Electric Power Generation from Earth's Rotation through its Own Magnetic Field

Christopher F. Chyba*

Department of Astrophysical Sciences and Woodrow Wilson School of Public and International Affairs, Princeton University, Princeton, New Jersey 08544, USA

Kevin P. Hand†

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109, USA

(Received 25 June 2015; revised manuscript received 16 November 2015; published 29 July 2016)

We examine electric power generation from Earth's rotation through its own nonrotating magnetic field (that component of the field symmetric about Earth's rotation axis). There is a simple general proof that this is impossible. However, we identify a loophole in that proof and show that voltage can be continuously generated in a low-magnetic-Reynolds-number conductor rotating with Earth, provided magnetically permeable material is used to ensure curl$(\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$ within the conductor, where $\mathbf{B_0}$ derives from the axially symmetric component of Earth's magnetic flux density, and $\mathbf{v}$ is Earth's rotation velocity at the conductor's location. We solve the relevant equations for one laboratory realization, and from this solution, we predict the voltage magnitude and sign dependence on system dimensions and orientation relative to Earth's rotation. The effect, which would be available nearly globally with no intermittency, requires testing and further examination to see if it can be scaled to practical emission-free power generation.

DOI: 10.1103/PhysRevApplied.6.014017

I. INTRODUCTION

Barnett [1] showed in 1912 that when an axially symmetric electromagnet rotates about its north-south axis, its magnetic field does not rotate with the magnet, resolving a controversy [2] that had its origins in Faraday's [3,4] interpretation of his rotating-disk experiment. In Sec. II, we first review Faraday's results and carefully address the definition of electromotive force (emf) and the historical meaning of saying a magnetic field "rotates with the magnet." We then describe the compelling experimental evidence for the nonrotation of axially symmetric magnetic fields. In Sec. III, we consider the particular case of the nonrotation of the axisymmetric component of Earth's magnetic field. The rotation of Earth's surface through that nonrotating component yields a steady $\mathbf{v} \times \mathbf{B}$ force that one might hope to use to generate electric power. However, in Sec. IV we present a simple and seemingly general proof that power generation in this way is impossible.

Nonetheless, in Sec. V we show that this proof has a loophole, suggesting that continuous power generation is possible if two unusual conditions are both met. These two conditions will not simultaneously hold in any typical natural or laboratory circuit, but it is possible to create them together. The first condition is that the current path must lie within a magnetically permeable conductor the topology of which is such that $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq \mathbf{0}$ in its interior, where $\mathbf{B}$ is the magnetic flux density, and $\mathbf{v}$ is the velocity of the conductor. The second is that this conductor must have a magnetic Reynolds number $R_m \ll 1$, which on a laboratory scale excludes all common metal and mu-metal conductors.

In Secs. VI–IX, we fully calculate one realization of such a system: a low-R_m magnetically permeable cylindrical shell. Section VI first considers the case when the shell is stationary ($\mathbf{v} = \mathbf{0}$) with respect to a constant background magnetic field (with zero background electric field). Then the current density $\mathbf{J} = \mathbf{0}$, and it is straightforward to derive the corresponding magnetic flux density $\mathbf{B_0}$ within the shell. We prove that, in general, $\nabla \times (\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$, so that if the shell's composition, dimensions, and velocity can be chosen to yield $R_m \ll 1$, the system will fulfill the necessary conditions for emf generation.

In Secs. VII–IX, we demonstrate that for this system these conditions are also sufficient. In Sec. VII, we show that if the shell is put into motion transverse to its long axis, $\mathbf{B_0}$ can no longer be a solution. We calculate the $\mathbf{B}$ that does satisfy the induction equation within the moving shell when $\mathbf{v} \neq \mathbf{0}$ and $R_m \ll 1$ and find that the time-dependent part of $\mathbf{B}$ goes to zero extremely rapidly. In Sec. VIII, we find that there remains a time-independent solution given by $\mathbf{B_0}$ plus a series of perturbation terms scaled by successive powers of R_m. We use these results in Sec. IX to derive an expression for the emf generated in the shell. Section X provides an intuitive discussion of the results of the previous calculations.

In Sec. XI, we present a parallel analysis in a frame comoving with the translating shell and demonstrate that there is a net nonzero Poynting vector flux delivering power

*cchyba@princeton.edu; http://wws.princeton.edu/faculty-research/faculty/cchyba

†Kevin.P.Hand@jpl.nasa.gov; https://science.jpl.nasa.gov/people/Hand/

into the shell. We show in Sec. XII that in the frame in which the shell is translating at $\mathbf{v}$, a magnetic braking term arises in Poynting's theorem, and the generated electrical power equals the braking loss from Earth's rotational kinetic energy.

We make quantitative predictions for this system in Sec. XIII, including the striking prediction that the voltage generated should change sign when the cylindrical shell (together with its attached leads and voltmeter) is rotated by 180°. Section XIV begins the discussion of whether such systems might be scaled up to generate useful amounts of electric power.

II. HISTORICAL BACKGROUND AND DEFINITIONS

In December 1831, Faraday [3,4] experimented with a conducting disk rotating near a magnet. The disk connected via brushes to a simple galvanometer, with leads running to the disk's axle and edge. The galvanometer circuit was stationary in the laboratory. Current flowed when the magnet was stationary and the disk rotated or when the disk and magnet rotated coaxially along the magnet's north-south axis of symmetry but not when the magnet rotated about this axis and the disk was stationary [3,5].

Faraday subsequently experimented with a rotating conducting magnet connected to a galvanometer via brushes on the magnet's axle and rim [5,6]. Current flowed when the magnet rotated around its north-south axis but the galvanometer circuit remained stationary, or when the magnet was stationary but the circuit rotated. Subsequent researchers have explored additional permutations in the configurations of Faraday's experiments [7].

In modern terms, the conducting magnet rotates at velocity $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$ (for angular velocity $\boldsymbol{\omega}$ and cylindrical radius $\boldsymbol{\rho}$) through its own magnetic field $\mathbf{H}$ (or, equivalently, through its own magnetic flux density $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability), generating a $\mathbf{v} \times \mathbf{B}$ Lorentz force that drives the current.

The emf around a path C with line element $\mathbf{dl}$ is given by [8–10]

$$\begin{aligned} \text{emf} &= \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} \\ &= \int_S [-\partial\mathbf{B}/\partial t + \nabla \times (\mathbf{v} \times \mathbf{B})] \cdot \mathbf{da}, \end{aligned} \tag{1}$$

where $\mathbf{E}$ is the electric field, and the area element $\mathbf{da}$ is right-hand normal to the surface S bounded by C. The second equality in Eq. (1) holds via Stokes's theorem and the Faraday's law Maxwell equation, provided there is no jump discontinuity on S [11]. This condition is met in our work below, and we calculate the emf using Eq. (1), which we take as the definition of the term.

For the Faraday disk, for which $\mathbf{B}$ is spatially constant and $\partial\mathbf{B}/\partial t = 0$, only the $\mathbf{v} \times \mathbf{B}$ term contributes to the integral. If the entire circuit rotates at constant $\boldsymbol{\omega}$, the curl of $\mathbf{v} \times \mathbf{B}$ will be zero and we will have emf $= 0$. But because the galvanometer circuit is stationary while the disk rotates in the laboratory frame, the line integral of $\mathbf{v} \times \mathbf{B}$ around C is nonzero. In any frame, at least part of C is in motion. A Poynting theorem analysis of the Faraday disk shows that the energy for the electric current flowing between the axle and rim in the disk comes from the disk's kinetic energy of rotation [12]. Taking into account the small magnetic perturbations to the applied $\mathbf{B}$ due to the current that flows in C does not change these conclusions [12].

The emf in Eq. (1) is often identical to an electromotive force defined by the "flux rule":

$$\text{emf}_\Phi = -d\Phi/dt, \tag{2}$$

where magnetic flux $\Phi = \int_S \mathbf{B} \cdot \mathbf{da}$. Inequality between the emf and emf_Φ in Eqs. (1) and (2) in certain circumstances gives rise to so-called Faraday paradoxes. Auchmann *et al.* [11], consistent with some earlier discussions [13], show that equality requires the path velocity of the moving surface S (and its boundary C) to be equal to the material velocity of the conducting medium in which S is embedded. Our applications below meet this requirement.

Faraday [3,4,6] concluded from his experiments that magnetic field lines do not rotate with a magnet when the magnet rotates around its axis of symmetry, but Preston [2] showed in 1885 that Faraday's results are equally explained if the magnetic field does rotate with the magnet, producing a $\mathbf{v} \times \mathbf{B}$ force on the stationary part of C, giving an emf identical to that of a nonrotating field with the rotating disk. The idea of the field "rotating with the magnet" was understood [2,14,15] to mean that a force $q\mathbf{v} \times \mathbf{B}$ will be experienced by an electric charge q if q has a velocity $\mathbf{v}$ relative to axes fixed in (so corotating with) the rotating magnet. This understanding differs from the current understanding of the $q\mathbf{v} \times \mathbf{B}$ force, in which $\mathbf{v}$ is the velocity of q in the frame in which the magnetic flux density is $\mathbf{B}$ [13,16].

Poincaré [17], among other contemporary scientists [18], asserted that since both the rotating and nonrotating pictures appeared to give identical results, the distinction between them was meaningless. But Barnett's [1] experiments in 1912 reproduced by Kennard [15] and improved upon by Pegram [19] in 1917 demonstrated a difference and resolved the question for the magnetic fields of electromagnets by using an open circuit. Barnett placed a cylindrical capacitor axially in the field of a solenoid (or, in analogous experiments, between two large iron flat-pole electromagnets); a thin wire connected the two concentric cylinders of the capacitor. Corotation of the cylinders and their connecting wire while holding the solenoid stationary charged the capacitor (due to the $\mathbf{v} \times \mathbf{B}$ force on the wire). After charging, the connecting wire was disconnected, the system despun, and opposite charges on the cylinders were measured by an electrometer. But rotating the solenoid (or flat-pole electromagnets) while holding the cylindrical capacitor and connecting wire stationary generated no charge. Corotation of the capacitor and connecting wire together with the solenoid charged the capacitor [19]. Barnett and his contemporaries [1,5,15,18,19] thereby proved that the field of a rotating axially symmetric electromagnet does not itself rotate.

III. NONROTATION OF EARTH'S AXISYMMETRIC FIELD

The Barnett [1], Kennard [15], and Pegram [19] experiments with electromagnets suggest that those components of Earth's magnetic field that are axisymmetric about Earth's rotation axis will be stationary with respect to (do not rotate with) the rotating Earth, understood in the sense described for rotating electromagnets in the previous section [20–22]. These components are, for example, the axially symmetric dipole, quadrupole, and octopole components, with coefficients in the usual Schmidt-normalized Legendre-function expansion of g_1^0, g_2^0, and g_3^0, respectively [23]. Components with coefficients g_n^m or h_n^m where $m \neq 0$ depend on azimuthal angle φ like $\cos(m\varphi)$ and $\sin(m\varphi)$, respectively, and, therefore, rotate with Earth.

Nonrotation of Earth's axisymmetric field is the conservative expectation given the experimental results for electromagnets [1,5,15,19], and this has been taken to be the case by many authors [20–22,24]. If the effect we predict in this paper is demonstrated, an ancillary consequence will be an experimental demonstration of the nonrotation of Earth's axisymmetric field.

Earth's rotation carries its surface through the nonrotating component of Earth's magnetic flux density $\mathbf{B}$ with an azimuthal speed $v = 465 \sin\theta$ m s^{-1} at colatitude θ. The resulting $\mathbf{v}\times\mathbf{B}$ force generates position-dependent volume charge densities (of order 1 e$^-$ m^{-3} [22]) whose electric field perfectly cancels $\mathbf{v}\times\mathbf{B}$ [20,24,25]. A resulting latitude-dependent surface charge density maintains overall charge balance, with a corresponding electric potential at Earth's surface. Any additional motion of individual conductors or conducting fluids leads to continuous extremely rapid charge redistribution with resulting perfect cancellation of fields.

However, Earth is surrounded by a conducting ionosphere corotating with Earth. Does this external conducting spherical shell mean that Earth's axisymmetric magnetic field is somehow "dragged" into corotation with Earth? One might imagine that this is an implication of Alfvén's "frozen-flux" theorem [26], which considers Ohm's law (for current density $\mathbf{J}$) for a moving conductor

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{J}/\sigma \tag{3}$$

in the limit $\sigma \to \infty$ (a so-called perfect conductor), so that $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$. Then, Eqs. (1) and (2) imply that the magnetic flux Φ cannot change through the surface S as C moves along—in the usual picturesque language, the flux is "frozen in."

But consider an axially symmetric conductor rotating with angular speed ω about the axis of Earth's axisymmetric magnetic field. Clearly, $\partial\mathbf{B}/\partial t = 0$ in such a case. This term is the first term in the integrand of the surface integral in Eq. (1). In spherical polar coordinates (r, θ, φ), we have $\mathbf{v} = \omega r \sin\theta\hat{\varphi}$ and find

$$\nabla\times(\mathbf{v}\times\mathbf{B}) = -\omega\partial\mathbf{B}/\partial\varphi \tag{4}$$

using $\nabla\times(\mathbf{v}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{B} + \mathbf{v}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{v})$ and $\nabla\cdot\mathbf{B} = 0$. Therefore, the second term in the integrand in Eq. (1) is also zero due to axisymmetry, and by Eq. (2), this means $d\Phi/dt = 0$. But this has nothing to do with $\sigma \to \infty$; it would be just as true for a very poor conductor as for a perfect conductor. It is, therefore, not a consequence of the frozen-flux theorem. It is simply a consequence of the symmetry involved, and there is no reason to view the field as somehow being dragged around with the ionosphere.

In fact, there are well-known examples where translating or rotating conductors do not "drag" magnetic fields at all and cause no distortions in the magnetic fields through which they are moving [22,27,28]. This is because the background field is modified only if a current density $\mathbf{J}$ is induced in the conductor; $\mathbf{J}$ then induces a magnetic field of its own via Ampère's law, and it is this induced field that distorts the shape of the overall $\mathbf{B}$ field away from the background field. This distortion often, though not always [27], leads to field lines that have the appearance of being dragged by the moving conductor. But Van Bladel [29] has proven that it is impossible to induce a nonzero $\mathbf{J}$ for any axially symmetric conductor rotating in an axially symmetric field. By this theorem, a conducting ionosphere rotating about Earth's axially symmetric field components cannot induce a $\mathbf{J}$ and, therefore, cannot distort (drag along with it) Earth's axially symmetric field. On the contrary, Appendix A describes how it is Earth's nonrotating axially symmetric field that brings charged particles in a conducting plasma around Earth into corotation [24].

IV. A PROOF THAT ELECTRIC POWER GENERATION IS IMPOSSIBLE

Can we construct a circuit C in the lab whose rotation along with Earth's surface through Earth's axially symmetric field generates a continuous electric current via the $\mathbf{v}\times\mathbf{B}$ force? The emf around any path C is given by Eq. (1). The $\mathbf{v}\times\mathbf{B}$ force experienced as the conductor C rotates through Earth's magnetic field drives electron redistribution until the resulting electrostatic field $\mathbf{E}$ perfectly cancels the $\mathbf{v}\times\mathbf{B}$ field: $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$ everywhere within C [5,22,30]. The redistribution of charge occurs extremely rapidly, on a classical charge relaxation time scale $\tau_e \sim \epsilon_0/\sigma \approx 10^{-11}(1\ \mathrm{S\,m^{-1}}/\sigma)$ s [31]. For very good conductors such as typical metals for which $\sigma \sim 10^7$ S m^{-1}, the relaxation time is given by the electron-collision time scale $\tau_c \sim 10^7 \tau_e$ or $\sim 10^{-11}$ s [32]. Since charge redistributes rapidly and continuously to maintain $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, emf $= 0$ by Eq. (1) always. Electric power generation, therefore, appears impossible for uniform rotation about an axially symmetric field.

However, this argument contains hidden assumptions. The electric field of a static charge distribution may always be written as a potential of a scalar field: $\mathbf{E} = -\nabla V$. But since $\nabla \times \nabla V = 0$ always, the equation $\mathbf{E} = -\nabla V = -\mathbf{v} \times \mathbf{B}$ can hold only if $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$. We use magnetically permeable materials to violate this requirement, providing a necessary, but not sufficient, condition for generating a nonzero emf.

Of course, one can always choose to transform to a gauge in which the transformed scalar potential $\tilde{V} = 0$. But then the vector potential transforms to $\tilde{\mathbf{A}} = (\nabla V)t$, and $\mathbf{E} = -\partial \tilde{\mathbf{A}}/\partial t - \nabla \tilde{V} = -\nabla V$, as before [33]. So once again, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ can hold only if $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$.

V. THE LOOPHOLE IN THE PROOF

Magnetically permeable materials channel magnetic flux and can be used to alter $\mathbf{B}$ to give $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq 0$. This inequality guarantees that the electrons in such a conductor cannot rearrange themselves to generate an electrostatic field $\mathbf{E} = -\nabla V$ that satisfies $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ in Eq. (1). This inequality is the first of our two necessary conditions for electric power generation. But one can still have $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ if $\mathbf{E}$ is no longer purely electrostatic, i.e., if one had $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla V = -\mathbf{v} \times \mathbf{B}$, where $\mathbf{A}$ is the magnetic vector potential. If this equality is required to hold for our system, power generation will still be impossible. Are there circumstances where this equality can be circumvented? This may be answered using the advection-diffusion equation for $\mathbf{A}$, to which we now turn.

Consider two inertial frames. In frame K at infinity, there is a constant background magnetic flux density ($\mathbf{B}_\infty$) and no electric field ($\mathbf{E}_\infty = \mathbf{0}$). A conductor is moving at constant velocity $\mathbf{v} = v\hat{\mathbf{y}}$ in K. Frame K' is the frame comoving with the conductor. Frame K' approximates our frame on Earth's surface (the laboratory frame), translating through the nonrotating component of Earth's field. Frame K approximates a nonrotating frame fixed at Earth's center and moving with Earth in its orbit.

Frames K and K' are not exactly related by a Lorentz boost because of Earth's rotation. In K', Maxwell's equations incorporate rotation via the metric tensor $g_{\mu\nu}$, introducing factors $\sqrt{g_{00}} \approx 1 - \frac{1}{2}(v/c)^2$ when $(v/c) \ll 1$ [29]. For $v = 465$ m s^{-1}, $(v/c)^2 \approx 10^{-12}$. We show below that these corrections are negligible compared to the effects of interest. We assume $(v/c)^2 \ll 1$ throughout. We may, therefore, approximate K and K' as two inertial frames in relative linear motion.

Coordinates in the two frames are then related by $t' = t$, $x' = x$, $y' = y - vt$, and $z' = z$. We have $\partial x^\mu/\partial x^\nu = \delta^\mu_\nu$ and $\partial x'^\mu/\partial x'^\nu = \delta^\mu_\nu$, $\partial t/\partial t' = 1$, $v = \partial y/\partial t'$, $\partial/\partial t' = \partial/\partial t + v\partial/\partial y$, and $\partial/\partial x' = \partial/\partial x$, $\partial/\partial y' = \partial/\partial y$, $\partial/\partial z' = \partial/\partial z$, so $\nabla'^2 = \nabla^2$. The fields are related by

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{5}$$

$\mathbf{B}' = \mathbf{B}$, and $\mathbf{A}' = \mathbf{A}$. For our system, the curl of Eq. (5) yields $\partial \mathbf{B}/\partial t' = \partial \mathbf{B}/\partial t + v\partial \mathbf{B}/\partial y$; i.e., the curl of the field transformation for $\mathbf{E}'$ is just the advective derivative for $\mathbf{B}$. While $\mathbf{B} = \mathbf{B}'$, Eq. (5) means that $\mathbf{E} = \mathbf{0}$ in K implies $\mathbf{E}' = \mathbf{v} \times \mathbf{B}$ in K'.

We begin with an analysis in frame K and examine results in K' in Sec. XI. Ohm's law in K' is $\mathbf{E}' = J'/\sigma$, so by Eqs. (3) and (5), $\mathbf{J}' = \mathbf{J}$. We have $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{B}' = \mu\mathbf{H}'$ [29]. Using $\mathbf{E} = -\nabla V - \partial \mathbf{A}/\partial t$ and $\mathbf{J} = \nabla \times \mathbf{H}$, Eq. (3) yields the advection-diffusion equation for $\mathbf{A}$ in K:

$$-\nabla V - \partial \mathbf{A}/\partial t + \mathbf{v} \times (\nabla \times \mathbf{A}) = \eta \nabla \times \nabla \times \mathbf{A}, \tag{6}$$

where $\eta = (\sigma\mu)^{-1}$ is the magnetic diffusivity, here assumed constant. The displacement current does not appear in Ampère's law because $|\epsilon_0 \partial \mathbf{E}/\partial t| \ll |\mathbf{J}|$ (ϵ_0 is the vacuum permittivity) for time scales $t \gg \tau_e$ [10,34]. Ohm's law in K' also yields Eq. (6) because of the field transformation Eq. (5).

The curl of Eq. (6) yields the advection-diffusion equation for $\mathbf{B}$ or "induction equation":

$$-\partial \mathbf{B}/\partial t + \nabla \times (\mathbf{v} \times \mathbf{B}) = -\eta \nabla^2 \mathbf{B}. \tag{7}$$

Integrated over S, Eq. (7) is identical to Eq. (1). Therefore,

$$\text{emf} = -\eta \int_S \nabla^2 \mathbf{B} \cdot \mathbf{da} = \eta \oint_C (\nabla \times \mathbf{B}) \cdot \mathbf{dl}. \tag{8}$$

Whether $\eta \nabla^2 \mathbf{B}$ is negligible in Eq. (7) depends on the magnetic Reynolds number $R_m = \tau_D/\tau_v = \sigma\mu v\xi$, where $\tau_D = \xi^2/\eta$ is the magnetic diffusion time, and $\tau_v = \xi/v$ the transport time for a system that varies over a characteristic length scale ξ. Then, $|\eta \nabla^2 \mathbf{B}| \sim \eta B/\xi^2$ and $|\nabla \times (\mathbf{v} \times \mathbf{B})| \sim vB/\xi$ so $R_m = |\nabla \times (\mathbf{v} \times \mathbf{B})|/|\eta \nabla^2 \mathbf{B}|$ [35,36]. If $R_m \gg 1$, $\eta \nabla^2 \mathbf{B}$ is negligible in Eq. (7), so emf $= 0$. If $R_m \ll 1$, however, we may have emf $\neq 0$. $R_m \ll 1$ is the second of our two necessary conditions for electric power generation.

In Eq. (1), consider a path C lying within a conducting slab made, say, of aluminum for which $\sigma = 4 \times 10^7$ S m^{-1} and relative permeability $\mu_r = 1$ ($\mu_r = \mu/\mu_0$ where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}) [37]. Then for $v = 465$ m s^{-1}, $R_m \gg 1$ if $\xi > 1$ mm so emf $= 0$. We instead explore a system satisfying Ohm's law with $R_m \ll 1$ and $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq 0$. As we show below, one realization is a path C lying within a long cylindrical shell made of an appropriate magnetically permeable MnZn ferrite [38].

We first consider this system at rest in a frame K in which $\mathbf{B}_\infty$ is constant and there is no background electric field ($\mathbf{E}_\infty = 0$), in which case emf $= 0$. We then give this system a velocity $\mathbf{v}$ and show that a nonzero emf will be generated. Such a system at rest in a laboratory on Earth's surface would, therefore, generate electrical power as Earth rotates.

VI. MAGNETICALLY PERMEABLE CYLINDRICAL SHELL

Consider an infinitely long magnetically permeable conducting cylindrical shell with axis of symmetry along the z axis, and inner and outer radii a and b, respectively. The background fields at infinity are $\mathbf{B}_\infty = B_\infty \hat{\mathbf{x}}$ and $\mathbf{E}_\infty = 0$ in a frame in which the shell has $\mathbf{v} = \mathbf{0}$. Of course, $\mathbf{v} \times \mathbf{B} = \mathbf{0}$, and with $\mathbf{E}_\infty = \mathbf{0}$ we must have by Eq. (3) $\mathbf{J} = \mathbf{0}$. Therefore, $\nabla \times \mathbf{H} = \mathbf{J} = \mathbf{0}$, so $\mathbf{H} = -\nabla W$, where W is a magnetic potential. We designate $\mathbf{H}$ when $\mathbf{v} = \mathbf{0}$ as $\mathbf{H}_0$ (and define $\mathbf{B_0} = \mu \mathbf{H}_0$) so $\nabla \cdot \mathbf{H_0} = \nabla^2 W = 0$, whose solution for a magnetically permeable cylindrical shell for all space is well known in cylindrical (ρ, ϕ, z) coordinates [39,40]. In Cartesian coordinates, the resulting magnetic flux densities exterior to the shell, within its conducting body, and within its hollow interior are

$$B_{0x}(\rho > b) = B_\infty + \beta_3 (b/\rho)^2 \cos 2\phi, \tag{9a}$$

$$B_{0y}(\rho > b) = \beta_3 (b/\rho)^2 \sin 2\phi; \tag{9b}$$

$$B_{0x}(a \le \rho \le b) = \beta_1 - \beta_2 (a/\rho)^2 \cos 2\phi, \tag{10a}$$

$$B_{0y}(a \le \rho \le b) = -\beta_2 (a/\rho)^2 \sin 2\phi; \tag{10b}$$

and

$$B_{0x}(\rho < a) = 2\beta_1 (\mu_r + 1)^{-1}, \tag{11a}$$

$$B_{0y}(\rho < a) = 0. \tag{11b}$$

Here,

$$\beta_1 = 2B_\infty \mu_r (\mu_r + 1)\zeta, \tag{12}$$

$$\beta_2 = 2B_\infty \mu_r (\mu_r - 1)\zeta, \tag{13}$$

$$\beta_3 = B_\infty [1 - (a/b)^2](\mu_r^2 - 1)\zeta, \tag{14}$$

and

$$\zeta = [(\mu_r + 1)^2 - (a/b)^2 (\mu_r - 1)^2]^{-1}. \tag{15}$$

If $a = 0$, Eq. (10) collapses to that for a solid magnetically permeable cylinder:

$$\mathbf{B}(\rho \le b) = \beta_1 (a = 0)\hat{\mathbf{x}} = 2\mu_r (\mu_r + 1)^{-1} B_\infty \hat{\mathbf{x}}, \tag{16}$$

for which the magnetic field is constant in the interior, although of course the exterior ($\rho > b$) field is distorted according to Eq. (9) with $a = 0$.

Because $B_z = 0$, only the z component of $\mathbf{A}$ is nonzero [9,27], so

$$B_x = \partial A_z / \partial y \tag{17a}$$

and

$$B_y = -\partial A_z / \partial x. \tag{17b}$$

Equations (9) to (11) then correspond to the vector potential $\mathbf{A}_0 = A_0 \hat{\mathbf{z}}$ with

$$A_0(\rho > b) = B_\infty y + \beta_3 (b^2/\rho) \sin \phi, \tag{18}$$

$$A_0(a \le \rho \le b) = \beta_1 y - \beta_2 (a^2/\rho) \sin \phi, \tag{19}$$

and

$$A_0(\rho < a) = 2\beta_1 (\mu_r + 1)^{-1} \rho \sin \phi, \tag{20}$$

with the usual gauge ambiguity allowing the addition of a gradient of a single-valued function. Moreover, because of Eq. (17), any function of z alone may be added to $\mathbf{A_0}$ without affecting $\mathbf{B_0}$ (or $\mathbf{E}$). $\mathbf{A_0}$ must be continuous across the boundaries at $\rho = a$ and $\rho = b$; this is easy to verify for Eqs. (18)–(20). This requirement means that a choice of gauge on one side of a boundary restricts the choice of gauge on the other [41]: one cannot arbitrarily assign different gradient terms (or functions of z) to A_0 in each of Eqs. (18)–(20), and this proves important in Appendix B.

From Eq. (19), we see that a solid permeable cylinder has $A_0(\rho \le b) = \beta_1 y$ in its interior. That is, the first term on the right in Eq. (19) is that for a solid cylinder; when $a \ne 0$, a second term enters as a modification of this first $a = 0$ term.

For the region within the body of the cylindrical shell ($a \le \rho \le b$), we find by Eq. (10):

$$\nabla \times (\mathbf{v} \times \mathbf{B_0}) = 2v\beta_2 a^2 \rho^{-3} [(3 \sin \phi - 4\sin^3 \phi)\hat{\mathbf{x}} + (3 \cos \phi - 4\cos^3 \phi)\hat{\mathbf{y}})] \ne 0 \tag{21}$$

using $\partial\rho/\partial x = \cos\phi$, $\partial\phi/\partial x = -\rho^{-1} \sin\phi$, $\partial\rho/\partial y = \sin\phi$, and $\partial\phi/\partial y = \rho^{-1} \cos\phi$. Such a translating shell if made out of conducting material satisfying $R_m \ll 1$ will, therefore, satisfy our two necessary criteria for electric power generation. We show below that in this case, these conditions are also sufficient.

Instead of an infinite shell, consider a finite shell lying along the z axis from $-L/2$ to $L/2$, with $L \gg 2b$. The magnetic field in the interior ($\rho < a$) of a finite permeable cylindrical shell may be written as the sum of two contributions: the field corresponding to the shielded interior of an infinitely long shell plus a contribution from the field penetrating in from the openings [42]. For $\rho < a$

near $z = \pm L/2$, $\mathbf{B}$ deviates from $\mathbf{B_0}$, but moving inward, this deviation falls off rapidly like $\exp(-3.83z/a)$ [42], so in the interior, the result for an infinite shell should hold for a finite shell provided $|z| \lesssim L/2 - a$. For this result to hold for $\rho < a$, the field in the region $a \le \rho \le b$ must be similarly undisturbed, so we take the result for a finite shell in this region to correspond to those for an infinite shell provided $|z| \lesssim L/2 - a$.

VII. TIME-DEPENDENT SOLUTION FOR $\mathbf{v} \neq 0$ AND $R_m \ll 1$

It is clear that $\mathbf{B_0}$ in Eq. (10) can no longer be a solution for our system once $\mathbf{v} \neq 0$, since $\mathbf{B_0}$ can only solve Eq. (7) were $\nabla \times (\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$ in contradiction to Eq. (21). We show in Appendix B that $\mathbf{B_0}(x, y') = \mathbf{B_0}(x, y - vt)$, i.e., the advecting version of Eq. (10), also cannot be a general solution when $\mathbf{v} \neq \mathbf{0}$. Any traveling wave solution of the form $\mathbf{B}(x, y - vt)$ solves the transport equation so will solve Eq. (7) in the limit $R_m \gg 1$. We wish to solve Eq. (7) for smaller R_m when the diffusion term is not negligible.

It is easiest first to solve for $\mathbf{A}$. We, therefore, solve Eq. (6) explicitly to find $\mathbf{A}$ (and so $\mathbf{B}$) for the magnetically permeable cylindrical shell in the case $\mathbf{v} \neq 0$ (Fig. 1) with $R_m \ll 1$. We impose the requirement that in the limit $\mathbf{v} \to \mathbf{0}$ we must have $\mathbf{A} \to \mathbf{A_0}$ and $\mathbf{B} \to \mathbf{B_0}$. We first work in K and examine the picture in K' in Sec. XII.

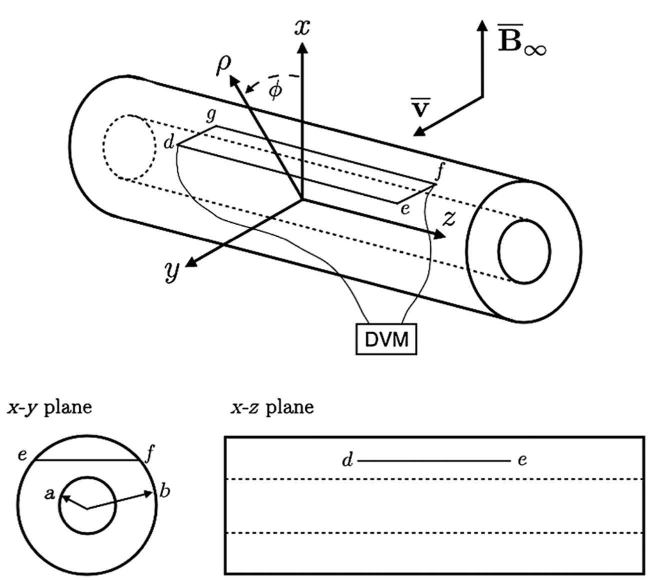

FIG. 1. A magnetically permeable, low-R_m cylindrical shell with inner and outer radii a and b and length L is moving at velocity $\mathbf{v} = v\hat{\mathbf{y}}$ through background fields $\mathbf{B}_\infty = B_\infty \hat{\mathbf{x}}$ and $\mathbf{E}_\infty = 0$. A rectangular current path C with vertices d, e, f, g is embedded in, and translating with, the shell. An emf is generated around C according to Eq. (63). A digital voltmeter (DVM) measures half this emf between d and f. C lies in the plane $x = b\cos\phi_0 \equiv x_0$, with $|x_0| \ge a$. It has right-angle vertices at $d = (x_0, y_0, -l/2)$, $e = (x_0, y_0, l/2)$, $f = (x_0, -y_0, l/2)$, and $g = (x_0, -y_0, -l/2)$, where $y_0 = b\sin\phi_0$ and $l/2 < L/2 - a$.

Our calculations can be facilitated by a choice of gauge to simplify Eq. (6). We choose a gauge sometimes used in eddy current [43] or magnetohydrodynamic (MHD) [44] applications that relates the potentials by the gauge condition:

$$\nabla \cdot \mathbf{A} = -V/\eta. \tag{22}$$

Because Eq. (22) is less familiar than the more commonly used Lorenz ($\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \partial V/\partial t$) or Coulomb ($\nabla \cdot \mathbf{A} = \mathbf{0}$) gauges [45], we discuss it further in Appendix C. In the gauge of Eq. (22), Eq. (6) simplifies to

$$-\partial \mathbf{A}/\partial t + \mathbf{v} \times (\nabla \times \mathbf{A}) = -\eta \nabla^2 \mathbf{A} \tag{23}$$

using the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{24}$$

A complete solution to the system is given by Eqs. (22) and (23) together. We have $\mathbf{A} = A_z \hat{\mathbf{z}}$ and $\nabla \cdot \mathbf{A} = \partial A_z/\partial z$. Because $\mathbf{v} \times (\nabla \times \mathbf{A}) = -v \partial A_z/\partial y \hat{\mathbf{z}}$, Eq. (23) reduces to a single nontrivial equation:

$$\partial A_z/\partial t + v \partial A_z/\partial y = \eta \nabla^2 A_z. \tag{25}$$

By Eq. (17), a function $f(z)$ may be added to A_z without altering $\mathbf{B}$, so $f(z)$ may be chosen to yield in Eq. (22) the appropriate V expected by physical arguments. However, by Eq. (1), the emf around C is independent of V, so for the emf, it is enough to solve Eq. (23).

If $R_m \gg 1$, $|\eta \nabla^2 A_z| \ll |v \partial A_z/\partial y|$ and Eq. (25) collapses to a transport equation whose solution is a function of the form $A_z(x, y - vt)$. We are interested in the case $R_m \ll 1$, for which we expect diffusion to be important. The advection term $v\partial A_z/\partial t$ in Eq. (25) cannot be neglected even with $R_m \ll 1$ because $\eta \nabla^2 A_0 = 0$, and by analogy to MHD [10,34,41], we expect (or at least must not exclude *ab initio*) $|v \partial A_0/\partial t| \sim |\eta \nabla^2 A_1|$, where A_1 is a small perturbation term satisfying $|A_1| \sim R_m |A_0|$. We seek a solution A_z to Eq. (25) that holds when $R_m \ll 1$ for any $\mathbf{v} = v\hat{\mathbf{y}}$ with the requirement $A_z \to A_0$ as $\mathbf{v} \to \mathbf{0}$. Henceforth, we set $\xi = b$ as the relevant diffusion length scale, so put $R_m = \mu\sigma v b$, with advection time scale $\tau_v = b/v$ and diffusion time scale $\tau_D = b^2/\eta = R_m \tau_v$.

We solve Eq. (25) exactly using cylindrical coordinates in K with the origin centered in the shell at some particular instant; such a solution will hold only for a time short compared with τ_v, after which the shell will have moved sufficiently far from the origin that the cylindrical symmetry assumed in Eqs. (9)–(11) is broken. However, we will see that the system reaches a steady-state extremely rapidly with $\tau_D \ll \tau_v$, meaning that $\mathbf{B}$ extremely rapidly adapts itself via diffusion to the shell's motion [34,41]. For any location of the translating shell, we may choose the origin in K to coincide with the center of the shell at that instant. Since there was nothing special about the instant chosen, this should represent the steady state for the system.

The solution to Eq. (25) may, in general, be written as

$$A_z = A_s(\rho, \phi) + A_t(\rho, \phi, t), \tag{26}$$

where $A_s(\rho, \phi)$ solves the steady-state equation

$$v\partial A_s/\partial y = \eta \nabla^2 A_s, \tag{27}$$

and $A_t(\rho, \phi, t)$ solves the time-dependent equation

$$\partial A_t/\partial t = -v\partial A_t/\partial y + \eta \nabla^2 A_t. \tag{28}$$

When $R_m \ll 1$, naive inspection of Eq. (28) suggests A_t will exponentially decay away on a time scale of approximately τ_D [9,44]. We explicitly solve Eq. (28) by separation of variables using $A_t = G(\rho, \phi)W(t)$. With separation constant $-\alpha^2$, this gives

$$\eta^{-1}\partial W(t)/\partial t = -\alpha^2 W(t) \tag{29}$$

and

$$\nabla^2 G - (v/\eta)\partial G/\partial y + \alpha^2 G = 0. \tag{30}$$

By Eq. (29),

$$W(t) = C_0 e^{-\eta\alpha^2 t}, \tag{31}$$

where all C_i's are constants. The alternative choice of separation constant $+\alpha^2$ yields an A_t (hence, $\mathbf{B}$) exponentially growing with time, so we exclude this solution on physical grounds. If $R_m \gg 1$, Eq. (28) becomes the transport equation for which the separation of variables yields a traveling wave solution.

Putting $G = g(\rho, \phi)e^{ky}$ (a standard technique from MHD [10,27]) in Eq. (30) yields

$$\nabla^2 g + \lambda^2 g = 0 \tag{32}$$

with

$$k = v/2\eta \tag{33}$$

and $\lambda^2 = \alpha^2 - k^2$. Therefore, Eq. (31) becomes

$$W(t) = C_0 e^{-\eta(k^2+\lambda^2)t}. \tag{34}$$

Solving Eq. (32) by putting $g = m(\rho)n(\phi)$ with separation constant ν^2 yields

$$m(\rho) = C_1 J_\nu(\lambda\rho) + C_2 Y_\nu(\lambda\rho) \tag{35}$$

and

$$n(\phi) = C_3 \cos(\nu\phi) + C_4 \sin(\nu\phi), \tag{36}$$

where the J_ν and Y_ν are Bessel functions of the first and second kinds of order ν. Therefore,

$$A_t = C_0 m(\rho)n(\phi)e^{k\rho\sin\phi}e^{-\eta(k^2+\lambda^2)t}. \tag{37}$$

Since $\eta(k^2 + \lambda^2) > 0$ always, in Eq. (37) A_t decays exponentially, and the system over time goes to the steady-state solution $A_s(\rho, \phi)$ in Eq. (26). We, therefore, cannot choose the trivial solution $A_s = 0$ in Eqs. (26) and (27) since for $v = 0$ we must have $A_z = A_0$ with A_0 given by Eq. (19). Therefore, $A_s(v=0) = A_0$. The condition $R_m \gg 1$ requires $v \neq 0$, so solutions for $R_m \gg 1$ need not satisfy this constraint.

We use boundary conditions and Eqs. (35) to (37) to solve for λ. This allows us to show explicitly that the exponential in Eq. (37) does indeed decay on a time scale (even faster than) approximately τ_D, consistent with more general arguments [9,44]. In Eq. (35), we set $C_2 = 0$ so that our solutions remain bounded in the case $a \to 0$. By Gauss's law, we know that B_ρ must be continuous across the boundary of the cylindrical shell at $\rho = a$. In the case of a static external transverse-magnetic field, $B_\rho(\rho < a) \sim 10^{-3}B_\infty$ for $\mu_r \sim 5 \times 10^3$, a value typical of the magnetically permeable materials we discuss here. That is, the shell acts as a magnetic shield for its hollow interior [39,40,42,46]. For time-varying fields, the shielding is as good or better than it is for the static field case [46–48]. We may then take as a boundary condition for the time-dependent part $\mathbf{B}_t$ of $\mathbf{B}$ that $B_{t\rho}(\rho = a) \to 0$ for all ϕ as one approaches the boundary from $\rho > a$ within the shell. Since $B_{t\rho} = \rho^{-1}\partial A_t/\partial\phi$ and A_t evolves independently of A_s, this boundary condition then implies that for all ϕ,

$$\left.\frac{1}{\rho}\frac{\partial A_t}{\partial \phi}\right|_{\rho=a} = 0. \tag{38}$$

One can try to satisfy Eq. (38) for all ϕ by setting the product of constants in Eq. (37), either $C_0C_1C_3$ or $C_0C_1C_4$, to be proportional to some negative power of μ_r that goes to zero for large μ_r. But this solution is an unphysical choice, since its effect is to force A_t to 0 for all ρ within $a \le \rho \le b$, meaning that the cylindrical shell magnetically shields itself throughout its entire volume as well as its hollow interior. If this unphysical choice is made nonetheless, it will render A_t negligible so that only the steady-state solution A_s would remain. This conclusion will be the same as that we obtain below, but below it is for the reason that A_t decays away extremely quickly.

Using Eqs. (35) with $C_2 = 0$ and Eq. (37), requiring that Eq. (38) be true for all ϕ requires

$$J_\nu(\lambda a) = 0 \tag{39}$$

for all ν. Choosing $\nu = 0$, the first zero of the Bessel function gives $\lambda = 2.40/a$, and Eq. (34) becomes

$$W(t) = C_0 e^{-(R_m/4)t/\tau_v} e^{-(2.4b/a)^2 t/\tau_D} \tag{40}$$

using the definitions of R_m, τ_v, and τ_D. The first exponential in Eq. (40) is a decay that is slow with respect to the translation time scale τ_v. The second exponential is a decay that is much faster than the diffusion time scale τ_D. Since $\tau_D = R_m \tau_v$, this decay for $R_m \ll 1$ is extremely fast with respect to τ_v. Choosing any higher value of ν (or values, if one makes the solution a series of terms in ν) yields larger values for λ, leading to even faster exponential decays in Eq. (40). In the special case $a = 0$ in Eq. (40), $W(t) = 0$ so $A_t = 0$, and the full solution is just the steady-state solution $A_s(a = 0)$ from Eq. (26).

VIII. TIME-INDEPENDENT SOLUTION FOR $\mathbf{v} \neq 0$, $R_m \ll 1$

Clearly, $W(t) \to 0$ as $t \to \infty$ in Eq. (40), with A_s in Eq. (26) the steady-state solution that remains. When $R_m \ll 1$, $W(t) \to 0$ on a time scale $< \tau_D \ll \tau_v$, so A_t in Eq. (37) decays rapidly away in the time that it takes the shell to move a distance $< v\tau_D$, where $v\tau_D \ll v\tau_v = b$. That is, we are, as expected, in a quasistationary situation where at any point in the shell's translation, $A_z = A_s(\rho, \phi)$ to a good approximation, with A_s given by Eq. (27). We now solve Eq. (27).

First consider the special case where our cylinder is solid ($a = 0$) and translating at $\mathbf{v} = v\hat{\mathbf{y}}$ through the background field $\mathbf{B}_\infty = B_\infty \hat{\mathbf{x}}$. Then $A_0(a = 0)$ must be given by Eq. (19) with $a = 0$, i.e., $A_0 = \beta_1 y + h(z)$, where $h(z)$ is an arbitrary function of z. This satisfies Eq. (27) provided $h(z) = k\beta_1 z^2$ with k given by Eq. (33), so that

$$A_0(a = 0) = k\beta_1 z^2 + \beta_1 y. \tag{41}$$

By the gauge condition Eq. (22), we then have

$$V(a = 0) = -v\beta_1 z. \tag{42}$$

For a finite solid cylinder, charges in the cylinder experience a $\mathbf{v} \times \mathbf{B}$ force and flow in response, redistributing on an extremely short time scale τ_e until an electric field $\mathbf{E} = -\nabla V$ is established that perfectly cancels $\mathbf{v} \times \mathbf{B}$. In particular, we see that Eq. (42) gives the physically correct answer for the special case of a translating finite solid cylinder.

Now what happens when our cylinder becomes a cylindrical shell with $a \neq 0$? We anticipate from Eq. (19) that $A_0(a \neq 0)$ will be given by $A_0(a = 0)$ plus additional terms. We solve Eq. (27) for the general ($a \neq 0$, $\mathbf{v} \neq \mathbf{0}$) case, with the requirements that we recover Eq. (41) when $a = 0$ and Eq. (19) when $\mathbf{v} = \mathbf{0}$.

Equation (27) is solved by $f(\rho, \phi)e^{ky}$, where the function f satisfies

$$\nabla^2 f - k^2 f = 0 \tag{43}$$

with k given by Eq. (33), so

$$f(\rho, \phi) = [C_5 \cos(\nu\phi) + C_6 \sin(\nu\phi)][C_7 I_\nu(k\rho) + C_8 K_\nu(k\rho)], \tag{44}$$

where the separation constant is ν^2, and I_ν and K_ν are modified Bessel functions of order ν of the first and second kind. We, therefore, write the general solution as

$$A_s = k\beta_1 z^2 + \beta_1 y + f(\rho, \phi)e^{ky}, \tag{45}$$

where the first two terms provide the solution to Eq. (27) for the case $a = 0$, and the final term modifies that solution analogously to Eq. (19) for the case $a \neq 0$. Equation (45) must go to Eq. (19) in the $v = 0$ limit. Noting that as $k\rho \to 0$ [49]:

$$K_1(k\rho) = (k\rho)^{-1} + k\rho(2\gamma - 1) + (k\rho/2)\ln(k\rho/2) + O(k\rho)^2, \tag{46}$$

where $\gamma = 0.5772\ldots$ is the Euler constant,

$$I_\nu(k\rho) = (k\rho)^\nu/(2^\nu \nu!) + O(k\rho)^{\nu+2}, \tag{47}$$

and

$$e^{ky} = 1 + k\rho\sin\phi + (1/2)(k\rho)^2 \sin^2\phi + O(k\rho)^3, \tag{48}$$

requiring Eq. (45) to equal Eq. (19) for $v = 0$ fixes in Eq. (44) $C_5 = 0 = C_7$ and $\nu = 1$, with $C_6 = 1$ and $C_8 = -\beta_2 k a^2$. Then the solution to Eq. (27) is

$$A_z(a \le \rho \le b) = A_s = k\beta_1 z^2 + \beta_1 y - \beta_2 k a^2 K_1(k\rho) e^{ky} \sin\phi. \tag{49}$$

For $k\rho \to 0$, Eq. (49) becomes

$$A_z(a \le \rho \le b) = A_s = A_0 + A_1 + O(R_m)^2, \tag{50}$$

where A_0 is given by Eq. (19),

$$A_1 = -(R_m/2)b^{-1}\beta_2 a^2 \sin^2\phi, \tag{51}$$

and $R_m = 2kb = \mu\sigma v b$. That is, when $R_m \ll 1$, A_z for $v \neq 0$ is perturbed away from the $v = 0$ solution (A_0) by a series whose terms are scaled by powers of R_m.

Finally, applying the gauge condition Eq. (22) to $\mathbf{A} = A_z\hat{\mathbf{z}}$ with Eq. (49), we find

$$V(a \le \rho \le b) = -v\beta_1 z, \qquad (52)$$

so that even when $a \ne 0$,

$$\nabla V = -v\beta_1\hat{\mathbf{z}}. \qquad (53)$$

IX. GENERATION OF AN EMF

Equations (17a) and (49) yield

$$B_x(a \le \rho \le b) = \beta_1 - \beta_2(a/\rho)^2 e^{ky} \times \{[k\rho\cos 2\phi + (k\rho)^2\sin\phi]K_1(k\rho) - (k\rho)^2\sin^2\phi K_0(k\rho)\} \qquad (54)$$

using the identities [50]

$$\partial K_1(k\rho)/\partial(k\rho) = -[K_0(k\rho) + K_2(k\rho)]/2 \qquad (55)$$

and

$$K_0(k\rho) - K_2(k\rho) = -(2/k\rho)K_1(k\rho), \qquad (56)$$

so that

$$\partial K_1(k\rho)/\partial(k\rho) = -K_0(k\rho) - (k\rho)^{-1}K_1(k\rho). \qquad (57)$$

Noting that [49]

$$K_0(k\rho) = -\gamma - \ln(k\rho/2) + O(k\rho), \qquad (58)$$

as $v \to 0$, we have by Eq. (54) for $R_m \ll 1$:

$$B_x(a \le \rho \le b) = B_{0x} + B_{1x} + O(R_m)^2 \qquad (59)$$

with

$$B_{1x} = -R_m b^{-1}\beta_2 a^2\rho^{-1}\sin\phi\cos^2\phi. \qquad (60)$$

Figure 2 shows for a particular case how B_x differs from B_{0x} to $O(R_m)$.

Similarly, Eqs. (17b) and (49) yield

$$B_y(a \le \rho \le b) = B_{0y} + B_{1y} + O(R_m)^2 \qquad (61)$$

with

$$B_{1y} = -R_m b^{-1}\beta_2 a^2\rho^{-1}\sin^2\phi\cos\phi. \qquad (62)$$

Figure 3 shows for a particular case how B_y differs from B_{0y} to $O(R_m)$.

We see that $B_{1x} = \partial A_1/\partial y$ and $B_{1y} = -\partial A_1/\partial x$. Equations (60) and (62) are readily checked to verify that they do solve Eq. (7), e.g., $v\partial B_{0x}/\partial y = \eta\nabla^2 B_{1x}$ as required. Equations (59) and (61) show that the effect of $v \ne 0$ for $R_m \ll 1$ is to perturb $\mathbf{B}$ away from $\mathbf{B_0}$ by a series scaled by successive powers of R_m.

The asymmetry of B_x about $y = 0$ leads to the continuous generation of an emf within the cylindrical shell. Consider the current path C in Fig. 1 for $x_0 = b\cos\phi_0$, and $y_0 = b\sin\phi_0$. Then, Eq. (8) for this path gives

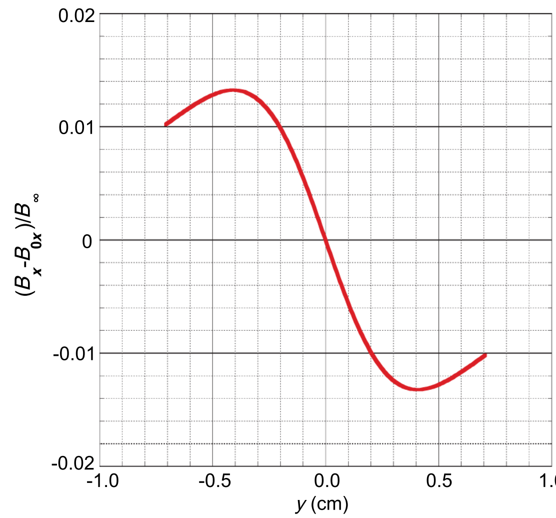

FIG. 2. Deviations in the x component B_x of the magnetic flux density [Eq. (59)] for the moving cylindrical shell from that for the stationary shell [Eq. (10)], relative to the flux density at infinity, for a shell made of MN60 material (see text) with $b = 1$ cm. These results are for the $x_0 = b/\sqrt{2}$ plane, with $(a/b) = 1/\sqrt{2}$.

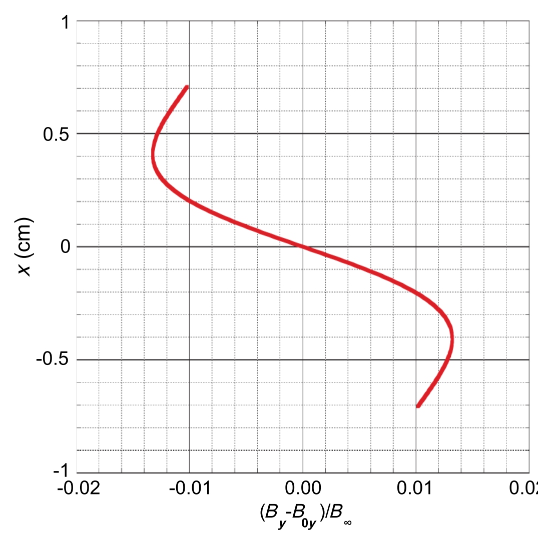

FIG. 3. Deviations in the y component B_y of the magnetic flux density [Eq. (61)] for the moving cylindrical shell from that for the stationary shell [Eq. (10)], relative to the flux density at infinity, for a shell made of MN60 material (see text) with $b = 1$ cm. These results are for the $y_0 = b/\sqrt{2}$ plane, with $(a/b) = 1/\sqrt{2}$.

$$\mathrm{emf}(x_0, y_0) = -\eta \oint_C \nabla^2 A_z \hat{\mathbf{z}} \cdot \mathbf{dl}$$
$$= -2R_m v\beta_2 l(a/b)^2 \sin\phi_0 \cos^2\phi_0 + O(R_m)^2 \quad (63)$$

using

$$\eta\nabla \times \mathbf{B} = -\nabla V - \eta\nabla^2 \mathbf{A} \quad (64)$$

from Eqs. (22) and (24), $A_z = A_s$, and Eqs. (27), (17), and (59). Equation (63) is valid only for $R_m \ll 1$; if $R_m \gg 1$, emf $= 0$ by Eq. (7). Even for $R_m \ll 1$, emf $= 0$ in Eq. (63) if $v = 0$, or $a = 0$, or $\mu_r = 1$. The emf in K' is the same as that in K provided $(v/c)^2 \ll 1$ [13].

The result in Eq. (63) is for one designated current path C. For an arbitrary C with segments parallel to the z axis, the integration underlying Eq. (63) leads to emf $\propto [B_x(x_1, y_1) - B_x(x_2, y_2)]$, where the coordinates designate the (x, y) coordinates of the two segments of C parallel to the z axis. (Arbitrary current paths can then be built up by fusing such rectangular subpaths.) Because of the symmetry in ϕ of B_{0x}, Eq. (10a), it is clear that for every such circuit with $0 < \phi < \pi$, there is a corresponding circuit with $\pi < \phi < 2\pi$ that yields an emf of opposite sign with respect to B_{0x}. This is because the scalar product $(\mathbf{v} \times \mathbf{B_0}) \cdot \mathbf{dl} = vB_{0x}\hat{\mathbf{z}} \cdot \mathbf{dl}$ has the opposite sign in the two cases, since the circuits are mirror reflections across the $y = 0$ plane, and in each case, C is traversed using a right-hand rule. Over the entire shell, these contributions, therefore, average to zero. It is the component of B_x of $O(R_m)$ that makes a nonzero contribution because of the asymmetry in ϕ of B_{1x}, with B_{1x} switching sign at the $y = 0$ plane (Fig. 2). With respect to B_{1x}, for every current path C with $0 < \phi < \pi$, there is a corresponding path with $\pi < \phi < 2\pi$ that yields an emf of identical sign, so the two do not cancel. To $O(R_m)$, therefore, the average emf around the shell cannot be zero. We will see in Sec. XII that consistent with a nonzero emf, there is a net absorption of power by the shell from Poynting vector inflow.

An infinite solid conducting bar moving through a background magnetic field will (in principle) generate a current due to $\mathbf{v} \times \mathbf{B}$ since its infinite extent prevents the accumulation of the charges at its ends that for a finite bar generates the electric field $\mathbf{E} = -\nabla V = -\mathbf{v} \times \mathbf{B}$. However, even for the infinite solid bar, $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$, so there will be emf $= 0$ about any closed path C lying within the bar. The nonzero emf in Eq. (63) is not, therefore, attributable to the fact that our formalism began with an infinite cylindrical shell.

X. INTUITIVE PHYSICAL PICTURE

A simple physical picture offers insight into why a magnetically permeable cylindrical shell moving at velocity $\mathbf{v}$ and satisfying $R_m \ll 1$ is expected to generate an emf according to Eq. (8). In frame K, picture a finite cylindrical shell moving transversely to its long axis (Fig. 1). Assume $a \neq 0$. By Eq. (21), we know that it is impossible for the shell's electrons to establish a configuration such that $-\nabla V = -\mathbf{v} \times \mathbf{B}$.

Imagine beginning with the cylindrical shell at rest and then placing it into motion at velocity $\mathbf{v}$. The magnetic flux density within the shell itself is initially $\mathbf{B_0}$ given by Eq. (10). $\mathbf{B_0}$ results from Maxwell's equations requiring the continuity of the normal component of $\mathbf{B}$ and tangential component of $\mathbf{H}$ at the surfaces $\rho = a$ and $\rho = b$. As the shell moves, it attempts, so to speak, to enforce $\mathbf{B} = \mathbf{B_0}$ throughout $a \le \rho \le b$. If $\tau_D/\tau_v = R_m \gg 1$, the diffusion time scale τ_D for the magnetic flux density is much longer than the advection time scale τ_v. That is, diffusion is negligible compared to advection, and the disturbance in the field moves along with the shell, so that $\mathbf{B} = \mathbf{B_0}$ to very high precision. (Alfvén's frozen-flux theorem [26] holds.) Since $\nabla \times \mathbf{B_0} = 0$, by Eq. (8) we must have emf $= 0$ when $R_m \gg 1$.

Contrast this limit with the case $\tau_D/\tau_v = R_m \ll 1$. Now the time scale τ_D for diffusion is much shorter than τ_v. That is, as the shell moves, the field's adjustment is dominated not by advection but by diffusion toward a field configuration at which diffusion stops, i.e., toward $\mathbf{B_0}(a \le \rho \le b)$, where the "destination" $\mathbf{B_0}$ is the value that applies for a stationary shell at the location to which the shell has just moved. The field can never reach this end point since the shell keeps moving even as the field diffuses, so a steady state is reached in which the diffusing field differs slightly from $\mathbf{B_0}$. The field within the shell does not adjust instantaneously to the shell's motion, so it never (unless the shell is brought to rest in K) fully "catches up" to that motion. A closed path C moving with the shell constantly experiences a field that is diffusing across its boundaries, and Eq. (8), in general, is nonzero.

XI. ANALYSIS IN THE LABORATORY FRAME

We now consider our system in the laboratory frame K', where $\mathbf{v} = \mathbf{0}$ so there is no magnetic $\mathbf{v} \times \mathbf{B}$ force, but there is, instead, an electric field given by the field transformation Eq. (5): $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. Ohm's law in K' is simply $\mathbf{E}' = \mathbf{J}'/\sigma$, which leads to the induction equation in K':

$$\partial\mathbf{B}/\partial t' = \eta\nabla^2\mathbf{B}. \quad (65)$$

Since $\mathbf{J}' = \mathbf{J}$ and $\mathbf{B}' = \mathbf{B}$ when $(v/c)^2 \ll 1$, the emf is given by

$$\mathrm{emf}' = \oint_C \mathbf{E}' \cdot \mathbf{dl}' = \sigma^{-1} \oint_C \mathbf{J} \cdot \mathbf{dl} = \eta \oint_C (\nabla \times \mathbf{B}) \cdot \mathbf{dl}. \quad (66)$$

We use the fact that $\mathbf{E}'$ and, therefore, $\mathbf{J}$ must be parallel to $\hat{\mathbf{z}}$, so the relevant part of $\mathbf{dl}$ is perpendicular to y, and, therefore, we can put $\mathbf{dl}' = \mathbf{dl}$. Equation (66) is identical to Eq. (8) and, therefore, to Eq. (63), so emf$'$ = emf, as expected [13]. Equation (66) is nonzero in K' provided the same conditions hold as those needed for Eq. (63) to give emf $\neq 0$.

It might, nevertheless, seem puzzling that an emf could be generated in K'. We intuitively expect $\partial\mathbf{B}/\partial t' = 0$ in a steady state, so that by Eqs. (8) and (65), emf$' = 0$. But care must be taken with $\mathbf{B}$ in Eq. (65): because $\mathbf{B}$ is not rotating with Earth, it cannot be treated implicitly as $\mathbf{B}(x, y')$, where $y' = y - vt$ relates the coordinates in K' and K.

If $\mathbf{B}$ rotates with Earth, then in K' we simply have $\mathbf{B} = \mathbf{B}(x, y')$ and $\partial\mathbf{B}(x, y')/\partial t' = (\partial\mathbf{B}/\partial y')(\partial y'/\partial t') = \mathbf{0}$ using the chain rule and $\partial y'/\partial t' = 0$. But treating $\mathbf{B}$ as rotating with Earth is inconsistent with the clear expectation from the results of the Barnett [1], Kennard [15], and Pegram [19] experiments. Rather, in K' we must treat $\mathbf{B}$ as advecting through the cylindrical shell at velocity $\mathbf{v} = -v\hat{\mathbf{y}}$, which we capture by writing $\mathbf{B} = \mathbf{B}(x, y)$ with $y = y' + vt$. The time dependence of $\mathbf{B}(x, y)$ is driven by the advection of $\mathbf{B}$ through the shell; this dependence is included implicitly by $y = y(y', t) = y' + vt$. Then by the chain rule and $\partial y/\partial t' = v$,

$$\partial\mathbf{B}/\partial t' = \partial\mathbf{B}[x, y(y', t)]/\partial t' = (\partial\mathbf{B}/\partial y)(\partial y/\partial t') = v\partial\mathbf{B}/\partial y. \tag{67}$$

That is, we *do* have $\partial\mathbf{B}(x, y')/\partial t' = \mathbf{0}$, but we also have $\partial\mathbf{B}(x, y)/\partial t' = v\partial\mathbf{B}/\partial y \neq \mathbf{0}$, and we must distinguish between the two. Only the second representation for $\mathbf{B}$ in K' is consistent with experiment.

Substituting Eq. (67) into (65) gives a time-independent equation for $\mathbf{B}$:

$$v\partial\mathbf{B}/\partial y' = \eta\nabla^2\mathbf{B}. \tag{68}$$

Recalling $\partial/\partial y' = \partial/\partial y$, Eq. (68) yields Eq. (27) for $\mathbf{A}$ given the gauge choice Eq. (22). Physically, the induction (advection-diffusion) equation concerns the steady state that is reached in $\mathbf{B}$ as it advects through the $R_m \ll 1$ cylindrical shell and undergoes concomitant diffusion; as a result, $\mathbf{B}$ [as we know from Eqs. (59)–(62)] is slightly perturbed away from $\mathbf{B_0}$. If $\mathbf{B}$ instead advects along with the shell, there will be no emf.

A Poynting vector and flux transport analysis [41,51,52] in K' make it clear that energy is flowing into our $R_m \ll 1$ cylindrical shell, providing the power required to sustain emf$' \neq 0$. The Poynting vector in K' is

$$\mathbf{S}' = \mu^{-1}(\mathbf{E}' \times \mathbf{B}) = \mu^{-1}\eta(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{69}$$

where we use $\mathbf{E}' = \mathbf{J}/\sigma$ and Ampère's law. If $\mathbf{B} = \mathbf{B}_0$, we will have $\mathbf{S}' = 0$ by $\nabla\times\mathbf{B_0} = 0$ in Eq. (69), and there will be no energy input to the cylindrical shell. However, $\nabla\times\mathbf{B_1} \neq 0$ and using Eqs. (24), (23), (53), (27), and (17), we find

$$\eta\nabla\times\mathbf{B} = v(\beta_1 - B_x)\hat{\mathbf{z}} \tag{70}$$

giving

$$\mathbf{S}' = v\mu^{-1}(\beta_1 - B_x)(B_x\hat{\mathbf{y}} - B_y\hat{\mathbf{x}}). \tag{71}$$

We perform our calculations at the instant at which the origins of the K' and K frames coincide. The net energy flux P'_S into the shell's surface within $l/2 \leq z \leq l/2$ (where $l/2$ is chosen to be sufficiently far in from the shell's edge at $L/2$) is given by

$$P'_S = \int_0^{2\pi}\int_{-l/2}^{l/2}\mathbf{S}'\cdot\hat{\rho}\rho d\phi dz, \tag{72}$$

where $\hat{\rho} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$ and the boundaries at both $\rho = b$ and $\rho = a$ must be taken into account by summing the contributions from evaluating Eq. (72) at $\rho = b$ and at $\rho = a$. The boundary at $\rho = a$ enters with a negative sign, opposite from that at $\rho = b$. The calculation is simplified by noting

$$\begin{aligned}(B_x\hat{\mathbf{y}} - B_y\hat{\mathbf{x}})\cdot\hat{\rho} &= -B_\mathbf{y}\cos\phi + B_x\sin\phi \\ &= [\beta_1 + \beta_2(a/\rho)^2]\sin\phi + O(R_m)^2;\end{aligned} \tag{73}$$

i.e., the $O(R_m)^1$ terms cancel in Eq. (73). In Eq. (72), nearly all terms integrate to zero and

$$P'_S = (\pi/4)\sigma v^2\beta_2^2 a^2[1 - (a/b)^2]l + O(R_m)^2. \tag{74}$$

$P'_S = 0$ if $v = 0$, or $\mu_r = 1$ (because then $\beta_2 = 0$), or $a = 0$. Otherwise, the Poynting vector $\mathbf{S}'$ in K' gives a net energy flow into the cylindrical shell that sustains the emf$'$.

It is interesting to ask which terms within $\mathbf{S}'$ provide this energy. Nearly every term in Eq. (71) makes zero contribution to Eq. (72) either because it cancels an identical term of opposite sign or because it integrates to zero over ϕ: the energy from most terms simply flows through the shell, with as much energy leaving as entering. The only term in $\mathbf{S}'$ that makes a nonzero contribution is

$$\mathbf{S'_x} = 2\sigma v^2\beta_2^2 a^4\rho^{-3}\sin^2\phi\cos\phi(4\cos^2\phi - 1)\hat{\mathbf{x}}. \tag{75}$$

Equation (65) can be written [41,52] as

$$\partial\mathbf{B}/\partial t' = \nabla\times(\mathbf{w}\times\mathbf{B}) \tag{76}$$

for an appropriate velocity $\mathbf{w}$. By Ampère's and Ohm's law in K', we have

$$\mathbf{E}' = \eta\nabla\times\mathbf{B}, \tag{77}$$

so $\mathbf{E}' \cdot \mathbf{B} = 0$ since $\mathbf{B} = (B_x, B_y, 0)$. When $\mathbf{E}' \cdot \mathbf{B} = 0$, $\mathbf{w}$ in Eq. (76) is [41,52]

$$\mathbf{w} = (\mathbf{E}' \times \mathbf{B})/B^2, \tag{78}$$

which is called the "flux-transporting velocity" [52], meaning that $\mathbf{w}$ for the case $\sigma \neq \infty$ preserves flux because it satisfies Eq. (76) in the same way that $\mathbf{v}$ satisfies Eq. (7) for the case $\sigma = \infty$. A contour C in the cylindrical shell will, therefore, have $d\Phi/dt = 0$ through its corresponding surface S if C is moving at $\mathbf{w}$, where $\mathbf{w}$ may vary point to point along C.

By Eqs. (69) and (78),

$$\mathbf{w} = \mu \mathbf{S}'/B^2. \tag{79}$$

Direct calculation using Eqs. (79) and (71) reveals $\mathbf{w}$ to be algebraically complicated with $\mathbf{w} \neq \mathbf{v}$, even while satisfying $\nabla \times (\mathbf{w} \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B})$, as it must. When $\mathbf{w} \neq \mathbf{v}$, then if C is simply being transported with the conductor at $\mathbf{v} \neq \mathbf{0}$, there must be an emf around C. This is the case, for example, for the contour C in Fig. 1. By Eq. (78), the zero-velocity solution $\mathbf{B_0}$ in K' is not transported through the shell because, in this case, $\mathbf{E}' = 0$ by Eq. (77), so $\mathbf{w} = 0$ in Eq. (78). Of course, $\mathbf{B_0}$ does not diffuse: $\nabla^2 \mathbf{B_0} = 0$. Only the perturbations $\mathbf{B_1}$ and higher orders will have $\mathbf{w} \neq 0$.

Equation (79) means that magnetic flux is transported proportionally to the transport of energy defined by the Poynting vector. Since $\mathbf{S}'$ integrated over the cylindrical shell is nonzero, there is a corresponding net flow of magnetic flux into the shell. We have a picture in K' in which by Eq. (75), near the $x = 0$ plane for $\pi/3 < \phi < 2\pi/3$ and again for $4\pi/3 < \phi < 5\pi/3$, magnetic field lines are diffusing (transported at velocity $\mathbf{w}$) in the x direction vertically toward the $x = 0$ plane from above and below. These lines annihilate [9,52,53] in the $x = 0$ plane, providing energy that drives the current flow in C. The cancellation (annihilation) of the magnetic filed lines in the $x = 0$ plane preserves the gradient, which, in turn, maintains the continuing inward diffusion of the field.

We note an analogy to the homopolar generator. By Eqs. (67), (68), and (76),

$$\text{emf}' = \oint_C (\mathbf{w} \times \mathbf{B}) \cdot \mathbf{dl}, \tag{80}$$

where $\mathbf{w}$ is given by Eq. (78). In the homopolar generator, the analog to Eq. (80) gives emf$' \neq 0$ because only part of C is rotating, so $\mathbf{v}$ varies (stepwise) around C, and $\mathbf{v} \times \mathbf{B}$ does not integrate to 0 around C. In the $R_m \ll 1$ cylindrical shell of Fig. 1, emf$' \neq 0$ because $\mathbf{w}$ varies around C according to Eq. (79), so $\mathbf{w} \times \mathbf{B}$ does not integrate to 0 around C.

XII. POYNTING'S THEOREM AND MAGNETIC BRAKING

When $R_m \ll 1$, in the steady state [e.g., Eq. (45)] we have $\partial \mathbf{A}/\partial t = \mathbf{0}$, so $\mathbf{E} = -\nabla V = v\beta_1 \hat{\mathbf{z}}$ by Eq. (53). Together with Ampère's law and Eq. (70), we have

$$\mathbf{E} \cdot \mathbf{J} = \sigma v^2 \beta_1 (\beta_1 - B_x). \tag{81}$$

By Eq. (3), we also have

$$\mathbf{E} \cdot \mathbf{J} = \sigma^{-1} J^2 + (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}. \tag{82}$$

Integrating Eq. (81) over the volume $dV = \rho d\phi d\rho dz$ gives zero, so Eq. (82) implies

$$\sigma^{-1} \int_V J^2 dV = -\int_V (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}\, dV, \tag{83}$$

where the integral on the right-hand side is the familiar expression for magnetic braking. In K, Joule heating, therefore, derives from the energy made available by magnetic braking of the cylindrical shell. Since the shell is being carried by Earth, it is clear that electrical power in our system derives ultimately from the kinetic energy of Earth's rotation. This is analogous to the Poynting theorem analysis of the homopolar generator [12].

Explicitly integrating $(\mathbf{J} \times \mathbf{B}) \cdot \mathbf{v}$ in Eq. (83) over the volume V of the shell with $\mathbf{J} = \mu^{-1} \nabla \times \mathbf{B}$ shows the power removed from Earth's rotational kinetic energy to be

$$\begin{aligned} P_k &= -\sigma v^2 l \int_a^b \int_0^{2\pi} B_x (\beta_1 - B_x) \rho d\rho d\phi \\ &= (\pi/2) \sigma v^2 \beta_2^2 l a^2 [1 - (a/b)^2] + O(R_m)^2. \end{aligned} \tag{84}$$

If $\mathbf{B} = \mathbf{B_0}$, we will have $P_k = 0$ since $\nabla \times \mathbf{B_0} = 0$. If $v = 0$ or $\mu_r = 1$ or $a = 0$, then $P_k = 0$. The power in K must equal that in the laboratory frame K' to $O(v/c)^2$ [13].

The manner in which this power arises in K' is of interest. Poynting's theorem [54] states that the rate at which work is done on the electrical charges within a volume V of surface area Σ is equal to the decrease in energy stored in the electric and magnetic fields minus the energy that flows out through the surface bounding the volume. In K', Poynting's theorem is

$$\int_V \mathbf{E}' \cdot \mathbf{J}\ dV = -\mu^{-1} \int_V \mathbf{B} \cdot \partial \mathbf{B}/\partial t' dV - \int_\Sigma \mathbf{S}' \cdot \mathbf{d\Sigma}, \tag{85}$$

where, as before, the displacement current is negligible. By $\mathbf{E}' = \sigma^{-1} J'$, $\mathbf{E}' \cdot \mathbf{J} = \sigma^{-1} J^2$. The second term on the right of Eq. (85) is just Eq. (72). The first term on the right may be evaluated using Eq. (65), a calculation most easily performed with $\mathbf{B}$ in cylindrical coordinates (Appendix D). The result is identical to Eq. (74).

Therefore, in K', Eq. (85) gives the power P'_P provided to the shell to be

$$P'_P = \sigma^{-1}\int_V J^2 dV = (\pi/2)\sigma v^2\beta_2^2 a^2[1-(a/b)^2]l + O(R_m)^2, \quad (86)$$

with the energy for electrical power provided in Poynting's theorem coming equally from Poynting vector inflow and the $\partial\mathbf{B}/\partial t'$ term. This result is, indeed, equal to the expression found in Eq. (84) by calculating in the K frame: $P_k = P'_P$. For a given choice of b, Eqs. (84) and (86) reach their maximum values for $a = b/\sqrt{2}$.

An important question is whether the slight despinning of Earth caused by the magnetic braking found here is consistent with angular momentum conservation. Mechanical systems can come in or out of rotation solely via the transfer of angular momentum between mechanical rotation and the electromagnetic field. (For the fully calculated example of a charged magnetized sphere exhibiting this behavior, see Refs. [55,56].) The angular momentum of the electromagnetic field is proportional to $\mathbf{r}\times\mathbf{S}$ (with $\mathbf{r}$ the usual radial component in a spherical coordinate system). For the system in our thought experiment, the analogous issue is linear momentum conservation. In K, the mechanical momentum of the cylindrical shell lies in the $\hat{\mathbf{y}}$ direction, and the braking force per unit volume given by $\mathbf{J}\times\mathbf{B}$ acts in the $-\hat{\mathbf{y}}$ direction. The momentum (per unit volume) of the electromagnetic field associated with this system is $\mathbf{p} = \varepsilon_0\mu_0\mathbf{S}$, with $\mathbf{S} = \mu^{-1}\mathbf{E}\times\mathbf{B} = (\mu\sigma)^{-1}\mathbf{J}\times\mathbf{B}$; i.e., $\mathbf{p}$ is proportional to and lies in the direction of the magnetic braking vector. Therefore, as the system is braked, positive mechanical linear momentum is lost from the cylindrical shell while negative linear momentum is lost from the electromagnetic field, and momentum conservation is possible.

XIII. EXPERIMENTAL PREDICTIONS

Predicting the emf to be measured in a laboratory test of these claims requires that part $\mathbf{B}_\infty$ of Earth's total field that is axially symmetric about the planet's rotation axis. This part is well approximated by summing the axisymmetric dipole, quadrupole, and octupole components of the total field to yield the northward (X) and downward (Z) components of $\mathbf{B}_\infty$ at a point on the surface of the Earth. These components are [23] at colatitude θ,

$$X = -g_1^0\sin\theta - 3g_2^0\sin\theta\cos\theta - (3/2)g_3^0\sin\theta(5\cos^2\theta - 1) \quad (87a)$$

and

$$Z = -2g_1^0\cos\theta - (3/2)g_2^0(3\cos^2\theta - 1) - 2g_3^0\cos\theta(5\cos^2\theta - 3) \quad (87b)$$

giving

$$B_\infty = (X^2 + Z^2)^{1/2}, \quad (87c)$$

where the Gauss coefficients $g_1^0 = -29\,496.5$ nT, $g_2^0 = -2396.6$ nT, and $g_3^0 = 1339.7$ nT [57]. Then for, say, Princeton's colatitude $\theta = 49°39'$ (for which $v = 354\ \mathrm{m\,s^{-1}}$) $B_\infty = 45\ \mu$T, pointed downward into Earth's surface at an angle (from the horizontal when facing the north geographic pole) $\tan^{-1}(Z/X) = 57.5°$.

Suppose that our cylindrical shell has dimensions $L = 20$ cm, $b = 1$ cm, and $a = b/\sqrt{2}$, and is made of MN60 MnZn ferrite, with data-sheet values given as $\mu_r = 6500 \pm 3000$ and $\sigma \approx 0.5\ \mathrm{S\,m^{-1}}$ [58]. Then $R_m = 1.4\times10^{-2} \ll 1$, while $R_m \gg (v/c)^2$ ensures that $\sqrt{g_{00}}$ effects are small compared to first-order perturbations scaled by R_m. For $\phi_0 = 45°$, Eq. (63) gives emf = 65 μV.

By inspection of the integral in Eq. (63), the emf should reverse sign when the shell (together with the attached measuring apparatus, a digital voltmeter; see Fig. 1) is rotated by 180°. This is a striking prediction that should separate an emf generated by the effect predicted here from other types of emf generation. Our derivation is valid only for $\mathbf{v}$ transverse to the shell, but the emf must pass through zero between the two transverse orientations that are separated by 180°. A voltmeter across d and f in Fig. 1 measures half the emf around C in Eq. (63). We caution that C may "choose itself" under rotation, and experiment will show whether a voltage measurement actually somehow averages over many possible current paths. If so, we may approximate the expected emf by averaging over ρ and ϕ in the calculation leading to Eq. (63):

$$\langle\mathrm{emf}\rangle = -\frac{1}{\pi(b-a)}\int_a^b\int_0^\pi vB_x l\,d\rho\,d\phi = -(4/3\pi)R_m v\beta_2 l(a/b)^2(1-a/b)^{-1}\ln(b/a), \quad (88)$$

which for the identical parameter values as above gives $\langle\mathrm{emf}\rangle = 46\ \mu$V. Once again, the emf measured as in Fig. 1 yields half this value, and the sign reverses under 180° rotation.

XIV. SCALING AND CONCLUSIONS

The cylindrical shell is chosen as an especially simple realization of a conductor with $\nabla\times(\mathbf{v}\times\mathbf{B}) \neq \mathbf{0}$, and MN60 material is chosen to provide $R_m \ll 1$ on a laboratory scale. For the MN60 device considered above, $P_k \approx 16$ nW by Eq. (86). By the maximum power transfer theorem, at most half of this power can be transferred to the load [59]. To be useful, the effect must be scaled up greatly in voltage and power. One way might be to maintain $R_m \ll 1$ while increasing σ, by decreasing μ_r, b, and, therefore, a.

Carbon nanotubes can be coated with materials such as iron [60,61], so very small low-R_m magnetically permeable tubes seem plausible. One must also consider resistance and Ohmic loss.

It should be possible to separate the magnetic shield producing $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq \mathbf{0}$ from the conductor providing $R_m \ll 1$. For example, note that the functional form of Eq. (9) for the magnetic flux density outside a magnetically permeable shell is identical to that of Eq. (10) for the flux density in the shell's interior. This guarantees that Eq. (21) holds outside the shell with the substitutions $\beta_2 \to -\beta_3$ and $a^2 \to b^2$. Therefore, we should be able to realize the effect using a magnetically permeable cylinder surrounded by an insulated concentric cylindrical shell of a nonpermeable low-R_m material and find results analogous to those found above. Graphite has $\sigma = 7.3 \times 10^4\ \mathrm{S\,m^{-1}}$ [37], giving $R_m \approx 2 \times 10^{-2}$ for $b = 1$ mm, so we can hope to realize the effect for a mu-metal or ferrous cylindrical core surrounded by a thin insulator with an overlying shell of graphite. Decreasing b to 5 μm allows copper ($\sigma = 6.0 \times 10^7\ \mathrm{S\,m^{-1}}$ [37]) or other common metals to be used for the outer layer, with obvious advantages. Altogether different topologies and materials are possible.

The effect predicted here would be available nearly globally and with no intermittency, but it requires testing and then further examination to see if it or some other configuration based on broadly similar principles can be scaled to practical emission-free power generation. Devices can have important practical implications even if only voltages of approximately 1 V can be achieved. Such a device would represent a small-application power supply whose lifetime will be limited only by material degradation. At the other extreme end of speculations regarding generated power, we note that global installed power-generation capacity is projected to grow to 10 700 GW by 2040 [62]. Imagine as an upper limit that human civilization generates this power entirely from Earth's rotation through its magnetic field. Over a century, the resulting kinetic energy loss will increase Earth's rotation period by 7 ms. This may be compared to fluctuations in the length of Earth's day of 10 ms over time intervals of several decades [63] and an observed long-term increase (dominated by lunar tidal recession) of 2.5 ms per century [64].

ACKNOWLEDGMENTS

We thank three anonymous referees for their reviews, and we are grateful for helpful discussions with T. H. Chyba, R. L. Garwin, M. J. Rees, P. J. Thomas, E. L. Turner, and three anonymous colleagues. We thank G. Z. McDermott and G. Cooper for administrative support, B. A. Lin for reference assistance, and M. Northrup of National Magnetics Group Inc. for help with materials. C. F. C. acknowledges research funds from the Woodrow Wilson School and the Department of Astrophysical Sciences at Princeton University. K. P. H. acknowledges support through the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration, and through the National Aeronautics and Space Administration Exobiology Program (Grant No. NNH09ZDA001N).

APPENDIX A: COROTATION OF THE IONOSPHERE

What happens to charged particles in a conducting plasma around Earth in the presence of Earth's nonrotating axially symmetric field? Hones and Bergeson [24] building on Davis [20,25] and Backus [30] examined this question for the complicated general case of magnetic fields with both axisymmetric and nonaxisymmetric components, treating the purely axisymmetric case as a special case of their more general result. Here we follow their overall logic but present the simpler calculation for the axisymmetric component only: the special case of a magnetic dipole aligned antiparallel to Earth's rotation axis.

An observer in a nonrotating frame sees a (nonrotating) dipole field antialigned with Earth's axis given by $B_r = -(2M/r^3)\cos\theta$, $B_\theta = -(M/r^3)\sin\theta$, and $B_\varphi = 0$, where M is a constant proportional to the magnetic dipole moment. In general, the electric field seen in this frame is given by $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla V$. However, for our nonrotating dipole, we can put $\partial\mathbf{A}/\partial t = 0$ so $\mathbf{E} = -\nabla V$. Earth's rotation through its own dipole field leads to an electrostatic field within the Earth that balances the resulting $\mathbf{v} \times \mathbf{B}$ force: $\mathbf{E} = -\nabla\mathbf{V} = -\mathbf{v} \times \mathbf{B}$, which gives $V = -(M\omega/r)\sin^2\theta$ and a surface potential for Earth (of radius $R_\oplus$) of $V(r = R_\oplus) = -(M\omega/R_\oplus)\sin^2\theta$. Fields $\mathbf{E}$ and $\mathbf{B}$ in the plasma must satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ due to the plasma's near-infinite conductivity parallel to the magnetic field lines. This condition gives $2\cos\theta\partial V/\partial r = -(1/r)\sin\theta\partial V/\partial\theta$. The solution consistent with the boundary condition at $r = R_\oplus$ is $V = -(M\omega/r)\sin^2\theta$, so that $E_r = (M\omega/r^2)\sin^2\theta$, $E_\theta = -(2M\omega/r^2)\sin\theta\cos\theta$, and $E_\varphi = 0$. (Note that these equations satisfy $\mathbf{E}_\infty = 0$.) Charged particles in this plasma drift azimuthally at a velocity $\mathbf{v} = (\mathbf{E} \times \mathbf{B})/B^2$; direct calculation gives $\mathbf{v} = -\omega r\sin\theta\hat{\phi}$ for particles of any charge or mass. That is, the ionosphere comes into corotation with Earth because the charged particles composing it acquire exactly the necessary corotation velocity from their interactions with Earth's nonrotating axially symmetric field together with the electric field induced in the ionosphere. Earth's rotation through the nonrotating axisymmetric component of its magnetic field drives ionospheric corotation.

The nonaxisymmetric components—those components that give Earth's magnetic field its tilt away from Earth's rotation axis—of course, do rotate with Earth. Since magnetic field lines are defined as lines everywhere tangent to the magnetic field, an observer well away from Earth who could somehow see field lines would see Earth's tilted dipole lines rotating with Earth—the rotating lines being the vector sum of a nonrotating azimuthally constant component plus a rotating azimuthally varying component.

There is a standard result that magnetic lines of force in a perfectly conducting fluid move with the fluid—the fluid is "line preserving" [9,51,65]. (However, magnetic field lines are not relativistically covariant [51], and their reality must be treated with care [21,66,67].) When we calculate the equations for the magnetic field lines of a tilted dipole, we find that these lines are described by axisymmetric time-independent terms (from the nonrotating axisymmetric dipole) plus terms sinusoidal in ωt, i.e., terms that rotate with Earth. The field lines do, indeed, vary sinusoidally with ωt due to the superposition of a rotating component on top of an underlying axially symmetric component.

Magnetic field lines must satisfy $\mathbf{dl} \times \mathbf{B} = 0$, where $\mathbf{dl}$ is the arc length. This leads to the usual condition

$$dr/B_r = rd\theta/B_\theta. \tag{A1}$$

Earth's magnetic potential U taking into account only the lowest-order terms for the axisymmetric dipole (g_1^0) and inclined dipole (g_1^1 and h_1^1) terms is [23]

$$U = g_1^0(a^3/r^2)\cos\theta + (a^3/r^2)(g_1^1\cos\varphi + h_1^1\sin\varphi)\sin\theta, \tag{A2}$$

where $g_1^0 = -29\,496.5$ nT, $g_1^1 = -1585.9$ nT, and $h_1^1 = 4945.1$ nT [57]. Because of Earth's rotation, a nonrotating observer co-orbiting with Earth will see $\varphi = \omega t$ where ω is Earth's angular speed. Using Eq. (A1) with $B_r = -\partial U/\partial r$ and $B_\theta = -r^{-1}\partial U/\partial\theta$, we find

$$B_r = 2g_1^0(a/r)^3\cos\theta + 2(a/r)^3(g_1^1\cos\varphi + h_1^1\sin\varphi)\sin\theta, \tag{A3}$$

$$B_\theta = g_1^0(a/r)^3\sin\theta - (a/r)^3(g_1^1\cos\varphi + h_1^1\sin\varphi)\cos\theta, \tag{A4}$$

and

$$\frac{dr}{r} = 2d\theta\frac{g_1^0\cos\theta + (g_1^1\cos\varphi + h_1^1\sin\varphi)\sin\theta}{g_1^0\sin\theta - (g_1^1\cos\varphi + h_1^1\sin\varphi)\cos\theta}. \tag{A5}$$

Now since $g_1^1/g_1^0 \approx 0.05$ and $h_1^1/g_1^0 \approx 0.17$, we may roughly approximate Eq. (A5) as

$$\frac{dr}{r} \approx 2d\theta\{\cot\theta + [(g_1^1/g_1^0)\cos\varphi + (h_1^1/g_1^0)\sin\varphi]\csc^2\theta\}. \tag{A6}$$

Integrating, then exponentiating both sides, and using a Taylor expansion yields

$$r \approx r_0\sin^2\theta - r_0[(g_1^1/g_1^0)\cos\varphi + (h_1^1/g_1^0)\sin\varphi]\sin 2\theta, \tag{A7}$$

where r_0 is a constant of integration and $\varphi = \omega t$. The first term in Eq. (A7) is identical to the usual equation for the field lines of an axisymmetric dipole field [9]. The next term gives the inclined dipole and its rotation with Earth. An observer rotating with Earth at a particular φ can interpret what he or she sees as corotating field lines with a shape specific to that value of φ. An observer looking back at Earth who could see field lines would see an inclined dipole rotating with Earth.

APPENDIX B: FAILURE OF THE $\mathbf{v} = \mathbf{0}$ SOLUTION

We demonstrate that the $\mathbf{v} = \mathbf{0}$ solution $\mathbf{B_0}$ [Eq. (10)] is no longer a solution for the magnetically permeable cylindrical shell once the shell is moving with $\mathbf{v} = v\hat{\mathbf{y}}$ in K (Fig. 1). We assume $\mathbf{B_0}$ (or, equivalently, $\mathbf{A}_0$ with allowance for gauge ambiguity) remains the solution even though $\mathbf{v} \neq \mathbf{0}$ and show that this leads to a contradiction.

When $\mathbf{v} = \mathbf{0}$, we have $\mathbf{B}_\infty(\rho \gg b) = B_\infty\hat{\mathbf{x}}$ and $\mathbf{E}_\infty(\rho \gg b) = \mathbf{0}$ in K. These must continue to hold once $\mathbf{v} \neq \mathbf{0}$, since the shell's distortion of the fields must go to zero at infinity.

First assume $\mathbf{B_0}(x, y, z, t)$ to be a solution for the $a \neq 0$ cylindrical shell for $\mathbf{v} \neq 0$. Inserting Eq. (10) into Eq. (7) requires $\nabla \times (\mathbf{v} \times \mathbf{B_0}) = 0$. But we know by Eq. (21) that this is false in general. Therefore, $\mathbf{B_0}(x, y, z, t)$ cannot be a solution when $\mathbf{v} \neq \mathbf{0}$.

Rather than assuming a solution $\mathbf{B_0}(x, y, z, t)$, we instead treat the disturbance in the background field $\mathbf{B}_\infty$ as moving together with the cylindrical shell at $\mathbf{v}$. We implement this in Eqs. (9)–(11) by referring the coordinates of $\mathbf{B_0}$ to the K' system $(x', y', z', t') = (x, y - vt, z, t)$. For example, when $\mathbf{v} \neq \mathbf{0}$, Eq. (9a) is

$$B'_{0x}(\rho' > b) = B_\infty + \beta_3(b/\rho')^2\cos 2\phi', \tag{B1}$$

where $\rho' = (x^2 + y'^2)^{1/2}$, $\phi' = \tan^{-1}(y'/x)$, and, of course, $y' = y - vt$. Correspondingly, Eq. (18) becomes

$$A'_0(\rho' > b) = B_\infty y' + \beta_3(b^2/\rho')\sin\phi'. \tag{B2}$$

Henceforth, in this appendix, primed field quantities are understood to be written in terms of the coordinate y'. In the limit $\mathbf{v} \to \mathbf{0}$, Eqs. (B1), (B2), and their analogs go to Eqs. (9) to (11) and (18) to (20), as required.

In this appendix only, we make the following simplifying choice of gauge [33,41]:

$$\mathbf{A}' \to \tilde{\mathbf{A}}' = \mathbf{A}' + \nabla'\int V'dt' \tag{B3a}$$

so that

$$V' \to \tilde{V}' = V' - \frac{\partial}{\partial t'} \int V' dt' = 0. \quad \text{(B3b)}$$

I.e., the corresponding gauge condition is $V' = 0$. Because $V' = V - vA_y = V$ since only A_z is nonzero, we have $V' = V = 0$. Henceforth, dropping the tilde on $\mathbf{A}'$, we have

$$\mathbf{E}' = -\partial \mathbf{A}'/\partial t'. \quad \text{(B4)}$$

Equations (B2) and (B4) give $\mathbf{E}'(\rho' > b) = 0$, so by Eq. (5), $\mathbf{E}(\rho > b) = vB_{0x}\hat{\mathbf{z}}$, where B_{0x} is given by Eq. (9a). But by taking $\rho \to \infty$ in Eq. (9a), this means $\mathbf{E}_\infty = vB_\infty\hat{\mathbf{z}}$, which contradicts our premise that $\mathbf{E}_\infty = \mathbf{0}$ in K. Therefore, Eq. (B2) cannot be a solution for the magnetically permeable cylindrical shell once $\mathbf{v} \neq \mathbf{0}$.

But perhaps we can add a piece to A_0' that preserves $\mathbf{B}_0'$ while giving $\mathbf{E}_\infty = 0$. (This will not be a gauge transformation, as we will be explicitly attempting to alter the field quantity $\mathbf{E}$ while preserving $\mathbf{B}_\mathbf{0}'$.) We now show that satisfying these conditions together is impossible so that there is no modification of Eqs. (18)–(20) that both maintains $\mathbf{B}' = B_\mathbf{0}'$ and is consistent with the requirement that $\mathbf{B}_\infty(\rho \gg b) = B_\infty\hat{\mathbf{x}}$ and $\mathbf{E}_\infty(\rho \gg b) = 0$. Whatever term is added to Eq. (B2) cannot vary with x or y', or else $\mathbf{B}_0'$ will change, in contradiction to our premise. If we try instead to add a spatially constant term $vB_\infty t'$ to Eq. (B2) to alter $\mathbf{E}'$ and thereby $\mathbf{E}_\infty$, by continuity of $\mathbf{A}'$ at $\rho' = b$ and Eq. (B4), $\mathbf{J}' = \sigma\mathbf{E}' = -\sigma v B_\infty \hat{\mathbf{z}} \neq 0$ for $a \leq \rho' \leq b$, which means $\mathbf{B}_0'(a \leq \rho' \leq b)$ cannot be the solution, again contradicting a premise. We, therefore, show that $\mathbf{B}_0{}'(x', y', z', t') = \mathbf{B}_0{}'(x, y - vt, z, t)$ cannot be a solution when $\mathbf{v} \neq \mathbf{0}$. In effect, the "solution" $\mathbf{B}_0'(x, y - vt, z, t)$ is incompatible with the premise that $\mathbf{B}_\infty$ does not rotate together with the frame K'.

APPENDIX C: CHOICE OF GAUGE

While the gauge condition Eq. (22) is cited in the literature [43,44], it is not included in lists of standard electrodynamics gauges [68]. We, therefore, discuss it further here and show that it satisfies the requirements of gauge invariance. A gauge transformation leaves $\mathbf{B}$ and $\mathbf{E}$ unchanged provided the transformed vector and scalar potentials satisfy

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi \quad \text{(C1)}$$

and

$$\tilde{V} = V - \partial\chi/\partial t. \quad \text{(C2)}$$

Now take the divergence of Eq. (C1), multiply it by η, and add to this Eq. (C2) to obtain

$$\tilde{V} + \eta\nabla\cdot\tilde{\mathbf{A}} = V + \eta\nabla\cdot\mathbf{A} + (-\partial\chi/\partial t + \eta\nabla^2\chi). \quad \text{(C3)}$$

The gauge condition Eq. (22), therefore, holds both before and after the gauge transformation Eqs. (C1) and (C2), provided χ satisfies the diffusion equation

$$\partial\chi/\partial t = \eta\nabla^2\chi. \quad \text{(C4)}$$

APPENDIX D: CYLINDRICAL COORDINATE REPRESENTATION

Some calculations are most easily performed with $\mathbf{B_0}$ and $\mathbf{B_1}$ in cylindrical coordinates. For convenience, we give this representation here. We have

$$B_\rho = B_x \cos\phi + B_y \sin\phi, \quad \text{(D1a)}$$

$$B_\phi = -B_x \sin\phi + B_y \cos\phi, \quad \text{(D1b)}$$

so that from Eq. (10),

$$B_{0\rho}(a \leq \rho \leq b) = [\beta_1 - \beta_2(a/\rho)^2]\cos\phi \quad \text{(D2a)}$$

and

$$B_{0\phi}(a \leq \rho \leq b) = [-\beta_1 - \beta_2(a/\rho)^2]\sin\phi. \quad \text{(D2b)}$$

Using Eq. (D2) and $\mathbf{v} = v\hat{\mathbf{y}} = v\sin\phi\hat{\rho} + v\cos\phi\hat{\phi}$ yields a simpler expression for Eq. (21):

$$\nabla\times(\mathbf{v}\times\mathbf{B_0}) = 2v\beta_2 a^2\rho^{-3}[\sin 2\phi\,\hat{\rho} + \cos 2\phi\,\hat{\phi}]. \quad \text{(D3)}$$

We also have from Eqs. (60) and (62):

$$B_{1\rho}(a \leq \rho \leq b) = -(1/2)R_m b^{-1}\beta_2 a^2\rho^{-1}\sin 2\phi \quad \text{(D4a)}$$

and

$$B_{1\phi}(a \leq \rho \leq b) = 0. \quad \text{(D4b)}$$

The first term on the right-hand side of Eq. (85) may then be evaluated via Eq. (65), and the vector Laplacian in cylindrical coordinates. The calculation is tedious but straightforward.

[1] S. J. Barnett, On electromagnetic induction and relative motion, Phys. Rev. **35**, 323 (1912).

[2] S. T. Preston, On some electromagnetic experiments of Faraday and Plücker, Philos. Mag. **19**, 131(1885).

[3] M. Farady, *Faraday's Diary of Experimental Investigations 1820–1862*, edited by T. Martin, (HR Direct, Riverton, UT, 2008), Vol. I.

[4] M. Faraday, Experimental researches in electricity, Phil. Trans. R. Soc. London **122**, 125 (1832).

[5] A. I. Miller, Unipolar induction: A case study of the interaction between science and technology, Annals of Science **38**, 155 (1981).

[6] M. Faraday, Experimental researches in electricity: Twenty-eighth series, Phil. Trans. R. Soc. London **142**, 25 (1852).

[7] A. G. Kelly, Unipolar experiments, Ann. Fond. Louis Broglie **29**, 119 (2004).
[8] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Elsevier, New York, 2004).
[9] G. K. Parks, *Physics of Space Plasmas* (Perseus, Reading, MA, 1991).
[10] P. A. Davidson, *An Introduction to Magnetohydrodynamics* (Cambridge University Press, Cambridge, England, 2001).
[11] B. Auchmann, S. Kurz, and S. Russenschuck, A note on Faraday paradoxes, IEEE Trans. Magn. **50**, 7025404 (2014).
[12] C. F. Chyba, K. P. Hand, and P. J. Thomas, Energy conservation and Poynting's theorem in the homopolar generator, Am. J. Phys. **83**, 72 (2015).
[13] P. J. Scanlon, R. N. Henriksen, and J. R. Allen, Approaches to electromagnetic induction, Am. J. Phys. **37**, 698 (1969).
[14] E. H. Kennard, Unipolar induction, Philos. Mag. **23**, 937 (1912).
[15] E. H. Kennard, On unipolar induction: Another experiment and its significance as evidence for the existence of the aether, Philos. Mag. **33**, 179 (1917).
[16] I. Galili and D. Kaplan, Changing approach to teaching electromagnetism in a conceptually oriented introductory physics course, Am. J. Phys. **65**, 657 (1997).
[17] H. Poincaré, Sur l'induction unipolaire, L'Eclairage Elec. **23**, 41 (1900).
[18] A. I. Miller, *Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905-1911)* (Springer, New York, 1998).
[19] G. B. Pegram, Unipolar induction and electron theory, Phys. Rev. **10**, 591 (1917).
[20] L. Davis, Stellar electromagnetic fields, Phys. Rev. **72**, 632 (1947).
[21] H. Alfvén, On frozen-in field lines and field-line reconnection, J. Geophys. Res. **81**, 4019 (1976).
[22] P. Lorrain, F. Lorrain, and S. Houle, *Magneto-Fluid Dynamics: Fundamentals and Case Studies of Natural Phenomena* (Springer, New York, 2006).
[23] W. D. Parkinson, *Introduction to Geomagnetism* (Scottish Academic Press, Edinburgh, 1983).
[24] E. W. Hones and J. E. Bergeson, Electric field generated by a rotating magnetized sphere, J. Geophys. Res. **70**, 4951 (1965).
[25] L. Davis, D4. Stellar electromagnetic fields, Phys. Rev. **73**, 536 (1948).
[26] H. Alfvén, On the existence of electromagnetic-hydrodynamic waves, Ark. Matematik Astron. Fys. **29B**, 1 (1942).
[27] J. A. Shercliff, *A Textbook of Magnetohydrodynamics* (Pergamon, New York, 1965).
[28] P. Lorrain, J. McTavish, and F. Lorrain, Magnetic fields in moving conductors: Four simple examples, Eur. J. Phys. **19**, 451 (1998).
[29] J. Van Bladel, *Relativity and Engineering* (Springer-Verlag, Berlin, 1984).
[30] G. Backus, The external electric field of a rotating magnet, Astrophys. J. **123**, 508 (1956).
[31] D. V. Redžić, Conductors moving in magnetic fields: Approach to equilibrium, Eur. J. Phys. **25**, 623 (2004).
[32] R. J. Gutmann and J. M. Borrego, Charge density in a conducting medium revisited, IEEE Trans. Antennas Propag. **22**, 635 (1974).
[33] D. H. Kobe, Can electrostatics be described solely by a vector potential?, Am. J. Phys. **49**, 1075 (1981).
[34] M. Kinet, Ph.D. thesis, Université Libre de Bruxelles, 2009.
[35] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).
[36] J. E. Allen, A note on the magnetic Reynolds number, J. Phys. D **19**, L133 (1986).
[37] J. Emsley, *The Elements*, 3rd ed. (Clarendon, Oxford, 1998).
[38] E. C. Snelling, *Soft Ferrites: Properties and Applications*, 2nd ed. (Butterworth, Washington, DC, 1988).
[39] T. Rikitake, *Magnetic and Electromagnetic Shielding* (Kluwer, Dordrecht, 1987).
[40] J. Prat-Camps, C. Navau, D.-X. Chen, and A. Sanchez, Exact analytical demagnetizing factors for long hollow cylinders in transverse field, IEEE Magn. Lett. **3**, 0500104 (2012).
[41] P. H. Roberts, *An Introduction to Magnetohydrodynamics* (Longmans, New York, 1967).
[42] A. J. Mager, Das Eindringen von Magnetfeldern in offene Abschirmzylinder, Z. Angew. Phys. **23**, 381 (1967).
[43] T. Morisue, A comparison of the Coulomb gauge and Lorentz gauge magnetic vector potential formulations for 3D eddy current calculations, IEEE Trans. Magn. **29**, 1372 (1993).
[44] R. Moreau, *Magnetohydrodynamics* (Kluwer, Dordrecht, 1990).
[45] J. D. Jackson, From Lorenz to Coulomb and other explicit gauge transformations, Am. J. Phys. **70**, 917 (2002).
[46] A. J. Mager, Magnetic shields, IEEE Trans. Magn. **6**, 67 (1970).
[47] J. F. Hoburg, Principles of quasistatic magnetic shielding with cylindrical and spherical shields, IEEE Transactions on Electromagnetic Compatibility **37**, 574 (1995).
[48] S. Celozzi, R. Araneo, and G. Lovat, *Electromagnetic Shielding* (Wiley, New York, 2008).
[49] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, England, 2010).
[50] D. H. Menzel, *Fundamental Formulas of Physics* (Dover, New York, 1960), Vol. 1.
[51] W. A. Newcomb, Motion of magnetic lines of force, Ann. Phys. (N.Y.) **3**, 347 (1958).
[52] A. L. Wilmot-Smith, E. R. Priest, and G. Hornig, Magnetic diffusion and the motion of field lines, Geophys. Astrophys. Fluid Dyn. **99**, 177 (2005).
[53] E. Priest and T. Forbes, *Magnetic Reconnection: MHD Theory and Applications* (Cambridge University Press, Cambridge, England, 2000).
[54] J. H. Poynting, On the transfer of energy in the electromagnetic field, Phil. Trans. R. Soc. London **175**, 343 (1884).
[55] N. L. Sharma, Field versus action-at-a-distance in a static situation, Am. J. Phys. **56**, 420 (1988).
[56] D. J. Griffiths, Note on "field versus action-at-a-distance in a static situation", Am. J. Phys. **57**, 558 (1989).
[57] C. C. Finlay, S. Maus, C. D. Beggan, T. N. Bondar, and A. Chambodut *et al.*, International geomagnetic reference field: The eleventh generation, Geophys. J. Int. **183**, 1216 (2010).
[58] National Magnetics Group, MN60 Material, http://www.magneticsgroup.com/pdf/MN60%20.pdf.

[59] J. C. McLaughlin and K. L. Kaiser, "Deglorifying" the maximum power transfer theorem and factors in impedance selection, IEEE Trans. Ed. **50**, 251 (2007).

[60] Y. Zhang, N. W. Franklin, R. J. Chen, and J. Dai, Metal coating on suspended carbon nanotubes and its implication to metal-tube interaction, Chem. Phys. Lett. **331**, 35 (2000).

[61] S. M. Tripathi, T. S. Bholanath, and S Shantkriti, Synthesis and study of applications of metal coated carbon nanotubes, Int. J. Control Auto. **3**, 53 (2010).

[62] "World Energy Outlook 2014," International Energy Agency, Paris, 2014.

[63] K. Lambeck, Changes in length-of-day and atmospheric circulation, Nature (London) **286**, 104 (1980).

[64] K. Lambeck, Effects of tidal dissipation in the oceans on the Moon's orbit and the Earth's rotation, J. Geophys. Res. **80**, 2917 (1975).

[65] T. J. Birmingham and F. C. Jones, Identification of moving magnetic field lines, J. Geophys. Res. **73**, 5505 (1968).

[66] J. Slepian, Lines of force in electric and magnetic fields, Am. J. Phys. **19**, 87 (1951).

[67] C. Fälthammar, On the concept of moving magnetic field lines, EOS Trans. Am. Geophys. Union **88**, 169 (2007).

[68] J. D. Jackson and L. B. Okun, Historical roots of gauge invariance, Rev. Mod. Phys. **73**, 663 (2001).