

Spectral Characteristics of a Fully Superconducting SQUIPT

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We consider properties of a fully superconducting variant of the superconducting quantum-interference proximity transistor, a magnetic-flux sensor. We study the density of states in a finite-size superconducting metal wire in the diffusive limit, and how it depends on the phase gradient of the order parameter. We describe the dependence on the junction length and the interface transparency, and we discuss properties relevant for using the structure in magnetic-flux-detection applications.

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I. INTRODUCTION

Superconductivity on the scale of the coherence length is sensitive to its surroundings. This knowledge can be used to modulate the density of states in a mesoscopic metal wire via magnetic flux by imposing a phase gradient via embedding the wire as a weak link in a superconducting ring. Detecting the modulation with a tunnel-junction probe attached to the weak link is the basis of the superconducting quantum-interference proximity transistor (SQUIPT), which can be used as a magnetic-field sensor [1–8].

Previous experiments mostly employed normal-state metal wires [1]. For practical purposes, it, however, can be advantageous if also the weak link is made of a superconducting rather than normal-state material: sample fabrication is simpler, quality of the contacts can be better, and the quality of the energy gap can improve, resulting in better sensitivity. The intrinsic superconductivity, however, modifies the current-phase relation, which determines what phase gradient can be imposed, and affects the detailed form of the density of states.

The current-phase relation in S - S' - S junctions (see Fig. 1) has been extensively studied in the past and is largely understood [9–13]. The density of states (DOS) is, moreover, well studied in the normal-state case [14–19]. The DOS in a phase-biased superconducting wire, which is the case relevant for the fully superconducting SQUIPT, appears to have received somewhat less comment [20–22].

In this work, we investigate the density of states and other characteristics of a phase-biased superconducting wire, such as those embedded as a part of a SQUIPT structure, as depicted in Fig. 1. We approach the problem of finding the current-phase relation and spectral characteristics from a functional-minimization perspective. We describe the evolution of the current-phase relation and the density of states between short- and long-junction limits. Finite-size effects in the DOS turn out to decay as a power law with an increasing system length, rather than

exponentially as they do for the current-phase relation. We discuss the effect of differing materials in the ring and wire parts, the impact of interface transparency, and some practical consequences for the tunnel-superconducting SQUIPT (SSQUIPT) application.

II. MODEL

We consider the SQUIPT setup depicted in Fig. 1, where a mesoscopic conventional superconductor (S) metal wire of length L and cross section \mathcal{A} is embedded as a part of a superconducting ring. The density of states in the wire is probed by the current-voltage characteristics of a normal (N) or superconducting tunnel junction. The critical temperature of the ring material is $T_{c,R}$, and that of the wire material is $T_{c,w}$, corresponding to the superconducting coupling constants λ_R, λ_c and the zero-temperature values $\Delta_{0,R}, \Delta_{0,w}$ of the energy gap via the BCS relations. Possible dependence of the superconducting coupling on the film thickness is included in these parameters. We assume that the dimensions of the wire are small compared to the ring, so that the presence of the wire has a small effect on the superconductivity of the ring.

The superconducting properties in the diffusive limit at equilibrium are conveniently described by the nonlinear σ model [23–26]. The approach encompasses the well-known quasiclassical Green's-function theory [27] of diffusive superconducting systems as a special case, but it can also be used to study, e.g., fluctuation effects. Here, we consider only equilibrium properties of quasi-1D systems in the semiclassical approximation—in this case, the advantage is in directly specifying the problem in variational form. At the classical saddle points, the free-energy contribution is $\delta F = \mathcal{A}\nu_F[F_b + \int_0^L dx \mathcal{F}_0]$, where the density reads (cf. Ref. [28])

$$\mathcal{F}_0 = \frac{|\Delta|^2}{\lambda} + 2\pi T \sum_{\omega_n > 0} \left\{ \frac{D(\partial_x \chi - 2A)^2}{2} \sin^2 \tilde{\theta} + \frac{D(\partial_x \tilde{\theta})^2}{2} + 2\omega_n(1 - \cos \tilde{\theta}) - 2|\Delta| \cos(\chi - \phi) \sin \tilde{\theta} \right\}. \quad (1)$$

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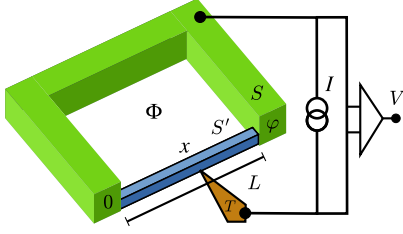


FIG. 1. Superconducting quantum-interference proximity transistor, consisting of a superconducting (S') weak link of length L embedded in a superconducting (S) SQUID ring. The magnetic flux Φ induces a difference of φ in the phase of the order parameter across the junction, which, in the weak-link limit, is $\varphi = 2\pi\Phi/\Phi_0$. The density of states in the weak link can be probed with a normal or superconducting tunnel probe (T).

Here, $\omega_n = 2\pi T(n + \frac{1}{2})$ are Matsubara frequencies, ν_F the DOS per spin at the Fermi level, and A the vector potential. D is the diffusion constant and $\Delta = |\Delta|e^{i\phi}$ the order parameter. Moreover, λ is the superconducting interaction constant. The parameters χ and $\tilde{\theta}$ are real valued and, at saddle points related to the quasiclassical anomalous Green's function, $f(x, \omega_n) = e^{i\chi(x, \omega_n)} \sin \tilde{\theta}(x, \omega_n)$ [27]. Here and below, we use the units $k_B = e = \hbar = 1$. For taking the weak-coupling limit $\lambda \rightarrow 0$, $\Delta_{w,0} = \text{const}$, it is useful to add and subtract terms to eliminate the implicit cutoffs,

$$\mathcal{F}_0 \simeq |\Delta|^2 \ln \frac{T}{T_c} + 2\pi T \sum_{\omega_n > 0} \left\{ \frac{|\Delta|^2}{\omega_n} + \frac{D(\partial_x \chi - 2A)^2}{2} \sin^2 \tilde{\theta} + \frac{D(\partial_x \tilde{\theta})^2}{2} + 2\omega_n(1 - \cos \tilde{\theta}) - 2|\Delta| \cos(\chi - \phi) \sin \tilde{\theta} \right\}. \quad (2)$$

The dependence on λ is now contained in T_c , assumed to be of the BCS weak-coupling form.

The connection of the wire to the superconducting ring is described via a tunneling boundary term [29,30],

$$F_b = \frac{2\pi TD}{r} \sum_{\omega_n > 0} \sum_{j=\pm} \{1 - \cos \tilde{\theta}(x_j) \cos \tilde{\theta}_{S_j} - \cos[\chi(x_j) - \chi_{S_j}] \sin \tilde{\theta}(x_j) \sin \tilde{\theta}_{S_j}\}, \quad (3)$$

where $r = 2R_I AD\nu_F$ is the ratio of interface resistance R_I to the resistance per length of the wire and $\tilde{\theta}_{S_\mp} = \arctan(\Delta_R/\omega_n)$. $\chi_{S-} = 0$ and $\chi_{S+} = \varphi$ are the values inside the ring at the left ($x_- = 0$) and right ($x_+ = L$) interfaces.

Requiring a variation of Eq. (2) vs $\tilde{\theta}$ and χ to vanish and an analytical continuation to the real axis $i\omega \mapsto E + i0^+$, then defining $\theta \equiv -i\tilde{\theta}$ produces the standard quasiclassical real-time description of the system via the Usadel equation [27,31], which can be written in the form

$$D\partial_x^2 \theta = -2iE \sinh \theta + \frac{D(\partial_x \chi)^2}{2} \sinh 2\theta + 2i|\Delta| \cos(\phi - \chi) \cosh \theta, \quad (4)$$

$$D\partial_x \cdot (\partial_x \chi \sinh^2 \theta) = -2i|\Delta| \sin(\chi - \phi) \sinh \theta. \quad (5)$$

The self-consistency equation for the order parameter Δ is obtained similarly,

$$|\Delta| \ln \frac{T}{T_c} = 2\pi i T \sum_{\omega_n > 0} \left[e^{i(\chi - \phi)} \sinh \theta - \frac{|\Delta|}{E} \right] \Big|_{E=i\omega_n}. \quad (6)$$

The boundary term generates the boundary conditions [29],

$$\mp r \partial_x \chi = \sin(\chi - \chi_{S_\mp}) \frac{\sinh \theta_{S_\mp}}{\sinh \theta}, \quad (7)$$

$$\mp r \partial_x \theta = \sinh \theta \cosh \theta_{S_\mp} - \cos(\chi - \chi_{S_\mp}) \cosh \theta \sinh \theta_{S_\mp}, \quad (8)$$

at the left ($-$) and right ($+$) interfaces. In the clean-interface limit $r \rightarrow 0$, these boundary conditions reduce to the continuity conditions $\chi(x_\pm) = \chi_{S_\pm}$, $\theta(x_\pm) = \theta_{S_\pm}$. Note also that, in this case, $F_b \rightarrow 0$ as $r \rightarrow 0$.

The reduced density of states $N(E, x) = \nu(E, x)/\nu_F = \text{Re} \cosh \theta(E, x)$ and the current $I(x) = -\frac{\delta F}{\delta A} \Big|_{A=0} = 4\pi T AD\nu_F \sum_{\omega_n > 0} \partial_x \chi \sin^2 \tilde{\theta}$ follow directly from the solutions of the equation set.

It is well known that there are multiple classical solutions, corresponding to different windings of the superconducting phase along the superconducting wire. In the numerical solution of the equation set, to handle this situation and to obtain also the solutions along the unstable branches, we use the pseudoarclength-continuation method applied on the self-consistency equation. This method is useful for tracing a continuous solution branch (φ, Δ) without requiring the existence of a single-valued function $\Delta(\varphi)$ (see Appendix A for details).

Within the approximations here, the configuration minimizing δF should be considered the stable solution. Along a continuous solution branch, δF can also be evaluated via a standard relation,

$$\delta F[X_*(\varphi); \varphi] - \delta F[X_*(0); 0] = \frac{1}{2} \int_0^\varphi d\varphi' I[X_*(\varphi'); \varphi'], \quad (9)$$

based on the current evaluated at a stationary point $X_* = (\tilde{\theta}_*, \chi_*, |\Delta_*|, \phi_*)$. This result follows from Eqs. (2) and (3) by noting the gauge transform $F[\tilde{\theta}, \chi, |\Delta|, \phi; \varphi, A] = F[\tilde{\theta}, \chi - \xi, |\Delta|, \phi - \xi; \varphi - \xi(L), 0]$ for $\xi(x) = 2 \int_0^x dx' A(x')$ and that $\delta F/\delta X|_{X=X_*} = 0$. Energy differences between disconnected branches, however, need to be determined from Eqs. (2) and (3).

III. DENSITY OF STATES AND CURRENT

The density of states in SSQUIPT is sensitive to the phase point where the system is biased. To specify this point, we need to comment on the supercurrent in the junction. The form of the current-phase relation (CPR) in superconducting strips is well studied under the approximations outlined above [9–13]. CPRs computed from the Usadel equation are illustrated in Fig. 2 for reference; see also Refs. [11,33]. For junctions that are short in comparison to the coherence length, $L \ll \xi_w \equiv \sqrt{D/(2\pi T_{c,w})}$, the CPR is a (deformed) sinusoid, described by a known analytical solution [9,32]. In long junctions, the CPR becomes multivalued, corresponding to multiple winding of the order-parameter phase. At temperatures close to T_c , the form can be found from Ginzburg-Landau equations [9,34]. For $L \gg \xi_{GL} = \sqrt{D\pi/[8(T_c - T)]}$, one expects a CPR $I = [A\sigma_N\pi\Delta^2/(4TL)]\varphi[1 - (\varphi\xi_{GL}/L)^2]$, up to the point $\varphi < \varphi_{\max} \approx L/(\sqrt{3}\xi_{GL})$. After this point, the solution transitions to a backward branch reaching to $\varphi = \pi$, $I = 0$,

$$I = \frac{A\sigma_N\pi\Delta^2}{2T\xi_{GL}} a\sqrt{a-k},$$

$$\varphi = \frac{L}{\xi_{GL}} \sqrt{a-k} + \frac{1}{a} \arcsin \sqrt{\frac{k}{a}}, \quad (10)$$

for $k \in [0, 1/2]$, $a = (k+1)/3$. The phase gradient becomes more nonuniform, corresponding to the formation of a phase-slip center in the middle of the junction [34,35].

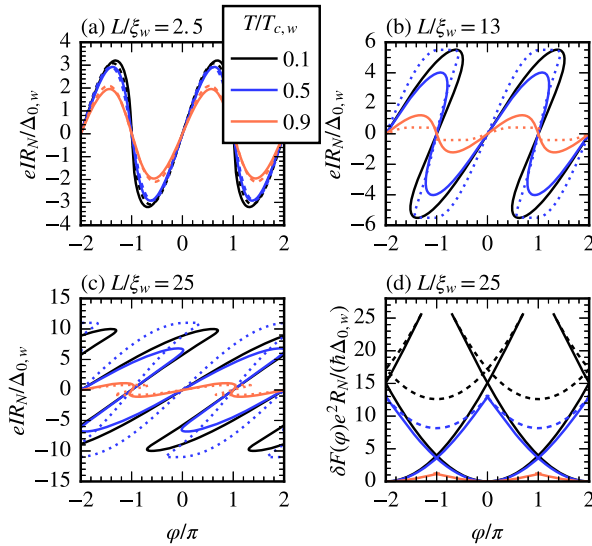


FIG. 2. (a)–(c) Current-phase relation vs junction length and temperature ($T/T_{c,w} = 0.1, 0.5, 0.9$) for $T_{c,R}/T_{c,w} = 1.5$. Here, $\xi_w = \sqrt{D/(2\pi T_{c,w})}$. Analytical short-junction result (dashed curves) [9,32] and the GL result discussed in the text (dotted curves) are also shown. (d) Free-energy change corresponding to the curves in (c), as obtained from Eq. (2) or (9). The backward branches are shown with dashed lines.

In long junctions, the CPR allows several saddle-point solutions for the current I when φ is fixed. The backward solution branches are unstable and not directly accessible. Which of the remaining multiple solutions are accessible depends on the rate Γ of phase slips by which the system can transition to lower-energy states [34,36]. The free-energy barrier U in the thermal-activation rate $\Gamma \propto e^{-U/T}$ can be estimated from the difference between the unstable and metastable-branch energies in Fig. 2(d) and is of the order $U \sim \hbar\Delta_{0,w}/(e^2R_\xi)$, $R_\xi \equiv \xi_w R_N/L$ [34–38]. The quantum phase-slip rate in the nanowires is $\Gamma \propto e^{-a\hbar/(e^2R_\xi)}$, with $a \sim 1$ [36]. For a typical SSQUIPT cross section 100×30 nm and a high-conductivity material, $R_\xi \sim 1 \Omega$, so that both rates are effectively zero, except close to $\varphi \approx \varphi_{\max}$ —where the CPR bends backward—or at high temperatures. For low relaxation rates, the experimental CPR has magnetic hysteresis, and the different branches can be accessed by sweeping the magnetic flux.

In short junctions where the CPR is single valued, the kinetic inductance of the wire decreases around $\varphi = \pi$ and can become small compared to that of the ring, inducing behavior similar to hysteretic rf SQUIDs [2]. To quantify this result, we show in Fig. 3 the Josephson inductance $\mathcal{L} = \hbar/(2e\partial_\varphi I)$ at $\varphi = \pi$ for different temperatures and wire lengths. In the short-junction limit $L \ll \xi_w$, from the known expression for the CPR [32], we have, after some rewriting,

$$\frac{1}{\mathcal{L}(\pi)} = -\frac{\pi\Delta_R}{\hbar R_N} \int_0^{\Delta_R/(2T)} dz \frac{\tanh z}{z}. \quad (11)$$

For $T \lesssim \Delta_R/2$, we have $\mathcal{L}(\pi)^{-1} \approx \frac{\pi\Delta_R}{\hbar R_N} \ln \frac{T}{2T_{c,R}}$. As shown in Fig. 3, for $L > \xi_w$, the normalized inductance further decreases, in general, from the short-junction value, regardless of the ratio of $T_{c,R}$ and $T_{c,w}$. However, the value is tunable by material choices.

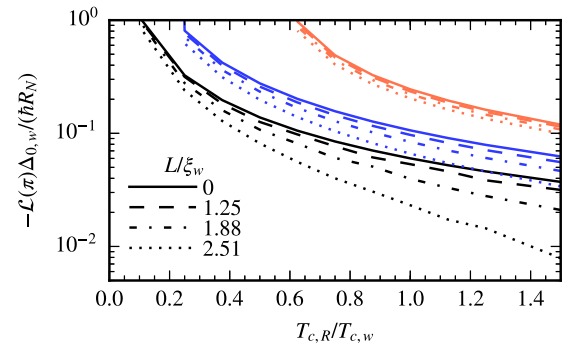


FIG. 3. Normalized inductance of the superconducting wire at $\varphi = \pi$, for different $T_{c,R}$'s and L 's, at $T/T_{c,w} = 0.01, 0.1, 0.5$ (from bottom to top). Solid lines denote the short-junction limit (11).

Let us now consider the DOS in the junction. Analytical solutions to Eq. (4) providing access to the DOS are known in the long-junction ($L \gg \xi$) and short-junction ($L \ll \xi$) limits. In the short-junction limit, we have a well-known result (see, e.g., Ref. [39]),

$$N(x, E) = \text{Re} \sqrt{\frac{E^2}{E^2 - \Delta_R^2 \cos^2 \frac{\varphi}{2}}} \cosh\left(\frac{2x - L}{L}\right) \times \text{arcosh} \sqrt{\frac{E^2 - \Delta_R^2 \cos^2 \frac{\varphi}{2}}{E^2 - \Delta_R^2}}, \quad (12)$$

independent of the superconductivity of the wire itself. For long wires ($L \gg \xi_w$), on the other hand, the result converges to the density of states of a bulk superconductor, affected by depairing from the supercurrent flow [12,13,20,40]. This result can be described as a depairing rate [41], $g^2 = D(\partial_x \phi)^2$, given a small enough constant phase gradient $\partial_x \phi$:

$$N(E) = \text{Re} \cosh \theta_0, \quad |\Delta| \coth \theta_0 = E + i \frac{g^2}{2} \cosh \theta_0. \quad (13)$$

For long junctions, the parameter can be estimated from Ginzburg-Landau (GL) solutions [34] to be $g^2 \approx D(\varphi/L)^2$ along the forward branch, and $g^2 \lesssim D \max(\partial_x \phi)^2 \approx D[1 + (L/\xi - 2\sqrt{2})^2/(\varphi - \pi)^2]/4$ on the backward branch close to $\varphi = \pi$. The approximation in Eq. (13) is, however, not expected to work as well along the backward branch, as the phase gradient is not uniform due to the formation of the phase-slip center.

The crossover from the short-wire to the long-wire limit is illustrated in Fig. 4. We can observe that the short-junction solution is fairly accurate up to $L \sim \xi_w$, except in narrow energy regions around the gap edges, $E = E_g(\varphi, L)$. As the length increases, the DOS converges towards that of a bulk superconductor affected by depairing from the superflow (the dashed lines). However, the peaks at the DOS gap edge vanish only rather slowly as $L/\xi_w \rightarrow \infty$ and are not present in the long-wire limit described by Eq. (13). The crossover is not very rapid; indeed, the rate of the decay is a power law rather than an exponential, as shown in Fig. 5(a).

The physical mechanism giving rise to the finite-size effects in the DOS are the Andreev reflections at the ring interfaces, $x = 0, x = L$. In a normal metal wire, this effect decoheres on a certain decay-length scale $\ell(E) \propto \sqrt{\hbar D/E}$, which can be understood to originate from energy-dependent phase shifts between electrons and the almost-retroreflected holes (cf., e.g., Ref. [42] for a review). The possibility of Andreev reflections also inside the superconducting wire itself, however, alters $\ell(E)$, e.g., inhibiting low-energy electrons from reaching the ring

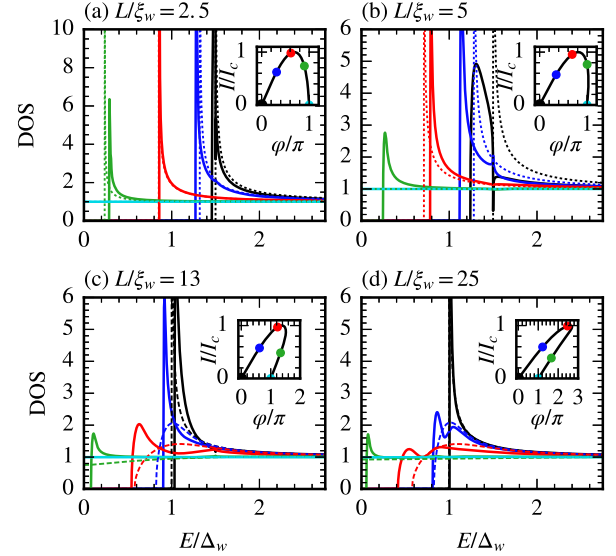


FIG. 4. Density of states in an S - S' - S junction at $x = L/2$, for different phase differences and junction lengths L . Here, $T_{c,R} = 1.5T_{c,w}$, and $T/T_{c,w} = 0.1$. The dotted lines in (a) indicate Eq. (12), and the dashed lines in (b)–(d) Eq. (13). (Insets) Current-phase relation and the points corresponding to the densities of states shown.

boundary. Moreover, in a superconductor, the decay length diverges around the gap edge, E_g [43], instead of around $E = 0$. This result can be seen by considering small perturbations $\theta = \theta_0 + \eta$, $\chi = \phi + \alpha$, $|\eta| \ll 1$, and $|\alpha| \ll 1$ around the uniform solution (θ_0, ϕ) described by Eq. (13). Substituting this ansatz into Eq. (4) and linearizing around (θ_0, χ_0) produces solutions of the form $\eta, \chi \propto e^{\pm x/\ell}$, where ℓ represents the decay lengths. Consequently, the factor $e^{-L/(2\ell)}$ indicates how much the boundary conditions affect the solution at the center of the wire. The linearized equation can be written here as $\partial_x(\partial_x \eta, \eta, \partial_x \alpha, \alpha)^T = M(\partial_x \eta, \eta, \partial_x \alpha, \alpha)^T$, where the matrix M is

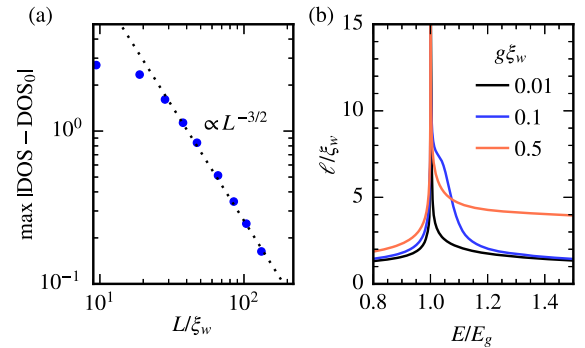


FIG. 5. (a) Maximum difference in density of states between Eq. (13) and the solution in the S - S' - S configuration, assuming $\Delta(x) = |\Delta_0|e^{igx}$, with $g = 0.1$. (b) Perturbation decay length ℓ .

$$M = \begin{pmatrix} 0 & \ell_0^{-2} & g \sinh 2\theta_0 & 0 \\ 1 & 0 & 0 & 0 \\ -2g \coth \theta_0 & 0 & 0 & -2i|\Delta| \operatorname{csch} \theta_0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (14)$$

where $\ell_0^{-2} = -2i|\Delta| \operatorname{csch} \theta_0 + g^2 \sinh^2 \theta_0$. The longest decay length is given by the eigenvalue with the smallest real part, $\ell^{-1} = \min |\operatorname{Re} \lambda|$. They are solutions to

$$\lambda^4 + (5g^2 \cosh^2 \theta_0 - g^2 - 2\ell_0^{-2})\lambda^2 = \frac{2i|\Delta|\ell_0^{-2}}{\sinh \theta_0}. \quad (15)$$

The energy dependence of $\ell(E)$ is shown in Fig. 5(b), with a divergence clearly visible. From Eq. (13), it follows that the gap edge is located at $E_g = \Delta[1 - (g^2/2\Delta)^{2/3}]^{3/2}$ [41], and $\theta_0(E_g) = \frac{-i\pi}{2} + \operatorname{arccosh}[(2\Delta g^{-2})^{1/3}]$. Consequently, $\ell_0^{-2}(E_g) = 0$, so that $\ell^{-2}(E_g) = 0$ also. Around the gap edge, the length scale diverges as $\ell \propto (E - E_g)^{-1/4}$. As the decay length is larger than L in an energy range of width $\delta E \propto L^{-4}$, the deviations from Eq. (13) at the middle of a long wire do not decay exponentially with an increasing L , and the scaling of Fig. 5(a) can occur.

A. Interface resistance

Imperfect interface transparency in SQUIPT influences both the CPR and the DOS. It particularly enables additional stationary solutions, where the phase drops across the barriers at the interfaces. With an increasing interface resistance r , the current-phase relation crosses over to $I(\varphi) \approx \pm I_c \sin(\varphi/2)$, that of two Josephson junctions in a series [22,29]. For two identical Josephson junctions and the superconducting wire in a series, with the phase drop ϕ_1 at the interfaces, the free energy is

$$\delta F(\varphi, \phi_1) \sim -2E_J(r) \cos(\phi_1) - E_{J,\text{wire}} \cos(\varphi - 2\phi_1). \quad (16)$$

For $r \rightarrow 0$ [$E_J(r) \gg E_{J,\text{wire}}$], lowest-energy solutions have $\phi_1 \approx 2\pi n$ and $I(\varphi) \approx I_{\text{wire}}(\varphi)$, whereas for $r \rightarrow \infty$ [$E_J(r) \ll E_{J,\text{wire}}$], $\phi_1 \approx \varphi/2 + \pi n$ and, consequently, $I(\varphi) \approx \pm I_J \sin(\varphi/2)$.

The crossover is illustrated in Figs. 6(a) and 6(b). A nonzero but small r only effectively adds to the length of the junction, as shown in Fig. 6(a), compared to Fig. 2(a). A larger r splits the solution to disconnected branches, as seen in Fig. 6(b), where there are three different solutions at $\varphi = 0$ yielding $I = 0$. These correspond to (i) no phase drop across the wire or interfaces, (ii) π phase drop at both interfaces, and (iii) similar to the second solution but with a smaller $|\Delta|$. As φ is varied continuously from 0 to 2π , solution (i) transforms to (ii) and vice versa. Solution (iii) is

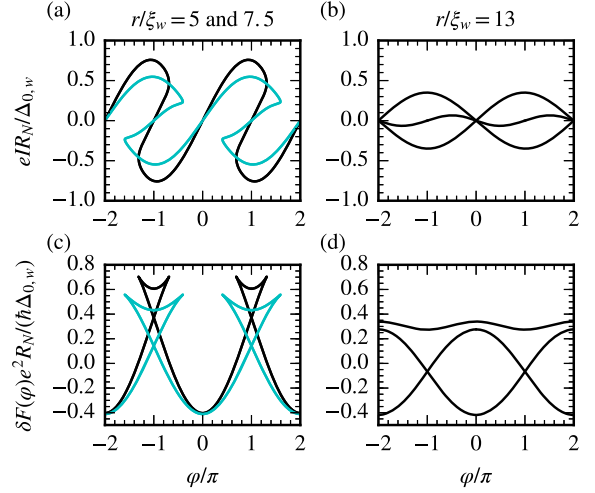


FIG. 6. (a),(b) CPR in an S - I - S' - I - S system for $L/\xi_w = 2.5$, $T/T_{c,w} = 0.1$, and $T_{c,R}/T_{c,w} = 1.5$. The dimensionless interface resistances are $r/\xi_w = 5$ (black), 7.5 (cyan) in the left panel, and 13 in the right. Higher-energy solutions may also exist, but they are not shown. (c),(d) The corresponding free-energy contribution from Eqs. (2) and (3).

not connected to the two; instead, when φ varies from 0 to π , a phase-slip center forms in the center of the wire; cf. also Ref. [22]. The evolution is somewhat more clear in the free energy shown in Figs. 6(c) and 6(d). The energy differences between the solutions in Fig. 6(d) must be obtained from Eqs. (2) and (3) instead of Eq. (9), as not all are continuously connected. Note that little qualitative change occurs in the lowest-energy solution.

The proximity effect from the ring diminishes with an increasing barrier height r , and controlling the properties of the weak link via the superconducting ring becomes less effective. This result is illustrated in Fig. 7, which shows the modulation of the density of states for a long junction with two different interface resistances. As r increases and the critical current of the interfaces becomes small compared to

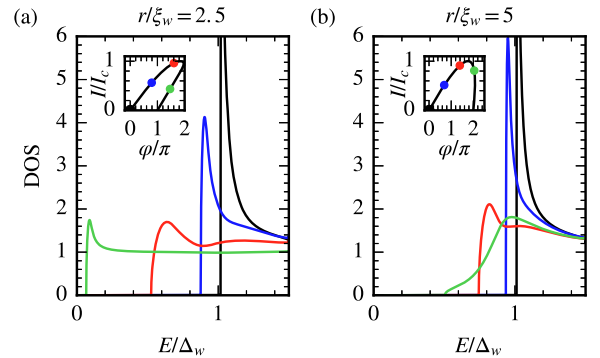


FIG. 7. DOS in an S - I - S' - I - S system for $L/\xi_w = 13$ and (a) $r/\xi_w = 2.5$ or (b) $r/\xi_w = 5$. Other parameters are as in Fig. 4. (Inset) The position along the solution branch.

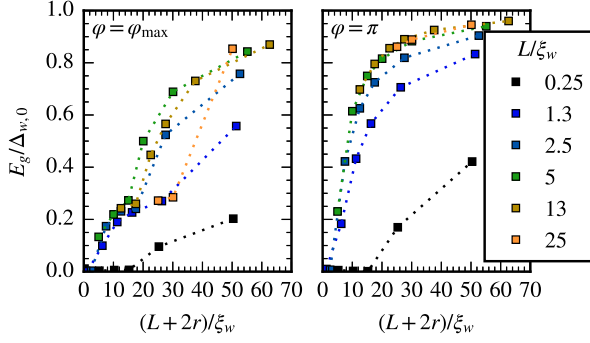


FIG. 8. Energy gap E_g achieved at $\varphi = \varphi_{\max}$, where CPR starts to bend back, and at $\varphi = \pi$ in the lowest-energy state. Wire lengths and interface resistances are varied; other parameters are as in Fig. 4.

that of the wire, the DOS approaches that of the bulk superconductor.

For the SSQUIPT device application, a relevant metric is how much the modulation of the superconducting gap is suppressed. In Fig. 8, we show the smallest magnitude of the energy gap achievable with phase biasing. Based on the above discussion, the minimum is achieved at the point $\varphi = \varphi_{\max}$, where the CPR transitions to the backward branch. In the lowest-energy state, the minimum gap is achieved at $\varphi = \pi$, which is also shown. We can note that, in Fig. 8, the curves for different wire lengths L tend to collapse onto a single curve for $L \gtrsim 2.5\xi_w$. The interface resistance in this case acts similarly as an extension of the junction length by $2r$, consistent with the increase in the total resistance, although, at $\varphi = \varphi_{\max}$, the behavior is complicated by the crossover illustrated in Fig. 6.

IV. DEVICE PERFORMANCE

We can now discuss the performance of the S - S' - S devices in a magnetometric measurement. In this mode, the tunnel contact [(T) in Fig. 1] connected to the S' wire is a current biased to a working point $I = I_{\text{bias}}$. The tunneling current depends on the DOS,

$$I(V, \Phi) = \frac{1}{R_T} \int_{-\infty}^{\infty} dE N(E, \Phi) N_{\text{probe}}(E - V) \times [f(E) - f(E - V)], \quad (17)$$

and the variation of the observed voltage $V(I_{\text{bias}}, \Phi)$ as a function of the flux Φ can be used to infer Φ . The sensitivity can be characterized by the flux-voltage transfer function

$$\mathcal{F}(\Phi) = \frac{dV}{d\Phi}. \quad (18)$$

The resolution is intrinsically limited by the voltage noise in the probe junction, which can be described by an equivalent flux noise,

$$S_{\Phi, T} = \frac{S_V}{\mathcal{F}(\Phi)^2}, \quad (19)$$

where $S_V = (dV/dI)^2 S_I$ and $S_I = 2eI \coth \frac{eV}{2k_B T}$ is the tunneling shot noise. Pre-amplifier voltage noise will also give a similar contribution.

The current-voltage characteristic $I(V, \Phi)$ is shown in Fig. 9 for the short $L = 2.5\xi_w$ and long $L = 13\xi_w$ junction cases. Here, we assume the tunnel probe is superconducting, with $T_{c, \text{probe}} = T_{c, R}$. For the short device fixed at $I = I_{\text{bias}}$, the external flux corresponds to a single measured voltage value in a wide bias range, shown in Fig. 10(a). The long-junction device on the other hand has magnetic hysteresis in the range $|\Phi - n\Phi_0| \gtrsim 0.25\Phi_0$, see Fig. 2, where each flux value is associated with two possible V 's. Which one is realized depends on the initialization of the device. Note that this deduction assumes that the relaxation rates $\Gamma(\Phi)$ of metastable states are very low, as was estimated in Sec. III. If $\Gamma(\Phi)$ is not low compared to measurement time scales, transitions can contribute additional voltage noise, which reduces the usefulness of the device. We will not consider this case here.

The voltage response at different bias currents and the corresponding transfer function are shown in Fig. 10(a) for a short-junction device. The overall behavior and magnitude of \mathcal{F} is relatively similar to a normal-metal SQUIPT in this case [3]. For long-junction devices, as is also visible in Fig. 9, we expect decreasing device sensitivity with increasing system length, as the gap suppression depends on the phase gradient which scales $\propto 1/L$.

Calculated equivalent flux noise from the tunnel junction is displayed in Fig. 10(b), where we choose representative values for R_T and $\Delta_{0, w}$. The tunnel junction noise is of the order of $S_{\Phi, T}^{1/2} \sim 10 \text{ n}\Phi_0/\sqrt{\text{Hz}}$ at bias currents of order 1 nA. Pre-amplifier voltage noise for a typical value of $S_{V, \text{pre}}^{1/2} \approx 1 \text{ nV}/\sqrt{\text{Hz}}$, on the other hand, yields $S_{\Phi, \text{pre}}^{1/2} \gtrsim 500 \text{ n}\Phi_0/\sqrt{\text{Hz}}$ for the parameters of Fig. 10(b). The results

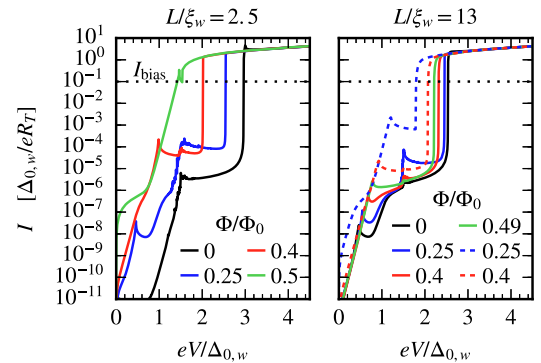


FIG. 9. Predicted I - V characteristics of devices with $L = 2.5\xi_w$ and $L = 13\xi_w$ junctions, at $T/T_{c, w} = 0.1$ and $T_{c, R} = 1.5T_{c, w}$. For $L = 13\xi_w$, the I - V characteristics have flux hysteresis for $|\Phi - n\Phi_0| \gtrsim 0.25\Phi_0$.

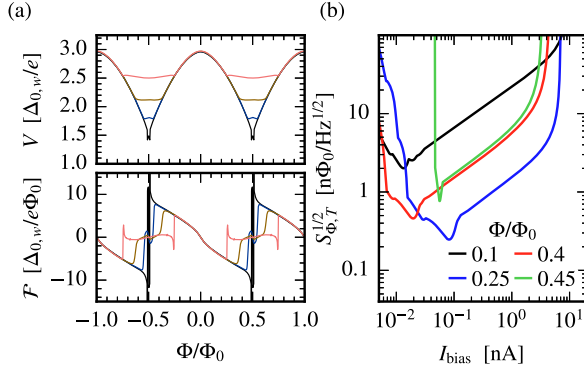


FIG. 10. (a) V vs Φ corresponding to Fig. 9 and $L = 2.5\xi_w$, for $eR_T I_{\text{bias}}/\Delta_{0,w} = 2, 1.5, 1, 0.2$ (from top to bottom), and the corresponding transfer function \mathcal{F} . (b) Equivalent flux noise $\sqrt{S_{\Phi,T}}$ from tunnel junction, for $\Delta_{0,w} = 200 \mu\text{eV}$ and $R_T = 100 \text{ k}\Omega$.

depend on the energy gap $\Delta_{0,w}$ via $S_{\Phi,T}^{1/2} \propto \sqrt{R_T/\Delta_{0,w}}$ and $S_{\Phi,\text{pre}}^{1/2} \propto 1/\Delta_{0,w}$, so that the performance is expected to improve with larger-gap superconductors. Based on these estimates, the performance of the device is, in practice, expected to be mostly limited by the external amplifier noise.

V. DISCUSSION AND CONCLUSIONS

For weak links that are short compared to the coherence length, there is little difference between the normal and superconducting cases, on the level of the present description. As the junction length increases, the change in the current-phase relation starts to limit the maximum achievable modulation of the density of states. This result is reflected in the decrease of the voltage modulation observed by the tunnel probe. Moreover, with increasing length of the junction, the gap-edge singularities of the BCS DOS transform to smaller peaks, in a way that is sensitive to the finite size of the weak link.

The long perturbation decay length at gap edges has an impact on how generic the results discussed here are in practice, even within the mean-field approximations. In reality, even if the superconducting wire is well described by the quasi-1D equations used above, the boundary conditions at $x = 0, L$ may not be as accurate. Namely, the contact region often has a nontrivial 3D structure, and the phase gradient also extends to the terminals. In such cases, we expect that, in the interior of the wire, the results will follow Eq. (13), but deviations appear at the gap edges, which, in practice, are likely to be sensitive to the details that are not necessarily described with a single parameter r .

In summary, we discuss the current-phase relation, density of states, and free energy of superconducting wires, focusing on points relevant for the SSQIPT application. The results point out that superconducting material is a viable choice, provided that the junction length does not

significantly exceed the coherence length, in order to retain sensitivity and avoid magnetic hysteresis.

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APPENDIX A: PSEUDOARCLENGTH CONTINUATION

For completeness, we describe here the application of pseudoarclength continuation on the self-consistency equation. The method generates a set of values (Δ_k, φ_k) , tracing a curve of solutions. The next point $(\Delta_{k+1}, \varphi_{k+1})$ is generated from the previous by solving

$$F'[\Delta_{k+1}, \varphi_{k+1}] = 0, \quad (\text{A1a})$$

$$s = (2 - \theta)\dot{\varphi}_k(\varphi_{k+1} - \varphi_k) + \theta \text{Re} \int_0^L dx \dot{\Delta}_k^*(x) [\Delta_{k+1}(x) - \Delta_k(x)]. \quad (\text{A1b})$$

Here, $F'[\Delta, \varphi] = 0$ denotes the set of equations (4)–(7), taking $\Delta(x)$ and the phase difference φ as unknowns. The value of φ_{k+1} is fixed by the pseudoarclength constraint (A1), where the parameter $s > 0$ is an arclength constant and $\theta \in [0, 2]$ a weight factor. The tangent approximants can be taken as $\dot{\varphi}_k = (\varphi_k - \varphi_{k-1})/\delta$, $\dot{\Delta}_k = (\Delta_k - \Delta_{k-1})/\delta$, with $\delta^2 = \theta \|\Delta_k - \Delta_{k-1}\|_2^2 + (2 - \theta) |\varphi_k - \varphi_{k-1}|^2$. Equation (A1) is of similar complexity as the self-consistency equation $F'[\Delta] = 0$, and it can be solved for $(\Delta_{k+1}, \varphi_{k+1})$ using standard nonlinear solvers, given a spatial discretization of Δ .

APPENDIX B: RICCATI PARAMETRIZATION

Equations (2) and (3) can be written in a Riccati parametrization, $e^{i\chi} \sin \hat{\theta} \equiv 2\gamma/(1 + |\gamma|^2)$:

$$\mathcal{F}_0 = |\Delta|^2 \ln \frac{T}{T_c} + 2\pi T \sum_{\omega_n > 0} \left\{ \frac{|\Delta|^2}{\omega_n} + \frac{2|\partial_x \gamma|^2}{(1 + |\gamma|^2)^2} + 4 \frac{\omega_n |\gamma|^2 - \text{Re}[\Delta^* \gamma]}{1 + |\gamma|^2} \right\}, \quad (\text{B1})$$

and

$$F_b = \frac{2\pi TD}{r} \sum_{\omega_n > 0} \sum_{j=\pm} \frac{2|\gamma(x_j) - \gamma_{Sj}|^2}{(1 + |\gamma(x_j)|^2)(1 + |\gamma_{Sj}|^2)}. \quad (\text{B2})$$

This form has some advantages for numerical implementation. Moreover, the well-known connection to the

Ginzburg-Landau functional [26,44] is straightforward in this form. The minimum of \mathcal{F}_0 vs γ is $\gamma(x, \omega_n) \approx \Delta(x)/(2\omega_n)$, up to corrections $\propto |\partial_x^2 \gamma|, |\Delta|^3$. Neglecting the corrections, substituting the leading term back in, and expanding in a small $|\Delta|$ yields

$$\mathcal{F}_0 \approx \frac{\pi}{8T} |\partial_x \Delta|^2 - |\Delta|^2 \ln \frac{T_c}{T} + \frac{7\zeta(3)}{16\pi^2 T^2} |\Delta|^4, \quad (\text{B3})$$

the GL functional.

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