delta x = 1 \mu\text{m}$. The DFT kernel becomes accurate only when $h > 1.7 \mu\text{m}$.

Here, \mathbf{J}_{true} is the actual current in sample A (as described in Sec. IV), and \mathbf{J}_λ is the current reconstructed from the calculated magnetic field at height h using either the analytical kernel (2.6) or the DFT kernel (2.22). Figure 1 shows that the DFT kernel introduces a large error and cannot be used for heights smaller than about twice the grid spacing, which, in this example, is $\Delta x = 1 \mu\text{m}$.

To summarize, the GI method constitutes an inversion of the magnetic field by having us take the following steps:

- (1) Estimate the regularization parameter λ as the minimizer of $G_1(\lambda)$ in Eq. (2.20).
- (2) Compute the stream function $g(\lambda)$ using Eq. (2.19).
- (3) Obtain the currents \mathbf{J}_λ by applying the Savitzky-Golay differentiation filter to g as described above, with $p = q = 2$.

In the following section, we develop an alternative method that does not require the intermediate computation of the stream function g .

III. THE DI OF THE BIOT-SAVART LAW

In this section, we introduce an alternative formulation of the inversion problem, which produces higher quality reconstructions, particularly in the presence of low noise.

In addition, we present both a reflection procedure for the reconstruction of currents crossing the image boundaries and a projected SURE method for determination of the regularization parameter in this case.

A. The forward problem

The forward problem of calculating the magnetic field, given distribution of the current, requires the solution of the Biot-Savart law, which is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}', \quad (3.1)$$

where \mathbf{r} is the observation coordinate, \mathbf{r}' is the source coordinate, and $\mathbf{J} = j_x(x, y)\hat{x} + j_y(x, y)\hat{y}$ is the current-density vector field. The z component of the magnetic field can be determined from Eq. (3.1) as

$$\begin{aligned} B_z(x, y, z) &= \frac{\mu_0}{4\pi} \int_{-(d/2)}^{d/2} \int_0^w \int_0^l \frac{(y - y')j_x(x', y') - (x - x')j_y(x', y')}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &\quad \times dx' dy' dz'. \end{aligned} \quad (3.2)$$

We can rewrite Eq. (3.2) as

$$\begin{aligned} B_z(x, y, z) &= \int_{-(d/2)}^{d/2} [A_1(x, y, z - z') * j_x(x, y) \\ &\quad + A_2(x, y, z - z') * j_y(x, y)] dz' \\ &= K_1(x, y, z, d) * j_x(x, y) + K_2(x, y, z, d) * j_y(x, y), \end{aligned} \quad (3.3)$$

where kernels A_1 and A_2 are given by

$$A_1(x, y, z) = \frac{\mu_0}{4\pi} \frac{y}{[x^2 + y^2 + z^2]^{3/2}}, \quad (3.4)$$

$$A_2(x, y, z) = \frac{\mu_0}{4\pi} \frac{-x}{[x^2 + y^2 + z^2]^{3/2}}, \quad (3.5)$$

and kernels K_1 and K_2 , which depend on the thin film thickness d , are given by

$$K_1(x, y, z, d) = \begin{cases} \frac{\mu_0}{4\pi} \frac{y}{x^2 + y^2} \left[\frac{z+d/2}{\sqrt{x^2 + y^2 + (z+d/2)^2}} - \frac{z-d/2}{\sqrt{x^2 + y^2 + (z-d/2)^2}} \right], & x^2 + y^2 > 0, \\ 0 & x^2 + y^2 = 0, \end{cases} \quad (3.6)$$

$$K_2(x, y, z, d) = \begin{cases} \frac{\mu_0}{4\pi} \frac{-x}{x^2 + y^2} \left[\frac{z-d/2}{\sqrt{x^2 + y^2 + (z-d/2)^2}} - \frac{z+d/2}{\sqrt{x^2 + y^2 + (z+d/2)^2}} \right], & x^2 + y^2 > 0, \\ 0 & x^2 + y^2 = 0, \end{cases} \quad (3.7)$$

for $|z| > d/2$. The analytical Fourier transforms of kernels A_1 and A_2 are

$$\hat{A}_1(u, v, z) = \begin{cases} -i \frac{\mu_0}{2} e^{-2\pi\sqrt{u^2+v^2}z} \frac{v}{\sqrt{u^2+v^2}}, & u^2 + v^2 > 0, \\ 0, & u^2 + v^2 = 0, \end{cases} \quad (3.8)$$

$$\hat{A}_2(u, v, z) = \begin{cases} i \frac{\mu_0}{2} e^{-2\pi\sqrt{u^2+v^2}z} \frac{u}{\sqrt{u^2+v^2}}, & u^2 + v^2 > 0, \\ 0, & u^2 + v^2 = 0, \end{cases} \quad (3.9)$$

while those of kernels K_1 and K_2 are

$$\hat{K}_1(u, v, z, d) = \begin{cases} -i \frac{\mu_0}{2\pi} e^{-2\pi\sqrt{u^2+v^2}z} \sinh(\pi d\sqrt{u^2+v^2}) \frac{v}{u^2+v^2}, & u^2 + v^2 > 0, \\ 0, & u^2 + v^2 = 0, \end{cases} \quad (3.10)$$

$$\hat{K}_2(u, v, z, d) = \begin{cases} i \frac{\mu_0}{2\pi} e^{-2\pi\sqrt{u^2+v^2}z} \sinh(\pi d\sqrt{u^2+v^2}) \frac{u}{u^2+v^2}, & u^2 + v^2 > 0, \\ 0, & u^2 + v^2 = 0. \end{cases} \quad (3.11)$$

If $h \gg d$, we can use the concept of sheet currents, in which case the magnetic field is given simply by

$$B_z(x, y, z) = A_1(x, y, z) * j_x(x, y) + A_2(x, y, z) * j_y(x, y), \quad (3.12)$$

without an integral in the z direction.

The relation (3.3) [or (3.12)] leads us to the following compatibility condition. By applying the Fourier transform to Eq. (3.12), we can present the relation for the zero mode ($u = v = 0$) as

$$\hat{B}_z(0, 0, z) = \hat{K}_1(0, 0, z) \hat{j}_x(0, 0) + \hat{K}_2(0, 0, z) \hat{j}_y(0, 0). \quad (3.13)$$

Since $\hat{K}_1(0, 0, z)$ and $\hat{K}_2(0, 0, z)$ are zero, the value of $\hat{B}_z(0, 0, z)$ should also be zero. In real space, the condition requires the mean value of B_z to be zero. Thus, the compatibility condition implies that the mean value of the currents cannot be deduced from the measured field since it does not contribute to this field. The reconstruction of the current is therefore possible only up to an additive constant that represents a uniform current in the physical space. On the positive side, Eq. (3.13) implies that our reconstruction is not sensitive to offsets in the magnetic field that are usually inflicted by external sources.

For a better understanding of the problem, we can find the length scale which characterizes kernels (3.8) and (3.9) on a grid. In the simple case of ($\Delta x = \Delta y = \Delta$), the kernels become dependent on one parameter, the ratio between the height z and the pixel size Δ . For kernels (3.10) and (3.11),

the same argument applies if the ratio between the height and the sample thickness d is kept constant. To see this dependence, we can rewrite our kernels in Fourier space outside the origin as

$$\hat{A}_1(u, v, z) = -i \frac{\mu_0}{2} e^{-2\pi\sqrt{k^2+l^2} \frac{z}{N\Delta}} \frac{l}{\sqrt{k^2+l^2}}, \quad (3.14)$$

where $k = 0, \dots, N$, $l = 0, \dots, M$, and Δ is the grid spacing. From Eq. (3.14), it is easy to see that, for fixed N and M values, it is only z/Δ that determines the decay of the kernel and the corresponding spatial resolution of the reconstructed currents. This finding is important because the kernel decay determines the smoothing effect of the kernel—and consequently the ill-conditioning and hence the difficulty of the reconstruction—as described in the next subsection.

B. The inverse problem

Equations (3.3) and (3.12) enable us to find the magnetic field from either the volume or the sheet-current distribution within the sample. The corresponding derivation of the currents from B_z thus requires solving the inversion problem with two kernels. This task may seem to be challenging and less controllable than the hitherto-used GI method, which involves only a single kernel. However, it is, in fact, more accurate, as it does not require the second Savitzky-Golay filter used in the GI method, thus enabling reconstruction of the finer details. In the following, we develop this alternative DI method and demonstrate its advantages.

Assuming an additive noise model as in Eq. (2.1), we can rewrite Eqs. (3.3) and (3.12) as

$$B_z(x, y) = K_1 * j_x(x, y) + K_2 * j_y(x, y) + N(x, y), \quad (3.15)$$

where K_1 and K_2 in Eqs. (3.6) and (3.7) can be replaced with kernels A_1 and A_2 in Eqs. (3.4) and (3.5), respectively, if $h \gg d$. The inversion of the Biot-Savart law (3.15) and determination of the currents $j_x(x, y)$ and $j_y(x, y)$ from Eq. (3.12) given B_z is ill posed. Therefore, similarly to Sec. II, we solve this problem using Tikhonov regularization by minimization of the following functional:

$$\min_{j_x, j_y} (\|K_1 * j_x + K_2 * j_y - B_z\|_2^2 + \lambda (\|L j_x\|_2^2 + \|L j_y\|_2^2)), \quad (3.16)$$

where the same parameter λ multiplies both penalty terms due to the lack of a directional preference in the problem, and we set $L = \nabla^2$ as in Sec. II B. For simplicity, we suppress in Eq. (3.16) the dependence of the kernels and of the currents on the variables x and y . The regularized solutions that minimize Eq. (3.16) are given by

$$j_x(\lambda) = \mathcal{F}^{-1} \left[\frac{\tilde{K}_1 \hat{B}_z}{|\hat{K}_1|^2 + |\hat{K}_2|^2 + \lambda (2\pi)^4 (u^2 + v^2)^2} \right], \quad (3.17)$$

$$j_y(\lambda) = \mathcal{F}^{-1} \left[\frac{\tilde{K}_2 \hat{B}_z}{|\hat{K}_1|^2 + |\hat{K}_2|^2 + \lambda (2\pi)^4 (u^2 + v^2)^2} \right]. \quad (3.18)$$

It is easy to verify that the reconstructed current (3.17) and (3.18) satisfies

$$\nabla \cdot \mathbf{J} = 0. \quad (3.19)$$

Owing to the compatibility condition (3.13), the dc components of the currents are not defined by Eqs. (3.17) and (3.18) and, as discussed above, cannot be reconstructed from the measured field. Therefore, we set them to zero, which is equivalent to assuming no uniform current flowing in the measurement window.

Discretizing Eqs. (3.17) and (3.18), as in the previous section, we obtain

$$j_{x, nm}(\lambda) = \text{DFT}^{-1} \left[\frac{\tilde{K}_{1, kl} \hat{B}_{kl}}{|\hat{K}_{1, kl}|^2 + |\hat{K}_{2, kl}|^2 + \lambda |\hat{\nabla}_{kl}^2|^2} \right]_{nm}, \quad (3.20)$$

$$j_{y, nm}(\lambda) = \text{DFT}^{-1} \left[\frac{\tilde{K}_{2, kl} \hat{B}_{kl}}{|\hat{K}_{1, kl}|^2 + |\hat{K}_{2, kl}|^2 + \lambda |\hat{\nabla}_{kl}^2|^2} \right]_{nm}, \quad (3.21)$$

where $\hat{K}_{j, kl} = \hat{K}_j[(k/N\Delta x), (l/M\Delta y), z]$ is given by the analytic expressions in either Eqs. (3.8) and (3.9) or

Eqs. (3.10) and (3.11), and the Laplacian is approximated by the central second-order finite-difference stencil.

In the presence of currents crossing the boundaries, a naive application of Eqs. (3.20) and (3.21) fails to produce an accurate solution due to the artifacts caused by the DFT, which assumes periodicity of the measured field, and due to the fact that these equations satisfy Eq. (3.19) everywhere, including the boundary. To overcome this problem, we apply a reflection rule to the measured magnetic field, as explained in the next subsection.

C. Reflection rule

In this section, we consider the inversion problem in the presence of currents flowing through the image boundary. An accurate reconstruction of the currents through the boundary requires knowledge of the magnetic field outside the imaged region. Absent such knowledge, the inversion equation (3.15) becomes underdetermined and does not have a unique solution. A naive application of the DFT, as in Eq. (2.19) or Eqs. (3.17) and (3.18), assumes periodic boundary conditions, extending the currents periodically to infinity. Since the measured field produced by currents that cross the boundary of the measurement window is, in general, nonperiodic, application of periodic boundary conditions in this case creates a discontinuity at the boundary. This discontinuity, in turn, causes Gibbs oscillations of the reconstructed current at the same boundary. In addition, the current conservation property (3.19), fulfilled by either Eq. (1.1) in the GI method or Eqs. (3.17) and (3.18) in the DI method, forces an incorrect closure of the current loops inside the reconstruction window if periodic boundary conditions are used. Thus, to handle current distributions extending beyond the measurement window, we must either supply information about the field outside the measurement window—which is not generally available—or implement more appropriate boundary conditions for the currents, which we develop in this section.

To implement more appropriate boundary conditions for the currents, we suggest replacing the image of the magnetic-field measurement with an extended image, such that the magnetic field outside the measurement area is a mirror image of the field inside the boundaries. Specifically, we symmetrically extend B_z by embedding it into a larger matrix \tilde{B}_z , such that

$$\tilde{B}_z = \begin{pmatrix} B_{rc} & B_c \\ B_r & B_z \end{pmatrix}, \quad (3.22)$$

where B_c is obtained from B_z by flipping its columns, B_r is obtained by flipping the rows, and B_{rc} is obtained by flipping both. The solution is then obtained by substituting \tilde{B}_z into Eqs. (3.17) and (3.18) for B_z and taking only the part of the Tikhonov solution in the original window.

The reflections in Eq. (3.22) ensure a continuous flow of the current across the different boundaries of the image by closing currents outside the measurement window, as shown in the following analysis. For simplicity of presentation, the analysis is carried out in continuous space. We examine first the effect of the reflection upon the reconstruction using the GI method. Since the reconstructed currents given by Eqs. (3.17) and (3.18) are translationally invariant due to their implicit periodic extension by the DFT, we consider here only two boundaries, $x = 0$ and $y = 0$, where the reflections in Eq. (3.22) ensure $B_z(-x, y) = B_z(x, y)$ and $B_z(x, -y) = B_z(x, y)$, respectively. We also need to recall that the convolution of two odd or two even functions is even, and the convolution of an odd and an even function is odd.

The kernel K , from either Eq. (2.3) or Eq. (2.7), is even in both the x and y directions. Disregarding the noise term in Eq. (2.1), we deduce that, if $B_z(-x, y) = B_z(x, y)$, then the function g_{true} should also be even [$g_{\text{true}}(-x, y) = g_{\text{true}}(x, y)$] and, since the derivative of an even function is odd, and vice versa, we obtain

$$j_x(-x, y) = j_x(x, y), \quad (3.23)$$

$$j_y(-x, y) = -j_y(x, y). \quad (3.24)$$

On the other hand, if $B_z(x, -y) = B_z(x, y)$, then $g_{\text{true}}(x, -y) = g_{\text{true}}(x, y)$. Using the same argument, we obtain

$$j_x(x, -y) = -j_x(x, y), \quad (3.25)$$

$$j_y(x, -y) = j_y(x, y). \quad (3.26)$$

Therefore, upon crossing the boundary, the component of the current perpendicular to the boundary remains unchanged, whereas the component parallel to the boundary changes its sign, as shown in Fig. 2.

To obtain similar results when the noise term is non-negligible, we recall that the inverse Fourier transform of a real and even function is even and that of an imaginary and odd function is odd. Next, we rewrite Eq. (2.11) used for reconstruction of g_λ as

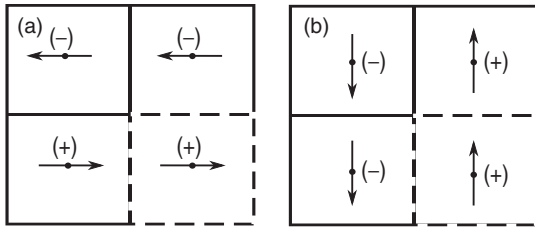


FIG. 2. A schematic description of the direction of currents (a) j_x and (b) j_y upon implementation of the symmetric extension of the field. The measurement window is marked by a dashed line.

$$g_\lambda(x, y) = \mathcal{F}^{-1} \left[\frac{\tilde{K}(u, v)}{|\hat{K}(u, v)|^2 + \lambda(2\pi)^4(u^2 + v^2)^2} \right] * B_z(x, y), \quad (3.27)$$

and since $\hat{K}(u, v)$ is even and real, we conclude that $g_\lambda(-x, y) = g_\lambda(x, y)$ and $g_\lambda(x, -y) = g_\lambda(x, y)$ for B_z even about $x = 0$ and $y = 0$, respectively. As a result, Eqs. (3.23)–(3.26) are satisfied by the Tikhonov solution g_λ .

Analysis of the effect of the symmetric extension on current reconstruction by the DI method is similar. Using either Eq. (3.3) or Eq. (3.12) and noting that K_1 and A_1 are even in x and odd in y , while K_2 and A_2 are odd in x and even in y , we deduce that if $B_z(-x, y) = B_z(x, y)$, then, using the aforementioned properties of convolution, we obtain Eqs. (3.23) and (3.24). Similarly, if $B_z(x, -y) = B_z(x, y)$, we find that Eqs. (3.25) and (3.26) are satisfied. This result is identical to the case of the GI method and is also illustrated by Fig. 2. Taking the noise into account, we use Eqs. (3.17) and (3.18) for reconstruction of the current field. Applying similar reasoning to that used above, we conclude that the currents obtained from Eqs. (3.17) and (3.18) also satisfy Eqs. (3.23)–(3.26).

D. Regularization parameter-choice methods

In the present section, we discuss the parameter-choice methods for the reconstruction of current densities using Eqs. (3.20) and (3.21). If the currents do not cross the boundaries, we can still use the GCV method similar to the one discussed in Sec. II B. The GCV for the DI method consists of minimization of the function

$$G_2(\lambda) = \frac{\|B_z - K_1 * j_x(\lambda) - K_2 * j_y(\lambda)\|_2^2}{T_2(\lambda)^2}, \quad (3.28)$$

where $T_2(\lambda)$ is formally given by

$$T_2(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - \hat{H}(\lambda)] dudv, \quad (3.29)$$

and

$$\hat{H}(\lambda) = \frac{|\hat{K}_1(u, v)|^2 + |\hat{K}_2(u, v)|^2}{|\hat{K}_1(u, v)|^2 + |\hat{K}_2(u, v)|^2 + \lambda(2\pi)^4(u^2 + v^2)^2}. \quad (3.30)$$

Similarly to Eq. (2.13), the function (3.28) is designed so that its minimum is close to the minimum of the PMSE, which is defined by

$$\text{PMSE}(\lambda) = \|B_{\text{true}} - B(\lambda)\|_2^2,$$

where $B_{\text{true}} = B_z - N$ is the true value of the magnetic field, $B(\lambda) = K_1 * j_x(\lambda) + K_2 * j_y(\lambda)$, and N is the

unknown noise. The discrete version of the GCV method is given by

$$G_2(\lambda) = \frac{\sum_{k=0}^{N-1} \sum_{l=0}^{M-1} [1 - \hat{H}_{kl}(\lambda)]^2 |\hat{B}_{kl}|^2}{[MN - \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{H}_{kl}(\lambda)]^2}, \quad (3.31)$$

where

$$\hat{H}_{kl}(\lambda) = \frac{|\hat{K}_{2,kl}|^2 + |\hat{K}_{1,kl}|^2}{|\hat{K}_{1,kl}|^2 + |\hat{K}_{2,kl}|^2 + \lambda |\hat{V}_{kl}^2|^2}. \quad (3.32)$$

The values of λ found using Eq. (3.28) are typically very close to optimal. However, they become unsatisfactory when currents cross the boundaries of the measurement window. Even though utilization of the reflection rule, achieved by substituting \tilde{B}_z [given by Eq. (3.22)] for B_z in Eq. (3.28) provides a significant improvement, the regularization may still be far from optimal. More accurate estimates of λ in this case can be obtained, however, by using the projected SURE method, which is similar to the method proposed in Refs. [64,65]. Particularly, let P denote a projection operator from the enlarged domain \tilde{B}_z to a region inside B_z . For example, in our numerical tests we choose the image of the projection P to contain the central 80% of the measured field B_z . To find the regularization parameter which gives the best reconstruction, we approximately minimize the projected PMSE norm $f(\lambda) = \|P[\tilde{B}_z - N - \tilde{B}(\lambda)]\|_2^2$, where $\tilde{B}(\lambda)$ is calculated using the currents $[j_x(\lambda)$ and $j_y(\lambda)]$ obtained by the inversion formulas (3.17) and (3.18) applied to the symmetrically extended magnetic field \tilde{B}_z . We can rewrite $f(\lambda)$ as

$$f(\lambda) = \|PN\|_2^2 + \|P[\tilde{B}_z - \tilde{B}(\lambda)]\|_2^2 - 2C(\lambda), \quad (3.33)$$

where, defining the ℓ_2 inner product by $\langle \dots, \dots \rangle_2$, the last term is given by

$$C(\lambda) = \text{Re}[\langle PN, P[\tilde{B}_z - \tilde{B}(\lambda)] \rangle_2]. \quad (3.34)$$

The first term on the right-hand side of Eq. (3.33) is independent of λ and therefore can be neglected. The second term in Eq. (3.33) can be easily calculated, while $C(\lambda)$ cannot be exactly calculated due to its dependance on an unknown noise N . However, it is possible to approximate $C(\lambda)$ as follows. First, we rewrite Eq. (3.34) as

$$C(\lambda) = \text{Re}[\langle PN, P(\tilde{B}_{\text{true}} + N) \rangle_2] - \text{Re}[\langle PN, P[B_{\text{true}}(\lambda) + N(\lambda)] \rangle_2], \quad (3.35)$$

where $N(\lambda) = \mathcal{F}^{-1}[\hat{H}(\lambda)\hat{N}]$, $B_{\text{true}}(\lambda) = \mathcal{F}^{-1}[\hat{H}(\lambda)\hat{B}_{\text{true}}]$, and \tilde{B}_{true} is the symmetrically extended version of B_{true} . We can then drop the terms $\text{Re}[\langle PN, P\tilde{B}_{\text{true}} \rangle_2]$ and

$\text{Re}[\langle PN, PB_{\text{true}}(\lambda) \rangle_2]$ as in Refs. [67,70] since their expected value is zero, so that

$$C(\lambda) \approx \hat{C}(\lambda) \equiv \|PN\|_2^2 - \text{Re}[\langle PN, PN(\lambda) \rangle_2]. \quad (3.36)$$

In the discrete version of the projected SURE method, we can approximate Eq. (3.36) by replacing the unknown noise N_{nm} with a known noise $N_{1,nm}$, which has the same mean and variance as N_{nm} [64]. Following Ref. [71], we choose $N_{1,nm}$ such that its components are either $+\sigma$ or $-\sigma$ with probability 0.5, where σ is the standard deviation of N_{nm} , which we estimate by a simple algorithm described in the Appendix. Using this method, the required regularization parameter can be found by minimizing the function

$$S(\lambda) = \left\| \text{PDFFT}^{-1} \left[\frac{\lambda |\hat{V}_{kl}^2|^2 (\hat{B}_{kl})}{|\hat{K}_{1,kl}|^2 + |\hat{K}_{2,kl}|^2 + \lambda |\hat{V}_{kl}^2|^2} \right] \right\|_2^2 - 2\hat{C}(\lambda), \quad (3.37)$$

where

$$\hat{C}(\lambda) = \|PN_1\|_2^2 - \text{Re}[\langle PN_1, \text{PDFFT}^{-1}[H_{kl}(\lambda)\hat{N}_{1,kl}] \rangle_2]. \quad (3.38)$$

The projected SURE for the GI-SURE scheme is obtained from Eqs. (3.37) and (3.38) by replacing $|\hat{K}_{1,kl}|^2 + |\hat{K}_{2,kl}|^2$ in Eq. (3.37) with $|\hat{K}_{kl}|^2$.

Thus, to apply the DI method, the following steps have to be taken:

- (1) Compute the extended field \tilde{B}_z using Eq. (3.22) (if currents cross the image boundary).
- (2) Compute λ by minimizing either Eq. (3.28) for DI GCV or Eq. (3.37) for DI SURE.
- (3) Obtain the currents using Eqs. (3.17) and (3.18), with \tilde{B}_z substituted for B_z .
- (4) Take only the currents lying inside the measurement window.

In Sec. II, we presented the algorithm for implementation of the GI method without discussing the possibility of symmetric extension of the field. Implementation of the GI method with the symmetric extension is similar to the DI method presented in this section in the sense that the calculations are performed using symmetrically extended data, and the result is taken from inside the measurement window. In the next section, we compare the performance of these two methods through several numerical examples.

IV. NUMERICAL RESULTS

In this section, we apply the above-proposed inversion algorithms to three examples of current distributions in thin films. Each example consists of a square sample with side length l_1 and a square hole in the center with side length l_2 . The circulating currents in the samples are determined by

numerically solving the Ginzburg-Landau equations in the presence of an applied external magnetic field [72]. The measured window, however, may contain only part of the sample and is further corrupted by noise,

$$B_{nm} = B_{\text{true},nm} + \sigma W_{nm}. \quad (4.1)$$

Here, $B_{\text{true},nm}$ is calculated using the Biot-Savart law while taking into account the currents that flow in the entire sample, and σW_{nm} is Gaussian white noise with the standard deviation $\sigma = s \max |B_{\text{true},nm}|$ and $s \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. Below, the magnetic field is given in units of gauss, the electric current in milliamperes, and the length in micrometers, so that $\mu_0 = 4\pi$. We set the grid step to $\Delta x = \Delta y = 1 \mu\text{m}$, the imaging height to $h = 1 \mu\text{m}$, and the thickness of the sample to $d = 0.05 \mu\text{m}$.

Sample *A* consists of a loop of outer and inner side lengths $l_1 = 161 \mu\text{m}$ and $l_2 = 83 \mu\text{m}$, respectively, and a clockwise current flow [see Fig. 3(a)]. The sample is entirely contained in the square measurement window of side length $223 \mu\text{m}$, making this example solvable without the reflection procedure.

Sample *B* consists of a square loop with $l_1 = 159 \mu\text{m}$ and $l_2 = 43 \mu\text{m}$ that carries a counterclockwise current flow in the inner part of the loop and a clockwise flow in the outer part, as shown in Fig. 3(d). The square measurement window with a side length $99 \mu\text{m}$ includes one corner of the loop only, making the use of reflection necessary for accurate current reconstruction.

The third example, sample *C*, shown in Fig. 3(g), consists of a square loop with $l_1 = 161 \mu\text{m}$ and $l_2 = 41 \mu\text{m}$. The loop has several vortices distributed in the sample. The measurement window of side length $151 \times 61 \mu\text{m}$ cuts the loop from all four sides. The bottom cut is very close to the cores of the vortices, representing a challenging inversion problem that can be dealt with by our reflection rule as demonstrated below.

The true current densities in samples *A*, *B*, and *C* are shown in Figs. 3(a), 3(d), and 3(g). Magnetic fields generated by the currents are corrupted by noise with $s = 10^{-1}$ and are shown in the central column [Figs. 3(b), 3(e), and 3(h)]. In the right-hand column [Figs. 3(c), 3(f), and 3(i)], we present the current densities reconstructed from the corresponding magnetic fields in the central column using the DI-SURE scheme. Comparing the left- and right-hand columns, we can conclude that the quality of the reconstruction is high, notwithstanding the high noise level in the central column. The success of current reconstruction in sample *C* is particularly impressive in view of the presence of vortices that are cut through by the measurement window, the reconstruction of which can be assumed to require more sophisticated boundary conditions.

In order to highlight the difference between the GI and DI methods, we present in Fig. 4 the results of current reconstruction in samples *A*, *B*, and *C* in the presence of low noise with $s = 10^{-3}$. The true current densities for these three samples are shown in Figs. 4(a), 4(e), and 4(i), and their reconstructions using a prior symmetric extension

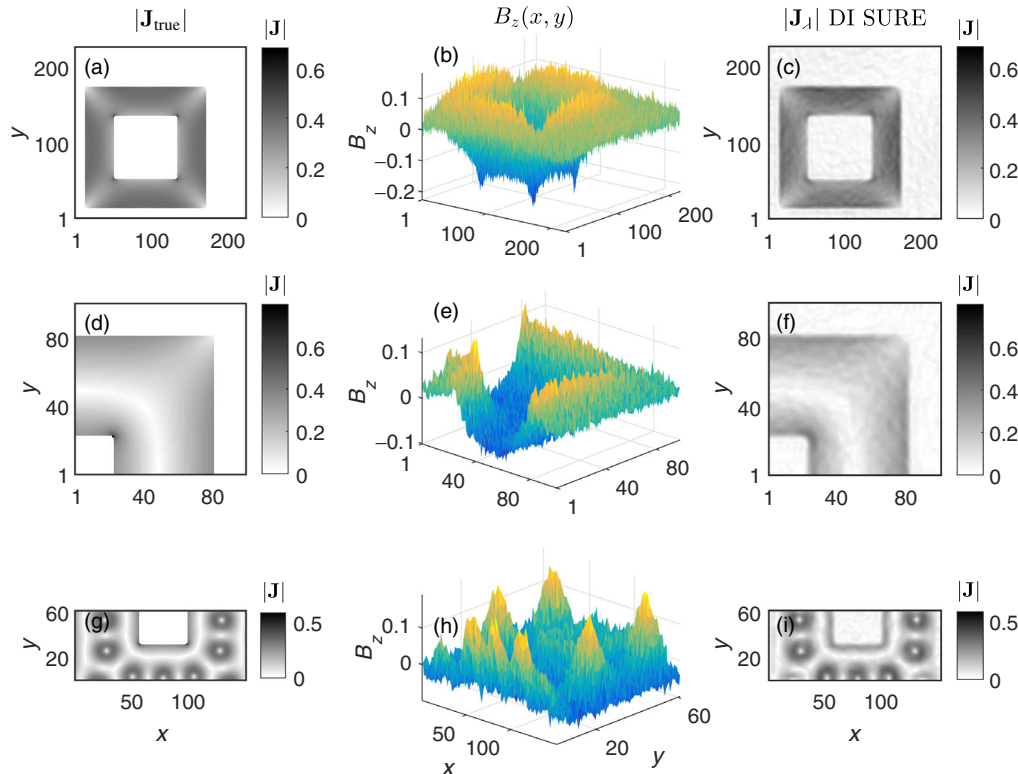


FIG. 3. (Left column) The true current density in samples *A* (top row), *B* (middle row), and *C* (bottom row). (Middle column) The corresponding calculated magnetic field at height $h = 1 \mu\text{m}$ perturbed by a noise of $s = 10^{-1}$. (Right column) The current density reconstructed from the noisy $B_z(x, y)$ using the DI-SURE scheme. The edges of the plots coincide with the measurement window. The x and y axes are in units of micrometers, the current density in $\text{mA}/\mu\text{m}^2$, and the field in gauss.

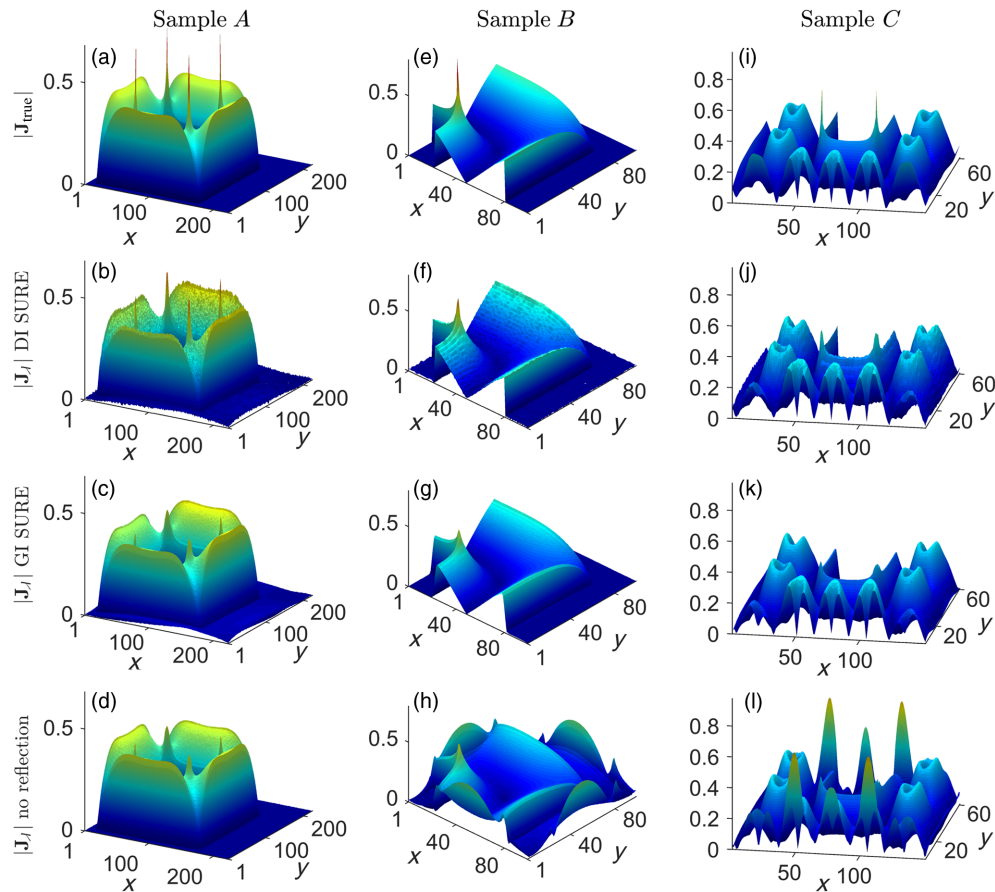


FIG. 4. (Top row) Surfaces of the true current density. (Second row) The current density reconstructed with DI SURE using the symmetric extension of the field. (Third row) The current density reconstructed with GI SURE using the symmetric extension of the field. (Bottom row) The current density reconstructed with GI SURE without using the symmetric extension of the field. The results are presented for $s = 10^{-3}$ in samples *A* (left column), *B* (middle column), and *C* (right column). The x and y axes are in units of micrometers and the current density in $\text{mA}/\mu\text{m}^2$.

of the magnetic field by either the DI-SURE or the GI-SURE method are shown in Figs. 4(b), 4(f), and 4(j) and Figs. 4(c), 4(g), and 4(k), respectively. Comparing these results, we observe that the GI method produces a smoother solution, while the DI method provides a better reconstruction of the sharp features in the current density. The reason for this behavior, as mentioned above, is the strong smoothing by the GI method, which uses two filters for current reconstruction. To emphasize the advantage of the reflection procedure, we present in the last row of the figure [Figs. 4(d), 4(h), and 4(l)] the results of reconstructions using GI SURE without a prior symmetric extension of the magnetic field. As expected, the reconstruction remains accurate for sample *A*, where the currents are closed within the measurement window. However, for samples *B* and *C*, the reconstruction is highly inaccurate, especially near the window boundary.

We now perform a quantitative analysis to compare the results of the four presented inversion methods by comparing their MSDs, which are defined in Eq. (2.23). We apply the inversion procedure to each sample 100 times, each time using a different noise realization, and present box plots of the MSD values in Fig. 5. The box plots graphically depict the results by splitting them into quartiles, so that each box spans the range that contains the second and third quartiles, termed the interquartile

range (i.e., the middle 50% of the data). The horizontal line in each box denotes the median, while the error bars span 150% of the interquartile range above the third quartile and below the second quartile. Any point outside this interval is denoted by “+” and considered an outlier.

The MSD of the reconstructions in Fig. 5 is given alongside the best possible MSD, which is calculated using the λ value that minimizes the MSD function (2.23). The accuracy of the reconstruction in sample *A*—where the current does not cross the image boundary and the symmetric extension is therefore not performed—is shown in Figs. 5(a)–5(c). By contrast, in samples *B* and *C*, the current crosses the measurement boundary, and a symmetric extension of the measured magnetic field is therefore necessary. The accuracy of the reconstruction of these samples is shown in Figs. 5(d)–5(f) and Figs. 5(g)–5(i). As can be seen, both methods are close to their optimum solutions, but the DI methods consistently achieve a lower MSD compared to the GI methods—in all examples. The advantage of the DI method becomes more significant at conditions of lower noise since, in contrast to the GI method, which uses the Savitzky-Golay filtering upon differentiating the g function, the DI method does not use an additional filter and thus preserves the sharp features of the solution. This effect is particularly pronounced in

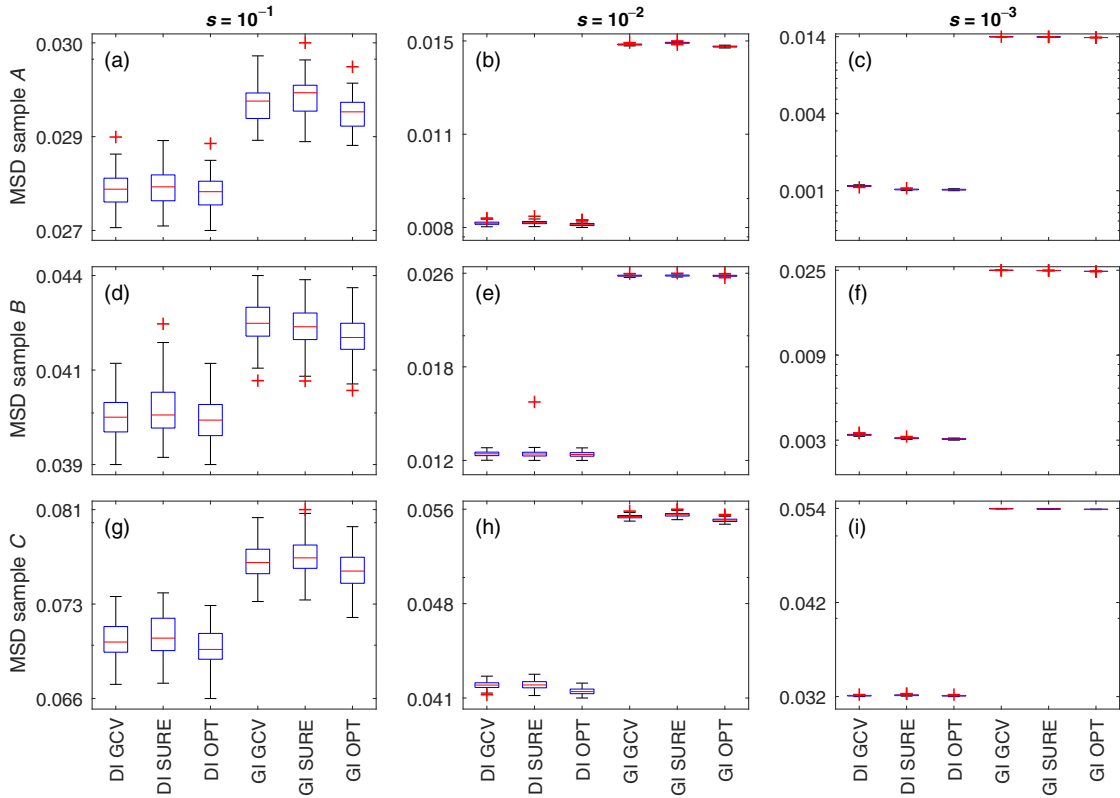


FIG. 5. Box plots of the MSDs of the currents reconstructed in samples *A* (top row), *B* (middle row), and *C* (bottom row) for noise levels of $s = 10^{-1}$ (left column), $s = 10^{-2}$ (middle column), and $s = 10^{-3}$ (right column). The MSD values for solutions obtained using the optimal values of λ are denoted DI OPT and GI OPT.

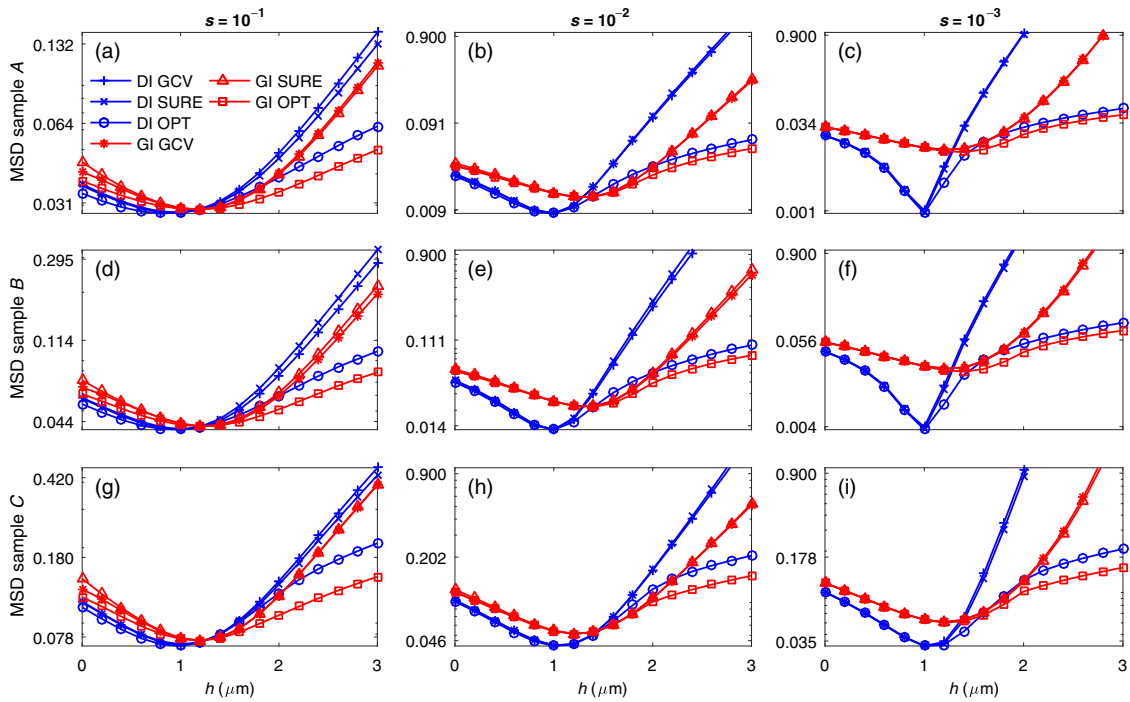


FIG. 6. Plots of the MSDs of current reconstructions as a function of the assumed height h , given the true height of $h_{\text{true}} = 1 \mu\text{m}$ in samples *A* (top row), *B* (middle row), and *C* (bottom row) for noise levels of $s = 10^{-1}$ (left column), $s = 10^{-2}$ (middle column), and $s = 10^{-3}$ (right column). The MSD values for solutions obtained using the optimal values of λ are denoted DI OPT and GI OPT.

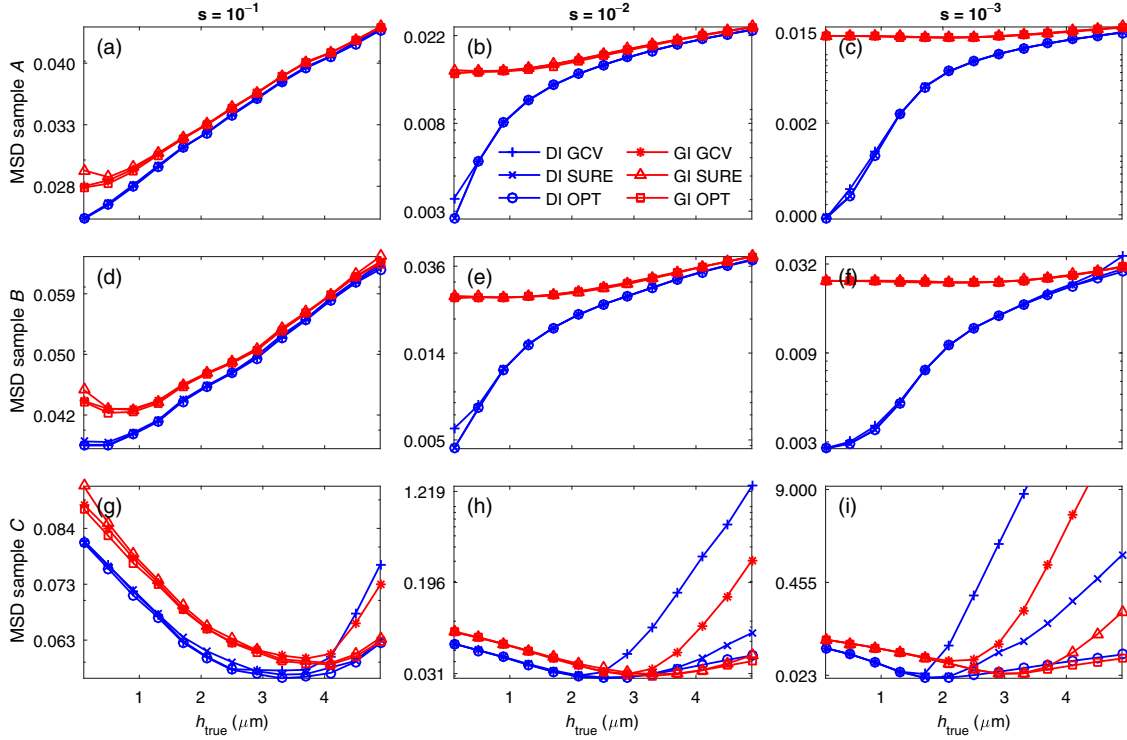


FIG. 7. Plots of the MSDs of current reconstructions as a function of the true height h_{true} in samples *A* (top row), *B* (middle row), and *C* (bottom row) for noise levels of $s = 10^{-1}$ (left column), $s = 10^{-2}$ (middle column), and $s = 10^{-3}$ (right column). The MSD values for solutions obtained using the optimal values of λ are denoted DI OPT and GI OPT.

Fig. 5(c), where the DI methods provide a MSD that is more than an order of magnitude lower.

Next, we use the magnetic field simulated at $h_{\text{true}} = 1 \mu\text{m}$ and assume, as in real measurements, that the exact values of the true height are not known. The accuracy of reconstructions of the currents, assuming different heights h , is shown in Fig. 6. The MSD curves in the presence of noise are not steep around the true height, and the reconstruction therefore remains reliable even with an incorrect estimation of h_{true} , with underestimation preferable to overestimation. The DI methods provides the lowest MSD value at h_{true} , while the lowest MSD values using the GI method are obtained for values slightly above the true height. For heights comparable to and lower than h_{true} , the DI method provides consistently lower MSD values, while for values above h_{true} , GI attains lower MSD values.

Finally, we analyze the accuracy of the inversion methods at different measurement heights, assuming that the reconstruction is performed at a true, variable height. In Fig. 7, we show the MSD curves of the reconstructed currents as a function of the true height, h_{true} . At low heights, the DI methods result in lower MSDs, but at large heights, the GI methods may have an advantage in some cases, as exemplified by sample *C*, in which highly irregular currents cross the measurement window. Another observation is that, when the currents are not closed within the measurement window, the GCV regularization

becomes ineffective for both the DI and the GI methods at larger heights, while the projected SURE regularization remains accurate for a much wider range of measurement heights.

V. CONCLUSIONS

Reconstruction of nanoscale electric-current distributions in thin samples is important for the characterization of low-dimensional materials and the evaluation of electric devices. A general scheme for the reconstruction of electric currents from a measured out-of-plane component of the magnetic field above the sample surface is presented. Our approach comprises three innovative parts: (1) a direct formulation of the inversion problem, (2) a symmetric extension of the measured magnetic field, and (3) an enhanced method for determination of the regularization parameter. Using the method of Feldmann [59] as reference, we show that direct formulation of the reconstruction problem allows us to improve the accuracy of current reconstruction, especially at regions that contain sharp alterations, while the symmetric extension of the measured magnetic field enables reconstruction of nonclosed currents. Finally, our scheme for determination of the regularization parameter makes current reconstruction possible over an extended range of measurement heights.

In this paper, we present several methods for current reconstruction. Two of the methods, DI GCV and DI

SURE, reconstruct current fields directly, while two other methods, GI GCV and GI SURE, reconstruct the stream function from which the current fields are obtained using a smoothed differentiation. The advantage of the DI methods is most pronounced at smaller heights, which are preferable for resolution of a local magnetic structure. At small heights (relative to the grid spacing), the DI methods indeed demonstrate a significant gain in accuracy in all of our numerical experiments. In the presence of large noise and at large heights, the DI schemes still outperform the GI schemes for samples with closed currents, albeit with a smaller gain in accuracy, with the GI schemes providing the extra benefit of a smoother solution. In our numerical simulations, the difference between GCV-based and SURE-based methods is very small. Therefore, for closed currents, we suggest using DI GCV at all times, unless a smooth solution is required and an accurate reconstruction of sharp changes of currents is not needed, in which case we suggest using GI GCV.

When currents cross the measurement boundary, a symmetric extension of the measured field is used to effectively approximate them and enable usage of the DFT, which requires periodicity. For small heights, the DI-GCV and DI-SURE schemes have a very similar accuracy which is superior to that of the GI schemes, which is similar to the case of closed currents. For large heights, however, GCV-based schemes become less accurate due to the poor reconstruction close to the measurement boundary, rendering the SURE-based schemes essential. In this case, the use of the GI-SURE method is recommended.

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APPENDIX: METHOD FOR VARIANCE ESTIMATION

An accurate estimation of the variance σ^2 of the measured magnetic field B_{nm} is important for a determination of the regularization parameter using the projected SURE method in Sec. III D. Here, we describe an algorithm which is based on the ideas developed in Ref. [67] and carried over to the discrete Fourier space using arguments given in Ref. [73]. Assuming that the image B_{nm} represents a smooth magnetic field corrupted by noise, the Fourier coefficients \hat{B}_{nm} can be divided into two parts, the part containing the information about the magnetic field and the other part containing the noise. We shift \hat{B}_{nm} such that the zero Fourier mode is located at $n = m = 1$, while the

high-frequency noise coefficients occupy the center of \hat{B}_{nm} . For a successful estimation, it is sufficient to find the region in the frequency space \hat{B}_{nm} which contains only noise and to compute its sample variance. For this purpose, we construct a nested sequence of rectangular submatrices $M_{j,nm}$ such that $M_{1,nm}$ equals the entire matrix \hat{B}_{nm} and $M_{q,nm}$ contains only the highest Fourier modes in the center of \hat{B}_{nm} . We then define the function

$$V(j) = \sum_{nm} |M_{j,nm}|^2 / N_j,$$

where N_j is the number of elements in $M_{j,nm}$. It is shown in Ref. [67] that the value of the curve $V(j)$, which approximates the expected value of $|M_{k,nm}|^2$, decreases and levels off at σ^2 . We thus find an index k_0 in the flat region of $V(k)$ by minimizing $\|\log[V(k+1)] - \log[V(k)]\|$ and obtain our estimate to be $\sigma^2 \approx V(k_0)$.

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